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Multimeasures and integration of multifunctions in Banach spaces

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INTRODUCTION

The general framework of this thesis is the *theory of integration for multifunctions and multimeasures.*

The theory of the integration for multifunctions has its origins in the pioneering works of Gérard Debreu and Robert Aumann, Nobel Prizes in Economics in 1983 and 2005, respectively, and it has found many applications in various fields of mathematics applied to economics, optimal control and optimization.

There is a great deal of literature on Bochner and Pettis integration of Banach space-valued multifunctions (see K. El Amri and C. Hess [24], B. Cascales, V. Kadets and J. Rodríguez [9, 10],...) of several types. In particular, quite recently nice characterizations of Pettis integrable multifunctions having their values convex weakly compact or compact subsets of a Banach space are presented ([24, Theorem 5.4] and [24, Theorem 5.5]).

The definitions of such integrals involve the Lebesgue integrability of the support functions. The theory of integration introduced by Lebesgue at the beginning of the twentieth century is a powerful tool which, perhaps because of its abstract character, does not have the intuitive appeal of the Riemann integral.

As Lebesgue himself observed in his thesis, his integral does not integrate all unbounded derivatives and so it does not provide a solution for the problem

of primitives, i.e. for the problem of recovering a function from its derivative. Moreover, the Lebesgue theory does not cover nonabsolutely convergent integrals.
In 1957 J. Kurzweil and, independently, in 1963 R. Henstock, by a simple modification of the Riemann's method, introduced a new integral, which is more general than the Lebesgue's one.
It retains the intuitive appeal of the Riemann definition and, at the same time, coincides with the Lebesgue integral on the class of the positive measurable functions. Moreover, it integrates all derivatives, so it solves the problem of the primitives.
For these reasons many mathematicians have been interested in integrals constructed by *Riemann sums* and in particular in the Henstock-Kurzweil integral. In the last fourty years the theory of nonabsolutely convergent integrals has gone on significantly, and the researches in this field are still active and far to be complete.
This is the motivation to consider, also in the case of multifunctions, the Henstock-Kurzweil integral in places where the Lebesgue integral used to be applied.
So, an obvious generalization of the Pettis integral of a multifunction is obtained by replacing the Lebesgue integrability of the support functions by their Henstock-Kurzweil integrability (such an integral is called Henstock-Kurzweil-Pettis). L. Di Piazza and K. Musiał proved in [21], in case of separable Banach spaces, a surprising and unexpected characterization of the Henstock-Kurzweil-Pettis integral in terms of the Pettis one: the Henstock-Kurzweil-Pettis integral is a translation of the Pettis integral. A similar result in case of Henstock integrable multifunction was proved in [22].
Moreover, the result proved in [21] has been generalized in [23] for an arbitrary Banach space.

The theory of multimeasures is a natural extension of the theory of vector measures. It can be viewed as a development of the theory of integration for multifunctions. As well as the multifunctions, the multimeasures are a useful analytical tool in mathematics applied to the economics; in particular in the equilibrium theory of production-exchange.

There are many pubblications concerning the Radon-Nikodým theorem for countably additive multimeasures. Pioneering results were established among others by Z. Artstein [2], A. Costé and R. Pallu de la Barrière [15].
Little or nothing exists in the literature concerning the Radon-Nikodým theorem in the finitely additive case. Moreover, the majority of results known so far requires the separability of the Banach space.
Nevertheless it is very recent the paper [8] of B. Cascales, V. Kadets and J. Rodríguez in which they proved two Radon-Nikodým theorems, using set-valued Pettis integrable derivatives, and with the absence of any separability assumptions.

The aim of this thesis is to add significant contributions to the theory of integrals of Henstock and Henstock-Kurzweil-Pettis. In particular, we try to extend in that area some of the results known in the literature for the integrals of Bochner and Pettis, or at least try to fill the gap.
This gap derives essentially from the fact that the primitives of Bochner and Pettis integrals are countably additive, while the Henstock and Henstock-Kurzweil-Pettis primitives are only finitely additive.
Moreover, we try to obtain some Radon-Nikodým theorems in the finitely additive case, using set-valued Henstock-Kurzweil-Pettis integrable derivatives and without the assumption of separability.

The thesis is organized as follows. The first chapter is devoted, on the one hand, to fix the notations and terminology used throughout all the thesis and, on the other hand, to give some preliminary notions and results that are a useful tool for the next chapters.
In particular, the notions of support function, Hausdorff distance, measurability of multifunctions, Pettis, Henstock and Henstock-Kurzweil-Pettis integrals and multimeasures are introduced.
Moreover, some representation theorems for Pettis, Henstock and Henstock-Kurzweil-Pettis integrable multifunctions are recalled. Such results are well known and are presented without proof.

The second chapter is devoted to study the decomposability for vector-valued functions integrable in the Henstock sense.
The notion of decomposability that is considered here presents a slight but

essential modification with respect to the classical notion of decomposability. Indeed, in the framework of Bochner and Pettis integrability the decomposability is defined on the σ-algebra of all measurable sets, while in this context it is defined on the ring generated by the intervals $[a, b) \subseteq [0, 1]$.
First we introduce some preliminary lemmas. Then we study some properties of decomposable subsets of the space of Henstock integrable functions and more in general of Henstock-Kurzweil-Pettis integrable functions. We give also a characterization of the separable Banach spaces with the Schur property (see Proposition 2.3.5). This result is a useful tool to prove a representation theorem for decomposable sets of Henstock-Kurzweil-Pettis integrable functions (see Theorem 2.3.2).
We prove also a relationship between decomposability and convexity in the space of Henstock integrable functions (see Theorem 2.3.1). Finally, we show a representation theorem for decomposable sets of Henstock integrable functions (see Theorem 2.3.3).

In the third chapter we study finitely additive interval multimeasures. In the first part of the chapter we find some properties and in particular we focus the attention to the existence of finitely additive vector valued selections. Then we extend to the multivalued case the notion of variational measure already known for vector valued interval measure. This measure is a very useful tool for our investigation.
In the final part of the chapter we show some Radon-Nikodým theorems for finitely additive interval multimeasures.
More precisely in the convex compact case we present a result for dominated interval multimeasures (Theorem 3.4.1) that improves [6, Theorem 3.1]. The main tool we use is an extension of a finitely additive multimeasure to a countably additive multimeasure defined in the σ-algebra of the Borel subsets of $[0, 1]$ (see Proposition 3.4.1). Then we show a generalization of Theorem 3.4.1 (see Theorem 3.4.2) valid for pointwise dominated interval multimeasures and a result (see Theorem 3.4.3) similar to that we have in case of X-valued functions (see [3, Theorem 3.6]) and where the interval multimeasure takes its values on convex compact subsets of the real line.
In the more general context of convex weakly compact valued multimeasures

we find a result that works for interval multimeasures with absolutely continuous variational measure (see Theorem 3.4.4). In such a case the Radon-Nikodým property is required to the Banach space, but not the separability.

The fourth chapter is devoted to study the differentiability of multifunctions. We consider the Hukuhara difference between two sets and the notion of Hukuhara differentiability for multifunctions.
We generalize to the multivalued case some results valid for vector-valued functions. In particular we prove the almost everywhere Hukuhara differentiability for a variational Henstock primitive (see Theorem 4.2.1) and the variational Henstock integrability of a Hukuhara derivative (see Theorem 4.2.2).
A characterization of variational Henstock primitives is also given (see Theorem 4.2.4). Moreover, as an application of the Hukuhara differentiability, we show that all the scalarly measurable selections of a variationally Henstock integrable multifunction are variationally Henstock integrable (see Theorem 4.3.1).
This result is similar to a known property of the selections of Pettis and Henstock-Kurzweil-Pettis integrable multifunctions (see [24] and [23], respectively).

CHAPTER 1

NOTATIONS AND PRELIMINARIES

The terminology used throughout this thesis is standard.
Let $[0,1]$ be the unit interval of the real line, endowed with the usual topology and the Lebesgue measure λ. By $\mathcal{A}$ we denote the ring generated by the subintervals $[a,b) \subseteq [0,1]$ (it is known that $\mathcal{A}$ is dense in $\mathcal{L}$, the class of measurable subsets of $[0,1]$, that is for every $A \in \mathcal{L}$ and for every $\varepsilon > 0$, there exists $B \in \mathcal{A}$ such that $\lambda(A \,\Delta\, B) < \varepsilon$ [16, Teorema 11, p. 42]). By $\mathcal{I}$ we denote the family of all non trivial closed subintervals of $[0,1]$.
X is a Banach space, whose norm is denoted by $||\cdot||$, with topologic dual X^*. We denote by $B(X^*)$ the closed unit ball of X^*. $\mathcal{B}(X)$ is the Borel σ-algebra of X. By 2^X we denote the family of all non-empty subsets of X. We define the following subfamilies of 2^X:

- $CL(X)$: closed subsets of X,

- $CC(X)$: closed convex subsets of X,

- $CB(X)$: closed bounded subsets of X,

- $CBC(X)$: closed bounded convex subsets of X,

- $CK(X)$: convex compact subsets of X,

- $CWK(X)$: convex weakly compact subsets of X.

If $A \in 2^X$, then $\overline{A}$ is its *closure*.
The set $co(A) = \{\sum_{i=1}^n a_i x_i : x_i \in A, a_i \in [0,1], \sum_{i=1}^n a_i = 1\}$ is the *convex hull* of A, while the set $\overline{co}(A) = \overline{co(A)}$ is the *closed convex hull* of A.
On 2^X we consider the *Minkowski addition* and the *scalar multiplication*, respectively defined by

$$C \dot{+} C' := \overline{\{x + x' : x \in C,\, x' \in C'\}} \quad \alpha C := \{\alpha x : x \in C\},$$

where $C, C' \in 2^X$ and $\alpha \in \mathbb{R}$.

1.1 The support function

Definition 1.1.1. Let $C \in 2^X$. The *support function* of C is denoted by $s(\cdot, C)$ and defined on X^* by

$$s(x^*, C) := \sup\{\langle x^*, x\rangle : x \in C\}, \text{ for every } x^* \in X^*,$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing.

The support function is an important tool in multivalued analysis and allows us to derive properties of closed convex sets.
In particular, for every $C \in CC(X)$, we have

$$C = \bigcap_{x^* \in X^*} \{x \in X : \langle x^*, x\rangle \leq s(x^*, C)\}.$$

The support function satisfies the following properties

$$s(x^*, C) = s(x^*, co(C)), \forall\, x^* \in X^*, \forall\, C \in 2^X,$$
$$s(x^*, C + C') = s(x^*, C) + s(x^*, C'), \forall\, x^* \in X^*, \forall\, C, C' \in 2^X.$$

It is homogeneous and subadditive

$$s(\alpha x^*, C) = \alpha s(x^*, C), \forall\, \alpha \geq 0, \forall\, x^* \in X^*, \forall\, C \in 2^X,$$
$$s(x_1^* + x_2^*, C) \leq s(x_1^*, C) + s(x_2^*, C), \forall\, x_1^*, x_2^* \in X^*, \forall\, C \in 2^X.$$

In particular,

$$s(x^*, C) + s(-x^*, C) \geq 0, \text{ for every } C \in 2^X.$$

We denote by w^* and τ respectively the *weak-star* topology and the *Mackey topology* on X^*. We recall that the Mackey topology on X^* is the topology of the uniform convergence on convex weakly compact subsets of X.
It is useful to recall some dual characterizations of the above classes of subsets in terms of support functions (see [24, Proposition 1.5]).

Proposition 1.1.1. *Let $C \in CC(X)$. Then the following equivalences hold:*

(a) *$C \in CBC(X)$ if and only if $s(\cdot, C)$ is strongly continuous on X^*,*

(b) *$C \in CWK(X)$ if and only if $s(\cdot, C)$ is τ-continuous on X^*,*

(c) *$C \in CK(X)$ if and only if the restriction of $s(\cdot, C)$ to $B(X^*)$ is w^*-continuous on X^*.*

1.2 Hausdorff distance

Definition 1.2.1. Let $x \in X$ and $A \in 2^X$. The *distance* of x from A is defined by $d(x, A) := \inf\{||x - a|| : a \in A\}$.

Definition 1.2.2. Let $A, B \in 2^X$.

(a) $e(A, B) := \sup\{d(x, B) : x \in A\}$ is the *excess* of A over B.

(b) $d_H(A, B) := \max\{e(A, B), e(B, A)\}$ is the *Hausdorff distance* between A and B.

It is easy to check that the following properties hold for any $A, B, C \in 2^X$:

(i) $d_H(A, A) = 0$, for every $A \in 2^X$.

(ii) $d_H(A, B) = d_H(B, A)$, for every $A, B \in 2^X$.

(iii) $d_H(A, C) \leq d_H(A, B) + d_H(B, C)$, for every $A, B, C \in 2^X$.

Hence d_H is an extended pseudometric on 2^X. We have $d_H(A, B) = 0$ if and only if $\overline{A} = \overline{B}$. Moreover, if both A and B are bounded, then $d_H(A, B)$ is

guaranteed to be finite. Hence $CB(X)$ endowed with the Hausdorff distance is a metric space.
Moreover, $(CB(X), d_H)$ is a complete metric space (see [32, Theorem 1.1.5] or [12, Theorem II.3]). Of particular interest are the following subspaces $CBC(X)$, $CK(X)$, $CWK(X)$ of $CB(X)$. Indeed they are closed, complete subsets of $(CB(X), d_H)$ (see [32, Proposition 1.1.8]).
If $A \in 2^X$, then we define

$$||A|| := \sup\{||x|| : x \in A\}.$$

From the definition of Hausdorff distance it follows that

$$||A|| = d_H(A, \{0\}). \tag{1.1}$$

It is useful for the applications the following *Hörmander formula* (see [32, Theorem 1.13]): for every $A, B \in CBC(X)$, one has

$$d_H(A, B) = \sup\{|s(x^*, A) - s(x^*, B)| : x^* \in B(X^*)\}. \tag{1.2}$$

It is easy to check that

$$d_H(A, B) = \sup\{s(x^*, A) - s(x^*, B) : x^* \in B(X^*)\}.$$

By (1.1) and (1.2), for every $A \in CBC(X)$, one has

$$||A|| = \sup\{|s(x^*, A)| : x^* \in B(X^*)\}.$$

Other properties of the Hausdorff distance are listed in the following proposition. The proof can be found in [17, pp.69-70].

Proposition 1.2.1. *Let $A, A_1, B, B_1 \in CB(X)$. Then*

(i) $d_H(tA, tB) = t d_H(A, B)$ *for all* $t > 0$,

(ii) $d_H(A \dotplus B, A_1 \dotplus B_1) \leq d_H(A, A_1) + d_H(B, B_1)$.

