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SOME REMARKS ON GEOMETRIC SIMPLE CONNECTIVITY IN DIMENSION FOUR

PART A

Abstract. The present paper contains some complements and comments to the longer article *Geometric simple connectivity in smooth four dimensional differential Topology, Part A*, by the first author. Its aim is to be a useful companion when reading that article, and also to help in understand how it fits into the first author's program for the Poincaré conjecture.

1. Introduction

This paper is a companion to the very long work [11] by the first author. Before to explain what this paper contains, it is useful to remind the reader the main results in [11].

For this purpose we will need to develop first some terminology. Let M^4 be any smooth compact bounded 4-manifold.

We consider some collar of the boundary $\partial M^3 \times [0, 1] \subset M^4$, such that $\partial M^4 \times 1 = \partial M^4$, and with this we define

$$M^4_{\text{small}} = M^4 - \partial M^3 \times (0, 1],$$

i.e. M^4_{small} is just another, diffeomorphic, copy of M^4 obtained by pushing M^4 away from its boundary, towards the interior.

A smooth compact bounded 4-manifold X^4 which possesses a smooth handlebody decomposition

$$X^4 = B^4 + \{\text{handles of index } \lambda = 2 \text{ and } \lambda = 3\},$$

will be said to be ***geometrically simply connected***. This notion immediately extends to non-compact manifolds, but then one has to insist that the handlebody decomposition be PROPER. In Morse-theoretical language this means that we can find a PROPER smooth function $V^n \xrightarrow{f} R_+$ such that

1) All the singularities of f are of the Morse type and contained inside intV^n. Moreover their indices λ are always $\lambda \neq 1$.

2) The function $f|\partial V^n$ is also Morse. Moreover for all of its ***non-fake*** critical points x_0 (i.e. those for which $f^{-1}(-\infty,\ f(x_0)-\varepsilon] \implies f^{-1}(-\infty,\ f(x_0)+\varepsilon]$ involves an actual change in topology) come with indices $\lambda \neq 1$

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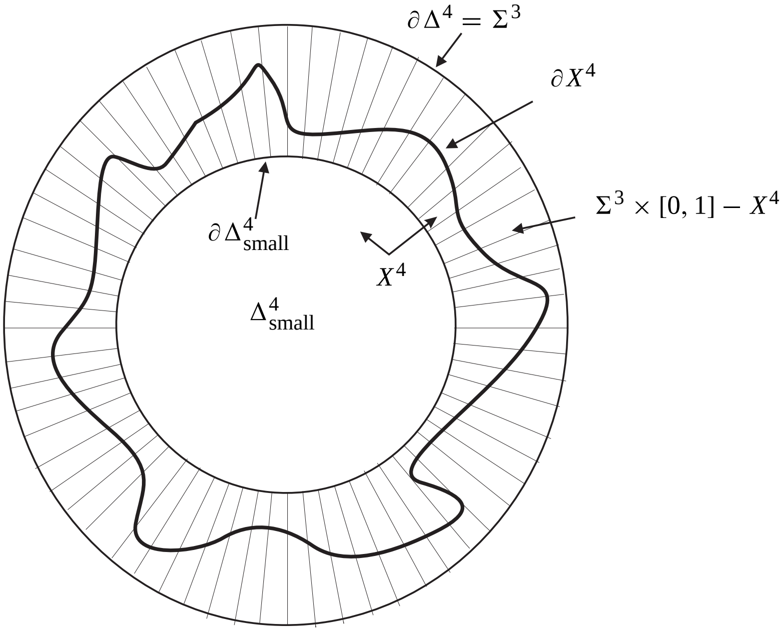


Figure 1: Here X^4 lives inside the fat contour (∂X^4) and it has no handles of index $\lambda = 1$. The collar $\Delta^4 - \Delta^4_{\text{small}}$ has an obvious product structure.

Next, let V^4 be a smooth 4-dimensional which is either open *or* compact bounded. Following a suggestion of Barry Mazur, we will say that V^4 is ***geometrically simply connected at long distance***, if any compact set $K \subset \text{int} V^4$ can be engulfed by a smooth compact codimension zero submanifold $X^4 \subset V^4$ which is geometrically simply connected. When V^4 is compact bounded, i.e. $\partial V^4 \neq \emptyset$, this means exactly that such an X^4 can be sandwiched between V^4_{small} and $V^{4\dagger}$

$$V^4_{\text{small}} \subset X^4 \subset V^4.$$

All this having been said, the main result in [11] (see also [12]) is the following

THEOREM 1. *Let Δ^4 be a smooth compact 4-manifold the boundary of which is a homology sphere Σ^3 and which is geometrically simply-connected at long distance. Then the open manifold*

$$Y^4 \overset{\text{def}}{=} \text{int}(\Delta^4 \# \infty \# (S^2 \times D^2)) \tag{1}$$

[†] There is also a weaker notion, where we only ask that K should be engulfable by a compact simply-connected subspace K_0. In dimension 3, this notion implies simple connectivity at infinity for open simply-connected manifolds, but this is part of an altogether different story, to be told elsewhere (see, for instance [10] and [19]. Revised versions of these papers are in preparation).

(there the infinite connected sum is taken along the boundaries) admits a smooth PROPER *handlebody decomposition* ***without handles of index*** $\lambda = 1$*, i.e.* Y^4 *is geometrically simply-connected.*

The proof of Theorem 1 is given in the paper [11].

Let now Δ^4 be the bounded 4-manifold which is geometrically simply-connected at long distance and which appears in Theorem 1. We denote $\Sigma^3 = \partial \Delta^4$. By hypothesis there is a geometrically simply-connected bounded 4-manifold X^4, such that

$$\Delta^4_{\text{small}} \subset X^4 \subset \Delta^4\text{small} \bigcup_{\partial \Delta^4_{\text{small}} = \Sigma^3 \times 0} (\Sigma^3 \times [0,1]).$$

We will write $X^4 = \Delta^4_{\text{small}} \cup N^4$ where our $N^4 \subset \Sigma^3 \times [0,1]$ will be, by definition

$$N^4 \overset{\text{def}}{=} X^4 \cap (\Sigma^3 \times [0,1]) \subset \Sigma^3 \times [0,1]. \tag{2}$$

Our collar $\Sigma^3 \times [0,1]$ above, comes with two obvious projection maps, namely

$$\begin{tikzcd} \Sigma^3 & \times \arrow[d, "\pi"'] & {[0,1]} \arrow[r, "\pi_0"] & {[0,1]} \\ & \Sigma^3 & & \end{tikzcd}$$

and this leads us to the restricted height function

$$\partial N^4 \xrightarrow{\pi_0 | \partial N^4} [0,1],$$

which we will assume to be Morse. We will distinguish, among the critical points of $\pi_0 | \partial N^4$ the fake ones and the ***non-fake*** ones. Here, by definition, a critical point p of $\pi_0 | \partial N^4$ is non-fake iff the passage

$$N^4 \cap \pi_0^{-1}[0, \pi_0(p) - \varepsilon] \implies N^4 \cap \pi_0^{-1}[0, \pi_0(p) + \varepsilon]$$

involves a change of topology. [The change in question consists, of course, exactly in adding a handle of index λ]. Alternatively again (and this is the good definition for our purposes), a critical point of $\pi_0 | \partial N^4$ is non-fake iff the passage of "raising the sea-level"

$$N^4 \cup \pi_0^{-1}[0, \pi_0(p) - \varepsilon] \implies N^4 \cup \pi_0^{-1}[0, \pi_0(p) + \varepsilon]$$

involves a change of topology. The change in question consists, this time, in the addition of a handle of index $\lambda + 1$.

Let us also consider the cobordism

$$\big(W^4 \overset{\text{def}}{=} \Sigma^3 \times (0,1] - \text{int} N^4;\ \partial^0 W^4 = \partial N^4,\ \partial^1 W^4 = \Sigma^3 \times \{1\} \big).$$

At first sight, it would seem that the obvious strategy for proving Theorem 1 should be the ***external*** road of coming to grips with W^4. Indeed it is very easy to see that IF

$\pi_0|\partial N^4$ has ***no non-fake minima***, then W^4, viewed as a cobordism $\partial^0 W^4 \longrightarrow \partial^1 W^4$ has no handles of index $\lambda = 1$; in turn, this would imply

$$\Delta^4 = X^4 + \{\text{handles of index } \lambda > 1\}. \tag{3}$$

This is more than what the conclusion of Theorem 1 tells us, indeed more than what we claim, hence the question mark.

Actually, the proof of Theorem 1 does not follows the external road (studying W^4) but a much more indirect and contorted ***internal*** road of COHERENT zipping and, moreover, achieving such a coherent zipping leads to INFINITE PROCESSES. It is not the place here to remind the reader what all this means, since it is already explained at length in [11], [12]. Let us just say that the $\infty\#(S^2 \times D^2)$ appearing in (1) stems from the infinite processes we just mentioned.

In fact, the proof of Theorem 1 provides us with an embedding, engulfing Δ^4_{small}

$$Y^4 = \text{int}(\Delta^4_{\text{small}}\#\infty\#(S^2 \times D^2)) \xrightarrow{j} \Delta^4 \tag{4}$$

and with a splitting by $\Sigma^3 \times 0 = \partial\Delta^4_{\text{small}} \subset Y^4$

$$Y^4 = \Delta^4_{\text{small}} \underbrace{\cup}_{\Sigma^3\times 0} (Y^4 \cap \Sigma^3 \times [0,1]).$$

The aim of the next section in this paper is to analyze the complementary region

$$\Sigma^3 \times [0,1] - Y^4. \tag{5}$$

We will actually show that (5) is ***wild***, in the sense that it is not locally simply-connected. We will also explain why this not only dooms the external road, mentioned above, but also other standard, classical algebraic topology type of thoughts which one might have, concerning the context of Theorem 1.

Section 3 explains the connection of Theorem 1 with the 3-dimensional Poincaré Conjecture. It also discusses the crucial concept of COHERENCE which plays such an important role in the first author's program for the Poincaré Conjecture.

Section 4 is essentially an updated review of zipping, which we hope to be useful not only for the readers of [11]. Zipping, of course, comes with a certain kind of 4-dimensional thickening and, in this connection, a certain specific claim is made in Lemma 2.3 of [11]. We will not review that statement now but the point is that, when we have an embedding {2-complex K^2} $\overset{f}{\hookrightarrow}$ $\Sigma^3 \times [0,1]$ such that $\pi \circ f$ is generic, we have not only the standard 4-dimensional regular neightborhood coming with f, but also the one coming with the zipping of $K^2 \xrightarrow{\pi\circ f} \Sigma^3$. Lemma 2.3 in [11] offers a comparison of these two items. For reasons of lack of space, the proof of the lemma in question has not found its place in [11], but it will be given in a sequel to the present paper, starting from the material in the present section 4.

In a similar vein as in section 4, the last section 5 picks up some zipping issues which have been left dangling in section VII of [11], and provides proof for them.

The first author is very grateful to David Gabai for help and encouragement. He also wants to thank Michael Freedman, Bary Mazur and Frank Quinn, the conversations with which have triggered many of the ideas in this paper.

2. On the untouched part of the collar $\Sigma^3 \times [0,1] - Y^4$

The point which we will try to make in the present section is that the infinitistic approach which takes, nevertheless, place inside a compact ambient space makes that the region complementary to our construction is ***wild***, leading to phenomena which, from the viewpoint of finite constructions leaving out compact 4-dimensional smooth cobordism, are unusual. We will start with the following trivial remark.

Consider the obvious splitting $X^4 = \Delta^4_{\text{small}} \cup N^4$ and assume that, starting with $(\Sigma \times I, N^4 \overset{\text{def}}{=} X^4 \cap (\Sigma^3 \times I))$, (see (2)) we would manage to construct some bounded manifold P^4 with $N^4 \subset P^4 + \{\text{handles of index } \lambda > 1\} \subset \Sigma^3 \times I$, such that the relative $\pi_1(\Sigma^3 \times I - \overset{\circ}{P}{}^4, \partial P^4) = 0$. This would be certainly the case, for instance, if by adding handles of index $\lambda > 1$ to N^4, in an embedded way, we could get to the ***standard*** pair $[\Sigma^3 \times I, \Sigma^3 \times [0,\varepsilon] \# k \# (S^2 \times D^2)]$, (with finite k and with the $\#(S^2 \times D^2)$'s being added along $\Sigma^3 \times \varepsilon$).

