# The explicative power of the vector potential for superconductivity: a path for high school

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Abstract

*In the classroom practice the notion of the magnetic vector potential is never introduced, both because it is not contained in secondary school textbooks and because teachers usually associate this concept with complex topics they dealt with in their university courses. In our experience instead, we have found that the introduction of the vector potential can be of great help in students' understanding of electromagnetism and modern physics topics. In this paper we will show how the use of the vector potential allows a phenomenological and consistent explanation of superconductivity at a level suitable for high school students. We will deal with the two main aspects of superconductivity: the resistivity of the superconductor that drops to zero at the critical temperature and the expulsion of the magnetic field from the bulk of a superconductor (Meissner effect). By the use of the vector potential, students can build a phenomenological interpretation of superconductivity, always remaining in the frame of electromagnetism and thus avoiding the use of too complicated mathematical tools that the explanation of the microscopic mechanism would require.*

Keywords: secondary education, magnetic vector potential, superconductivity

## Introduction

The physics education research group of the University of Milan has been dealing with superconductivity for 8 years with high school students and, despite superconductivity is a very difficult topic, in our experience students have been always interested and involved. Depending on the time that students can spend in our laboratory, we propose two types of interventions: a) an afternoon lab of about 4 hours, just to familiarize with low temperatures physics and some typical phenomena of superconductivity, or b) an educational path of at least 24 hours, including lectures and laboratory.

As superconductivity links electromagnetism, thermodynamics, waves and quantum physics, the educational path of type b), whose main theoretical part we describe in this work, is thought for the final year of a secondary school physics course ($12^{th}$ or $13^{th}$ grade). In fact, the complexity and the witchery of superconductivity can be a stimulus for students to take up topics covered till that moment, deepen them and frame them in a new light. Before entering the core of our proposal we resume the main prerequisites needed to face superconductivity.

1. Electromagnetism. In particular, the vector potential **A**, through which we can introduce the London equation, as we will see in the next sections.
2. Thermodynamics. In particular, the definition of a thermodynamic state as a function of some variables: temperature T, pressure P, volume V and also the magnetic field **B**.

3. Waves. In fact it is possible to describe the super-current in terms of (material) waves (i.e. to generalize the London equation and to get the quantization of the magnetic flux in a superconductor).

For what concerns this paper only prerequisite 1. is needed, while the others are important for a deeper approach. We have developed our educational path in the frame of electromagnetism, by the use the mathematical tools of flux and circulation that students should have already known from their study when dealing with the Maxwell's equations. In this same framework the introduction of the magnetic vector potential becomes very helpful. Although in this paper we will not treat explicitly how to introduce the vector potential to high school students, as it is described in [1,2], we will use it to develop a consistent phenomenological description of superconductivity.

## The two fluid model to explain the experimental evidences of superconductivity

The two main features of superconductivity are:

1. The resistivity ρ of the superconductor drops to zero below a particular temperature $T_C$, called critical temperature that is characteristic of the given superconductive material.
2. If a magnetic field $B_{app}$ is applied to a superconducting sample, the magnetic field in the bulk of the sample remains always zero, unless the applied field overcomes a critical intensity $B_C$. This experimental evidence that occurs with sufficiently low applied magnetic fields, is the well-known Meissner effect (discovered by Meissner and Ochsenfeld in 1933).

In order to construct a meaningful path for secondary school, we have to develop a model able to explain the previous features. The theory that inspired our work is the two-fluid theory that has been developed in 1934 by Gorter and Casimir. In this theory the superconductor is treated as containing a mixture of two fluids: the normal fluid, with the same properties of the ohmic electrical current in a metal, and the superconducting fluid, that flows without any friction (see Table 1). The two fluids are always present at the same time, therefore the superconducting sample can be thought as a circuit with two branches in parallel, the superconductive one and the normal one, as can be seen in Figure 1.

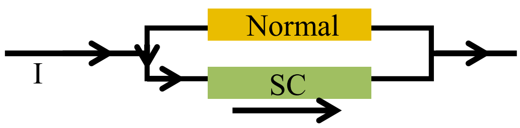


Figure1. Schematic representation of a superconductor, where SC stands for superconductor

Hence, if a constant current $I$ is flowing in the superconductor, than the presence of the superconducting branch shunts the circuit, so that all the current runs in that branch, as the arrows in Figure 1 show.