If $A, B \in CBC(X)$ and $C \in CB(X)$, then

(iii) $d_H(A \dotplus C, B \dotplus C) = d_H(A, B)$.

We recall also a fundamental result, known as *Rådstrom Embedding Theorem.*

Theorem 1.2.1 (Rådstrom Embedding Theorem)**.** *Let consider the map* $R : CWK(X) \to \ell_\infty(B(X^*))$ *given by* $R(C)(x^*) = s(x^*, C)$. *Then* R *satisfies the following properties:*

1. $R(C + D) = R(C) + R(D)$ *for every* $C, D \in CWK(X)$*;*

2. $R(\alpha C) = \alpha R(C)$ *for every* $\alpha \geq 0$ *and* $C \in CWK(X)$*;*

3. $d_H(C, D) = ||R(C) - R(D)||_\infty$ *for every* $C, D \in CWK(X)$*;*

4. $R(CWK(X))$ *is closed in* $\ell_\infty(B(X^*))$*.*

1.3 Measurable multifunctions

A *multifunction* is a map $F : [0, 1] \to 2^X$. We consider multifunctions which take their values on the above subcollections of 2^X.
Given a multifunction $F : [0, 1] \to 2^X$, we call a *selection* of F a function $f : [0, 1] \to X$ such that $f(t) \in F(t)$ for almost every $t \in [0, 1]$.
The set

$$G(F) := \{(t, x) \in [0, 1] \times X : x \in F(t)\}$$

is called the *graph* of F.
For every $B \in 2^X$, we set

$$F^-(B) := \{t \in [0, 1] : F(t) \cap B \neq \emptyset\}.$$

We start with the classical notion of measurability, known as the “Effros measurability”.

Definition 1.3.1. A multifunction $F : [0, 1] \to 2^X$ is said to be *Effros measurable* or simply *measurable* if for each open $O \in 2^X$, the set $F^-(O) \in \mathcal{L}$.

Theorem 2.1.35 in [32] gives a detailed description about measurability of closed-valued multifunctions into a separable Banach space.

Theorem 1.3.1. *Let X be a separable Banach space and let consider a multifunction $F : [0,1] \to CL(X)$. The following statements are equivalent:*

1. *F is measurable;*
2. *for each $B \in \mathcal{B}(X)$, $F^{-}(B) \in \mathcal{L}$;*
3. *for each $C \in CL(X)$, $F^{-}(C) \in \mathcal{L}$;*
4. *$G(F)$ is $\mathcal{L} \otimes \mathcal{B}(X)$-measurable.*

Properties *2*, *3*, *4*, of Theorem 1.3.1 are known, respectively, as the "Borel measurability", "strong measurability" and "graph measurability".
An important question concerning a measurable multifunction is the existence of measurable selections. One of most important results in this direction is the Kuratowski-Ryll Nardzewski Theorem (see [35]), which involves closed-valued multifunctions into a separable Banach space.

Theorem 1.3.2 (Kuratowski-Ryll Nardzewski)**.** *Let X be a separable Banach space and let $F : [0,1] \to CL(X)$ be a measurable multifunction. Then F admits a measurable selection.*

An application of the Kuratowski-Ryll Nardzewski Theorem allows to prove the following density result (cf. [32, Proposition 2.2.3]).

Theorem 1.3.3. *Let X be a separable Banach space and let consider a multifunction $F : [0,1] \to CL(X)$. The following two statements are equivalent:*

1. *F is measurable;*
2. *There exists a sequence $(f_n)_{n=1}^{\infty}$ of measurable X-valued functions on $[0,1]$ such that $F(t) = \overline{\{f_n(t) : n \geq 1\}}$ for every $t \in [0,1]$.*

Definition 1.3.2. A multifunction $F : [0,1] \to 2^X$ is said to be *scalarly measurable* if for every $x^* \in X^*$, the map $s(x^*, F(\cdot))$ is measurable.

The notion of scalar measurability is more appropriate than the classic one for the study of convex-valued multifunctions, because of the presence of

support function.
If X is separable, the scalar measurability of $CWK(X)$-valued multifunctions yields their measurability [32, Proposition 2.2.39]. The reverse implication is always true.
We conclude this section recalling the definition of Bochner measurability.

Definition 1.3.3. A multifunction $\Gamma : [0,1] \to 2^X$ is said to be a *simple multifunction* if there exists a finite collection $\{E_1, \ldots, E_p\}$ of measurable subsets of $[0,1]$, pairwise disjoint, such that Γ is constant on each E_j.
A multifunction $\Gamma : [0,1] \to 2^X$ is said to be *Bochner measurable* if there exists a sequence $(\Gamma_n)_{n=1}^{\infty}$ of simple multifunctions such that $\Gamma_n \to \Gamma$ almost everywhere, where the convergence is with respect to the Hausdorff metric.

1.4 Pettis type integration

Definition 1.4.1. A function $f : [0,1] \to X$ is said to be *scalarly integrable* if for every $x^* \in X^*$, the real function $\langle x^*, f(\cdot)\rangle$ is integrable.
A scalarly integrable function $f : [0,1] \to X$ is said to be *Pettis integrable* if for every $E \in \mathcal{L}$, there exists $x_E \in X$ such that

$$\langle x^*, x_E\rangle = \int_E \langle x^*, f\rangle \, d\lambda, \text{ for every } x^* \in X^*.$$

We call the x_E *Pettis integral* of f on E and we write $x_E := (P)\int_E f \, d\lambda$.

We denote by $P([0,1], X)$ the space of X-valued Pettis integrable functions on $[0,1]$.
A Pettis integrable function $f : [0,1] \to X$ is scalarly measurable (i.e. for every $x^* \in X^*$ the real function $\langle x^*, f(\cdot)\rangle$ is measurable).
If X is separable, then by the Pettis Measurability Theorem, scalar measurability and strong measurability are equivalent. We recall that a function $f : [0,1] \to X$ is strongly measurable if it is the limit of an almost everywhere convergent sequence of measurable simple functions.
It is useful to recall a result due to K. Musiał (see [41, Theorem 5.2]) which provides a characterization of the Pettis integrability for a scalarly integrable, strongly measurable function $f : [0,1] \to X$.

Theorem 1.4.1. *A strongly measurable and scalarly integrable function $f : [0,1] \to X$ is Pettis integrable if and only if the set $\{\langle x^*, f\rangle : x^* \in B(X^*)\}$ is uniformly integrable (i.e. $\lim_{\lambda(A)\to 0} \sup_{x^*\in B(X^*)} \int_A |\langle x^*, f\rangle| \, d\lambda = 0$).*

One can define a norm on $P([0,1], X)$ by

$$||f||_P := \sup_{x^*\in B(X^*)} \int_0^1 |\langle x^*, f\rangle| \, d\lambda.$$

An easy calculation shows that

$$\sup_{E\in\mathcal{L}} \left|\left|(P) \int_E f \, d\lambda\right|\right|$$

defines an equivalent norm on $P([0,1], X)$ (see [41, p. 198]).
We also consider in $P([0,1], X)$ the τ_P*-topology*, defined by the following convergence of nets:

$$f_\alpha \to f \Leftrightarrow ||\langle x^*, f_\alpha\rangle \to \langle x^*, f\rangle||_{L^1([0,1])}, \text{ for every } x^* \in B(X^*).$$

We finally consider the topology induced by the tensor product of $L^\infty([0,1])$ and $B(X^*)$. It is known as the *weak Pettis topology* and defined as:

$$f_\alpha \to f \Leftrightarrow \int_0^1 g \, \langle x^*, f_\alpha\rangle \, d\lambda \to \int_0^1 g \, \langle x^*, f\rangle \, d\lambda,$$

for every $x^* \in B(X^*)$ and every $g \in L^\infty([0,1])$.

Definition 1.4.2. A multifunction $F : [0,1] \to CC(X)$ is said to be *scalarly integrable* if for every $x^* \in X^*$ the real function $s(x^*, f(\cdot))$ is integrable.
A scalarly integrable multifunction $F : [0,1] \to CC(X)$ is said to be *Pettis integrable* in $CC(X)$ ($CBC(X)$, $CK(X)$, $CWK(X)$, respectively) if for every $E \in \mathcal{L}$, there exists $C_E \in CC(X)$ ($CBC(X)$, $CK(X)$, $CWK(X)$, respectively) such that

$$s(x^*, C_E) = \int_E s(x^*, F) \, d\lambda, \text{ for every } x^* \in X^*.$$

We call C_E the *Pettis integral* of F on E and we write $C_E := (P) \int_E F \, d\lambda$.

By the definition, it follows that Pettis integrable multifunctions are scalarly measurable.
Some authors (see for instance [24, 42]) use a more general definition of Pettis integrable multifunction. In particular, they use the condition of scalar quasi-integrability to define the Pettis integrability. In such a case we say that a multifunction F is *quasi-Pettis integrable*.
Part (iv) of [24, Example 3.3] shows that a multifunction $F : [0,1] \to CC(X)$ can be quasi-Pettis integrable in $CC(X)$ without being scalarly integrable. However, if F is a scalarly integrable multifunction with values in $CBC(X)$ and if F is quasi-Pettis integrable, then it is Pettis integrable in $CBC(X)$. This follows from the fact that a subset is bounded if and only if its support function is finite at each point of X^*.
Conversely, if F is quasi-Pettis integrable in $CBC(X)$, then F is scalarly integrable.
Given a multifunction F, by S_F^P we denote the family of all Pettis integrable selections of F.

Definition 1.4.3. A measurable multifunction $F : [0,1] \to CL(X)$ is said to be *Aumann-Pettis integrable* if $S_F^P \neq \emptyset$. In such case we define

$$(AP)\int_0^1 F\,d\lambda := \left\{(P)\int_0^1 f\,d\lambda : f \in S_F^P\right\}.$$

Proposition 2.2 in [24] indicates an important relationship between the scalar integrability of a multifunction and the scalar integrability of its measurable selections.

Proposition 1.4.1. *Let $F : [0,1] \to CB(X)$ be a measurable multifunction. The following statements are equivalent:*

1. *F is scalarly integrable;*
2. *for every $x^* \in B(X^*)$, the real function $s(x^*, F(\cdot))^+$ is Lebesgue integrable;*
3. *every measurable selection of F is scalarly integrable.*

The following result shows the relationship between Aumann-Pettis and Pettis integrability (see [24, Theorem 3.7]).

Theorem 1.4.2. *Let X be a separable Banach space. Let $F : [0,1] \to CC(X)$ be a measurable multifunction such that $\int_{[0,1]} s(x^*, F(t))^- \, dt < +\infty$. Consider the following statements:*

1. *F is Aumann-Pettis integrable;*
2. *the set $\{s(x^*, F(\cdot))^- : x^* \in B(X^*)\}$ is uniformly integrable;*
3. *F is quasi-Pettis integrable in $CC(X)$.*

Then, one has 1. $\Rightarrow$ 2. $\Rightarrow$ 3.

Moreover, the following characterization of $CWK(X)$-valued Pettis integrable multifunctions holds (see [24, Theorem 5.4]).

Theorem 1.4.3. *Assume that X is a separable Banach space.*
Let $F : [0,1] \to CWK(X)$ be a measurable and scalarly integrable multifunction. The following statements are equivalent:

1. *F is Pettis integrable in $CWK(X)$;*
2. *the set $\{s(x^*, F) : x^* \in B(X^*)\}$ is uniformly integrable;*
3. *each measurable selection of F is Pettis integrable;*
4. *for every $E \in \mathcal{L}$, $(AP) \int_E F \, d\lambda \in CWK(X)$ and*

$$s\left(x^*, (AP) \int_I F \, d\lambda\right) = (P) \int_I s(x^*, F) \, d\lambda, \text{ for every } x^* \in X^*.$$

The above theorem remains true if $CWK(X)$ is replaced by $CK(X)$ (see [24, Theorem 5.5]).

1.5 Henstock type integration

A *tagged interval* is a pair (I, t), where I is a compact interval of $[0, 1]$ and $t \in [0, 1]$.
Two compact intervals $I, J \subseteq [0, 1]$ are called *non-overlapping* if $\mathring{I} \cap \mathring{J} = \emptyset$, where $\mathring{I}$ denotes the interior of the interval I.
A finite collection $\{(I_j, t_j)\}_{j=1}^q$ of pairwise non-overlapping intervals is called a *partition in* $[0, 1]$.
Given a subset E of $[0, 1]$, we say that the partition $\{(I_j, t_j)\}_{j=1}^q$ is *anchored* on E if $t_j \in E$ for each $j = 1, \ldots, q$.
A partition $\{(I_j, t_j)\}_{j=1}^q$ in $[0, 1]$ such that $t_j \in I_j$ for every $j = 1, \ldots, q$ is called a *Perron partition in* $[0, 1]$.
A partition $\{(I_j, t_j)\}_{j=1}^q$ in $[0, 1]$ such that $\bigcup_{j=1}^q I_j = [0, 1]$ is called a *partition of* $[0, 1]$.
Similarly, a Perron partition $\{(I_j, t_j)\}_{j=1}^q$ in $[0, 1]$ such that $\bigcup_{j=1}^q I_j = [0, 1]$ is called a *Perron partition of* $[0, 1]$.
A *gauge on* $[0, 1]$ is a positive function $\delta : [0, 1] \to (0, +\infty)$.
Given a gauge δ on $[0, 1]$, we say that a tagged interval (I, t) is *δ-fine* if $I \subset (t - \delta(t), t + \delta(t))$.
A partition (or a Perron partition) $\{(I_j, t_j)\}_{j=1}^q$ is called *δ-fine* if all the tagged intervals (I_j, t_j), $j = 1, \ldots, q$ are δ-fine.
The following is well-known.

Lemma 1.5.1 (Cousin)**.** *Let δ be a gauge on $[0, 1]$. Then there exists a δ-fine Perron partition of $[0, 1]$.*

Now let us introduce the definition of the Henstock integral.

Definition 1.5.1. A function $f : [0, 1] \to X$ is said to be *Henstock integrable* on $[0, 1]$ if there exists $x \in X$ with the following property: for every $\varepsilon > 0$ there exists δ gauge on $[0, 1]$ such that

$$\left\| \sum_{i=1}^{q} f(t_j)|I_j| - x \right\| < \varepsilon,$$

for every δ-fine Perron-partition $\{(I_j, t_j)\}_{j=1}^q$ of $[0, 1]$.
We call x the *Henstock integral* of f and we set $x := (H) \int_0^1 f \, d\lambda$.