If all this would be the case, it would clearly imply that we had already, in the beginning,

$$\pi_1(\Sigma^3 \times (0,1] - \overset{\circ}{N}{}^4, \partial N^4) = 0.$$

But we certainly do ***not*** assume such a thing, in the context of Theorem 1.

Now, as already said, the proof of our theorem provides us, actually, with an embedding of the open manifold $Y^4 \overset{\text{def}}{=} \text{int}(\Delta^4_{\text{small}} \# \infty \# (S^2 \times D^2))$, engulfing Δ^4_{small} see (4),

$$Y^4 \xrightarrow{j} \text{int}\Delta^4. \tag{6}$$

But in view of our use and of {holes} (see [11]) and of the fact that our Morse theory from [11] takes us very far out of the collar, the proof in question, certainly does ***not*** provide us with a diffeomorphism taking the form

$$\begin{aligned} Y^4 - \Delta^4_{\text{small}} = M^4 + \{&\text{an infinite PROPER collection of} \\ &\text{handles of index } \lambda > 1, \text{ killing all the boundary}\}, \end{aligned} \tag{7}$$

where $M^4 \overset{\text{def}}{=} \{\text{regular neighbourhood in } \Sigma^3 \times I \text{ of } (\Sigma^3 \times 0) \cup \text{ (some 2-skeleton of } N^4)\} - (\Sigma^3 \times 0)$. It is a distinctive feature of [11], in contrast with the first author's earlier, abortive, (and unpublished) attempts on the same lines, that the whole geometrically simply-connected blub X^4 is used in the (infinite) handle-cancelling arguments, and not just the interaction of the blub in question with the collar. It is Lemma 2.5 in [11] with its subsequent collection of Whitney disks $D^2(i_1 \cdots i_{p+1})$ which is global and not just localized inside the collar. This forces, in turn, our whole "non compact

Morse theory" from [11] whether enhanced or not to be global too. In the present section, which is motivated by some questions raised by Frank Quinn concerning the $\pi_1(\Sigma^3 \times (0,1] - \overset{\circ}{N}{}^4, \partial N^4)$ issue, we will take, for the sake of the argument, a diffeomorphism (7) as a ***hypothesis*** and try to answer the question whether this would lead to some major contradiction.

We will make, also, the following assumptions

(a) The restriction of j (see (6)) to M^4 is the standard embedding $M^4 \subset \Sigma^3 \times I$.

(b) Let us start by considering the non-compact smooth 4-manifold with ("large") non-empty connected boundary

$$Z^4 \overset{\text{def}}{=} (\Sigma^3 \times (0,\varepsilon])\#(S_1^2 \times D_1^2)\#(S_2^2 \times D_2^2)\# \cdots ,$$

where $(S_1^2 \times D_1^2)$ is glued to $(\Sigma^3 \times (0,\varepsilon]) \supset \Sigma^3 \times \varepsilon$ along the 3-ball $d_1^3 \subset \Sigma^3 \times \varepsilon$, and where each $(S_{n+1}^2 \times D_{n+1}^2)$ is glued to $\partial(S_n^2 \times D_n^2) - d_n^3$ along some 3-ball d_{n+1}^3. [Remember here that in constructing a manifold like Z^4 one has to be precise about the infinite tree giving the recipe for attaching the successive $\#(S^2 \times D^2)$'s; the end-structure of the tree in question is visible in the diffeomorphism type of the resulting manifold with boundary. In our specific case, the tree on which Z^4 is modelled is R_+ (without any branches). Similarly

$$\partial Z^4 = (\Sigma^3 \times \varepsilon)\#(S_1^2 \times \partial D_1^2)\#(S_2^2 \times \partial D_2^2)\# \cdots ,$$

(infinite connected sum of closed manifolds). It should be noticed, finally, that when in a formula like Z^4 one takes the interiors, then the difference between the various possible infinite trees used for adding the $\#(S^2 \times D^2)$ to $\Sigma^3 \times (0,\varepsilon]$ gets washed out, and one gets a unique, canonical open manifold.]

With all this, we have a diffeomorphism

$$Y^4 \cap \Sigma^3 \times (0,1] = Y^4 - \Delta^4_{\text{small}} = \text{int}Z^4 \tag{8}$$

This (8) is provided by [11].

This ends the discussion of our features a) and b).

(c) But, in principle at least, it is not reasonable to assume that the proof of Theorem 1 might allow us to be more specific about the topology coming along with the embedding (with M^4 like in (7))

$$\begin{tikzcd}
Y^4 - \Delta^4_{\text{small}} = \text{int}Z^4 \arrow[rr, "j|(Y^4-\Delta^4_{\text{small}})"] & & \Sigma^3 \times (0,1] \\
& M^4. \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

What we could ***hope*** for, in the best of the worlds, would be for the following two items d) and e) also to happen and, so, we will assume this now too.

(d) The $j|(Y^4 - \Delta^4\text{small})$ extends to the ***standard embedding***

$$Z^4 \overset{\mathcal{J}}{\longrightarrow} \Sigma^3 \times (0,1) \subset \Sigma^3 \times (0,1], \tag{9}$$

with $\lim_{n=\infty} \mathcal{J}(S_n^2 \times D_n^2) = p_\infty \in \Sigma^3 \times (0,1) - \text{Im}\mathcal{J}$.

(c) (Second hope.) We have a PROPER smooth handlebody decomposition (analogous to (6) but now with boundaries being present)

$$Z^4 = M^4 + \{\text{an infinite PROPER collection of handles of index } \lambda > 1\}. \tag{10}$$

REMARK 1. Even in the best of all worlds, we still cannot have everything. The embedding $\mathcal{J}$ certainly cannot be PROPER, (we have made here just the simplest possible assumption, namely $\overline{\mathcal{J}Z^4} = \mathcal{J}Z^4 \cup \{p_\infty\}$), and the set $\Sigma^3 \times (0,1] - \mathcal{J}Z^4$ is neither open nor closed, in $\Sigma^3 \times (0,1]$. Even the closed set, complementary to our open Y^4 i.e. our $\Sigma^3 \times [0,1] - Y^4$ which, for the sake of the argument we write now as

$$Y_C^4 \overset{\text{def}}{=} \Sigma^3 \times (0,1] - j(\text{int}Z^4) \supset \mathcal{J}(\partial Z^4) \cup \{p_\infty\} \overset{\text{def}}{=} \mathcal{J}(\partial Z^4)^\wedge,$$

is ***not*** a manifold, actually it is not locally simply-connected at $p_\infty \in Y_C^3$. Moreover, even if we have (10) there is still no analogous property which we could prove, nor even reasonably state, concerning the "one point compactification" $\mathcal{J}Z^4 \cup \{p_\infty\}$.

So let us ***assume*** now (7), a),b),d),e), in addition to (6) from our theorem. If we would also know that $\pi_1(Y_C^4, \mathcal{J}(\partial Z^4)) = 0$, then our little argument with which we have started the whole discussion would imply that, to begin with, we had $\pi_1(\Sigma^3 \times (0,1] - \text{int}M^4, \partial M^4) = 0$, and hence we would have "proved" that $\pi_1(\Sigma^3 \times (0,1] - \text{int}N^4, \partial N^4) = \pi_1(\Delta^4 - \text{int}X^4, \partial X^4) = 0$ which, as already said, would signal that something was wrong, indeed. But we have the following fact, which is also the main result of the following

PROPOSITION 1. *Under our various assumptions, we have*

$$\pi_1(Y_C^4, \mathcal{J}(\partial Z^4)) \neq 0. \tag{11}$$

Proof. We will consider an infinite sequence of concentric 4-balls, all centered at p_∞, with the radii going to zero: $\Sigma^3 \times (0,1) \supset B_1^4 \supset B_2^4 \supset \cdots$ each $B_n^4 \subset \text{int}B_{n+1}^4$, $\cap_n B_n^4 = \{p_\infty\}$. It is assumed that

$$(B_n^4, S_n^3) \cap \mathcal{J}Z^4 = ((S_{n+1}^2 \times D_{n+1}^2)\#(S_{n+2}^2 \times D_{n+2}^2)\# \cdots d_{n+1}^3).$$

In the closed set

$$\mathcal{J}(\partial Z^4)^\wedge = \mathcal{J}\{(\Sigma^3 \times \varepsilon)\#(S_1^2 \times \partial D_1^2)\#(S_2^2 \times \partial D_2^2)\# \cdots\} \cup \{p_\infty\} \subset Y_C^4,$$

we consider, for each n, a closed loop based at p_∞

$$(S^1, *) \xrightarrow{\lambda_n} (\mathcal{J}(\partial Z^4)^\wedge) \cap B^4_{n-1}, p_\infty),$$

obtained by bringing the free loop $x_n \times \partial D^2_n (x_n \in S^2_n)$ to the base point p_∞ via an arc (see Figure 2). Inside $\mathcal{J}(\partial Z^4)^\wedge \subset Y^4_C$, we have $\lim\limits_{n=\infty} \lambda_n = p_\infty$, and so it makes sense to join all the λ_n's into a unique ***continuous*** loop based at p_∞ (see again Figure 2)

$$(S^1, *) \xrightarrow{\Lambda} (\mathcal{J}(\partial Z^4)^\wedge), p_\infty), \tag{12}$$

given by the following, obvious, completed infinite composition of loops $\Lambda = (\lambda_1 \cdot \lambda_2 \cdot \cdots) \cup \{p_\infty\}$. [The Λ we have just defined goes infinitely many times through p_∞, but for the game which we will play now, we could also use a continuous loop, slightly more sophisticated than Λ, going only once through p_∞.] Our claim (11) is certainly proved once we will have managed to show that ***there is no free homotopy, inside*** Y^4_C ***bringing*** Λ ***completely inside*** $\mathcal{J}(\partial Z^4)$.

So let us assume that such a free homotopy exists, and we denote by ξ the corresponding, hypothetical, closed loop of $\mathcal{J}(\partial Z^4)$. At this point, the heuristic idea is that the homology class $[\xi]$ cannot exist, since any smooth 1-cycle representing it would have infinite length; so Λ has to stay hooked at p_∞.

In order to formalize this little heuristic argument, we consider the following commutative diagram and its effect on the singular homology $H_1(\cdots, Z)$ (actually any kind of coefficients would also do).

$$\begin{tikzcd}
 & Y^4_C \arrow[dl] \arrow[d] \arrow[drr] & & & & \\
Y^4_C/B^4_1 & Y^4_C/B^4_2 \arrow[l] & \cdots \arrow[l] & Y^4_C/B^4_n \arrow[l] & \cdots \arrow[l] & \\
\partial Z^4/B^4_1 \arrow[u, hook] & \partial Z^4/B^4_2 \arrow[l] \arrow[u, hook] & \cdots \arrow[l] & \partial Z^4/B^4_n \arrow[l] \arrow[u, hook] & \cdots \arrow[l] & \\
 & \partial Z^4 \arrow[ul] \arrow[u] \arrow[urr] \arrow[uuu, bend right=90, "\mathcal{J}"'] & & & &
\end{tikzcd}$$

Here Y^4_C/B^4_n (respectively $\partial Z^4/B^4_n$) really means the quotient-space

$$\overline{\mathcal{J}} Y^4_C / (\overline{\mathcal{J}} Y^4_C \cap B^4_n)$$

where $\overline{\mathcal{J}}$ is the natural embedding $Y^4_C \subset \Sigma^3 \times I$ (respectively $\mathcal{J}(\partial Z^4)/(\mathcal{J}(\partial Z^4) \cap B^4_n)$. Notice that $H_1(\partial Z^4/B^3_n) = H_1(\Sigma^3) + \sum_1^n H_1(S^2_1 \times \partial D^2_i)$, and that all the (vertical) inclusion maps

$$H_1(\partial Z^4/B^4_n) \overset{\approx}{\to} H_1(Y^4_C/B^4_n)$$

are isomorphisms (actually, since (9) is standard, even at the π_1-level, the corresponding maps are isomorphisms). So, using any of the two infinite horizontal lines of the diagram above we can define the pro-group

$$\mathcal{H}_1 \overset{\text{def}}{=} \underset{\overleftarrow{n=\infty}}{\lim} H_1(Y^4_C/B^4_n) = \underset{\overleftarrow{n=\infty}}{\lim} H_1(\partial Z^4/B^4_n). \tag{13}$$

[We will keep here the algebraic topology at its most elementary level, without bothering to consider $\mathcal{H}_1$ (or the $\mathcal{H}^*_1$ below) in any of their, other, homological or cohomological possible interpretations.]