In this paper we suppose that the properties of the normal fluid have already been treated in the lessons on electrical conduction, as its behaviour is explained by the Ohm laws, that can be summarized in the local formulation $\mathbf{J}_n = \sigma\mathbf{E}$, where $\mathbf{J}_n$ is the density current and $\mathbf{E}$ is the electric field. The behaviour of the superconducting fluid is, instead, the main fact to be modelled.

### Description of the superconducting fluid

In 1935, the brothers F. and H. London proposed for the first time a phenomenological equation for the frictionless motion of the super-current [3]:

$$\mathbf{J}_s + k\mathbf{A} = 0. \quad (1)$$

In fact, taking the time derivative of eq. (1) we get:

$$\frac{\partial}{\partial t}\mathbf{J}_s = k\left(-\frac{\partial \mathbf{A}}{\partial t}\right) = k\mathbf{E}, \quad (2)$$

where **E** is the electric field inside the sample.

The Ohm equation (describing the normal fluid) states that the current density $\mathbf{J}_n$ is proportional to the electric field, so that it is the velocity of the fluid that is proportional to the force acting on it: motion in presence of friction. On the contrary, from the equations (1) and (2) we see that it is the time derivative of the current density that is proportional to the electric field, so that it is the acceleration of the superfluid that is proportional to the force acting on it and the superfluid is thus frictionless ($\rho = 0$).

It is interesting to observe that the frictionless condition described by eq.(2) is not equivalent to the London condition given by eq.(1). In fact to deduce eq.(1) from eq.(2) we have to choose the integration constant equal to zero. This is a new phenomenological assumption, made by the London brothers, that is needed to explain the Meissner effect.

It is also necessary to stress that eq.(1) is valid only if the superconductor is simply connected (it has no holes), and therefore it describes only a narrow, although very important, set of experimental situations: roughly speaking, very weak magnetic fields applied to samples without holes. In the following, we will hint the necessity of a generalization of eq. (1) while, in Table 1, we summarize what we have found so far.

Table 1. Characterization of the two fluids

| | | | | |
|---|---|---|---|---|
| *Normal fluid* | $\mathbf{v}_n \propto \mathbf{E}$ | friction | $\rho \neq 0$ | $\mathbf{J}_n - \sigma\mathbf{E} = 0$ |
| *Superconducting fluid* | $\mathbf{a}_s \propto \mathbf{E}$ | no friction | $\rho = 0$ | $\mathbf{J}_s + k\mathbf{A} = 0$ |

## The London equation and the Meissner effect

### An approach for teachers

Starting from the London equation, through the application of differential operators such as the curl, we can immediately obtain a relation that explains the expulsion of the magnetic field from the bulk of the superconductor. Since differential operators are not known to secondary school students, we address this part only to teachers.

Taking the curl of both members of eq.(1) and using the relation $\mathbf{B} = \nabla \times \mathbf{A}$, we have:

$$\nabla \times \mathbf{J}_s = -k\mathbf{B}. \quad (3)$$

If we now consider the Maxwell's equation: $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$ from eq.(3), we obtain:

$$\nabla \times \nabla \times \mathbf{B} = -\mu_0 k\mathbf{B}. \quad (4)$$

Hence, recalling that:

$$\nabla \times \nabla \times \mathbf{B} = \nabla\nabla \cdot \mathbf{B} - \nabla^2 \mathbf{B} \tag{5}$$

and taking into account the solenoidality of the magnetic field, from eqs.(4) and (5) we have:

$$\nabla^2 \mathbf{B} = \mu_0 k \mathbf{B}. \tag{6}$$

Starting from eq.(6) we can discuss the penetration of a magnetic field inside a superconductor. We do this in a mono-dimensional case of very simple geometry: the superconducting sample, in an applied magnetic field, fills a semi-space and its surface is in the *yz* plane, while the field and the super-current depend only on *x*, as Figure 2 shows. In this case the solution of eq.(6) is:

$$\mathrm{B}(x) = \mathrm{B}(0)e^{-\frac{x}{\lambda_L}}, \tag{7}$$

where $\lambda_L = \sqrt{1/\mu_0 k}$ is a phenomenological parameter, called *penetration length*, that is a measure of how much the magnetic field penetrates the superconductor before vanishing. Eq.(7) shows that the magnetic field goes exponentially to zero inside the superconductor, as it is sketched in Figure 2.