If $X = \mathbb{R}$, then f is said to be *Henstock-Kurzweil integrable* or simply *HK-integrable* on $[0, 1]$ and the *Henstock-Kurzweil integral* (or *HK-integral*) is denoted by $x := (HK) \int_0^1 f \, d\lambda$.
It is well known that if $f : [0, 1] \to X$ is Henstock integrable on $[0, 1]$ and $I \in \mathcal{I}$, then also the function $f\chi_I$ is Henstock integrable on $[0, 1]$ [49, Theorem 3.3.4]. We say in such a case that f is *Henstock integrable on* I.
We denote by $\mathcal{H}([0, 1], X)$ the space of all X-valued Henstock integrable functions on $[0, 1]$. The space of *HK*-integrable functions on $[0, 1]$ is denoted by $\mathcal{HK}([0, 1])$.
It is clear that every Riemann integrable function is also *HK*-integrable. In that case the gauge δ on $[0, 1]$ is a constant function. In general the class of Riemann integrable function is strictly contained in the class of *HK*-integrable function, as the following example shows.

Example 1.5.1. The function $f : [0, 1] \to \mathbb{R}$ defined by $f(t) := \chi_{[0,1]\cap\mathbb{Q}}$ is *HK*-integrable but not Riemann integrable.

The next theorems show some properties of the primitives of *HK*-integrable functions and some relationship between the *HK*-integral and the Lebesgue integral.

Theorem 1.5.1 (Theorem 9.12,[29])**.** *Let $f : [0, 1] \to \mathbb{R}$ be HK-integrable on $[0, 1]$ and let $F(t) := (HK) \int_0^t f \, d\lambda$ for every $t \in [0, 1]$. Then*

(a) *F is continuous on $[0, 1]$,*

(b) *F is differentiable almost everywhere on $[0, 1]$ and $F' = f$ almost everywhere on $[0, 1]$,*

(c) *f is measurable.*

Theorem 1.5.2 (Theorem 9.1,[29])**.** *Let $f : [0, 1] \to \mathbb{R}$ be HK-integrable on $[0, 1]$.*

(a) *If f is non-negative on $[0, 1]$, then f is Lebesgue integrable on $[0, 1]$.*

(b) *If f is HK-integrable on every measurable subset of $[0, 1]$, then f is Lebesgue integrable on $[0, 1]$.*

Pettis integrability can be generalized by replacing Lebesgue integral with HK-integral for the dual product $\langle \cdot, \cdot \rangle$.

Definition 1.5.2. A function $f : [0,1] \to X$ is said to be *scalarly HK-integrable* if for every $x^* \in X^*$, the real function $\langle x^*, f(\cdot)\rangle$ is *HK*-integrable. A scalarly *HK*-integrable function $f : [0,1] \to X$ is said to be *Henstock-Kurzweil-Pettis integrable* or simply *HKP-integrable* on $[0,1]$ if for every interval $I \in \mathcal{I}$, there exists $x_I \in X$ such that

$$\langle x^*, x_I\rangle = (HK) \int_I \langle x^*, f\rangle \, d\lambda, \text{ for every } x^* \in X^*.$$

We call x_I the *HKP-integral* of f on I and we write $x_I := (HKP) \int_I f \, d\lambda$.

The space of X-valued *HKP*-integrable functions on $[0,1]$ is denoted by $\mathcal{HKP}([0,1], X)$. It is clear that $P([0,1], X) \subseteq \mathcal{HKP}([0,1], X)$. Moreover, $\mathcal{H}([0,1], X) \subset \mathcal{HKP}([0,1], X)$ and in general the inclusion is proper, as the following example shows (see [21, Example 1, p. 171]).

Example 1.5.2. Let $I_n = [a_n, b_n]$ be a sequence of subintervals of $[0,1]$ such that $a_1 = 0$, $b_n < a_{n+1}$ for every n and $\lim_{n\to+\infty} b_n = 1$. Let us define the function $f : [0,1] \to c_0$ by

$$f(t) = \left(\frac{1}{2|I_{2n-1}|} \chi_{I_{2n-1}}(t) - \frac{1}{2|I_{2n}|} \chi_{I_{2n}}(t) \right)_{n=1}^{\infty}.$$

This function is *HKP*-integrable but not Henstock integrable.

In $\mathcal{HKP}([0,1], X)$ we define the *Alexiewicz norm* by

$$||f||_A := \sup_{[a,b]\subseteq[0,1]} \left|\left| (HKP) \int_a^b f \, d\lambda \right|\right|.$$

We also consider in $\mathcal{HKP}([0,1], X)$ the τ_{HKP}*-topology*, defined by the following convergence of nets:

$$f_\alpha \to f \Leftrightarrow ||\langle x^*, f_\alpha\rangle \to \langle x^*, f\rangle||_A, \text{ for every } x^* \in B(X^*).$$

Moreover, we consider in $\mathcal{HKP}([0,1], X)$ the topology induced by the tensor product of the space of real-valued functions of bounded variation and $B(X^*)$. It is known as *w-HKP topology*, and defined as:

$$f_\alpha \to f \Leftrightarrow (HK) \int_0^1 g \, \langle x^*, f_\alpha\rangle \, d\lambda \to (HK) \int_0^1 g \, \langle x^*, f\rangle \, d\lambda,$$

for every $x^* \in B(X^*)$ and every $g : [0,1] \to \mathbb{R}$ of bounded variation.
We can generalize the multivalued Pettis integration in a similar way to vectorial case.

Definition 1.5.3. A multifunction $F : [0,1] \to CL(X)$ is said to be *scalarly HK-integrable* if for every $x^* \in X^*$, the real function $s(x^*, F(\cdot))$ is *HK*-integrable.
A scalarly *HK*-integrable multifunction $F : [0,1] \to CC(X)$ is said to be *Henstock-Kurzweil-Pettis integrable* or simply *HKP-integrable* in $CC(X)$ ($CBC(X)$, $CK(X)$, $CWK(X)$, respectively) if for every interval $I \in \mathcal{I}$, there exists $C_I \in CC(X)$ ($CBC(X)$, $CK(X)$, $CWK(X)$, respectively) such that

$$s(x^*, C_I) = (HK) \int_I s(x^*, F)\, d\lambda, \text{ for every } x^* \in X^*.$$

C_I is called the *HKP-integral* of F over I and we set $C_I := (HKP) \int_I F\, d\lambda$.

Definition 1.5.4. A multifunction $F : [0,1] \to CBC(X)$ is said to be *Henstock integrable* (resp. *McShane integrable*) if there exists $W \in CBC(X)$ with the following property: for every $\varepsilon > 0$ there exists δ gauge on $[0,1]$ such that, for every δ-fine Perron-partition (risp. partition) $\{(I_1, t_1), \ldots, (I_p, t_p)\}$ of $[0,1]$, we have

$$d_H\left(W, \sum_{j=1}^{p} F(t_j)|I_j|\right) < \varepsilon.$$

W is called the *Henstock-integral* (resp. *McShane-integral*) of F and we denote it $W := (H) \int_I F\, d\lambda$ (resp. $W := (Ms) \int_E F\, d\lambda$).

By the Hörmander equality, one has

$$d_H\left(K, \sum_{j=1}^{q} F(t_j)|I_j|\right) = \sup_{x^* \in B(X^*)} \left| s(x^*, K) - \sum_{j=1}^{q} s(x^*, F(t_j))|I_j| \right|,$$

for every partition $\{(I_j, t_j)\}_{j=1}^{q}$ of $[0,1]$ and for every $K \in CWK(X)$.
Consequently each Henstock integrable multifunction is also *HKP*-integrable and each McShane integrable multifunction is also Pettis integrable.
Given a multifunction F, we denote by S_F^H and S_F^{HKP} the families of all measurable selections of F that are respectively Henstock integrable and *HKP*-integrable.

Definition 1.5.5. A multifunction $F : [0,1] \to CBC(X)$ is said to be *variationally Henstock integrable*, or simply *variationally H-integrable*, if there exists a finitely additive multifunction $\Phi : \mathcal{I} \to CBC(X)$, satisfying the following condition: given $\varepsilon > 0$ there exists a gauge δ on $[0,1]$ such that

$$\sum_{j=1}^{q} d_H(F(t_j)|I_j|, \Phi(I_j)) < \varepsilon, \tag{1.3}$$

for every δ-fine Perron-partition $\{(I_j, t_j)\}_{j=1}^{q}$ of $[0,1]$.
We set $\Phi(I) = (vH)\int_I F\, d\lambda$ and call Φ the *variational Henstock primitive* of F.

Obviously, each variationally Henstock integrable multifunction is also Henstock integrable.

Definition 1.5.6. A measurable multifunction $F : [0,1] \to CL(X)$ is said to be *Aumann-Henstock-Kurzweil-Pettis integrable* or simply *AHKP-integrable* if $S_F^{HKP} \neq \emptyset$. In this case we define

$$(AHKP)\int_0^1 F\, d\lambda := \left\{ (HKP)\int_0^1 f\, d\lambda : f \in S_F^{HKP} \right\}.$$

F is said to be *Aumann-Henstock integrable* if $S_F^{H} \neq \emptyset$. In this case we define

$$(AH)\int_0^1 F\, d\lambda := \left\{ (H)\int_0^1 f\, d\lambda : f \in S_F^{H} \right\}.$$

Next theorem due to L. Di Piazza and K. Musiał states the following characterization of $CWK(X)$-valued *HKP*-integrable multifunctions. It was established in [21, Theorem 1] for separable Banach spaces and in [23, Theorem 1] for an arbitrary Banach space.

Theorem 1.5.3. *Let $F : [0,1] \to CWK(X)$ be scalarly HK-integrable. Then the following statements are equivalent:*

1. *F is HKP-integrable in CWK(X).*
2. *S_F^{HKP} is non-empty and for every $f \in S_F^{HKP}$ there esists a multifunction $G : [0,1] \to CWK(X)$ such that $F = G + f$ and G is Pettis integrable in CWK(X).*

3. *Each scalarly measurable selection of F is HKP-integrable.*

4. *For every $I \in \mathcal{I}$, $(AHKP)\int_I F\, d\lambda \in CWK(X)$ and*

$$s\left(x^*, (AHKP)\int_I F\, d\lambda\right) = (HK)\int_I s(x^*, F)\, d\lambda,$$

for every $x^* \in X^*$.

By [21, Remark 1] and [23, Theorem 2], the above theorem remains true if $CWK(X)$ is replaced by $CK(X)$. Moreover, *AHKP*-integral and *HKP*-integral coincide.
The following equivalence between Pettis and McShane integrability in $CK(X)$ holds when X is separable.

Proposition 1.5.1 (Proposition 2, [22]). *Let $F : [0,1] \to CK(X)$ be a multifunction. Then F is Pettis integrable in CK(X) if and only if it is McShane integrable.*

Moreover, we recall the following characterization in separable case.

Theorem 1.5.4 (Theorem 2, [22]). *Let $F : [0,1] \to CK(X)$ be scalarly HK-integrable. Then the following statements are equivalent:*

1. *F is Henstock integrable.*

2. *S_F^H is non-empty and for every $f \in S_F^H$ there esists a multifunction $G : [0,1] \to CK(X)$ such that $F = G + f$ and G is McShane integrable.*

3. *Each measurable selection of F is Henstock integrable.*

If X does not contain any copy of c_0, then the above conditions are equivalent also to:

4. *S_F^H is non-empty.*

By previous result it follows that, if X does not contain any copy of c_0, then a measurable $CK(X)$-valued multifunction is Henstock integrable if and only if it is Aumann-Henstock integrable.

1.6 Multimeasures and their selections

The theory of multimeasures is a natural extension of the classical theory of vector measures. It can be viewed as an outgrowth of integration theory of multifunctions. Multimeasures are a useful analytical tool in mathematical economics, in statistics and in control theory.

In this section we present the different notions of multimeasure existing in literature and compare them.

We recall that a series $\sum_{n=1}^{\infty} x_n$ is said to be *unconditionally convergent* if for every one-to-one map f from $\mathbb{N}$ onto itself the series $\sum_{n=1}^{\infty} x_{f(n)}$ is convergent.

Definition 1.6.1. Given a sequence $(C_n)_{n=1}^{\infty} \subset 2^X$, the infinite sum $\sum_{n=1}^{\infty} C_n$ is defined by

$$\sum_{n=1}^{\infty} C_n := \left\{ x \in X : x = \sum_{n=1}^{\infty} x_n \text{ (uncond. convergent)}, x_n \in C_n \right\} \tag{1.4}$$

A first definition of multimeasure refers to the summability notion induced by (1.4).

Definition 1.6.2. A multifunction $M : \mathcal{L} \to 2^X$ is said to be a *strong multimeasure* if for every sequence $(A_n)_{n=1}^{\infty} \subset \mathcal{L}$ of pairwise disjoint sets, we have $M(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} M(A_n)$.

Example 1.6.1. Let C be a nonempty bounded subset of X and let $\mathcal{S}$ be a collection of X-valued measures such that $m(A) \in \lambda(A)C$ for every $m \in \mathcal{S}$ and every $A \in \mathcal{L}$. Let define $M : \mathcal{L} \to 2^X$ by

$$M(A) := \left\{ \sum_{k=1}^{n} m_k(A_k) : \{A_k\}_{k=1}^{n} \ \mathcal{L}\text{-partition of } A, \{m_k\}_{k=1}^{n} \subset \mathcal{S}, n \geq 1 \right\}.$$

Then M is a strong multimeasure.

A second definition of multimeasure uses the Hausdorff distance on the space of closed sets.

Definition 1.6.3. $M : \mathcal{L} \to CL(X)$ is said to be a *d_H-multimeasure* if for every sequence $(A_n)_{n=1}^{\infty} \subset \Sigma$ of pairwise disjoint sets with $A = \bigcup_{n=1}^{\infty} A_n$, we have $d_H(M(A), \sum_{k=1}^{n} M(A_k)) \to 0$ as $n \to +\infty$.

Example 1.6.2. Let X be a separable Banach space. Let consider a multifunction $F : [0,1] \to CB(X)$ graph measurable and integrably bounded (i.e., the real function $t \mapsto ||F(t)||$ is Lebesgue integrable).
Define $M : \mathcal{L} \to CL(X)$ by $M(A) := \overline{(A)\int_A F\,d\lambda}$. It is easy to check that M is a d_H-multimeasure.

A third definition involves support functions. It is the most popular and flexible definition.

Definition 1.6.4. A multifunction $M : \mathcal{L} \to CL(X)$ is said to be a *weak multimeasure* or simply a *multimeasure* if for every $x^* \in X^*$, $A \mapsto s(x^*, M(A))$ is a real valued measure.

Example 1.6.3. Let X be a separable Banach space and let $F : [0,1] \to 2^X$ be a graph measurable and Aumann-Pettis integrable function.
Define $M : \mathcal{L} \to CL(X)$ by $M(A) := \overline{(AP)\int_A F\,d\lambda}$. Then M is a multimeasure.