With this, we get a commutative diagram

$$\begin{tikzcd}
 & & H_1 Y^4_C \arrow[lld, "\alpha_n"'] \arrow[ld, "\alpha_\infty"] & \ni & {[\Lambda] = \gamma\,[\xi]} \\
H_1(\partial Z^4/B^4_n) & \mathcal{H}_1 \arrow[l, two heads] & & & \\
 & & H_1(\partial Z^4) \arrow[llu, "\beta_n"] \arrow[lu, "\beta_\infty"'] \arrow[uu, "\gamma"] & \ni & {[\xi],} \arrow[uu, mapsto]
\end{tikzcd}$$

where $\alpha_n[\Lambda] = \sum_1^n [x_n \times \partial D^2_n]$. This completely determines $\alpha_\infty[\Lambda]$, for which we will use, accordingly, the notation

$$\alpha_\infty[\Lambda] = \sum_1^\infty [x_n \times \partial D^2_n]. \tag{14}$$

Starting with the same diagram at page 320, but using this time singular cohomology, we define

$$\mathcal{H}^*_1 \overset{\text{def}}{=} \underset{\overrightarrow{n=\infty}}{\lim} H^1(\partial Z^4/B^4_n) \xrightarrow{\beta^*_\infty} H^1(\partial Z^4).$$

Inside $\mathcal{H}^*_1$ we consider the class

$$\eta = \sum_1^\infty [S^2_n \times y_n], y_n \in \partial D^2_n \tag{15}$$

where the orientations are chosen in such a way that for every fixed n and for the canonical coupling

$$H^1(\partial Z^4/B^4_{n+1}) \times H_1(\partial Z^4/B^4_{n+1}) \xrightarrow{<,>} Z$$

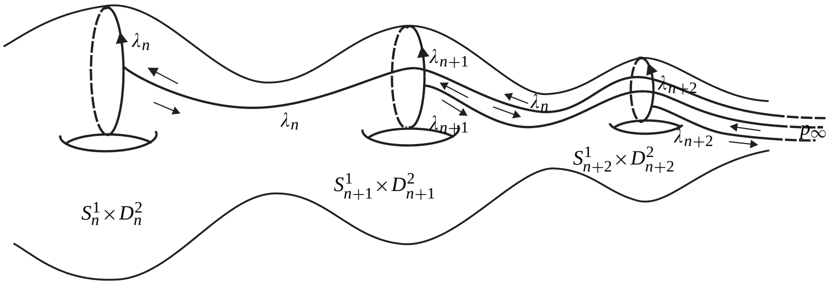


Figure 2: We can visualize $\Lambda = (\lambda_1 \cdot \lambda_2 \cdots) \cup \{p_\infty\}$ in a 3-dimensional model. We start by considering the standard embedding $(S^1_1 \times D^2_1)\#(S^1_2 \times D^2_2)\# \cdots \overset{K}{\subset} S^3$, with $\lim\limits_{n\to\infty} K(S^1_n \times D^2_n) = p_\infty \in S^3$. With this, the local picture of $(\Sigma^3 \times (0,1], \mathcal{J}Z^4)_{p_\infty}$ is the suspension of the local picture of $(S^3, \mathrm{Im}K)_{p_\infty}$. We can then, as well, construct our Λ in $(\partial\, Im\, K)^\wedge = ((S^1_1 \times \partial D^1_1)\#(S^1_2 \times \partial D^2_2)\# \cdots) \cup \{p_\infty\}$.

we get, for all $i, j \leq n$, and for the classes used in (14), (15)

$$< [S^2_i \times y_i], [x_j \times \partial D^2_j] >= \delta_{ij}.$$

Now, for any homology class $[\xi] \in H_1(\partial Z^4)$ the sequence of values $< \eta, \alpha_n[\xi] >\in Z$ stabilizes for high n's, so that

$$< \beta^*_\infty \eta, [\xi] >= \lim_{n=\infty} < \eta, \alpha_n[\Lambda] > . \tag{16}$$

But, with all our choices, the right hand side in (16) blows up to infinity; this means that $[\xi]$ cannot exist. □

This finishes the proof of our claim (11), and so we cannot "deduce" automatically that $\pi_1(\Delta^4 - \mathrm{int}X^4, \partial X^4) = 0$ just from (5) together with be other assumptions, (7), a), b), d), e) which we have so liberally made. This is in sharp contrast with what would have happened in a compact context. One should also notice the following fact. As long as we insist for our continuous loop (12) to live inside $Y^4_C \supset \mathcal{J}(\partial Z^4)^\wedge$ it cannot be but very singular. But as soon as we allow it to move inside the larger $\Sigma^3 \times (0,1] - \mathrm{int}M^4$, we can change it, via homotopy, into a very smooth loop Λ_1. What our argument shows, is that apriori at least, there is no way to guarantee the existence of a free homotopy inside $\Sigma^3 \times (0,1] - \mathrm{int}M^4$ which would bring this Λ_1 into ∂M^4, just on formal grounds.

Concerning the issue of staying localized inside the collar, we have mentioned already that Frank Quinn had pointed out in connection with an earlier, wrong start for what eventually became the [11], that ***if*** our arguments, which at that time took completely place inside the collar, would have also "proved" something like $\pi_1(\Sigma^3 \times I - \text{int}X^4, \partial X^4) = 0$ (which is certainly ***not*** an assumption in our Theorem 1), then something would have been be quite wrong, indeed. At a different moment, Michael Freedman also pointed out that if in any way our arguments would "prove" that a contractible compact 4-manifold Δ^4 with non-simply-connected boundary has no handles of index one then we would again be in deep trouble. (Remember that Andrew Casson has shown that if $\pi_1 \partial \Delta^4$ has a nontrivial representation into a compact Lie group, like it is for instance the case for the manifolds of Barry Mazur [3] and Po [5], then one cannot kill the 1-handles of Δ^4 (see for instance [4], [2]). Both of these cautionary remarks have been extremely useful in helping the first author to stay on the right track, while writing [11].

3. Connection with the 3-dimensional Poincaré Conjecture

The aim of the present section is to explain how the Theorem 1 from the introduction (i.e. the main result in [11] fits into the first author's Program for the Poincaré Conjecture, see also [1], [6] [7], [8],[14].)

We start with some background material from [18].

By definition, ***a sort of links*** is a smooth noncompact 4-manifold V^4 with non empty boundary, which is such that

$$\text{int}V^4 = R^4_{\text{standard}}, \ \partial V^4 = \sum_1^{\alpha} S^1_i \times \overset{\circ}{D}{}^2_i \ \text{(with } 1 \leq \alpha \leq \infty).$$

It will be always assumed that the parametrization $S^1 \times \text{int}D^2$ of the boundary corresponds to the null-framing. A sort of link V^4 will be said to be smoothly tame if there is a smooth embedding $V^4 \subset B^4_{\text{standard}}$ inducing a diffeomorphism between the interiors and sending boundary to boundary, i.e. if V^4 can be smoothly compactified to the standard 4-ball. Any Casson handle is a sort of link with $\alpha = 1$ and those C.H. which are not smoothly standard are also ***smoothly wild***. In [18], one proves the following result

THEOREM 2 (Smooth tameness Theorem). *For any homotopy 3-sphere Σ^3, we can find a sort of link V^4 with infinitely many boundary components $\partial V^4 = \sum_1^\infty S^1_i \times$ intD^2_i such that*

A *We have a diffeomorphism between the following open 4-manifolds*

$$(17) \quad int\{(\Delta^3 \times I)\#(\infty\#(S^2 \times \overset{\circ}{D}{}^2_i))\} =$$
$$= V^4 + \{\text{all the 2-handle } D^2_i \times \overset{\circ}{D}{}^2_i \ \text{defined by } \partial V^4\}.$$

B *For any* $N < \infty$, *the truncation*

$$V^4|N = V^4 - \sum_{N+1}^{\infty} S_i^1 \times \overset{\circ}{D}{}_i^2$$

is smoothly tame.

Here is a first corollary of our theorem above.

COROLLARY 1. *The left-hand side of* (17) *is geometrically simply connected at long distance.*

Proof. Let $(B^4, \sum_1^N S_i^1 \times D_i^2)$ be the smooth compactification of $V^4|N$. For this pair, we have the canonically attached compact 4-manifold with boundary

$$W^4(N) \overset{\text{def}}{=} B^4 + \{\text{the 2-handles } \sum_1^N D_i^2 \times D_i^2\}.$$

Any compact K contained inside the right- hand side of (17) is also contained inside some open subset

$$X^4(N) \overset{\text{def}}{=} V^4|N + \{\text{the 2-handles } D_i^2 \times \overset{\circ}{D}{}_i^2 \text{ defined by } \partial(V^4|N)\},$$

and inside $X^4(N)$, our K can be engulfed by an embedded $W^4(N) \subset X^4(N)$. □

We also have a corollary of this corollary, namely

COROLLARY 2. $\Delta^3 \times I$ *itself is geometrically simply connected at long distance.*

Proof. Let us denote by Z^4 the left-hand side of (17). Our $\Delta^3 \times I$ is a compact subset of Z^4 and hence, itself it can be engulfed inside another compact submanifold which is geometrically simply connected

$$\Delta^3 \times I \subset Q^4 \subset Z^4.$$

By compacteness, Q^4 can be engulfed inside another compact manifold $Q^4 \subset P^4 \subset Z^4$ such that

$$P^4 \overset{\text{DIFF}}{=\!=} (\Delta^3 \times I)\#(S^2 \times D^2)\# \cdots \#(S^2 \times D^2),$$

and that the inclusion $\Delta^3 \times I$ is the standard map

$$(\Delta^3 \times I)_{\text{small}} \subset (\Delta^3 \times I)\#(S^2 \times D^2)\# \cdots \#(S^2 \times D^2). \tag{18}$$

The result follows by capping off the $S^2 \times D^2$ factors from the right hand side of (18) with obvious 3-handles so as to change (18) into the standard inclusion $(\Delta^3 \times I)_{\text{small}} \subset (\Delta^3 \times I)$. □

It follows from Corollary 2 that, if we take $\Delta^4 = \Delta^3 \times I$, then all the hypothesis of our Theorem 1 are satisfied. So, we can plug Theorem 1 in (and this is how it fits into the Poincaré Program) and deduce the following

COROLLARY 3. *The open 4-manifold*

$$int((\Delta^3 \times I)\#\infty\#(S^2 \times D^2))$$

is geometrically simply-connected.

This ends the first step of the Poincaré Program, the proof being completely covered by the papers [6], [7], [8], [9], [11]. At the time when these lines are being written, the next step is by now completely worked out too. This is the following result, contained in the long (and not yet typed) manuscript [13].

THEOREM 3. *For any homotopy* 3*-ball* Δ^3 *which is such that the open smooth manifold open* 4*-manifold*

$$int((\Delta^3 \times I)\#\infty\#(S^2 \times D^2))$$

is geometrically simply connected, the smooth compact bounded manifold $\Delta^3 \times I$ *is also geometrically simply connected.*

The technology of [13] has important consequences for the smooth 4-dimensional Schoenflies problem. This is object of joint work of the first author with David Gabai; we will not give more details here.

The third and last step of the Program, is the following theorem, proved in [13]

THEOREM 4. *Let* Δ^3 *be a homotopy 3-ball such that* $\Delta^3 \times I$ *is geometrically simply-connected, then* $\Delta^3 = B^3$.

In all the three steps above, the notion of COHERENCE (described in [1], [6], [7]) is crucial. This notion is, among other things, strongly connected with the 4-dimensional geometric simple connectivity, another key concept for the whole approach. The connection is made transparent by the following result, which is one of the first lemmas in the proof of Theorem 4.

LEMMA 1. *Let* Δ^3 *be a homotopy 3-ball. If* $\Delta^3 \times I$ *is geometrically simply-connected, then for the associated homotopy 3-sphere* Σ^3*, there i a* ***collapsible*** *pseudo-spine representation* $K^2 \xrightarrow{f} \Sigma^3$*, a* ***desingularization*** φ *of* K^2 *and a* ***strategy*** *for exhausting the double points of* f *by a sequence of* $O(i)$ *moves, such that all the* $O(3)$ *moves are* COHERENT. *Conversely, if for some* Σ^3 *there exists such a* COHERENT *strategy, then the* $\Delta^3 \times I$ *is geometrically simply-connected.*

The various terms used in this statement are explained in [1], [6], [7]. The proof, which is assumed in [14], will be given somewhere else.

The notion of COHERENCE entails some subtleties which we will try to explain in the rest of this section. Let us also say that what comes next was triggered, in part, by various questions which Barry Mazur and Ofer Gabber have asked the first author. We wish to thank then here too.