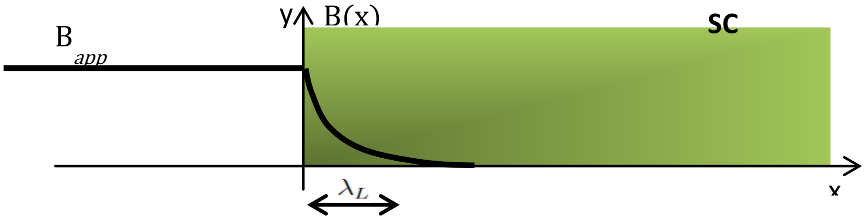


Figure 2. Intensity of the magnetic field: B is uniform outside the superconductor ($B_{app}$) while it decreases exponentially inside the superconductor

Since $\lambda_L$ is of the order of 50 nm for most superconductors, we argue that the magnetic field is present only on the surface of the superconductor itself. Being the Meissner effect a fundamental property of superconductors, in the next paragraph we propose a way to describe it, with a simpler mathematical formalism that is suitable also for high school students.

**An approach for students**

Students can have a mathematical explanation of the Meissner effect using integral operators, such as circulation and flux that they should have already used in their previous path of electromagnetism. It should be convenient to divide this explanation in two parts: (1) the first part is preparatory to the second and pertains the fact that an applied magnetic field orthogonal to the superconductor surface cannot penetrate the sample; (2) the second part pertains the expulsion from the bulk of the superconductor of a magnetic field parallel to the surface.

To follow the sequence that we propose, students have already to know some properties of the magnetic vector potential and some basic properties of the magnetic field that we resume below:

1. $C_\gamma(\mathbf{A}) = \Phi_{S,open}(\mathbf{B})$ (8)

   where $C_\gamma(\mathbf{A})$ indicates the circulation of the vector potential along a closed line γ, while $\Phi_{S,open}(\mathbf{B})$ indicates the flux of the magnetic field through the surface that has γ as a boundary. This property must have already been explained within a path on the vector potential which we cannot treat in this work [2].

2. $C_\gamma(\mathbf{B}) = \mu_0 I$ (9)

that is the Ampère-Maxwell law, where $I$ is the current passing through γ.

3. $\Phi_{S,closed}(\mathbf{B}) = 0$ (10)

that is the Maxwell's equation that states the solenoidality of the magnetic field.

(1) An orthogonal magnetic field cannot penetrate the superconductor.

We suppose that a uniform magnetic field **B** is applied orthogonally to the superconductor surface, as shown in Figure 3(a). We also suppose that the magnetic field can penetrate the superconductor. If the superconductor is homogeneous, isotropic and infinitely extended along the *yz* plane, then we can conclude that the problem has a symmetry for translation along the *y* and *z* axes. For this reason the magnetic field will be uniform over planes parallel to the *yz* plane, also inside the superconductor. If we consider a closed surface *S*, for example a parallelepiped, whose section is represented in Figure 3(a), and apply eq.(10), we can immediately deduce that the intensity of the magnetic field **B** is also invariant along *x*, and the magnetic field must therefore be uniform inside the entire superconductor.

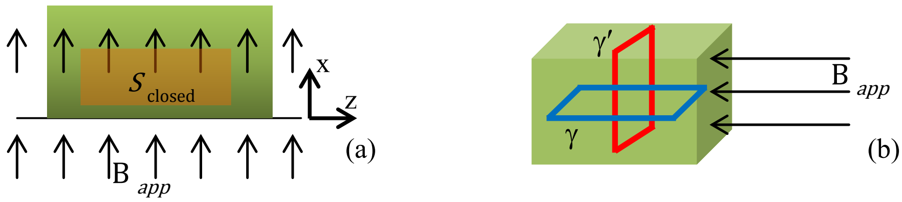


Figure 3. (a) The superconductor, in green, is infinitely extended along the x axis. It is also shown a section of the closed surface *S* and the arrows indicate the direction of the magnetic field. (b) A portion of the infinite superconductor and the two paths γ and γ′.