The main connections between the above three definitions are provided in the next proposition [32, Proposition 8.4.7].

Proposition 1.6.1. *(a) If $M : \mathcal{L} \to 2^X$ is a strong multimeasure, then the map $A \mapsto \overline{M(A)}$ is a d_H-multimeasure.*

(b) If $M : \mathcal{L} \to CL(X)$ is a d_H-multimeasure, then M is a multimeasure.

Example 1.6.4. An integrably bounded multifunction is Aumann-Pettis integrable, but the converse is false. Indeed, let us consider the measurable multifunction F defined by

$$F(t) = B(0, r(t)) = \text{the closed ball of radius } r(t) \text{ centered at the origin},$$

where $r : [0,1] \to (0, +\infty)$ is a given nonintegrable measurable function. F is Aumann integrable (hence Aumann-Pettis integrable) and its Aumann integral over $[0,1]$ is equal to X. But F is not integrably bounded.
Consequently the map $M(A) := (AP)\int_A F\,d\lambda$ is a multimeasure but not a d_H-multimeasure.

The three notions coincide whenever the multimeasure takes its values in $CWK(X)$ (see [32, Theorem 8.4.10]).

Theorem 1.6.1. *If $M : \mathcal{L} \to CWK(X)$, then M is a d_H-multimeasure if and only if M is a multimeasure.*

Definition 1.6.5. Let $M : \mathcal{L} \to 2^X$ be a multimeasure. A vector measure $m : \mathcal{L} \to X$ such that $m(A) \in M(A)$ for every $A \in \mathcal{L}$ is called a *selection* of M.

Definition 1.6.6. Let $M : \mathcal{L} \to 2^X$ be a multimeasure. We say that M is λ*-continuous* and denote it by $M << \lambda$ if $\lambda(A) = 0$ yields $M(A) = \{0\}$.

Definition 1.6.7. Let $M : \mathcal{L} \to 2^X$ be a multimeasure. For every $A \in \mathcal{L}$ we define

$$|M|(A) := \sup \sum_i ||M(A_i)||,$$

where the supremum is taken over all finite partitions $(A_i)_i$ of A in $\mathcal{L}$.
We say that M is of *finite variation* if $|M|([0,1]) < +\infty$.
We say that M is of σ*-finite variation* if there exists a sequence $(A_n)_{n=1}^{\infty} \subset \mathcal{L}$ of pairwise disjoint sets covering $[0,1]$ and such that $|M|(A_n) < +\infty$ for every positive integer n.

We end this section recalling that a Banach space X is said to have the *Radon-Nikodým property* (shorty *RNP*) if for every X-valued measure $m : \mathcal{L} \to X$ which is of finite variation and with $m << \lambda$ there exists a Bochner integrable function $f : [0,1] \to X$ such that m coincides with the Bochner integral of f. It is known that reflexive Banach spaces and separable dual spaces have the RNP. Other equivalent formulations of the RNP can be found in [18, pp. 217-219].

CHAPTER 2

DECOMPOSABILITY IN THE SPACE OF *HKP*-INTEGRABLE FUNCTIONS

2.1 Introduction

In this chapter we study the notion of decomposability for vector-valued functions integrable in Henstock sense.
This notion was introduced by R. T. Rockafellar (see [44]) for vector valued measurable functions. Later, it was extended to Bochner integrable and to Pettis integrable functions. Formally, the definition of a decomposable set resembles that of a convex set. The difference is that instead of constants $\alpha \in [0,1]$, in the definition of decomposability we have a characteristic function χ_E, with $E \subseteq [0,1]$ measurable.
Decomposability and convexity are in relationship. In particular, for the Banach valued Pettis integrable functions defined on $[0,1]$ and endowed with the Alexiewicz topology, any decomposable closed set is convex [48, Theorem 11].

The decomposability is a fundamental concept in multivalued analysis. In fact, there are several results of representation of decomposable sets in terms of selections of a suitable multifunction.
F. Hiai and H. Umegaki [31] proved that any decomposable norm-closed subset of $L^1([0,1],X)$, the space of Bochner integrable functions taking values on a separable Banach space X, is exactly the family of all Bochner integrable selections of a closed-valued Aumann integrable multifunction.
Assuming norm separability in $P([0,1],X)$, the space of X-valued Pettis integrable functions, C. Godet-Thobie and B. Satco proved that every nonempty norm-closed decomposable subset of $P([0,1],X)$ coincides with the norm-closure of the Pettis integrable selections of an Aumann-Pettis integrable multifunction F [28, Theorem 25].
Imposing more conditions on the target Banach space X, as well as on a given decomposable norm-closed convex subset K of $P([0,1],X)$, N. D. Chakraborty and T. Choudhury [13, Theorem 3.3.1] improved the result of C. Godet-Thobie and B. Satco. In fact, they showed that K coincides with the family of all the selections of a Pettis integrable multifunction.
We want to gain insight on the concept of decomposability in the space of Henstock integrable functions and more in general in the space of *HKP*-integrable functions. This involves a slight but essential modification to the definition of decomposability. Indeed, the primitives of Bochner and Pettis integrable functions are countably additive, while the Henstock type primitives are finitely additive interval functions. So, in the framework of Bochner or Pettis integrability the decomposability is defined with respect to the σ-algebra of all measurable sets, while in our case we consider the decomposability with respect to the ring $\mathcal{A}$ generated by the subintervals $[a,b) \subseteq [0,1]$.
In this chapter, first we introduce some preliminary lemmas. Then we study some properties of decomposable subsets of the space of Henstock integrable functions and more in general of Henstock-Kurzweil-Pettis integrable functions. We give also a characterization of the separable Banach spaces with the Schur property (see Proposition 2.3.5). This result is a useful tool to prove a representation theorem for decomposable sets of Henstock-Kurzweil-Pettis integrable functions (see Theorem 2.3.2).

We prove also a relationship between decomposability and convexity in the space of Henstock integrable functions (see Theorem 2.3.1). Finally, we show a representation theorem for decomposable sets of Henstock integrable functions (see Theorem 2.3.3).

2.2 Basic Facts

It is useful to recall two fundamental theorems of the Banach spaces theory (see [18, p. 51]).

Theorem 2.2.1 (Krein-Smulian). *Let W be a weakly compact subset of a Banach space X. Then also $\overline{co}(W)$ is weakly compact.*

Theorem 2.2.2 (Mazur). *Let K be a compact subset of a Banach space X. Then also $\overline{co}(K)$ is compact.*

The proofs of Lemma 2.2.1 and 2.2.2 below are essentially in [28], Lemma 3 and Theorem 24 (first part). We prefer to reproduce them for seek of completeness.

Lemma 2.2.1. *Let $G : [0,1] \to CL(X)$ be an Aumann-Pettis integrable multifunction. Then there exists a sequence of functions $(g_n)_{n=1}^{\infty} \subseteq S_G^P$ such that $G(t) = \overline{\{g_n(t) : n \geq 1\}}$ for every $t \in [0,1]$.*

Proof. By Aumann-Pettis integrability assumption, G admits a Pettis integrable selection g. By Theorem 1.3.3, there exists a sequence $(f_n)_{n=1}^{\infty}$ of measurable selections of G such that $G(t) = \{f_n(t) : n \geq 1\}$ for every $t \in [0,1]$.
For each $m, n \geq 1$, we set $E_{n,m} := \{t \in [0,1] : m-1 \leq ||f_n(t)|| < m\}$ and $g_{n,m} := f_n \chi_{E_{n,m}} + g\chi_{E^c_{n,m}}$. Each $E_{n,m} \in \mathcal{L}$. Moreover, for every $n \geq 1$ the $E_{n,m}$ are pairwise disjoint and $\bigcup_{m=1}^{\infty} E_{n,m} = [0,1]$.
For every $m, n \geq 1$, $g_{n,m}$ is Pettis integrable because $g\chi_{E^c_{n,m}}$ is Pettis integrable and $f_n\chi_{E_{n,m}}$ is bounded, hence Pettis integrable. Moreover, by definition, each $g_{n,m}$ is a selection of G. Thus $g_{n,m} \in S_G^P$ for every $m, n \geq 1$.
Let $t \in [0,1]$. We prove that $\{g_{n,m}(t) : n, m \geq 1\}$ is dense in $G(t)$.
For every $x \in G(t)$ and every $\varepsilon > 0$, $||x - f_n(t)|| < \varepsilon$ for some $n \geq 1$.

$f_n(t) \in [0,1] = \bigcup_{m=1}^{\infty} E_{n,m}$, so $f_n(t) \in E_{n,m}$ for some $m \geq 1$. Consequently $f_n(t) = g_{n,m}(t)$ and $||x - g_{n,m}(t)|| < \varepsilon$. $\square$

Lemma 2.2.2. *Let $G : [0,1] \to CL(X)$ be an Aumann-Pettis integrable multifunction and let $(g_n)_{n=1}^{\infty} \subseteq S_G^P$ be such that $G(t) = \overline{\{g_n(t) : n \geq 1\}}$ for every $t \in [0,1]$. Then for every $g \in S_G^P$, for every $\varepsilon > 0$, there exists a finite collection $\{A_1, \ldots, A_n\} \subset \mathcal{L}$ of pairwise disjoint sets, with $\bigcup_{j=1}^{n} A_j = [0,1]$, such that*

$$\left|\left| g - \sum_{j=1}^{n} \chi_{A_j} g_j \right|\right|_P < \varepsilon.$$

Proof. Let $g \in S_G^P$ and let $\varepsilon > 0$.
For every $n \geq 1$ we set $E_n := \{t \in [0,1] : ||g(t) - g_n(t)|| < \frac{\varepsilon}{2}\}$. Clearly $E_n \in \mathcal{L}$ for every $n \geq 1$ and $\bigcup_{n=1}^{\infty} E_n = [0,1]$. We may assume without restrictions that the E_n are pairwise disjoint sets. Since g and g_1 are Pettis integrable, by Theorem 1.4.1, the set $\{\langle x^*, g - g_1 \rangle : x^* \in B(X^*)\}$ is uniformly integrable. So there exists $m \geq 1$ such that $||(g - g_1)\chi_{\bigcup_{n \geq m+1} E_n}||_P < \frac{\varepsilon}{2}$.
Put $A_1 = E_1 \cup \bigcup_{n \geq m+1} E_n$ and $A_j = E_j$ for $j = 2, \ldots, m$. The sets A_j are measurable, pairwise disjoint and $\bigcup_{j=1}^{m} A_j = [0,1]$.
We have

$$\begin{aligned}
\left|\left| g - \sum_{j=1}^{m} \chi_{A_j} g_j \right|\right|_P &= \left|\left| \sum_{j=1}^{m} (g - g_j)\chi_{A_j} \right|\right|_P \leq \sum_{j=1}^{m} \left|\left|(g - g_j)\chi_{A_j}\right|\right|_P \\
&= \left|\left|(g - g_1)\chi_{\bigcup_{j \geq m+1} E_j}\right|\right| + \sum_{j=1}^{m} \left|\left|(g - g_j)\chi_{E_j}\right|\right|_P < \varepsilon.
\end{aligned}$$

$\square$

It is possible to obtain a result similar to Lemma 2.2.1 for *AHKP*-integrable multifunctions.

Lemma 2.2.3. *Let $F : [0,1] \to CL(X)$ be an AHKP-integrable multifunction. Then there exists a sequence of functions $(f_n)_{n=1}^{\infty} \subseteq S_F^{HKP}$ such that $F(t) = \overline{\{f_n(t) : n \geq 1\}}$ for every $t \in [0,1]$.*

Proof. Since F is *AHKP*-integrable, $S_F^{HKP} \neq \emptyset$.
Let $h \in S_F^{HKP}$ and consider the multifunction $G : [0,1] \to CL(X)$ defined by

$G(t) := F(t) - h(t)$, $t \in [0,1]$.
G is Aumann-Pettis integrable (indeed $g \equiv 0$ is a Pettis integrable selection of G). Consequently by Lemma 2.2.1, there exists a sequence $(g_n)_{n=1}^{\infty} \subseteq S_G^P$ such that $G(t) = \overline{\{g_n(t) : n \geq 1\}}$ for every $t \in [0,1]$.
Put now $f_n = g_n + h$, $n \geq 1$. Each f_n is an *HKP*-integrable selection of F. Moreover, $F(t) = \overline{\{f_n(t) : n \geq 1\}}$, as required. □

Corollary 2.2.1. *Let $F_1, F_2 : [0,1] \to CL(X)$ be AHKP-integrable multifunctions. If $S_{F_1}^{HKP} = S_{F_2}^{HKP}$ then $F_1 = F_2$.*

Proof. By Lemma 2.2.3, there exist $(f_{1,n})_{n=1}^{\infty} S_{F_1}^{HKP}$ and $(f_{2,n})_{n=1}^{\infty} S_{F_2}^{HKP}$ such that $F_1(t) = \overline{\{f_{1,n}(t) : n \geq 1\}}$ and $F_2(t) = \overline{\{f_{2,n}(t) : n \geq 1\}}$ for all $t \in [0,1]$.
Since $S_{F_1}^{HKP} = S_{F_2}^{HKP}$, $f_{1,n} \in S_{F_2}^{HKP}$ and $f_{2,n} \in S_{F_1}^{HKP}$ for every $n \geq 1$.
Consequently, for every $n \geq 1$ and for every $t \in [0,1]$ we have $f_{1,n}(t) \in F_2(t)$, and so $F_1(t) \subseteq F_2(t)$. Similarly $F_2(t) \subseteq F_1(t)$.
Then $F_1(t) = F_2(t)$ for every $t \in [0,1]$ and the two multifunctions coincide. □

Moreover, it is possible to improve Lemma 2.2.2 in the sense that each Pettis integrable selection of an Aumann-Pettis integrable multifunction can be approximated by a combination of the type $\sum_{j=1}^{n} \chi_{B_j} g_j$ where the sets B_j are pairwise disjoint and belong to $\mathcal{A}$.

Lemma 2.2.4. *Let $G : [0,1] \to CL(X)$ be an Aumann-Pettis integrable multifunction and let $(g_n)_{n=1}^{\infty} \subseteq S_G^P$ be such that $G(t) = \overline{\{g_n(t) : n \geq 1\}}$ for every $t \in [0,1]$. Then for every $g \in S_G^P$ and for every $\varepsilon > 0$, there exists a finite $\mathcal{A}$-partition $\{M_1, \ldots, M_{s+1}\}$ of $[0,1]$ such that*

$$\left\| g - \sum_{j=1}^{s+1} \chi_{M_j} g_j \right\|_P < \varepsilon.$$

Proof. Let $g \in S_G^P$, $\varepsilon > 0$. By [13, Lemma 3.3.1], there exists a finite collection $\{A_1, \ldots, A_s\} \subset \mathcal{L}$ of pairwise disjoint sets, with $\bigcup_{j=1}^{s} A_j = [0,1]$, such that $\left\| g - \sum_{j=1}^{s} \chi_{A_j} g_j \right\|_P < \varepsilon/2$.