We will start with some abstract nonsense. Let $X \xrightarrow{g} Y$ be some map; for our present purposes X will be a simplicial complex, Y a smooth manifold of dimension higher than X and g a piecewise smooth generic immersion. We define the n-multiple point set $M_n(g) \subset X$ as consisting of those $x \in X$ such that $\text{card}(g^{-1}gx) \geq n$. We also consider $M^2(g) \subset X \times X$ which is, by definition, the set of pairs (x_1, x_2) with $x_1 \neq x_2$ and $gx_1 = gx_2$. The projection on the first factor induces a map $M^2(g) \xrightarrow{p_0} M_2(g)$; if X, Y are smooth manifolds with $\dim X < \dim Y$ and g a generic smooth immersion, then p_0 is exactly the ***desingularization*** of the branched set $M_2(g) \subset X$. There is also an obvious $(Z/2Z)$-principal fibration

$$M^2(g) \to M^2(g)/(Z/2Z) \subset \underbrace{X \times X/(Z/2Z)}_{\text{the symmetrized } X\times X} \tag{19}$$

and we will introduce the following "quantity" $\mu_2(g)$ where, by definition

$$\mu_2(g) = 0 \Leftrightarrow \text{ the fibration (19) is a product}$$

respectively

$$\mu_2(g) > 0 \Leftrightarrow \text{ the fibration (19) is not a product.}$$

If $M_2(g)/(Z/2Z)$ has, let us say c connected components, then $\mu_2(g) > 0$ means that at least over one of them (19) is a 2-sheeted covering. Obviously, $\mu_2(g)$ is the first obstruction for lifting g to an embedding into $Y \times R$. Now, if $\mu_2(g) = 0$, we have 2^c possible order relations on the fibers $p^{-1}(y)$ $(y \in M^2(g)/(Z/2Z))$, varying continuously with y. Once we have granted that $\mu_2(g) = 0$ the necessary and sufficient condition for g to be *liftable* to an embedding into $Y \times R$, is that among these 2^c possible orders there is at least one which has no mismatch at the level of the triple points (x_1, x_2, x_3) (see [10], [14] for more details, in particular, in these papers, another "cohomological" invariant, similar to μ_2, but involving this time the triple points is introduced, call it ν_3. The necessary and sufficient condition for liftability turns out to be $\mu_2 = \nu_3 = 0$). All this is somewhat similar to orientability and spin structure.

We consider now a singular 2-dimensional polyhedron $K^2 \xrightarrow{f} M^3$, which is such that f can be exhausted by O(i) moves. We will again introduce a "quantity" $\bar{\mu}_2(f)$ defined by

$$\bar{\mu}_2(f) = 0 \Leftrightarrow \left\{ \begin{array}{l} \text{there exists both a desingularization } \varphi \text{ for } K^2 \textit{ and } \text{a} \\ \text{strategy for exhausting the double points of } f \text{ by } O(i) \\ \text{moves, such that all the } O(3) \text{ moves are COHERENT} \end{array} \right\}$$

respectively

$$\bar{\mu}_2(f) > 0 \Leftrightarrow \{\text{the condition above for } \bar{\mu}_2(f) = 0 \text{ is violated}\}.$$

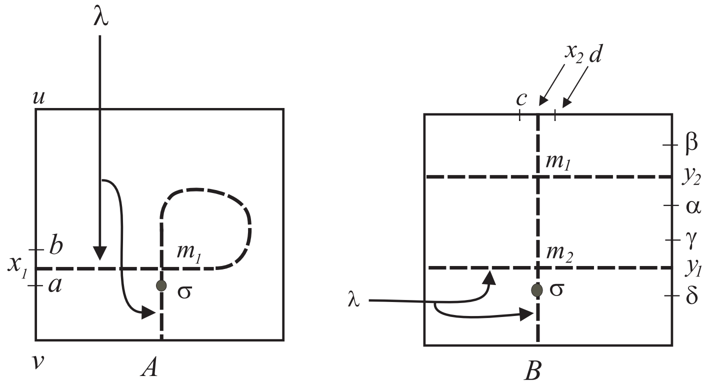


Figure 3: At the source. The dotted line is $M_2(f_0) \subset A \cup_\lambda B$.

Consider the generic immersion

$$K^2 - \mathrm{Sing}(f) \xrightarrow{f = f|(K^2 - \mathrm{Sing}(f))} M^3. \tag{20}$$

It is very easy to see that $\bar{\mu}_2(f) = 0$ implies that (20) is liftable. So $\bar{\mu}_2(f) = 0$ implies $\mu_2(f) = 0$, which we might as well write $\mu_2(f) \leq \overline{\mu}_2(f)$. What we want to show next is that, generally speaking, this inequality is ***strict***.

More explicitly, the point which we want to make now is that (20) *can be liftable without, at the same, time* having $\overline{\mu}_2(f) = 0$.

We will explicitly construct a 2-dimensional singular polyhedron $P^2 \xrightarrow{f} R^3$ as follows. We start by considering the two pieces A, B from Figure 3, which we glue along the dotted lines denoted by λ so as to get a singular space $P_0^2 = A \bigcup_\lambda B$. We will define a map $P_0^2 \xrightarrow{f_0} R^2$ with a unique singularity at σ and which is such that $f_0|A$ is an embedding and $f_0|B$ an immersion without triple points. At the source (see Figure 3), the double points of f_0 are dotted and (m_1, m_2, m_3) corresponds to a triple point. Also $f_0 x_1 = f_0 x_2$, $f_0 y_1 = f_0 y_2$. So the graphical convention is that points at the source which differ only by their subscript (like the m_i's) have the same image, at the target. Figure 3 presents the full image of $f_0 B \subset R^3$, and a small piece of $f_0 A \subset R^3$ can be seen too. All this should make $P_0^2 \xrightarrow{f_0} R^3$ completely explicit. We can glue next, staying always inside R^3, a small neighborhood of x' in $\partial f_0 P_0^2$ to a small neighborhood of y' in $\partial f_0 P_0^2$. This gives a quotient space of P_0^2, generically denoted by P^2 which comes equipped with a map $P^2 \xrightarrow{f} R^3$ induced from f_0. The singular 2-dimensional polyhedron $P^2 \xrightarrow{f} R^3$ has a unique singularity σ, hence a unique desingularization φ, up to a global change $S \longleftrightarrow N$, and a unique strategy involving a unique O(3) movement. After a possible 90° twist in our gluing, this

O(3) can be made NON-COHERENT. Explicitly, we will glue (see Figure 3) (a', x', b') to (a', y', β') and (c', x', d') to (γ', y', δ'). This fixes completely the definition of $P^2 \xrightarrow{f} R^3$. The following proposition is left as an exercise to the reader.

Proposition 2. *The singular 2-dimensional polyhedron $P^2 \xrightarrow{f} R^3$, which has a unique singularity σ, possesses the following two properties:*

1. *We have $\bar{\mu}_2(f) > 0$; there is actually an essentially unique possible strategy and this is not* COHERENT.
2. *Nevertheless, $P^2 - Sing(f) \xrightarrow{f} R^3$ can be lifted to an embedding into $R^3 \times R$, i.e. $\mu_2(f) = 0$ (and $\nu_3(f) = 0$ too).*

Remark 2. (I) As already noticed, μ_2 and liftability are analogous to the first two Stiefel-Whitney classes, and are probably amenable to a more or less standard cohomological treatment. From this viewpoint, $\bar{\mu}_2$ appears as a much more exotic thing; unlike μ_2 and/or liftability it is strategy-dependent, i.e. it pays attention to the *order* via which one exhaust the double points of f (by O(i)-moves), in other words it has a basic noncommutativity built into it.

(II) Although μ_2, $\bar{\mu}_2$ might look superficially similar, their invariance properties

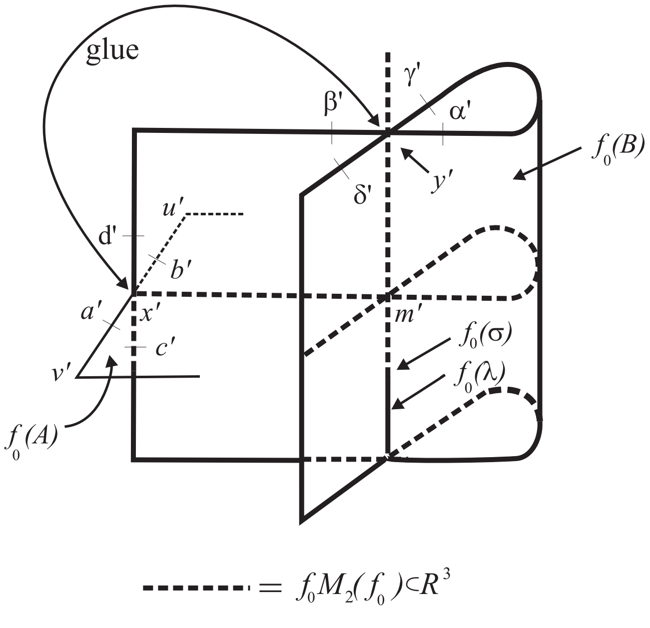


Figure 4: At the target. The dotted line is $f_0 M_2(f_0) \subset R^3$. The convention here is $f_0(m_i) = m'$, $f_0(b) = b'$.

are quite different. When we restrict f to a part of its source, μ_2 decreases (i.e. if we start with $\mu_2 = 0$ we do not loose this property). But this is not so for $\bar{\mu}_2$; on the other hand, $\bar{\mu}_2$ is invariant for the honeycomb calculus transformations. The "honeycomb calculus" is developed at length in [7], [8] (see [1] too). In one first variant, it is used for changing a general collapsible pseudo-spine representation into a honeycomb representation (which is a very special kind of collapsible pseudospine representation). But then, the honeycomb calculus can be used to change the honeycomb representations around and, infinite iteration of *this* process, eventually leads to our Smooth Tameness Theorem 2 above. Now, when representation (which is a very special kind of collapsible pseudo-spine representation). But then, the honeycomb we change $K^2 \xrightarrow{f} M^3$ by the operations of honeycomb calculus $\bar{\mu}_2$ stays the same. But this, in turn, is false for μ_2 which might increase under the effect of the honeycomb calculus. This should be enough for giving a flavor of the subtilities involved with $\overline{\mu}_2/\mu_2$.

4. A Review of the basic technology of zipping, desingularizations

We will start with a list of concepts from [6], but see also [1], which will be very useful. The notion of singular 2-dimensional polyhedron is to be found in section 1 (of [6]). The admissible singularities are defined in section 1, and displayed in Figure 1 of the same paper. After [6] was already in print, Barry Mazur suggested the name "undrawable" for these singularities. The $\mathrm{O}(i)$ moves are defined in section 1, and displayed in the Figures 2 to 7. The pseudo-spine representations are defined in Section 2.1, (see also Theorem 1, and Theorems 2.4, 2.5). The notion of almost collapsibility is defined in Section 2.1. Desingularizations and COHERENCE are defined in Section 4. The reader is strongly advised to omit all the rest of [6], as far as the present paper is concerned. In particular, section 2 (the Z-topology), Sections 2.11, 2.III and 3 can be skipped. If $K^2 \xrightarrow{f} M^2$ is a singular 2-dimensional polyhedron, then "$\Phi(f) = \Psi(f)$" will mean that f can be exhausted by $\mathrm{O}(i)$ -moves. The equivalence relations $\Psi(f) \subset \Phi(f) \subset K^2 \times K^2$ are defined in [5]. In this section, among other things, we will reproduce, with a minimum of editing, some pieces from section 2 of [7]. The numbering of the formulas or of the figures, used below, is not related to the notations from [11]. An alternative exposition for all the material coming next is [1], which one might want to read first.

A singular 2-dimensional polyhedron is a triple $(K^2,\ f, M^3)$ where

1. M^3 is a smooth 3-manifold without boundary, K^2 is a finite 2-dimensional simplicial complex and f is a simplicial map whose restriction to any simplex $\sigma \subset K^2$ is a smooth embedding. Any point in K^2 belongs to some 2-simplex.
2. There is a finite set of vertices of K^2 denoted by $\mathrm{Sing}(f) \subset K^2$, such that $f|(K^2 - \mathrm{Sing}(f))$ is an immersion (i.e. locally injective). Moreover, this immersion is generic, in a sense which we explain later on.
3. For each *singularity* $\sigma \in \mathrm{Sing}(f)$, there is a coordinate neighborhood $R^3 = (x, y, z) C M^3$, containing $f(\mathbb{S}) = (0, 0, 0)$, and a very precise *local model* for

$f^{-1}R^3 \xrightarrow{f} R^3$, which we will describe now. The open set $f^{-1}R^3 C K^2$ is the union of two "branches" P_1 and P_2, each of which is a copy of R^2, endowed with coordinates (u_1, v_1) and (u_2, v_2) respectively.