Now, if the magnetic field is uniform, for every closed loop $\gamma$

$$C_\gamma(\mathbf{B}) = 0, \tag{11}$$

and, therefore, if we take a little loop $\gamma$ orthogonal to *J*, from eq. (9) we get:

$$0 = \mu_0 I = JS, \tag{12}$$

where $I$ is the current passing through the line $\gamma$ along which we calculate the circulation, that is represented in Figure 3(b) and $S$ is the surface having $\gamma$ as boundary. Thus, from eq.(12) we have:

$$J = 0. \tag{13}$$

If we recall the London equation (1), we immediately get:

$$\mathbf{A} = 0. \tag{14}$$

We can now apply eq.(8), choosing a closed line $\gamma'$ parallel to the surface of the superconductor. Being $C_{\gamma'}(\mathbf{A}) = 0$, we obtain:

$$\Phi_{S'}(\mathbf{B}) = 0, \tag{15}$$

where the surface $S'$ has $\gamma'$ as a boundary. Therefore **B** = 0.

Since the surface $S'$ can be taken next to that surface as you want, eq.(15) states that no orthogonal magnetic field can penetrate the sample.

To have a picture of the behaviour of a superconductor it is necessary to consider together the London equation and the equations of the electromagnetism. This is what we did in the previous part (1); we will do the same in the next part (2). We stress this concept here because it is important that students understand that the London equation is the key point in the development of their path on superconductivity and it is important that they are able to recognize where this equation is implied.

(2) The expulsion of a parallel magnetic field from the bulk of a superconductor.

We suppose that a magnetic field is applied parallel to the superconductor surface, as in Figure 4 and that it enters the superconductors.


Figure 4. Section of the superconductor, in green. The red arrow represents the magnetic field applied parallel to the surface.

First of all, if the superconductor is isotropic the field inside will have the same direction as that applied. We want to demonstrate that the inside field cannot be uniform. Let us suppose for absurd that the field inside of the superconductor is uniform, as represented in Figure 5(a).

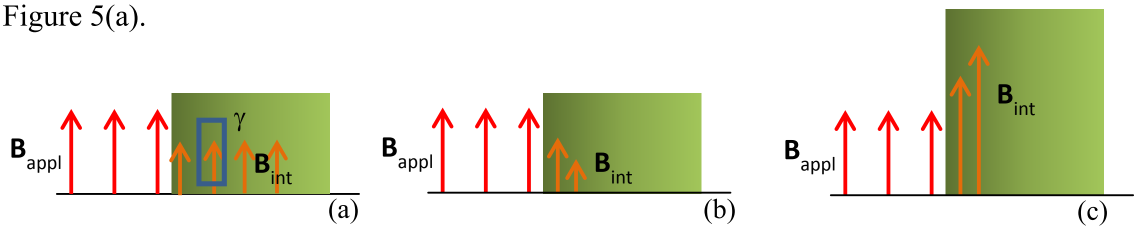


Figure 5. The magnetic field applied is represented in red, while the magnetic field that enters the superconductor is represented in yellow. The three figures show three hypothetical possibilities for the inside field.

If we apply eq.(9) to the closed narrow line γ represented in Figure 5(a) we have, for the uniformity of the magnetic field, that $C_\gamma(\mathbf{B}) = 0$ and hence:

$$\mu_0 I = 0. \tag{16}$$

Now, we rewrite the current $I$ in terms of the current density $J$ and apply the London equation (1), so to get:

$$-\mu_0 kSA = 0. \tag{17}$$

We must conclude that $A = 0$.

But $A = 0$ implies that also $B = 0$, because we can apply eq.(8). In fact, from $A = 0$ we have $C(\mathbf{A}) = 0$ and thus:

$$\Phi_{S,open}(\mathbf{B}) = 0. \tag{18}$$

Since $S$ is an open surface that can be chosen as you want, then eq.(18) implies that the magnetic field must be zero.