By the separability of X and the uniform integrability of the family of functions $\{\langle x^*, g_j\rangle : x^* \in B(X^*), j = 1, \ldots, s\}$ (see [42, Theorem 5.2]), there exists $\delta > 0$ such that, if $\lambda(A) < \delta$, then

$$||\chi_A g_j||_P = \sup_{x^* \in B(X^*)} \int_A |\langle x^*, g_j\rangle| \, d\lambda < \frac{\varepsilon}{4s}, \text{ for every } j = 1, \ldots, s.$$

For each $j = 1, \ldots, s$ there exists $B_j \in \mathcal{A}$ such that $\lambda(A_j \,\Delta\, B_j) < \frac{\delta}{4s^2}$ (see [29, Theorem 1.13]).

Now let consider $M_1, \ldots, M_s$, where $M_1 = B_1$ and $M_{j+1} = B_{j+1} \setminus \bigcup_{i=1}^{j} B_i$ for $j = 1, \ldots, s-1$. Clearly $M_1, \ldots, M_s \in \mathcal{A}$, are pairwise disjoint and $\bigcup_{j=1}^{s} M_j = \bigcup_{j=1}^{s} B_j$. We claim that $\lambda(A_j \,\Delta\, M_j) < \frac{\delta}{s}$ for every $j = 1, \ldots, s$.

In fact, since $\lambda(A_j \,\Delta\, B_j) < \frac{\delta}{4s^2}$, $|\lambda(A_j) - \lambda(B_j)| < \frac{\delta}{4s^2}$. Hence for every $i \neq j$,

$$\lambda(B_i \cap B_j) = \lambda(B_i) + \lambda(B_j) - \lambda(B_i \cup B_j) < \lambda(A_i) + \lambda(A_j) + \frac{\delta}{2s^2} - \lambda(B_i \cup B_j).$$

Moreover, for every $i \neq j$,

$$\lambda((A_i \cup A_j) \,\Delta\, (B_i \cup B_j)) \leq \lambda(A_i \,\Delta\, B_i) + \lambda(A_j \,\Delta\, B_j) < \frac{\delta}{2s^2}.$$

Thus $\lambda(B_i \cup B_j) \geq \lambda(A_i) + \lambda(A_j) - \frac{\delta}{2s^2}$.

It follows that for $i \neq j$, $\lambda(B_i \cap B_j) < \frac{\delta}{s^2}$. Moreover, for each j, $B_j \,\Delta\, M_j = B_j \setminus M_j = \bigcup_{i<j}(B_i \cap B_j)$. Thus $\lambda(B_j \,\Delta\, M_j) \leq \sum_{i<j} \lambda(B_i \cap B_j) < (s-1)\frac{\delta}{s^2}$.

Finally for every j we have $\lambda(A_j \,\Delta\, M_j) \leq \lambda(A_j \,\Delta\, B_j) + \lambda(B_j \,\Delta\, M_j) < \frac{\delta}{s}$.

Now set $M_{s+1} := [0,1] \setminus \bigcup_{j=1}^{s} M_j$. By definition, $M_{s+1} \in \mathcal{A}$ and is disjoint to each M_j. So $\{M_1, \ldots, M_s, M_{s+1}\}$ is a finite $\mathcal{A}$-partition of $[0,1]$. Moreover, $\lambda(M_{s+1}) < \delta$. In fact,

$$\lambda(M_{s+1}) = \lambda\left(\bigcup_{j=1}^{s} A_j \,\Delta\, \bigcup_{j=1}^{s} M_j\right) \leq \sum_{j=1}^{s} \lambda(A_j \,\Delta\, M_j) < s\frac{\delta}{s} = \delta.$$

Consequently, $||\chi_{M_{s+1}} g_{s+1}||_P < \frac{\varepsilon}{4s} \leq \frac{\varepsilon}{4}$.

Finally we have

$$
\begin{aligned}
&\left\|g - \sum_{j=1}^{s+1} \chi_{M_j} g_j\right\|_P \\
&\leq \left\|g - \sum_{j=1}^{s} \chi_{A_j} g_j\right\|_P + \left\|\sum_{j=1}^{s} (\chi_{A_j} - \chi_{M_j}) g_j\right\|_P + \left\|\chi_{M_{s+1}} g_{s+1}\right\|_P \\
&\leq \frac{\varepsilon}{2} + \left\|\sum_{j=1}^{s} (\chi_{A_j \setminus M_j} - \chi_{M_j \setminus A_j}) g_j\right\|_P + \frac{\varepsilon}{4} \\
&\leq \frac{\varepsilon}{2} + \left\|\sum_{j=1}^{s} \chi_{A_j \Delta M_j} g_j\right\|_P + \frac{\varepsilon}{4} \\
&\leq \frac{\varepsilon}{2} + \sum_{j=1}^{s} \left\|\chi_{A_j \Delta M_j} g_j\right\|_P + \frac{\varepsilon}{4} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon,
\end{aligned}
$$

as required. $\square$

2.3 Decomposability in $\mathcal{HKP}([0,1], X)$ - Main Theorems

We are going to introduce the notion of decomposability with respect to $\mathcal{A}$.

Definition 2.3.1. A set $K \subseteq \mathcal{HKP}([0,1], X)$ is said to be *decomposable with respect to the ring* $\mathcal{A}$ or simply *decomposable* if for all $f_1, f_2 \in K$ and for all $E \in \mathcal{A}$, $f_1\chi_E + f_2\chi_{E^c} \in K$.

Proposition 2.3.1. *Let $F : [0,1] \to CWK(X)$ be AHKP-integrable (resp. Aumann-Henstock integrable). Then S_F^{HKP} (resp. S_F^H) is decomposable and convex.*

Proof. Since F is $CWK(X)$-valued, it is clear that S_F^{HKP} (resp. S_F^H) is convex.
Let $f_1, f_2 \in S_F^{HKP}$ (resp. S_F^H) and let $E \in \mathcal{A}$. Rewrite $E = \bigcup_{j=1}^q I_j$, where the I_j are pairwise disjoint intervals. $E^c \in \mathcal{A}$ and in particular, $E^c = \bigcup_{i=1}^p J_i$, where the J_i are pairwise disjoint intervals. Clearly $\bigcup_{j=1}^q I_j \cup \bigcup_{i=1}^p J_i = [0,1]$.

So $f = f_1\chi_E + f_2\chi_{E^c} = f_1\chi_{I_1} + \cdots + f_1\chi_{I_q} + f_2\chi_{J_1} + \cdots + f_1\chi_{J_p}$. Therefore f is *HKP*-integrable (resp. Henstock integrable).
Since f_1 and f_2 are selections of F, also f is a selection of F. $\square$

Proposition 2.3.2. *Let $K \subseteq \mathcal{HKP}([0,1], X)$ be decomposable. Then also $\overline{K}^{\|\ \|_A}$ is decomposable.*

Proof. Let $f, g \in \overline{K}^{\|\ \|_A}$ and let $E \in \mathcal{A}$. We may assume that $E = \bigcup_{j=1}^{q} I_j$ and $E^c = \bigcup_{i=1}^{p} J_i$, where $\{I_j\}_{j=1}^{q}$ and $\{J_i\}_{i=1}^{p}$ are finite collections of pairwise disjoint intervals. For each $\varepsilon > 0$, $\|f - f_\varepsilon\|_A < \frac{\varepsilon}{2k}$ and $\|g - g_\varepsilon\|_A < \frac{\varepsilon}{2k}$ for some $f_\varepsilon, g_\varepsilon \in K$, where $k = \max\{p, q\}$. Thus $\|(f - f_\varepsilon)\chi_E\|_A < \frac{\varepsilon}{2}$ and $\|(g - g_\varepsilon)\chi_{E^c}\|_A < \frac{\varepsilon}{2}$. Since K is decomposable, $f_\varepsilon\chi_E + g_\varepsilon\chi_{E^c} \in K$. Moreover, $\|(f\chi_E + g\chi_{E^c}) - (f_\varepsilon\chi_E + g_\varepsilon\chi_{E^c})\|_A \leq \|(f - f_\varepsilon)\chi_E\|_A + \|(g - g_\varepsilon)\chi_{E^c}\|_A < \varepsilon$. We conclude that $f\chi_E + g\chi_{E^c} \in \overline{K}^{\|\ \|_A}$. $\square$

In general, the family of all *HKP*-integrable selections of a given multifunction F is not $\|\ \|_A$-closed in $\mathcal{HKP}([0,1], X)$. In the next proposition, we show that if F is $CWK(X)$-valued and *HKP*-integrable in $CWK(X)$, then S_F^{HKP} is $\|\ \|_A$-closed in $\mathcal{HKP}([0,1], X)$.

Proposition 2.3.3. *If the multifunction $F : [0,1] \to CWK(X)$ is HKP-integrable in CWK(X), then S_F^{HKP} is $\|\ \|_A$-closed in $\mathcal{HKP}([0,1], X)$.*

Proof. By Theorem 1.5.3, S_F^{HKP} is non-empty and for a fixed $\gamma \in S_F^{HKP}$ there exists $G : [0,1] \to CWK(X)$ Pettis integrable in $CWK(X)$ such that $F(t) = \gamma(t) + G(t)$ for every $t \in [0,1]$.
Let $(f_n)_{n=1}^{\infty}$ be a sequence of *HKP*-integrable selections of F $\|\ \|_A$-converging to $f \in \mathcal{HKP}([0,1], X)$.
For every $n \geq 1$, let g_n be the Pettis integrable selection of G defined by $g_n := f_n - \gamma$.
By [11, Proposition 3.4], there exists a subsequence $(g_{n_k})_{k=1}^{\infty}$ of $(g_n)_{n=1}^{\infty}$ that converges in the weak Pettis topology to a Pettis integrable selection g of G.
In particular, for every $x^* \in B(X^*)$ and every $I \in \mathcal{I}$

$$\int_I \langle x^*, g_{n_k} \rangle \, d\lambda \to \int_I \langle x^*, g \rangle \, d\lambda.$$

By hypothesis, $||f_n - f||_A \to 0$. So $||g_n - (f-\gamma)||_A \to 0$. Consequently, $(g_n)_{n=1}^{\infty}$ converges to $f-\gamma$ in the *w-HKP* topology. In particular, for every $x^* \in B(X^*)$ and every $I \in \mathcal{I}$

$$(HK)\int_I \langle x^*, g_n\rangle \, d\lambda \to (HK)\int_I \langle x^*, (f-\gamma)\rangle \, d\lambda.$$

It follows that $\int_0^1 \langle x^*, g\rangle \, d\lambda = (HK)\int_0^1 \langle x^*, (f-\gamma)\rangle \, d\lambda$, for every $x^* \in B(X^*)$ and every $I \in \mathcal{I}$. By [29, Theorem 9.12], $\langle x^*, g\rangle = \langle x^*, (f-\gamma)\rangle$ a.e. for every $x^* \in B(X^*)$, with the null-set depending on x^*. By [18, Corollary 7, p. 48], one obtains that $g = f-\gamma$ a.e.
Since g is a Pettis integrable selection of G, f is an *HKP*-integrable selection of F, thus S_F^{HKP} is $||\ ||_A$-closed in $\mathcal{HKP}([0,1],X)$. □

Using the previous proposition we also obtain

Proposition 2.3.4. *If the multifunction $F : [0,1] \to CK(X)$ is Henstock integrable, then S_F^H is $||\ ||_A$-closed in $\mathcal{H}([0,1],X)$.*

Proof. Let $(f_n)_{n=1}^{\infty}$ be a sequence of Henstock integrable selections of F $||\ ||_A$-converging to $f \in \mathcal{H}([0,1],X)$.
Since $CK(X) \subset CWK(X)$, F is *HKP*-integrable in $CWK(X)$. Moreover, $S_F^H \subset S_F^{HKP}$. So by Proposition 2.3.3, f is an *HKP*-integrable selection of F. But f is Henstock integrable. Hence $f \in S_F^H$ and S_F^H is $||\ ||_A$-closed in $\mathcal{H}([0,1],X)$. □

2.3.1 A relationship between decomposability and convexity in $\mathcal{H}([0,1],X)$

In this subsection we are going to prove the convexity of a decomposable set in $\mathcal{H}([0,1],X)$.
Let $K \subseteq \mathcal{HKP}([0,1],X)$. The *decomposable hull* of K is the smallest decomposable set containing K, it is denoted by $dec(K)$. The $||\ ||_A$*-closed decomposable hull* of K is the smallest $||\ ||_A$-closed decomposable set containing K and it is denoted by $\overline{dec}^{||\ ||_A}(K) := \overline{dec(K)}^{||\ ||_A}$.
We begin with an easy lemma.

Lemma 2.3.1. *Let $K \subseteq \mathcal{HK}([0,1],X)$ be convex. Then also $dec(K)$ is convex.*

Proof. Let us consider $\lambda \in [0,1]$ and $f, g \in dec(K)$.
Then there exist $\{M_i\}_{i=1}^n \subseteq \mathcal{A}$ with $\bigcup_{i=1}^n M_i = [0,1]$, $\{f_i\}_{i=1}^n, \{g_i\}_{i=1}^n \subset K$ such that $f = \sum_{i=1}^n \chi_{M_i} f_i$ and $g = \sum_{i=1}^n \chi_{M_i} g_i$.
Then by convexity of K,

$$\alpha f + (1-\lambda)g = \alpha \left(\sum_{i=1}^n \chi_{M_i} f_i \right) + (1-\alpha) \left(\sum_{i=1}^n \chi_{M_i} g_i \right)$$
$$= \sum_{i=1}^n (\alpha f_i + (1-\alpha) g_i) \chi_{M_i} \in dec(K).$$

Hence $dec(K)$ is convex. □

A key lemma is the following that is similar to [48, Lemma 6] in case of the Pettis integral.