At the source K^2, each point $(u_1 = 0, v_1 = t) \in P_1$ *with* $t \leq 0$, is identified to $(x_2 = 0, v_2 = t) \in P_2$; the singularity is represented by $(u_1 = v_1 = 0) \equiv (u_2 = v_2 = 0)$.

The maps $f_1 | P_1$ and f_2 are given respectively by $(x = 0, y = u_1, z = v_1)$ and $(x = u_2, y = 0, z = v_2)$ (see Figure 3). In other words, at the level of K_2, P_1 and P_2 are glued together *along a common half-line,* the restrictions $f|P_i (i = 1, 2)$ are injective, and at the target $f P_1$ and $f P_2$ *meet transversally*. There is a line of double points of f, which starts at the singularity $\mathbb{S}$. These kind of singularities, which are represented the best we can in Figure 5 will be called ***undrawable singularities*** (or, sometimes, admissible singularities). We will denote by $\mathrm{Sing}K^2$ (or sometimes just by sK^2) the set of singularities of K^2. One should notice that, outside $\mathrm{Sing}K^2$, the map f gives a well-defined recipe for constructing a 3-dimensional regular neighborhood of K^2. It will also be convenient to introduce the following kind of notations. If X, Y, Z are three spaces and $Z \xhookrightarrow{i_1} X$, $Z \xhookrightarrow{i_2} Y$ inclusion maps, we will denote by $X \underset{Z}{\oplus} Y$ the quotient space of disjoined union $X + Y$ obtained by identifying, for each $z \in Z$ the points $i_1(z) \in X$ and $i_2(z) \in Y$. With this notation, the singular local model for our 2-dimensional singular polyhedra (K^2, f, M^3) can also be expressed as

$$P(s) = P_1 \bigoplus_{\frac{1}{2}L} P_2, \tag{21}$$

where $P_1 = \{$the plane $x = 0$ in $R^3\}$, $P_2 = \{$the plane $y = 0$ in $R^3\}$ and $\frac{1}{2}L = \{$the half-line $(x = y = 0, z \leq 0)\}$ (see Figure 5).

We have already said that outside $sK^2 \subset K^2$, we have a very precise rule for building a 3-dimensional regular neighborhood for K^2, given by the immersion $f|(K^2 - sK^2)$. When we try to extend this to the whole of K^2, we are naturally led to the following items. There is, to begin with, a 3-dimensional version of (21), namely

$$K'(\mathbb{S}) = A'_1 \bigoplus_{\mathcal{C}'} A'_2 \tag{22}$$

where $\mathbb{S}$ is the singular little square which replaces the singular point s and $\mathcal{C}'$ a semi-infinite 3-dimensional column replacing $\frac{1}{2}L$. This $K'(\mathbb{S})$ comes with an obvious map into the coordinate neighbourhood R^3, call it again f. We will denote by $K(\mathbb{S})$ the compact truncation $K'(\mathbb{S}) \cap f^{-1}$(a ball of high radius). This comes with a splitting analogous to (22), namely

$$K(\mathbb{S}) = A_1 \underset{\mathcal{C}}{\oplus} A_2. \tag{23}$$

This $K(\mathbb{S})$ is split from the rest of the world (let us say from the rest of $K'(\mathbb{S})$), by a surface which is a punctured torus

$$\delta K(\mathbb{S}) = S^1 \times S^1 - \overset{\circ}{D}{}^2.$$

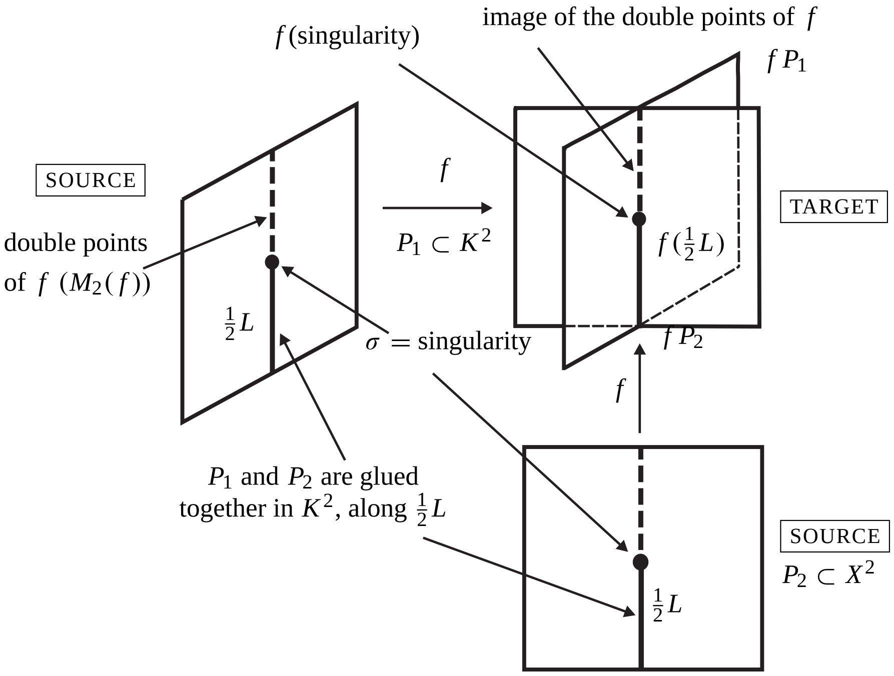


Figure 5: An ***undrawable*** (or admissible) singularity $\sigma \in \text{Sing}(f)$. We have here the following local model: there is a coordinate neighbourhood $R^3 = \{(x, y, z)\} \subset M^3$, with $f(\sigma) = (0, 0, 0)$. The open set $f^{-1}R^3 \subset X^2$ consists of the two planar "branches" P_1, P_2, glued at the source along the half-line $\frac{1}{2}L$ (ending at σ), and injected by f, transversally through each other.

The various (22) and/or (23) can be used for defining a canonical singular 3-dimensional thickening of K^2 which we will denote by $\Theta^3(K^2)$ and this kind of object will be referred to as a ***3-dimensional singular manifold***. But a singular 3-manifold X^3 does not necessarily take the form $X^3 = \Theta^3(K^2)$.

Our X^3 has, let us say the singularities $\mathbb{S}_1, \cdots, \mathbb{S}_r$, each coming with its own $K(\mathbb{S}_i)$. For each $K(\mathbb{S}_i) = A_1 \oplus A_2$ and for each of the two $j = 1$ or $j = 2$, inside $\delta K(\mathbb{S}) = \delta K(\mathbb{S}_i)$ we will consider the circle $c_j \subset \delta K(\mathbb{S}) \cap A_j$ going through the middle of $\partial A_j = c_j \times [-\varepsilon, \varepsilon]$. The circles c_1, c_2 have exactly *one* point in common and $\delta K(\mathbb{S})$ collapses onto the wedge $c_1 \cup c_2$. The various $\delta K(\mathbb{S}_i) \subset X^3$ induce a ***canonical splitting***

$$X^3 = X^3(\text{smooth}) \cup K(\mathbb{S}_1) \cup \cdots \cup K(\mathbb{S}_r) \tag{24}$$

where, by definition X^3(smooth)=Cl($X^3 - \bigcup_i K(\mathbb{S}_i)$), with Cl=closure.

A (combinatorial) ***desingularization*** of X^3 is, by definition, a collection φ of

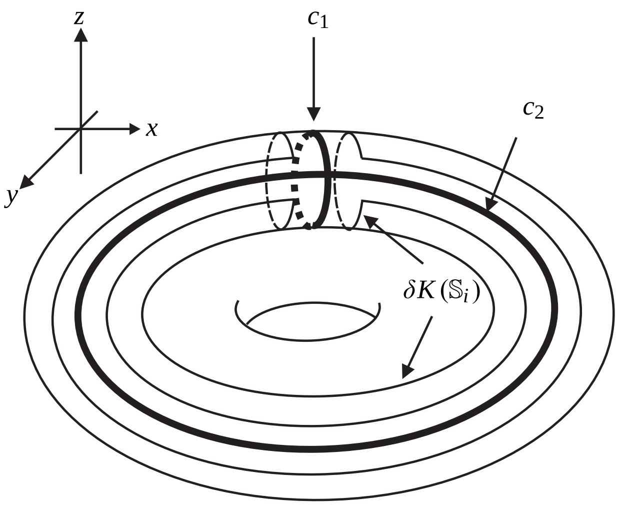


Figure 6: $\check{K}(\mathbb{S}_i) \subset S^3 = \partial B^4$. Here we see $\check{K}(\mathbb{S}_i)$ where $\varphi(A_1) = N$, $\varphi(A_2) = S$; we have denoted by $c_j\,(j = 1, 2)$ the circle going through the middle of $A_j \cap \delta K(\mathbb{S}_i)$.

bijections, defined for each $\mathbb{S}_i$

$$\{(A_1 \text{ branch of } K(\mathbb{S}_i)), (A_2 \text{ branch of } K(\mathbb{S}_i))\} \xrightarrow{\varphi_{\mathbb{S}_i}} \{S, N\}.$$

Here the set $\{S, N\}$ is given once and for all, and the letter S means *specified* while N means *non specified*. So there are 2^r such desingularizations. To each φ, we attach a geometrical desingularization

$$\check{X}^3 = \check{X}^3(\varphi) \xrightarrow{\pi=\pi(\varphi)} X^3, \tag{25}$$

where $\check{X}^3$ is smooth, π is surjiective, $\pi|(\check{X}^3 - \pi^{-1}sX^3)$ is bijective, and π blows-up every square $\mathbb{S}_i$ into a cylinder, inside the ***specified*** branch. This means that for each $K(\mathbb{S}_i) = A_1 \oplus A_2 \subset X^3$, where we choose the notation such that $\varphi_{\mathbb{S}_i}(A_1) = S$, one has to *unglue* $A_1 \cap (z \geq 0)$, from the rest of $K(\mathbb{S}_i)$ along $\mathbb{S}_i \cap (-\varepsilon < y < \varepsilon)$. In any case, $\check{K}^3(\mathbb{S}_i) \overset{\text{def}}{=} \pi^{-1}K(\mathbb{S}_i)$ is a solid torus of genus one, containing $\delta K(\mathbb{S}_i) = \pi^{-1}\delta K(\mathbb{S}_i)$ in its boundary. Figures 6 and 7 show the two possible cases. One should regard (25) and (26) as being essentially the same thing; sometimes one interpretation will be more convenient, sometimes the other.

The definition which follows now is very important not only for [11], but also for all the other papers concerning the Po's Program for the Poincaré Conjecture (see the bibliography); a simpler version of it is present in [1]. But the one presented here has a number of features which we need.

Let X^3 be a singular 3-manifold and φ a desingularization. We will attach canonically to (X^3, φ) a smooth compact 4-dimensional manifold with boundary, denoted by $\Theta^4(X^3, \varphi)$ and a canonical smooth embedding $\theta = \theta(\varphi)$ of X^3 into $\partial\Theta(X^3, \varphi)$.

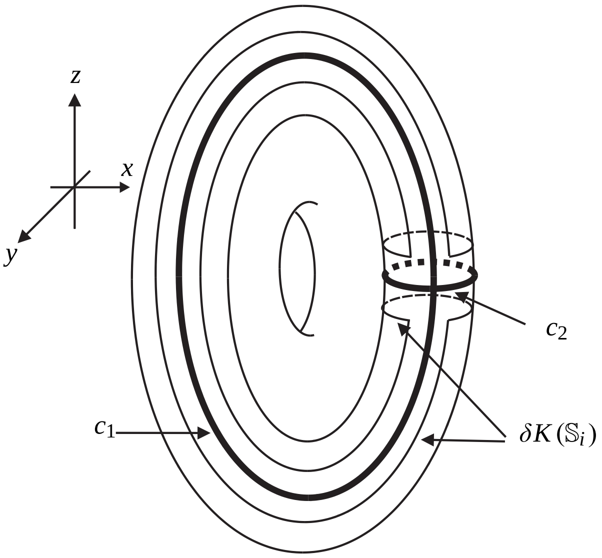


Figure 7: $\check{K}(\mathbb{S}_i) \subset S^3 = \partial B^4$. We see here $\check{K}(\mathbb{S}_i)$ where $\varphi(A_1) = S, \varphi(A_2) = N$.