In other words, there can be no region inside the superconductor where the magnetic field is uniform, or tends asymptotically to a finite value different from zero. Then it is reasonable to suppose that the field increases or decreases with the distance from the surface. For obvious reasons related to the conservation of the energy density, we have to reject the case of an indefinitely increasing field, as represented in Figure 5(b). Therefore the only possibility is represented in Figure 5(c) in which the magnetic field decreases to zero. Referring to what we have just said, we can notice that the slower the spatial variation of the magnetic field, the more the intensity of the field approaches zero. Taking into account that the intensity of $\mathbf{B}$ is different from zero at the surface, it turns out that $\mathbf{B}$ must vanish quickly inside the superconductor. Therefore, the magnetic field $\mathbf{B}$ will be present only very close to the surface where there will be also the current density $\mathbf{J}$, the source of the magnetic field that the superconductor produces in opposition to the applied field. In fact, we note that the relationship $\mathbf{J} = -k\mathbf{A}$ contains the minus sign just because the current density $\mathbf{J}$ that is established on the superconductor surface, flows in such a way to generate just the magnetic field opposite to the applied field in order to cancel the field inside the superconductor.

By the use of integral tools we have seen that the magnetic field must vanish in a small distance very close to the surface, and this can be considered a meaningful result for the secondary school, although it is not so precise from the mathematical point of view. Nonetheless the concept of penetration depth $\lambda_L$ can be easily introduced to students all the same. Moreover we note that in the practice the penetration depth is evaluated by experiments because its mathematical expression is often unusable.

**A brief comment on the validity of the London equation**

As it is well-known, there are many cases in which a magnetic field can penetrate a superconductive sample without destroying superconductivity, giving the so called mixed state [4]. Therefore, the situation described above of the completely expulsion of the magnetic field from the bulk of the sample, is only a limit case that holds for weak magnetic fields and for particular geometries and dispositions of the sample in the field. Most of the lab experiences for secondary school students, that are commonly called Meissner effect, are performed in situations in which the magnetic field does indeed penetrate the sample [5,6] although very weakly. In order to explain these more complex situations, a generalization of the simple London equation (1) is needed. In this paper we have not room to discuss this point but we strongly believe that this generalization is necessary even in secondary school, otherwise the phenomena observed by students could not have a complete explanation.

## Conclusions

Educational paths on superconductivity are often based on experimental approaches which are accompained by a popular explanation of the well-known BCS theory, a theory whose mathematical framework is really beyond the possibilities of secondary school students. For this reason we propose in this work an educational path based on a model (the two fluid model) that we have developed by mean of mathematical tools suited for a secondary school and that is able to explain most of the lab experiences that students perform.

The model proposed is fully set in the framework of electromagnetism and is based on the magnetic vector potential that can be certainly accessible to secondary school students [1] and very useful to better understand electromagnetism, even if, traditionally, it is never treated before the university courses. In our experience, high school students are very involved in superconductivity and for this reason we think that it is possible to exploit this occasion to deepen many topics covered in their physics courses until that moment and also become familiar with the important tool of the vector potential [7].

In this paper, we have listed among the prerequisites for superconductivity the knowledge of the wave behaviour of matter, for which we have developed a separate educational path [8] in order to present the modern physics in the secondary school. The wave behaviour of matter becomes the basis for a deeper understanding of superconductivity in which the superconducting fluid is described by a complex wave field, as in the Ginzburg-Landau theory. Even without taking into account any other equation, the description of the superconducting fluid as a wave field allows a generalization of the London equation in a quite clear and easily understandable way. Moreover, from the same generalization, it becomes also possible the discussion of the quantization of the magnetic field in a superconductor, and the introduction of the concept of fluxoid. We have not enough place here to discuss this topic, but it is part of a more complete path on superconductivity we are working on. In fact, through this last part, students can have an explanation of the phenomenology that they observe in lab, when they perform experiments using YBCO samples in which the magnetic field is not always expelled as in the Meissner effect, but can penetrate in fluxoids throughout the sample in the mixed state.

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