Lemma 2.3.2. *Let $\{f_i\}_{i=1}^{n+1}$ be a finite collection of functions in $\mathcal{H}([0,1],X)$ and let $\{\lambda_i\}_{i=1}^n$ be a finite set of real positive numbers with $\sum_{i=1}^n \lambda_i = 1$. Then, for every $\varepsilon > 0$, there exists a finite $\mathcal{A}$-partition $\{M_i\}_{i=1}^{n+1}$ of $[0,1]$ such that*

$$\left\| \sum_{i=1}^n \lambda_i f_i - \sum_{i=1}^{n+1} \chi_{M_i} f_i \right\|_A < \varepsilon.$$

Proof. Fix $\varepsilon > 0$.
At first we consider the case when the X-valued functions $f_1, \ldots, f_n, f_{n+1}$ are Pettis integrable.
Then by [48, Lemma 6], we can find a $\mathcal{L}$-partition $\{A_i\}_{i=1}^n \subset \mathcal{L}$ of $[0,1]$ such that $|| \sum_{i=1}^n \lambda_i f_i - \sum_{i=1}^n \chi_{A_i} f_i ||_A < \frac{\varepsilon}{2}$.
By using the same techniques of the proof of Lemma 2.2.4, we can find an $\mathcal{A}$-partition $\{M_i\}_{i=1}^{n+1}$ of $[0,1]$ such that $|| \sum_{i=1}^n (\chi_{A_i} - \chi_{M_i}) f_i ||_A < \frac{\varepsilon}{4}$ and $||\chi_{M_{n+1}} f_{n+1}||_A < \frac{\varepsilon}{4}$.
Finally we obtain $|| \sum_{i=1}^n \lambda_i f_i - \sum_{i=1}^{n+1} \chi_{M_i} f_i ||_A \leq || \sum_{i=1}^n \lambda_i f_i - \sum_{i=1}^n \chi_{A_i} f_i ||_A + || \sum_{i=1}^n (\chi_{A_i} - \chi_{M_i}) f_i ||_A + ||\chi_{M_{n+1}} f_{n+1}||_A < \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon$.
In the general case, let consider the $CK(X)$-valued multifunction defined by

$F(t) := co\{f_1(t)\dots, f_n(t), f_{n+1}(t)\}$.
By the definition of F, the family $\{s(x^*, F(\ )) : x^* \in B(X^*)\}$ is Henstock equiintegrable.
By [22, Proposition 1], F is Henstock integrable, hence *HKP*-integrable in $CWK(X)$.
The X-valued functions $f_1, \dots, f_n, f_{n+1}, f = \sum_{i=1}^{n} \lambda_i f_i$ are Henstock integrable selections of F, hence *HKP*-integrable. By Theorem 1.5.3, the functions $f - f_1, \dots, f - f_{n+1}$ are Pettis integrable.
Hence there exists an $\mathcal{A}$-partition $\{M_i\}_{i=1}^{n+1}$ of $[0,1]$ such that

$$\left|\left|\sum_{i=1}^{n+1} \chi_{M_i}(f - f_i) - \sum_{i=1}^{n} \lambda_i(f - f_i)\right|\right|_A < \varepsilon.$$

But $\sum_{i=1}^{n} \lambda_i(f - f_i) = 0$ and $\sum_{i=1}^{n+1} \chi_{M_i}(f - f_i) = f - \sum_{i=1}^{n+1} \chi_{M_i} f_i$.
Therefore $||f - \sum_{i=1}^{n+1} \chi_{M_i} f_i||_A < \varepsilon$ and the proof is over. □

Theorem 2.3.1. *Let $\emptyset \neq K \subseteq \mathcal{H}([0,1], X)$ be $||\ ||_A$-closed and decomposable. Then K is convex.*

Proof. Since X is separable, by [3, Proposition 1], also $\mathcal{H}([0,1], X)$ is separable. By hypothesis, K is closed and decomposable. So there exists a sequence $(f_n)_{n=1}^{\infty}$ of functions in $\mathcal{H}([0,1], X)$ such that

$$K = \overline{\{f_n : n \geq 1\}} = \overline{dec}^{||\ ||_A}(\{f_n : n \geq 1\}).$$

We prove that $co\{f_n : n \geq 1\} \subseteq \overline{dec}^{||\ ||_A}(\{f_n : n \geq 1\})$.
For this purpose, let $f \in co\{f_n : n \geq 1\}$ and fix $\varepsilon > 0$. By Lemma 2.2.4, there exists a finite $\mathcal{A}$-partition $\{M_k\}_{k=1}^{N}$ of $[0,1]$ such that $||f - f_\varepsilon||_A < \varepsilon$, where $f_\varepsilon := \sum_{k=1}^{N} \chi_{M_k} f_k \in dec(\{f_n : n \geq 1\})$.
Consequently, $f \in \overline{dec}^{||\ ||_A}(\{f_n : n \geq 1\})$.
By passing to the closed decomposable hull, we easily obtain the inclusion $\overline{dec}^{||\ ||_A}(co\{f_n : n \geq 1\}) \subseteq \overline{dec}^{||\ ||_A}(\{f_n : n \geq 1\})$. Since the opposite inclusion is obvious, we have $\overline{dec}^{||\ ||_A}(co\ \{f_n : n \geq 1\}) = \overline{dec}^{||\ ||_A}(\{f_n : n \geq 1\}) = K$.
By Lemma 2.3.1, the set $dec(co\{f_n : n \geq 1\})$ is convex. We conclude that $K = \overline{dec}^{||\ ||_A}(\{f_n : n \geq 1\})$ is convex. □

2.3.2 Characterization of the decomposable subsets of $\mathcal{HKP}([0,1],X)$ and $\mathcal{H}([0,1],X)$

Our aim is to characterize $\|\ \|_A$-closed, decomposable and convex subsets of $\mathcal{HKP}([0,1],X)$ in terms of *HKP*-integrable selections of a suitable *HKP*-integrable multifunction.

It was proved in [31, Theorem 3.1] that the decomposable norm-closed subsets of $L^1([0,1],X)$ are the families of all Bochner integrable selections of a suitable multifunction.

Assuming norm separability of $P([0,1],X)$, C. Godet-Thobie and B. Satco [28, Theorem 25] proved that every nonempty norm-closed decomposable subset K of $P([0,1],X)$ coincides with the closure (in $P([0,1],X)$) of S_F^P, where F is an Aumann-Pettis integrable multifunction F.

Imposing more conditions on the Banach space X as well as on the subset K of $P([0,1],X)$, N. D. Chakraborty and T. Choudhury [13, Theorem 3.3.1] improved the result of C. Godet-Thobie and B. Satco. They showed the existence of a $CWK(X)$-valued Pettis integral multifunction F such that $K = S_F^P$.

Our main result of decomposability (Theorem 2.3.2) is shown below, assuming that the Banach space is separable and has the Schur property.

We recall that a Banach space X has the *Schur property* if weak and norm sequential convergence coincide in X, i.e., a sequence $(x_n)_{n=1}^{\infty}$ in X converges to 0 weakly if and only if $(x_n)_{n=1}^{\infty}$ converges to 0 in norm.

The property above was named "Schur" in honour of Issai Schur who showed in 1921 that ℓ^1 has that property (see [49] and [1, Theorem 2.3.6]).

In general, weak and norm topologies are always distinct in infinite dimensional Banach spaces. Nevertheless, if X is a Banach with the Schur property, then every weakly compact subset of X is norm compact [1, Theorem 2.3.7]. Moreover, any Banach space with the Schur property does not contain copies of c_0 [1, Proposition 2.3.12].

We start with the following characterization.

Proposition 2.3.5. *Let X be a Banach space. The following assertions are equivalent:*

1. *X is separable and has the Schur property.*

2. *$\mathcal{HKP}([0,1],X)$ is separable.*

Proof. Assume that X is separable and has the Schur property.
Let $f \in \mathcal{HKP}([0,1],X)$ and let $F(t) = (HKP)\int_0^t f\,d\lambda$. F is weakly continuous on $[0,1]$, moreover X has the Schur property. Therefore F is continuous on $[0,1]$. Moreover, since F is defined on $[0,1]$, it is uniformly continuous.
Let us fix $\varepsilon > 0$. By the uniform continuity of F, there exists $\delta_\varepsilon > 0$ such that $|t-s| < \delta_\varepsilon$ implies $||F(t) - F(s)|| < \frac{\varepsilon}{2}$.
Now let us consider $0 = t_0 < t_1 < \ldots < t_N = 1$ such that $|t_{k+1} - t_k| < \delta_\varepsilon$ for $k = 0, \ldots, N-1$. Let us define $I_1 = [0, t_1]$ and for $k = 2, \ldots, N$ $I_k = (t_{k-1}, t_k]$.
Let $F_\varepsilon : [0,1] \to X$ be defined by

$$F_\varepsilon(t) = F(t_k) + \frac{F(t_{k+1}) - F(t_k)}{t_{k+1} - t_k}(t - t_k), \text{ if } t \in I_{k+1}.$$

We claim that $\sup_{t\in[0,1]} ||F(t) - F_\varepsilon(t)|| < \varepsilon$. If $t \in [0,1]$, then $t \in I_{k+1}$ for some k. So by the uniform continuity of F,

$$\begin{aligned} ||F(t) - F_\varepsilon(t)|| &= \left|\left| F(t) - F(t_k) - (F(t_{k+1}) - F(t_k))\frac{t - t_k}{t_{k+1} - t_k} \right|\right| \\ &\leq ||F(t) - F(t_k)|| + ||F(t_{k+1}) - F(t_k)|| \frac{t - t_k}{t_{k+1} - t_k} < \varepsilon. \end{aligned}$$

Now let us consider the step function defined by $f_\varepsilon := \sum_{k=1}^N x_k \chi_{I_k}$, where $x_k := \frac{F(t_{k+1}) - F(t_k)}{t_{k+1} - t_k}$.
Clearly f_ε is Bochner integrable and F_ε is its primitive. Finally

$$\begin{aligned} ||f - f_\varepsilon||_A &= \sup_{[a,b]} ||F_\varepsilon(b) - F_\varepsilon(a) - (F(b) - F(a))|| \\ &\leq \sup_{[a,b]} (||F_\varepsilon(b) - F(b)|| + ||F_\varepsilon(a) - F(a)||) < 2\varepsilon. \end{aligned}$$

Therefore the step functions f_ε approximate f in the Alexiewich norm. By separability of X and the fact that we can use intervals with rational endpoints, we get the separability of $\mathcal{HKP}([0,1],X)$.
Conversely, suppose that $\mathcal{HKP}([0,1],X)$ is separable. Clearly X is separable because the set of constant functions is separable and isomorfic to X.

Now assume that X does not have the Schur property. Then there exists a sequence $(x_n)_{n=1}^{\infty} \subset X$ such that $\lim_n \langle x^*, x_n\rangle = 0$ for every $x^* \in X^*$ and $||x_n|| = 1$ for every $n \geq 1$.
Now let $(I_n)_{n=1}^{\infty}$ be a sequence of mutually disjoint intervals, ordered in the sense of the real line, whose union is equal to $[0,1]$. Let divide each interval I_n in two disjoint equal parts I_n^+ and I_n^- and let define $g_n : I_n \to X$ by $g_n(t) = x_n$ if $t \in I_n^+$ and $g_n(t) = -x_n$ if $t \in I_n^-$.
Finally for every $A \subset \mathbb{N}$, set $f_A := \sum_{k\in A} \frac{1}{|I_k|} g_k$.
We are going to prove that $f_A \in \mathcal{HKP}([0,1], X)$ and $(HKP)\int_0^1 f_A\, d\lambda = 0$.
Let us fix $t \in [0,1)$. Since there is only a finite number of intervals I_k that lie in the closed interval $[0,t]$, then by definition, the restriction of f_A to $[0,t]$ is a step function. In particular, for every $x^* \in X^*$, the restriction of $\langle x^*, f_A\rangle$ to $[0,t]$ is also a step function. Therefore it is HK-integrable in $[0,t]$.
Moreover, $|(HK)\int_0^t \langle x^*, f_A\rangle\, d\lambda| \leq |\langle x^*, x_m\rangle|$, where m is the unique natural number such that $t \in I_m$. Since $\lim_n \langle x^*, x_n\rangle = 0$, then by [29, Theorem 9.21], $\langle x^*, f_A\rangle$ is HK-integrable and

$$(HK)\int_0^1 \langle x^*, f_A\rangle\, d\lambda = \lim_{t\to 1}(HK)\int_0^t \langle x^*, f_A\rangle\, d\lambda = 0.$$

Therefore f_A is HKP-integrable and $(HKP)\int_0^1 f_A\, d\lambda = 0$.
The set $\{f_A : A \subset \mathbb{N}\}$ is uncountable and satisfies the following inequality

$$||f_A - f_B||_A \geq \frac{1}{2}, \text{ for every } A \neq B.$$

In fact, suppose that $m \in A$ and $m \notin B$. Then
$(HKP)\int_{I_m^+} f_A\, d\lambda = \frac{x_m}{|I_m|}|I_m^+| = \frac{||x_m||}{2}$ and $(HKP)\int_{I_m^+} f_B\, d\lambda = 0$. Therefore

$$||f_A - f_B||_A \geq \left|\left|(HKP)\int_{I_m^+} f_A\, d\lambda - (HKP)\int_{I_m^+} f_B\, d\lambda\right|\right| = \frac{||x_m||}{2} = \frac{1}{2}.$$

But this contradicts the separability hypothesis of $\mathcal{HKP}([0,1], X)$. So we conclude that X has the Schur property. □

Lemma 2.3.3. *Let X be a separable Banach space with the Schur property and let $F : [0,1] \to CK(X)$ be a measurable and Aumann-Pettis integrable multifunction such that $\int_0^1 s(x^*, F)^-\, d\lambda < +\infty$. Then F is scalarly integrable.*

Proof. By [24, Theorem 3.7], F is quasi-Pettis integrable in $CC(X)$. Moreover, by [24, Theorem 3.9], for every $x^* \in X^*$ and every $E \in \mathcal{L}$ one has $s(x^*, (AP) \int_E F \, d\lambda) = \int_E s(x^*, F) \, d\lambda$.
We check that for every $E \in \mathcal{L}$, $(AP) \int_E F \, d\lambda$ is convex and norm compact. As the convexity is obvious we will try to prove the compactness of $(AP) \int_E F \, d\lambda$. To do it take a sequence $(x_n)_{n=1}^{\infty}$ of $(AP) \int_E F \, d\lambda$. Then there exists $(f_n)_{n=1}^{\infty} \subset S_F^P$ such that $x_n = (P) \int_E f_n \, d\lambda$.
Since F is $CK(X)$-valued, by [13, Theorem 3.4.1], S_F^P is convex and sequentially compact with respect to the weak Pettis topology of $P([0,1], X)$. Hence there exists a subsequence $(f_{n_k})_{k=1}^{\infty}$ of $(f_n)_{n=1}^{\infty}$ such that $f_{n_k} \to f$ in the weak Pettis topology.
In particular,

$$\int_E \langle x^*, f_{n_k} \rangle \, d\lambda \to \int_E \langle x^*, f \rangle \, d\lambda, \text{ for each } x^* \in X^*.$$

This means that

$$\langle x^*, x_{n_k} \rangle \to \langle x^*, x \rangle, \text{ for each } x^* \in X^*,$$

where $x = (P) \int_E f \, d\lambda$. Since X has the Schur property, $||x_{n_k} - x|| \to 0$. Therefore $(AP) \int_E F \, d\lambda$ is norm compact.
In particular, we have $||(AP) \int_E F \, d\lambda|| < +\infty$ for each $E \in \mathcal{L}$. Therefore $\int_E s(x^*, F) \, d\lambda < +\infty$ for every $x^* \in X^*$ and every $E \in \mathcal{L}$. We conclude that F is scalarly integrable. $\square$

Theorem 2.3.2. *Let X be a separable Banach space with the Schur property and let $\emptyset \neq K \subseteq \mathcal{HKP}([0,1], X)$ be decomposable, convex and $|| \, ||_A$-closed. Assume that for each $t \in [0,1]$ the set $K(t) = \{f(t) : f \in K\}$ is relatively norm compact.*
Then there exists a multifunction $F^* : [0,1] \to CK(X)$ HKP-integrable in $CK(X)$ such that $K = S_{F^*}^{HKP}$.