Figures 6 and 7 define for us a ***standard embedding*** $\check{K}(\mathbb{S}_i) \subset R^3 \subset R^3 \cup (\infty)$. We will consider this S^3 as the boundary of a copy of B^4_{standard} which we denote by $B^4(\mathbb{S}_i)$. Let us also consider tubular neighbourhoods $\delta K(\mathbb{S}_i) \times [0,1] \subset S^3 - \text{int}\check{K}(\mathbb{S}_i)$, with $\delta K(\mathbb{S}_i) = \delta K(\mathbb{S}_i) \times \{0\}$. So, in S^3, we have $\check{K}(\mathbb{S}_i) \cap (\delta K(\mathbb{S}_i) \times [0,1]) = \delta K(\mathbb{S}_i)$; in other words, $\delta K(\mathbb{S}_i) \times [0,1] \subset S^3$ is OUTGOING with respect to $\check{K}(\mathbb{S}_i) \subset S^3$.

REMARK 3. Figures 6 and 7 define two pairs $(S^3, \delta K(\mathbb{S}))$ which are in fact isotopic; but the corresponding pairs $(S^3, \delta K(\mathbb{S}) \times [0,1])$ are not diffeomorphic, unless we allow to reverse the parametrization of $[0,1]$ which is equivalent to the change $S \leftrightarrow N$. So if φ^* is the dual desingularization to φ obtained by changing $S \leftrightarrow N$, then $\Theta^4(X^3, \varphi) \overset{\text{DIFF}}{=} \Theta^4(X^3, \varphi^*)$. But, in general, for distinct φ_1 and φ_2, the two Θ^4 's will not be diffeomorphic.

Notice next that we have a splitting of $\check{X}^3$ by the $\delta K(\mathbb{S}_i)$'s completely analogous to (24)

$$\check{X}^3 = X^3(\text{smooth}) \cup \check{K}(\mathbb{S}_1) \cup \cdots \cup \check{K}(\mathbb{S}_r). \tag{26}$$

With this we have by definition

$$\Theta^4(X^3, \varphi) = (X^3(\text{smooth}) \times [0,1]) \oplus B^4(\mathbb{S}_1) \oplus \cdots \oplus B^4(\mathbb{S}_r),$$

where the embeddings

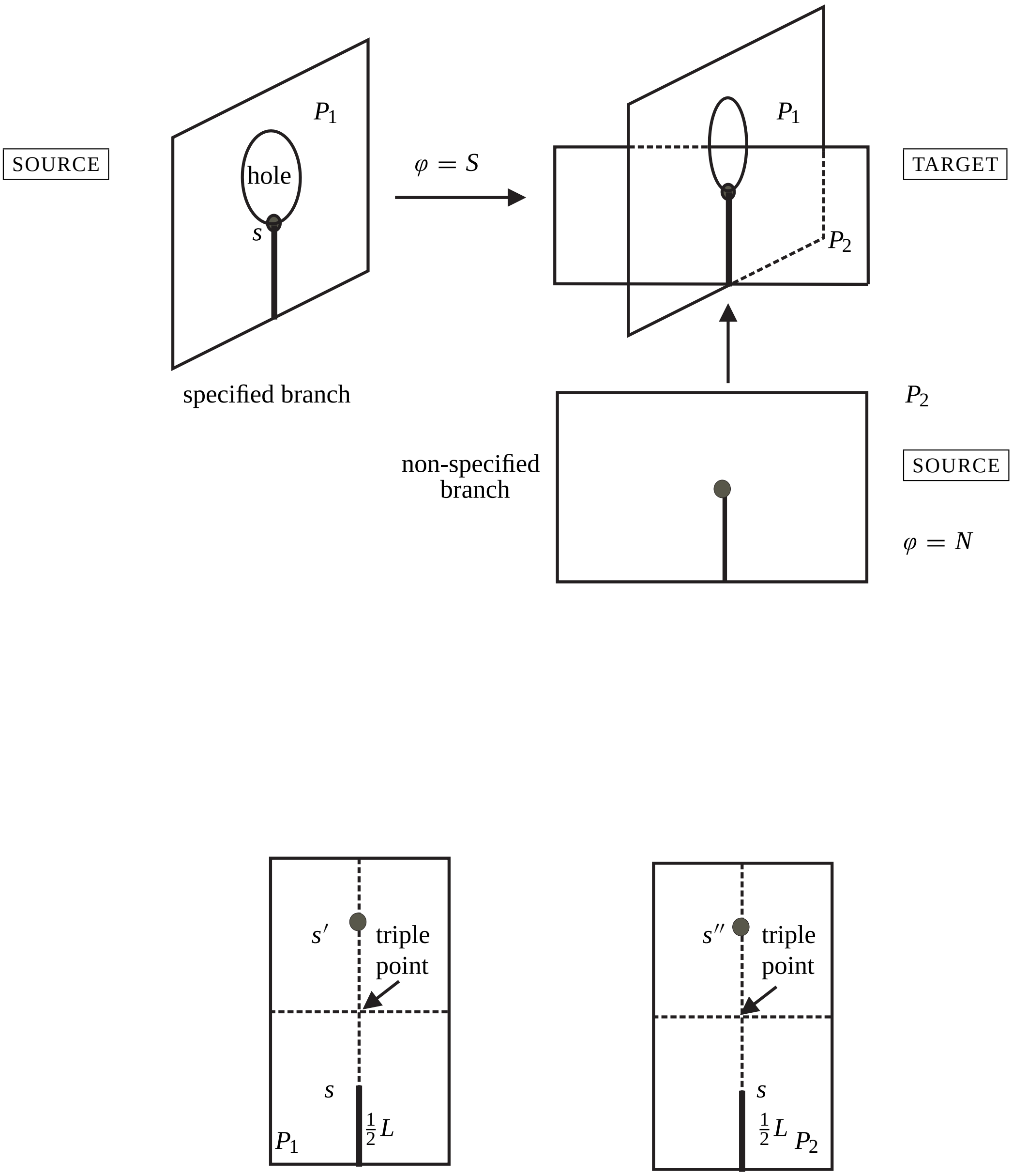


Figure 8: The picture above represents the desingularization of Figure 5. The one below shows that for the O(0)-movement g, we have $M_2(g) = [s, s'] \cup [s, s'']$.

$$\begin{tikzcd}
 & \partial(X^3(\text{smooth}) \times [0,1]) \\
\delta K(\mathbb{S}_i) \times [0,1] \arrow[ur, hook] \arrow[dr, hook] & \\
 & \partial B^4(\mathbb{S}_i)
\end{tikzcd}$$

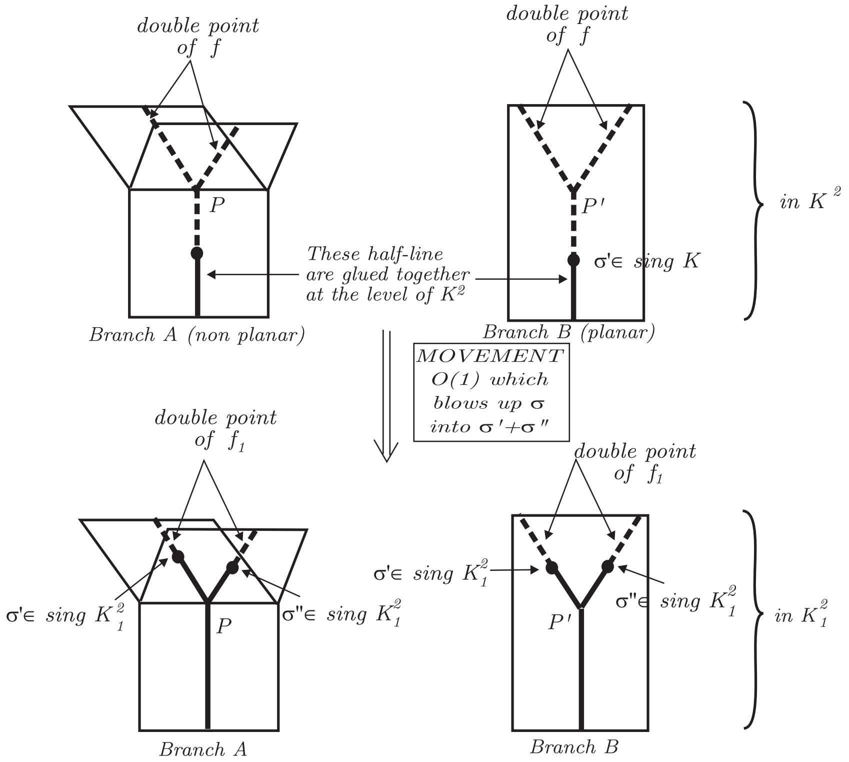


Figure 9: Movement $O(1)$. The figure is at the source.

are used for gluing $B^4(\mathbb{S}_i)$ to X^4(smooth) $\times$ [0, 1]. The three-manifold

$$(X^3(\text{smooth}) \times \{0\}) \oplus \check{K}(\mathbb{S}_1) \oplus \cdots \oplus \check{K}(\mathbb{S}_r),$$

which is an isomorphic copy of X^3 (via (2.9)), lives canonically inside $\partial\Theta^4(X^4, \varphi)$, and this defines the embedding $\theta = \theta(\varphi)$.

LEMMA 2. *The manifold $\Theta^4(X^3, \varphi)$ is a 4-dimensional regular neighbourhood of X^3 and this implies, in particular, that if X^3 is collapsible,then $\Theta^4(X^3, \varphi) \overset{\text{DIFF}}{=} B^4$ for all φ's.*

So, each desingularization φ of X^3 picks up a ***canonical 4-dimensional thickening*** $\Theta^4(X^3, \varphi)$ among the (possibly infinity many) 4-dimensional regular neighborhoods of X^3. Since $\Theta^4(\cdots)$ is defined by *gluing* together local pieces, it behaves well with respect to *splittings*. More precisely, if X_1^3, X_2^3 are *singular 3-manifolds*, with desingularizations φ_1, φ_3 and if Σ is a bounded surface given with embeddings

$\Sigma \to \partial X_i^3 - s(X_i^3)$ then we have

$$\Theta^4\Big(X_1^3 \bigoplus_{\Sigma} X_2^3, \varphi_1 \bigoplus \varphi_2\Big) = \Theta^4(X_1^3, \varphi_1) \bigoplus_{\Sigma \times [0,1]} \Theta^4(X_2^3, \varphi_2),$$

and a similar formula holds for $\theta(\varphi_1 \oplus \varphi_2)$.

In an appropriate sense (Θ^4, θ) is a *functor* from the category of desingularizations $\check{X}^3 \to X^3$ and their embeddings to the category of pairs $(V^4, V^3 \subset \partial V^4)$ (where $V^n =$ bounded DIFF n-manifold, for $n = 3, 4$) and their embeddings.

If (K^2, f, M^3) is a singular 2-dimensional polyhedron, any desingularization φ of the singular 3-manifold $\Theta^3(X^2)$ can be defined directly at the level of (K^2, f, M^3). The combinatorial version is a collection of bijections, defined for each singularity s (exactly like in (25)).

$$\big\{(P_1 \text{ branch of } P(s)), (P_2 \text{ branch of } P(s))\big\} \xrightarrow{\varphi_s} \{S, N\};$$

the geometrical version of the desingularization φ is a map $\check{K}^2 \xrightarrow{\pi=\pi(\varphi)} K^2$ which *blows-up* the singularity s into a circle, *inside the specified branch* $P_i \subset P(s) = P_1 \oplus P_2$. So, Figure 5 is changed, when we pass from K^2 to $\check{K}^2$ into Figure 8 above.

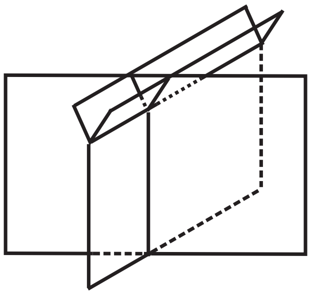
ELEMENTARY MOVEMENTS. In [6], we have defined the elementary movements O(i), $(i = 1, 2, 3)$ for 2-dimensional singular polyhedra, as commutative diagrams

$$\begin{tikzcd}
K^2 \arrow[rr, "f"] \arrow[dr, "g"'] & & M^3 \arrow[dl, "f_1"] \\
& K_1^2 &
\end{tikzcd}$$

where g is a quotient-space projection which, starting at some singularity of K^2, *zips* some of the double points of f.