Proof. By Proposition 2.3.5, we have that $\mathcal{HKP}([0,1], X)$ is separable. Since K is closed, there exists a sequence $(f_n)_{n=1}^{\infty} \subset \mathcal{HKP}([0,1], X)$ such that $K = \overline{\{f_n : n \geq 1\}}^{|| \, ||_A}$. Let us consider the multifunction $F : [0,1] \to CL(X)$

defined by $F(t) = \overline{\{f_n(t) : n \geq 1\}}$. Since each f_n is *HKP*-integrable (and in particular, measurable) and F is $CL(X)$-valued, by Theorem 1.3.3, F is measurable. Moreover, $S_F^{HKP} \neq \emptyset$. Therefore F is *AHKP*-integrable.
Now let define the multifunction $F^* : [0,1] \to 2^X$ by

$$F^*(t) := \overline{co}(F(t)),\ t \in [0,1].$$

First, we prove that F^* is $CK(X)$-valued.
Let $t \in [0,1]$. By definition, $F^*(t)$ is closed convex. Moreover, $F^*(t) = \overline{co}\{f_n(t) : n \geq 1\} \subseteq \overline{co}(K(t))$. By hypothesis, $K(t)$ is relatively norm compact, so by Theorem 2.2.2, $\overline{co}(K(t))$ is norm compact. Hence also $F^*(t)$ is norm compact and therefore $F^*(t) \in CK(X)$.
We observe moreover that for every $t \in [0,1]$, $F^*(t) = \overline{\{h(t) : h \in U\}}$, where $U = \{\sum_i \lambda_i f_i : \lambda_i \geq 0, \text{rational and } \sum_i \lambda_i = 1\}$. Therefore by Theorem 1.3.3, F^* is measurable. Moreover, by definition, F^* is *AHKP*-integrable.
Fix now $f \in S_{F^*}^{HKP}$ and define the multifunction $G^* := F^* - f$.
We observe that for all $t \in [0,1]$, $G^*(t) = \overline{\{h(t) - f(t) : h \in U\}}$. In fact, $x \in G^*(t)$ iff $x + f(t) \in F^*(t)$ iff $x + f(t) = \lim_k h_k(t)$ with $(h_k)_k \subseteq U$ iff $x = \lim_k (h_k(t) - f(t))$.
So G^* is measurable. Moreover, G^* is also $CK(X)$-valued, because it is a translation of F^*.
For every $x^* \in X^*$, $s(x^*, G^*) = s(x^*, F^*) - \langle x^*, f\rangle \geq 0$. So $s(x^*, G^*)^- \equiv 0$ and $\int_0^1 s(x^*, G^*)^- \, d\lambda < \infty$.
Moreover, since the function $g \equiv 0$ is a Pettis integrable selection of G^*, G^* is Aumann-Pettis integrable.
By Lemma 2.3.3, G^* is scalarly integrable. Moreover, by Theorem 1.4.2, G^* is quasi-Pettis integrable in $CC(X)$. Therefore by [42, Proposition 1.3], G^* is Pettis integrable in $CBC(X)$. In particular, each measurable selection of G^* is scalarly integrable. Since X has the Schur property, X does not contain copies of c_0. So by [18, Theorem 7, p. 54], each measurable selection of G^* is Pettis integrable. By [24, Theorem 5.3], we obtain that G^* is Pettis integrable in $CK(X)$. An application of Theorem 1.5.3 produces the *HKP*-integrability of F^* in $CK(X)$.
It remains to prove that $K = S_{F^*}^{HKP}$. Since the inclusion $K \subseteq S_{F^*}^{HKP}$ is triv-

ial, it is enough to show that $S_{F^*}^{HKP} \subseteq K$.
Let $f^* \in S_{F^*}^{HKP}$ and let $\varepsilon > 0$. The function $g^* = f^* - f \in S_{G^*}^{P}$. So by Lemma 2.2.4, there exist $h_1, \dots, h_n \in U$ and $B_1, \dots, B_n \in \mathcal{A}$ with $B_i \cap B_j \neq \emptyset$ such that $\left|\left|g^* - \sum_{j=1}^{n} \chi_{B_j}(h_j - f)\right|\right|_P < \varepsilon$.
Since in $P([0,1], X)$ the Alexiewicz norm topology is weaker than Pettis norm topology, $\left|\left|g^* - \sum_{j=1}^{n} \chi_{B_j}(h_j - f)\right|\right|_A < \varepsilon$. So $g^* \in \overline{dec}^{|| \, ||_A} U - f$. It follows that $f^* \in \overline{dec}^{|| \, ||_A} U \subseteq \overline{dec}^{|| \, ||_A} K = K$. Therefore $S_{F^*}^{HKP} \subseteq K$ and the proof is complete. $\square$

It is possible also to obtain a characterization of $|| \, ||_A$-closed and decomposable subsets of $\mathcal{H}([0,1], X)$ in terms of Henstock integrable selections of a suitable Henstock integrable multifunction. In such a case, the convexity hypothesis of K (see Theorem 2.3.2) can be dropped. Moreover, the Schur property is not required, provided that X does not contain copies of c_0.

Theorem 2.3.3. *Let X be a separable Banach space not containing copies of c_0. Let $\emptyset \neq K \subseteq \mathcal{H}([0,1], X)$ be decomposable and $|| \, ||_A$-closed. Assume that for every $t \in [0,1]$ the set $K(t) = \{f(t) : f \in K\}$ is relatively compact. Then there exists a multifunction $F^* : [0,1] \to CK(X)$ Henstock integrable such that $K = S_{F^*}^{H}$.*

Proof. Since X is separable, by [3, Proposition 1] also $\mathcal{H}([0,1], X)$ is separable. Since K is decomposable, by Theorem 2.3.1, K is convex. By hypothesis, K is closed. So there exists a sequence $(f_n)_{n=1}^{\infty} \subset \mathcal{H}([0,1], X)$ such that $K = \overline{\{f_n : n \geq 1\}}^{|| \, ||_A}$. Let us consider the multifunction $F : [0,1] \to CL(X)$ defined by $F(t) = \overline{\{f_n(t) : n \geq 1\}}$. Since each f_n is Henstock integrable (in particular, measurable) and F is $CL(X)$-valued, by Theorem 1.3.3, F is measurable. Moreover, $S_F^H \neq \emptyset$. Therefore F is Aumann-Henstock integrable.
Now define the multifunction $F^* : [0,1] \to 2^X$ by

$$F^*(t) := \overline{co}\, F(t), \; t \in [0,1].$$

We prove that F^* is $CK(X)$-valued.
Let $t \in [0,1]$. By definition, $F^*(t)$ is closed convex. Moreover, $F^*(t) = \overline{co}\{f_n(t) : n \geq 1\} \subseteq \overline{co}(K(t))$. By hypothesis, $K(t)$ is relatively compact, so

by Theorem 2.2.2, $\overline{co}(K(t))$ is compact.
Hence also $F^*(t)$ is compact and therefore $F^*(t) \in CK(X)$.
We observe moreover that for every $t \in [0,1]$, $F^*(t) = \overline{\{h(t) : h \in U\}}$, where $U = \{\sum_i \lambda_i f_i : \lambda_i \geq 0, \text{rational and } \sum_i \lambda_i = 1\}$. Therefore F^* is measurable by Theorem 1.3.3 and by definition, F^* is Aumann-Henstock integrable.
Now fix $f \in S^H_{F^*}$ and define the multifunction $G^* := F^* - f$.
G^* is $CK(X)$-valued, because it is a translation of F^*.
As in the proof of Theorem 2.3.2, we get that G^* is Aumann-Pettis integrable.
By Lemma 2.3.3, G^* is scalarly integrable, moreover by Theorem 1.4.2, G^* is quasi-Pettis integrable in $CC(X)$. Therefore by [42, Proposition 1.3], G^* is Pettis integrable in $CBC(X)$.
With the same arguments used in the proof of Theorem 2.3.2, we get that each measurable selection of G^* is Pettis integrable. So by [24, Theorem 5.3], G^* is Pettis integrable in $CK(X)$. An application of Proposition 1.5.1 and Theorem 1.5.4 produce the Henstock integrability of F^*.
It remains to prove that $K = S^H_{F^*}$.
Since the inclusion $K \subseteq S^H_{F^*}$ is trivial, it is enough to show that $S^H_{F^*} \subseteq K$.
Now let $f^* \in S^H_{F^*}$ and let $\varepsilon > 0$. The function $g^* = f^* - f \in S^P_{G^*}$. So by Lemma 2.2.4, there exist $h_1, \ldots, h_n \in U$ and there exist $B_1, \ldots, B_n \in \mathcal{A}$ with $B_i \cap B_j \neq \emptyset$ such that $\left\|g^* - \sum_{j=1}^n \chi_{B_j}(h_j - f)\right\|_P < \varepsilon$.
Since in $P([0,1], X)$ the Alexiewicz norm topology is weaker than Pettis norm topology, $\left\|g^* - \sum_{j=1}^n \chi_{B_j}(h_j - f)\right\|_A < \varepsilon$. So $g^* \in \overline{dec}^{\|\,\|_A} U - f$. It follows that $f^* \in \overline{dec}^{\|\,\|_A} U \subseteq \overline{dec}^{\|\,\|_A} K = K$. So $S^H_{F^*} \subseteq K$. $\square$

CHAPTER 3

RADON-NIKODÝM THEOREMS FOR FINITELY ADDITIVE MULTIMEASURES

3.1 Introduction

One of the most fascinating problems arising when we deal with multimeasures is the representation of a multimeasure as an integral, i.e., the existence of a Radon-Nikodým derivative.
Several papers concerning this question appeared since the 1970's where pioneering results have been established amongst others by Z. Artstein [2], A. Costé [14], A. Costé and R. Pallu de la Barrière [15]. These papers deal with countably additive multimeasures and use classical notions of integral existing in literature.
In the 1990's other results dealing with finitely additive multimeasures have been obtained by A. Martellotti, K. Musiał and A. R. Sambucini (see [38, 39]). In particular, they have been extended the trattation beyond the Banach spaces (in particular to locally convex spaces), but also in this case

classical integrals are used for the representation.
In general the results existing in literature use multimeasures defined on a σ-algebra. Moreover, most of them uses the separability assumption.
In this chapter we deal with the Radon-Nikodým problem for multimeasures defined on the family $\mathcal{I}$ of all non trivial closed subintervals of $[0, 1]$ and consequently we look for Radon-Nikodým derivatives of Henstock type.
Our starting point is the remarkable recent article of B. Cascales, V. Kadets and J. Rodríguez [8], where they obtain two Radon-Nikodým theorems for countably additive multimeasures without any separability assumption.
Here we go on in such kind of investigation and we consider finitely additive multimeasures defined on $\mathcal{I}$, taking convex compact values or more in general taking convex weakly compact values, in an arbitrary Banach space X.
In the first part of the chapter we focus the attention to the existence of finitely additive vector valued selections.
Then we extend to the multivalued case the notion of variational measure already known for vector valued interval measure. This measure is a very useful tool for our investigation. We recall also the variationally Henstock integral and prove the absolute continuity of the variational measures generated by the variational Henstock primitives.
In the final part of the chapter we show the main results.
In the convex compact case we find a Radon-Nikodým theorem for dominated interval multimeasures (see Theorem 3.4.1) that improves Theorem 3.1 of [8].
To get our goal we use an extension of a finitely additive multimeasure to a countably additive multimeasure defined in the σ-algebra of the Borel subsets of $[0, 1]$ (see Proposition 3.4.1). In Theorem 3.4.2 we generalize the previous result to the pointwise dominated interval multimeasures.
In the more general context of convex weakly compact valued multimeasures we find an *HKP*-integrable derivative under the hypothesis of absolute continuity for the associated variational measure (see Theorem 3.4.4). Also in such a case we do not require the separability to the target Banach space X, but we assume that X possesses the RNP.

3.2 Interval multimeasures and their selections

In the following by the symbol $C(X)$ we denote one of the families $CWK(X)$ or $CK(X)$. We start with the following definitions.

Definition 3.2.1. An interval multifunction $\Phi : \mathcal{I} \to C(X)$ is said to be *finitely additive* if for every non-overlapping intervals $I_1, I_2 \in \mathcal{I}$ such that $I_1 \cup I_2 \in \mathcal{I}$ we have $\Phi(I_1 \cup I_2) = \Phi(I_1) + \Phi(I_2)$.
An additive interval function $\phi : \mathcal{I} \to X$ is said to be a *selection* of Φ if $\phi(I) \in \Phi(I)$ for every $I \in \mathcal{I}$.

Remark 3.2.1. The primitives of Henstock or *HKP*-integrable multifunctions are finitely additive. Moreover, it is known that if a multifunction $F : [0,1] \to C(X)$ is Pettis integrable in $C(X)$, then its primitive is σ-additive (see [14]). If we set $\Phi(I) := \nu(I)$, $I \in \mathcal{I}$, then Φ is finitely additive.

Definition 3.2.2. A multifunction $\Psi : \mathcal{A} \to C(X)$ is said to be a *finitely additive multimeasure* if for every $A_1, A_2 \in \mathcal{A}$ such that $\mathring{A}_1 \cap \mathring{A}_2 = \emptyset$ we have $\Psi(A_1 \cup A_2) = \Psi(A_1) + \Psi(A_2)$.
A finitely additive measure $\phi : \mathcal{A} \to X$ is said to be a *selection* of Ψ if $\psi(I) \in \Psi(I)$ for every $A \in \mathcal{A}$.