We have given in [6] *local models* for each O(i). We remind the reader that for $i \geq 2$, O(i) is acyclic, i.e. it is a homotopy-equivalence. Movement O(0) destroys a triple point of f, but creates a branching point for the set of double points. Figure 8 (below) shows the support of the movement O(0), K^2 and K_1^2 are actually isomorphic, it is at the level of $M_2(f) \Rightarrow M_2(f_1)$ that the transformation is non-trivial. Movement O(1) destroys a branching point for $M_2(f)(\overset{\text{def}}{=}$ the set of double points of $f)$ but increases the number of singularities (see Figure 11), while movement O(2) kills a singularity by bringing it to the boundary of K^2 (see Figure 4 above) Movement O(3) is homotopically non-trivial; it kills simultaneously two singularities by bringing them together (see Figure 4 below) and so, homotopically speaking it corresponds to adding a 2-dimensional cell (so it decreases π_1 and/or increases the π_2).

Each of the Figures 9, 4 is supposed to show us *compact* pieces $k \subset K^2, k_1 \subset K_1^2$ such that $k_1 = gk$ and $M_2(g) \subset \text{int}k$. So we have a canonical isomorphism



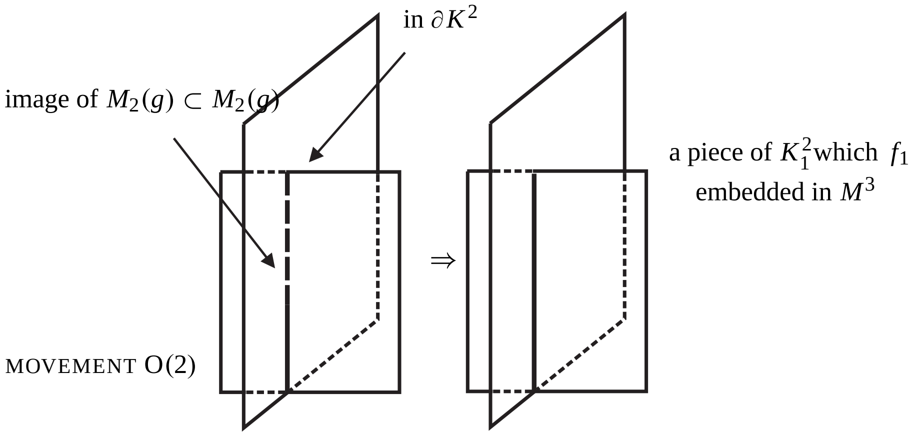


Figure 10.

$\partial k = \partial k_1$ and ∂k (respectively ∂k_1) splits k (respectively k_1) from the rest of K^2 (of K_1^2).

We can define these same movements O(i) ($i \leq 3$) for singular 3- manifolds $X^3 \xrightarrow{G} X_1^3$. We simply apply Θ^3 to the local model for g to get a local model for G. It will be understood that $\partial k = k \cap \partial\Theta^3(k)$; we will denote by $\delta\Theta^3(k) \subset \partial\Theta^3(k) - s\Theta^3(k)$ a thin regular neighborhood of ∂k. *In our local model,* $\delta\Theta^3(k)$ *splits* $\Theta^3(k) \subset X^3$ from the rest of X^3, and similarly $\delta\Theta^3(k_1)$ (which is canonically diffeomorphic to $\delta\Theta^3(k)$) splits $\Theta^3(k_1) \subset X_1^3$, from the rest of X_1^3.

So, any O(i)-movements *for singular 3-manifolds* takes the following form. We have a local model $\Theta^3(k) \xrightarrow{G} \Theta^3(k_1)$ inducing an isomorphism $\delta\Theta^3(k) \approx \delta\Theta^3(k_1)$. We are also given an embedding

$$\delta\Theta^3(k_1) = \delta\Theta^3(k) \to \partial Z^3 - s(Z^3),$$

where Z^3 is some given singular 3-manifold. Our global O(i)-movement is simply obtained by gluing

$$X^3 = Z^3 \bigoplus_{\delta\Theta^3(k)} \Theta^3(k) \xrightarrow{\mathrm{id}(Z^3)\oplus G} Z^3 \bigoplus_{\delta\Theta^3(k_1)} \Theta^3(k_1) = X_1^3. \tag{27}$$

Whenever there is no danger of confusion, we will write G instead of $\mathrm{id}(Z^3) \oplus G$.

If $(K^2, f, M^3) \overset{g}{\Longrightarrow} (K_1^2, f_1, M^3)$ is a movement O(i) with $i \leq 2$ and if φ is a desingularization for K^2, then we have a *canonically* induced desingularization φ^1 for K_1^2. For O(0), O(2), this is completely obvious, and as far as O(1) is concerned then, in the context of Figure 4, we have

$$\varphi^1_{\mathbb{S}'}(A) = \varphi^1_{\mathbb{S}''} = \varphi_{\mathbb{S}}(A) \ \ \text{(and hence also } \varphi^1_{\mathbb{S}'}(B) = \varphi^1_{\mathbb{S}''}(B) = \varphi_{\mathbb{S}}(B)).$$

NOTATIONAL REMARK. Whenever, there is no danger of confusion, we will simply write φ instead of φ^1.

DEFINITION 1. *Let φ be a desingularization of (K^2, f, M^3) and let*

$$(K^2, f, M^3) \overset{g}{\Longrightarrow} (K_1^2, f_1, M^3)$$

*be an O(3)-movement, as defined by the local model from Figure 11. We will say that g is **coherent** (with respect to φ) iff (see Figure 11)*

$$\varphi_{\mathbb{S}'}(C) = \varphi_{\mathbb{S}''}(C) \ \ \textit{(and hence also } \varphi_{\mathbb{S}'}(D) = \varphi_{\mathbb{S}''}(D)). \tag{28}$$

In the contrary case, the O(3)-movement will be said to be non-coherent. In both cases of the Definition 1, we will consider the induced desingularization φ_1 of K_1^2, defined simply by restricting φ to $sK^2 - \{\mathbb{S}', \mathbb{S}''\} = sK_1^2$. The definition of the induced desingularization and of the coherence for O(3)-moves extend immediately to the context of singular 3-manifolds.

We consider now the O(i)-movement for singular 3-manifolds given by (27) and a desingularization φ for X^3. We will assume that O(i) is either acyclical (i.e. with $i \leq 2$) or a *coherent* O(3)-movement. Under these conditions, if $p \in M_2(G) - X^3 - s(X^3)$ is a double point for G, it makes sense to define, in quite an obvious way, an $\{S, N\}$-value for p call it $\varphi(p) \in \{S, N\}$.

NOTATIONAL REMARK. Whenever there is no danger of confusion we will write $\Theta^4(k)$ instead of $\Theta^4(\Theta^3(k), \varphi)$. Similar notations will be used at the level of k_1. The next lemma will give some elementary invariance properties of the thickening operation $\Theta^4(\cdots)$, with respect to the O(i)-moves.

LEMMA 3 (Invariance properties for the O(i) movements).

1. *If* (27) *is an acyclic O(i)-movement, then we have a diffeomorphism*

$$\Theta^4(X^3, \varphi) = \Theta^4(X_1^3, \varphi_1), \tag{29}$$

which is the identity on $\Theta^4(Z^3, \varphi)$.

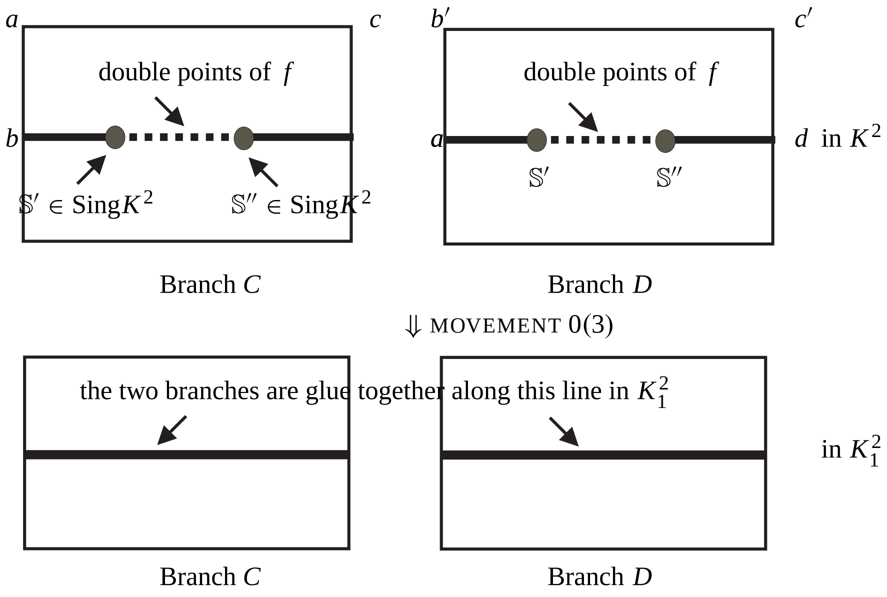


Figure 11: At target and/or in K_1^2 one has the situation from Figure 12.

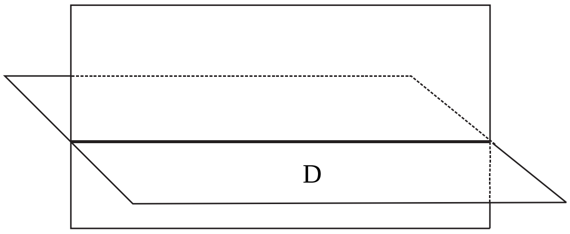


Figure 12.

2. *In the context of 1), we consider a compact subset* $E \subset X^3 - sX^3$ *and its obvious lift* $E \subset \check{X}^3$. *We will assume that* $G|E$ *is an embedding into* $X_1^3 - sX_1^3$ *so that* E *also lifts to* $E \subset X^3$. *Making use of* $\theta(\cdots)$ *from the construction* (21)*, we can define the two pairs* $(\Theta^4(X^3, \varphi), E)$*, and* $((\Theta^4(X_1^3, \varphi^1), E)$.

   If for every $p \in E \cap M_2(G)$ *we have* $\varphi(p) = N$ *, then* (29) *extends to a diffeomorphism of pairs*

$$(\Theta^4(X^3, \varphi), E) = (\Theta^4(X_1^3, \varphi^1), E).$$

   which is the identity on E.

3. *We consider now the case when (27) is a* COHERENT $O(3)$*-movement. There is*

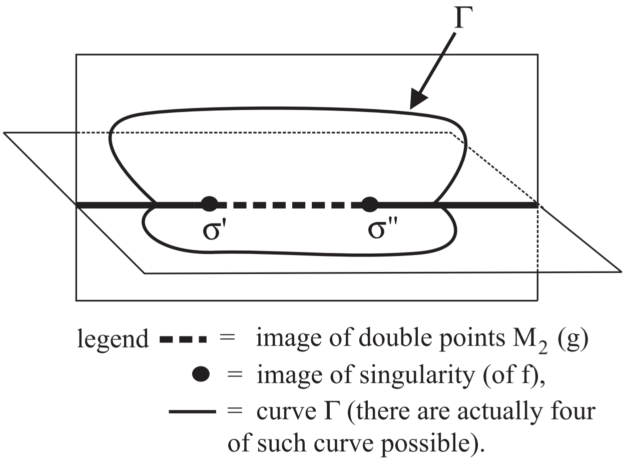


Figure 13.

a canonical embedding

$$\Theta^4(X^3,\varphi) \xhookrightarrow{\mathrm{id}(Z^3)\oplus j} \Theta^4(X_1^3,\varphi^1).$$

By pushing the image of $\Theta^4(Z^3,\varphi) \subset \Theta^4(X^3,\varphi)$ *into the interior of* $\Theta^4(X_1^3,\varphi^1)$, *we obtain a canonical smooth 4-dimensional cobordism*

$$\Big(W^4, \partial^0 W^4 = \partial\Theta^4(X^4,\varphi), \partial^1 W^4 = \partial\Theta^4(X_1^3,\varphi^1)\Big)$$

connecting $\Theta^4(X^3,\varphi) \subset int\Theta^4(X_1^3,\varphi^1)$ *to* $\Theta^4(X_1^3,\varphi^1)$. *This cobordism is of the type adding a handle of index two and, of course*

$$\Theta^4(X_1^3,\varphi^1) \overset{\mathrm{DIFF}}{=} \Theta^4(X^3,\varphi) \bigoplus_{\partial^0 W^4} W^4.$$

The statements 1), 2), 3) above are global, but there is also a list of statements of the same type, which are local ***and relative***. Since there are very important for us, we will explain them. One, actually, proves Lemma 3 by proving first, the local Lemma 4 below. Some notations will be introduced, first. Each of the Figures 8, 9,... is supposed to show us ***compact*** pieces $k \subset K^2, k_1 \subset K_1^2$ such that $k_1 = gk$ and $M_2(g) \subset \mathrm{int}k$. So we have a canonical isomorphism $\partial k = \partial k_1$, and ∂k (respectively ∂k_1) splits k (respectively k_1) from the rest of K^2 (of K_1^2). When we apply Θ^3 to the local model for g so as to get a local model for G, it will be understood that $\partial k = k \cap \partial\Theta^3(k)$; we will denote by $\delta\Theta^3(k) \subset \partial\Theta^3(k) - s\Theta^3(k)$ a thin regular neighborhood ∂k. *In our local model,* $\delta\Theta^3(k)$ *splits* $\Theta^3(k) \subset X^3$ from the rest of X^3, from the rest of X_1^3. So, any O(i)-movement for a *singular 3-manifold* takes the following form

We have a local model $\Theta^3(k) \xrightarrow{G} \Theta^3(k_1)$ inducing an isomorphism $\delta\Theta^3(k) \approx \delta\Theta^3(k_1)$. We are also given an embedding

$$\delta\Theta^3(k_1) = \delta\Theta^3(k) \longrightarrow \partial Z^3 - s(Z^3),$$

where Z^3 is some given singular 3-manifold. Our global O(i)-movement is simply obtained by gluing

$$X^3 = Z^3 \underset{\delta\Theta^3(k)}{\oplus} \Theta^3(k) \xrightarrow{\mathrm{id}(Z^3)\oplus G} Z^3 \underset{\delta\Theta^3(k_1)}{\oplus} \Theta^3(k_1) = X^3_1. \tag{30}$$

We consider now the O(i)-movement for singular 3-manifolds given by (30) and a desingularization φ for X^3. We will assume that O(i) is either acyclical (i.e. with $i \leq 2$) or a *coherent* O(3)-movement. Under these conditions,if $p \in M_2(G) - X^3 - s(X^2)$ is a double point for G it makes sense to define, in quite an obvious way, an $\{S, N\}$ -value for p call it $\varphi(p) \in \{S, N\}$.

LEMMA 4. (Invariance properties for the O(i)-movements in relative and local form.)

1. *If* (30) *is an acyclic O*(i)*-movement, then*

   $$\Theta^4(k) \overset{\mathrm{DIFF}}{=\!=} B^4_{\text{standard}} \overset{\mathrm{DIFF}}{=\!=} \Theta^4(k_1). \tag{31}$$

   We consider $\check{\Theta}^3(k) \xrightarrow{\pi(\varphi)} \Theta^3(k)$ *the embedding* $\check{\Theta}^3(k) \xrightarrow{\theta(\varphi)} \partial\Theta^4(k)$ *and the tubular neighbourhood* $\delta\Theta^3(k)\times[0,1] \subset \partial\Theta^4(k)$*, with* $\delta\Theta^3(k)\times\{0\} = \delta\Theta^3(k_1)$*, which is outgoing with respect to Im*($\theta(\varphi)$)*. We will define similarly* $\delta\Theta^3(k_1) \times [0,1] \subset \delta\Theta^4(k_1)$*. With this, we have a diffeomorphism*

   $$(\Theta^4(k), \delta\Theta^3(k) \times [0,1]) = (\Theta^4(k_1), \delta\Theta^3(k_1) \times [0,1]),$$

   which restricts to the obvious identification of $\delta\Theta^3(k) \times [0,1]$ *with* $\delta\Theta^3(k_1) \times [0,1]$.

2. *(Consequence of 1.) If* (30) *is an acyclic O*(i)*-movement, then we have a diffeomorphism*

   $$\Theta^4(X^4, \varphi) = \Theta^4(X^3_1, \varphi_1), \tag{32}$$

   which is the identity on $\Theta^4(Z^3, \varphi)$ *(see the gluing formula* (12)*).*

3. *In the context of 1) and 2), we will denote by* $\check{X}^3 \subset \Theta^4(X^3, \varphi)$ *the image* $\theta(\varphi)\check{X}^3$*, and similarly for* $\check{X}^3_1$*. With this, there is a passage*

$$\begin{tikzcd}
(\partial\Theta^4(X^3, \varphi), \check{X}^3) \arrow[r, Rightarrow] \arrow[r, bend right, "\approx"'] & (\partial\Theta^4(X^3_1, \varphi^1), \check{X}^3_1)
\end{tikzcd}$$

consisting of embedded surgeries (this passage can easily be described explicitly).

4. *(Important complement to 3) In the context of 1), 2), 3) we consider a compact subset* $E \subset X^3 - sX^3$ *and its obvious lift* $E \subset \check{X}^3$*. We will assume that* $G|E$ *is an embedding into* $X^3 - sX^3$ *so that E also lifts to* $E \subset \check{X}^3$*. So, making use of* $\theta(\cdots)$*, we can define the two pairs* $(\Theta^4(X^3, \varphi), E)$*, and* $(\Theta^4(X^3_1, \varphi^1), E)$*.*

   If for every $p \in E \cap M_2(G)$ *we have* $\varphi(p) = N$*, then* (32) *extends to a diffeomorphism of pairs*

   $$(\Theta^4(X^3, \varphi), E) = (\Theta^4(X^3_1, \varphi^1), E),$$

   which is the identity on E.

5. *We consider now the case when* (30) *is a* COHERENT$O(3)$*-movement. Here* $\Theta^3(k_1)$ *is non-singular. In fact* $\Theta^3(k_1) = B^3$ *and* $\Theta^4(k_1) = \Theta^3(k_1) \times [0,1] = B^4$*. There is a canonical smooth embedding*

   $$(\Theta^4(k), \delta\Theta^3(k) \times [0,1] \xrightarrow{j} (\Theta^4(k_1), \partial\Theta^4(k_1)))$$

   (with $j(\delta\Theta^3(k) \times [0,1]) = \delta\Theta^3(k_1) \times [0,1]$*) such that one goes from*

   $$(\Theta^4(k), \delta\Theta^3(k) \times [0,1])$$

   to $(\Theta^3(k_1), \delta\Theta^3(k_1) \times [0,1])$ *by adding to* $\Theta^4(k)$ *a smooth* ***handle of index two****, not touching* $\delta\Theta^3(k) \times [0,1]$*. (It will be assumed, in principle, that* $j(\Theta^4(k) - \delta\Theta^3(k) \times [0,1]) \subset int\Theta^4(k_1)$*.)*

6. *(Complement to 5)) Let us consider the bidimensional local model for the* $O(3)$*-move,* $k \xrightarrow{g} k_1$*, as given by Figure 12. In this figure, we see a simple closed curve* $\Gamma \subset k - \delta k - sk - M_2(g)$ *(there are exactly four such curves, depending of which sector we choose and what follows is valid for any one of them). There is an obvious lift* $\Gamma \subset \check{\Theta}^3(k) \subset \partial\Theta^4(k) - \delta\Theta^3(k) \times [0,1]$*. This simple closed curve* $\Gamma \subset \Theta^4(k)$ *is endowed with a canonical framing: one vector is tangent to* $\partial\check{\Theta}^3$ *and the other one orthogonal and outgoing. The handle of index two from 5) is defined by*

   $$\{\Gamma, \textit{the } \text{CANONICAL FRAMING}\}.$$

(7) *(Global consequence of the local statement 5)) There is, in the context of 5), a canonical embedding*

$$\Theta^4(X^3, \varphi) \xhookrightarrow[\Theta^4]{\mathrm{id}(Z^3) \oplus j} (X^3_1, \varphi^1).$$

By pushing the image of $\Theta^4(Z^3, \varphi) \subset \Theta^4(X^3, \varphi)$ *towards the interior of* $\Theta^4(X^3_1, \varphi^1)$*, we obtain a canonical smooth 4-dimensional cobordism*

$$(W^4, \partial^0 W^4 = \partial\Theta^4(X^3, \varphi), \partial^1 W^4 = \partial\Theta^4(X^2_1, \varphi^1)),$$

connecting $v\Theta^4(X^3,\varphi) \subset int\Theta^4(X_1^3,\varphi^1)$ *to* $\Theta^4(X_1^3,\varphi^1)$. *This cobordism is of the type adding a handle of index two and*

$$\Theta^4(X_1^3,\varphi^1) \overset{\text{DIFF}}{=\!=} \Theta^4(X^3,\varphi) \bigoplus_{\partial^0 W^4} W^4.$$

The proof of Lemma 4 is given in [7]. The global Lemma 3 is proved using Lemma 4, via (30).

The preceding theory was developed above in the context of a compact source for mappings like $K^2 \xrightarrow{f} M^3$. But since everything is really constructed by gluing together local pieces, it extends without any trouble to the case when K^2 is locally compact, but remains ***locally finite***, with an f which is PROPER i.e. f^{-1}(compact) = compact. Under these conditions, the Lemma 3 continues to be true; the proof uses again Lemma 4 which is local, anyway. The paper [11] needs ***this*** extension. But then, next, there is also the issue of the infinite PROPER zipping. We will come back to it in the subsequent paper which we have already mentioned in the introduction.

References

[1] GABAI D., *Valentin Poénaru's Program for the Poincaré Conjecture*, in: "Geometry, Topology and Physics for Raoul Bott" (S.T. Yau Ed.), International Press 1994, 139–166

[2] GERSTENHABER M. AND ROTHAUS O., *The solution of sets of equations in groups*, Proc. Nat. Acad. USA **48** (1962), 1531–1533.

[3] MAZUR B., *A note on some contractible 4-manifolds*, Ann. of Math. **73** 2 (1961), 221–228.

[4] MANDELBAUM R., *Four-dimensional topology: an introduction*, BAMS **1** 2 (1980), 1–159.

[5] POÉNARU V., *Les décompositions de l'hypercube en produit topologique*, Bull. Soc. Math. France **88** (1960), 113–129.

[6] POÉNARU V., *Infinite processes and the 3-dimensional Poincaré Conjecture I: the collapsible pseudo-spine representation theorem*, Topology **31** 3 (1992), 625–656.

[7] POÉNARU V., *Infinite processes and the 3-dimensional Poincaré Conjecture II: the Honeycom representation theorem*, Prépublications d'Orsay, 93-14 (1994).

[8] POÉNARU V., *Infinite processes and the 3-dimensional Poincaré Conjecture III: the algorithm*, Prépublication d'Orsay 92-10 (1992).

[9] POÉNARU V., *Processus infinis et Conjecture de Poincaré en dimension trois IV: le théorème de non sauvagerie lisse (The smooth tameness theorem)*, Part A, Orsay Preprint, 92-83; Part B, Orsay Preprint, 95-33 (1995).

[10] POÉNARU V., $\pi_1^\infty \widetilde{M}^3$ *(A short outline of the proof)*, Prèpublication d'Orsay 98-63 (1998).

[11] POÉNARU V., *Geometric simple connectivity in smooth four dimensional topology*, Part A, Préprint IHES/M/01/45 (2001).

[12] POÉNARU V., *Geometric simple connectivity in smooth four dimensional topology*, an outline, Prèpublication d'Orsay 98-63 (98).

[13] POÉNARU V., *Geometric simple connectivity in smooth four-dimensional topology*, Part B (manuscript 2005).

[14] POÉNARU V., *The strange compactification theorem*, Part A. IHES, Preprint M97-43 (1997); Part C, IHES Preprint M97-43 (1997); Part E is in the process of being typed.

[15] POÉNARU V., *Some invariants of generic immersions and geometric applications*, B.A.M.S. **81** (1975), 1079–1082.

[16] POÉNARU V., *Homotopie règuliére et isotopie*, Rend. Mat. Torino (1979), 15–56.

[17] POÉNARU V., *On the equivalence relation forced by the singularities of a non degenerate simplicial map*, Duke Math. Journal **63** 2 (1991), 421–429.

[18] POÉNARU V., *Processus infinis et conjecture de Poincaré en dimension trois IV: le Théorème de non sauvagerie lisse (The smooth tameness theorem)*, Part A, Orsay Preprint, 92-83 (1993); Part B, Orsay Preprint, 95-33 (1995).

[19] POÉNARU V., *Universal covering spaces of closed 3-manifolds are simple connected at infinity (with an appendix by L. Funar)*, Prèpublication d'Orsay 2000-20 (2000).

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