Remark 3.2.2. In the following, given a finitely additive interval multifunction $\Phi : \mathcal{I} \to C(X)$, we identify it with the finitely additive multimeasure $\Psi : \mathcal{A} \to C(X)$ defined by $\Psi(A) := \sum_{j=1}^{q} \Phi(I_j)$, where $A = \bigcup_{j=1}^{q} I_j$ and $I_1, \ldots, I_q$ are pairwise disjoint subintervals of $[0,1]$. We use a similar identification for the corresponding selections.
Hence we call *interval multimeasure* every finitely additive interval multifunction and *interval measure* every finitely additive interval function.
Moreover, we observe that if $\Phi : \mathcal{I} \to C(X)$ is an interval multimeasure, then for every $x^* \in X^*$, $s(x^*, \Phi(\cdot))$ is a real-valued interval measure.

An important question for an interval multimeasure is the existence of finitely additive selections. In Proposition 3.2.1 and in Corollary 3.2.1 below we prove that the answer is affirmative for $CK(X)$-valued and $CWK(X)$-valued interval multimeasures. We use a technique similar to that in [32] where the

case of σ-additive multifunctions is considered.
We need the following definitions.
We recall that for $\emptyset \neq K \subset X$, we say that $x \in K$ is an *exposed point* of K if there exists $x^* \in X^*$ such that $\langle x^*, x\rangle > \langle x^*, y\rangle$ for every $y \in K \setminus \{x\}$.
We say that x is a *strongly exposed point* of K if there exists $x^* \in X^*$ such that $\langle x^*, x\rangle > \langle x^*, y\rangle$ for every $y \in K \setminus \{x\}$ and such that, if $(x_n)_{n=1}^{\infty} \subset K$ and $\langle x^*, x_n\rangle \to \langle x^*, x\rangle$, then $||x_n - x|| \to 0$.
We denote by $exp(K)$ (resp. $str\, exp(K)$) the set of the exposed points (resp. strongly exposed points) of K.
It is known by the Krein-Milman Theorem (see [40, Theorem 2.10.6]), that if $K \in CK(X)$, then $exp(K) \neq \emptyset$ and $K = \overline{co}(exp(K))$. This result was improved by Lindenstrauss (see [36]), who showed that if $K \in CWK(X)$, then $str\, exp(K) \neq \emptyset$ and $K = \overline{co}(str\, exp(K))$.

Proposition 3.2.1. *Let $\Psi : \mathcal{A} \to CK(X)$ be a finitely additive multimeasure. If $x_0 \in exp(\Psi([0,1]))$ then there exists a selection $\psi : \mathcal{A} \to X$ of Ψ such that $\psi([0,1]) = x_0$ and $\psi(A) \in exp(\Psi(A))$ for every $A \in \mathcal{A}$.*

Proof. Let $x_0^* \in X^*$ be such that $\langle x_0^*, x_0\rangle > \langle x_0^*, y\rangle$ for all $y \in \Psi([0,1]) \setminus \{x_0\}$.
Given $A \in \mathcal{A}$, we have $\Psi([0,1]) = \Psi(A) + \Psi(A^c)$. So $x_0 = x_A + x_{A^c}$ with $x_A \in \Psi(A)$ and $x_{A^c} \in \Psi(A^c)$.
Since

$$\langle x_0^*, x_A\rangle + \langle x_0^*, x_{A^c}\rangle = \langle x_0^*, x_0\rangle = s(x_0^*, \Psi([0,1])) = s(x_0^*, \Psi(A)) + s(x_0^*, \Psi(A^c)),$$

we have $\langle x_0^*, x_A\rangle = s(x_0^*, \Psi(A))$ and $\langle x_0^*, x_{A^c}\rangle = s(x_0^*, \Psi(A^c))$.
Moreover, $\langle x_0^*, x_A\rangle > \langle x_0^*, z\rangle$ for every $z \in \Psi(A) \setminus \{x_A\}$ (indeed, if $\overline{x}_A \in \Psi(A)$ is such that $\langle x_0^*, \overline{x}_A\rangle \geq \langle x_0^*, x_A\rangle$, setting $\overline{x}_0 = \overline{x}_A + x_{A^c}$ we get $\overline{x}_0 \in \Psi([0,1])$ and $\langle x_0^*, \overline{x}_0\rangle = \langle x_0^*, \overline{x}_A\rangle + \langle x_0^*, x_{A^c}\rangle \geq \langle x_0^*, x_A\rangle + \langle x_0^*, x_{A^c}\rangle = \langle x_0^*, x_0\rangle$, clearly impossible).
Similarly, $\langle x_0^*, x_{A^c}\rangle > \langle x_0^*, z\rangle$ for every $z \in \Psi(A^c) \setminus \{x_{A^c}\}$.
Thus it has been proved that for every $A \in \mathcal{A}$, there exists a unique point $x_A \in \Psi(A)$ such that $\langle x_0^*, x_A\rangle = s(x_0^*, \Psi(A))$. Moreover, $x_A \in exp(\Psi(A))$.
Now let $\psi : \mathcal{A} \to X$ be defined by $\psi(A) := x_A$. It is clear that $\psi(A) \in \Psi(A)$ for every $A \in \mathcal{A}$ and $\psi([0,1]) = x_0$. It remains to prove that ψ is finitely

additive.
Let $A_1, A_2 \in \mathcal{A}$ be disjoint and let $A = A_1 \cup A_2$. It is clear the fact that $\psi(A_1) + \psi(A_2) \in \Psi(A)$. Moreover, $\psi(A)$ is the unique element of $\Psi(A)$ such that $\langle x_0^*, \psi(A)\rangle = s(x_0^*, \Psi(A))$ and for $i = 1, 2$, $\psi(A_i)$ is the unique element of $\Psi(A_i)$ such that $\langle x_0^*, \psi(A_i)\rangle = s(x_0^*, \Psi(A_i))$. So it is enough to prove that $\langle x_0^*, \psi(A)\rangle = \langle x_0^*, \psi(A_1)\rangle + \langle x_0^*, \psi(A_2)\rangle$.
But $\langle x_0^*, \psi(A_1)\rangle + \langle x_0^*, \psi(A_2)\rangle = s(x_0^*, \Psi(A_1)) + s(x_0^*, \Psi(A_2)) = s(x_0^*, \Psi(A)) = \langle x_0^*, \psi(A)\rangle$. So $\psi(A) = \psi(A_1) + \psi(A_2)$.
We conclude that ψ is a selection of Ψ. $\square$

Corollary 3.2.1. *Let $\Phi : \mathcal{I} \to CK(X)$ be an interval multimeasure. If $x_0 \in exp(\Phi([0,1]))$ then there exists a selection $\phi : \mathcal{I} \to X$ of Φ such that $\phi([0,1]) = x_0$ and $\phi(I) \in exp(\Phi(I))$ for every $I \in \mathcal{I}$.*

As a consequence of Proposition 3.2.1 and Corollary 3.2.1, a $CK(X)$-valued interval multimeasure possesses finitely additive selections.
With similar arguments, we obtain

Proposition 3.2.2. *Let $\Psi : \mathcal{A} \to CWK(X)$ be a finitely additive multimeasure. If $x_0 \in str\,exp(\Psi([0,1]))$ then there exists a selection $\psi : \mathcal{A} \to X$ of Ψ such that $\psi([0,1]) = x_0$ and $\psi(A) \in str\,exp(\Psi(A))$ for every $A \in \mathcal{A}$.*

Corollary 3.2.2. *Let $\Phi : \mathcal{I} \to CWK(X)$ be an interval multimeasure. If $x_0 \in str\,exp(\Phi([0,1]))$ then there exists a selection $\phi : \mathcal{I} \to X$ of Φ such that $\phi([0,1]) = x_0$ and $\phi(I) \in str\,exp(\Phi(I))$ for every $I \in \mathcal{I}$.*

Also in this case, as natural consequence of Proposition 3.2.2 and Corollary 3.2.2, we have that every interval multimeasure with values in $CWK(X)$ possesses finitely additive selections.
If $\Psi : \mathcal{A} \to C(X)$ (resp. $\Phi : \mathcal{I} \to C(X)$) is a finitely additive multimeasure (resp. an interval multimeasure), we denote by S_Ψ (resp. S_Φ) the set of all selections of Ψ (resp. Φ).
We can see S_Ψ as a subset of $X^{\mathcal{A}}$, the set of all X-valued functions defined on $\mathcal{A}$, endowed with the topology τ of the pointwise convergence.

Proposition 3.2.3. *Let $\Psi : \mathcal{A} \to CK(X)$ be a finitely additive multimeasure. Then for every $A \in \mathcal{A}$, $\Psi(A) = \{\psi(A) : \psi \in S_\Psi\}$. Consequently, for every $A \in \mathcal{A}$ and every $x \in \Psi(A)$, there exists $\psi \in S_\Psi$ such that $\psi(A) = x$.*

Proof. Define $\Gamma(A) := \{\psi(A) : \psi \in S_\Psi\}$. We prove that $\Psi(A) = \Gamma(A)$.
It is clear that $\Gamma(A) \subseteq \Psi(A)$. So it is enough to show that $\Psi(A) \subseteq \Gamma(A)$.
First, we claim that S_Ψ is τ-closed. For this purpose, let $(\psi_\alpha)_\alpha$ be a net in S_Ψ and assume that $\psi_\alpha \to \psi$. Then for every $A \in \mathcal{A}$, $\psi_\alpha(A) \to \psi(A)$ with respect to the norm of X. Since $\psi_\alpha(A) \in \Psi(A)$ for every $A \in \mathcal{A}$ and every α, then $\psi(A) \in \Psi(A)$ for every $A \in \mathcal{A}$. So $\psi \in S_\Psi$.
Moreover, the set $\prod_{A\in\mathcal{A}} \Psi(A)$ is τ-compact, because for each $A \in \mathcal{A}$ the set $\Psi(A)$ is compact in X.
Since $S_\Psi \subseteq \prod_{A\in\mathcal{A}} \Psi(A)$, it follows that S_Ψ is τ-compact.
Now for every $A \in \mathcal{A}$, let us consider the map $\gamma_A : S_\Psi \to X$ defined by $\gamma_A(\psi) := \psi(A)$. γ_A is τ-continuous and $\gamma_A(S_\Psi) = \Gamma(A)$. Since S_Ψ is convex and τ-compact, then $\Gamma(A)$ is convex and compact. Moreover, by Proposition 3.2.1, $exp(\Psi(A)) \subseteq \Gamma(A)$. Thus $co(exp(\Psi(A))) \subseteq \Gamma(A)$ and an application of the Krein-Milman Theorem gives $\Psi(A) = \overline{co}(exp(\Psi(A))) \subseteq \Gamma(A)$. □

Corollary 3.2.3. *Let $\Phi : \mathcal{I} \to CK(X)$ be an interval multifunction. Then for every $I \in \mathcal{I}$, $\Phi(I) = \{\phi(I) : \phi \in S_\Phi\}$. Consequently, for every $I \in \mathcal{I}$ and every $x \in \Phi(I)$, there exists $\phi \in S_\Phi$ such that $\phi(I) = x$.*

We observe that Proposition 3.2.3 and Corollary 3.2.3 remain true if $CK(X)$ is replaced by $CWK(X)$.

Definition 3.2.3. Let $\emptyset \neq K \subseteq X$ and let $x^* \in X^*$. We set

$$K^{|x^*} := \{x \in K : \langle x^*, x\rangle = s(x^*, K)\}.$$

Then we denote by $att(K)$ the set of those $x^* \in X^*$ that attain their supremum on K, that is $att(K) := \{x^* \in X^* : K^{|x^*} \neq \emptyset\}$.

It is important to recall this characterization of weakly compact subset of a Banach space X [34, Theorem 5].

Theorem 3.2.1. *A weakly closed subset K of a Banach space X is weakly compact if and only if each continuous linear functional on X attains its supremum on K.*

Proposition 3.2.4. *Let $\Psi : \mathcal{A} \to CWK(X)$ be a finitely additive multimeasure. Then for every $x^* \in X^*$, the multifunction $\Psi^{|x^*} : \mathcal{A} \to CWK(X)$ defined by $\Psi^{|x^*}(A) := \Psi(A)^{|x^*}$ is a finitely additive multimeasure.*

Proof. Since Ψ is $CWK(X)$-valued, by Theorem 3.2.1, we have that for every $A \in \mathcal{A}$, $att(\Psi(A)) = X^*$. Therefore $\Psi(A)^{|x^*}$ is non empty for every $x^* \in X^*$ and every $A \in \mathcal{A}$.
Let $A_1, A_2 \in \mathcal{A}$ be disjoint and let $A = A_1 \cup A_2$. It is enough to prove that $\Psi^{|x^*}(A) = \Psi^{|x^*}(A_1) + \Psi^{|x^*}(A_2)$.

($\subseteq$) Let $x \in \Psi^{|x^*}(A) \subseteq \Psi(A) = \Psi(A_1) + \Psi(A_2)$. So $x = x_1 + x_2$ with $x_1 \in \Psi(A_1)$ and $x_2 \in \Psi(A_2)$. Moreover, $\langle x^*, x_1\rangle + \langle x^*, x_2\rangle = \langle x^*, x\rangle = s(x^*, \Psi(A)) = s(x^*, \Psi(A_1)) + s(x^*, \Psi(A_2))$.
Thus $\langle x^*, x_1\rangle = s(x^*, \Psi(A_1))$ and $\langle x^*, x_2\rangle = s(x^*, \Psi(A_2))$ (in fact, if $\langle x^*, x_1\rangle < s(x^*, \Psi(A_1))$ then $\langle x^*, x_2\rangle > s(x^*, \Psi(A_2))$, a contradiction). Therefore $x_1 \in \Psi^{|x^*}(A_1)$ and $x_2 \in \Psi^{|x^*}(A_2)$.

($\supseteq$) Let $x \in \Psi^{|x^*}(A_1) + \Psi^{|x^*}(A_2)$. Then $x = x_1 + x_2$ with $x_1 \in \Psi^{|x^*}(A_1)$ and $x_2 \in \Psi^{|x^*}(A_2)$. Clearly $x \in \Psi(A)$. Moreover, $\langle x^*, x_1\rangle = s(x^*, \Psi(A_1))$ and $\langle x^*, x_2\rangle = s(x^*, \Psi(A_2))$. Thus $\langle x^*, x\rangle = s(x^*, \Psi(A))$ and therefore $x \in \Psi^{|x^*}(A)$.

Corollary 3.2.4. *Let $\Phi : \mathcal{I} \to CK(X)$ be an interval multimeasure. Then for every $x^* \in X^*$, the interval multifunction $\Phi^{|x^*} : \mathcal{I} \to C(X)$ defined by $\Phi^{|x^*}(I) := \Phi(I)^{|x^*}$ is an interval multimeasure.*

3.3 Variational meaures. The variational Henstock integral

Now we extend the notion of variational measure to additive interval multimeasures. This notion is a useful tool to study the primitives of real valued or, more in general, vector valued integrable functions.

Definition 3.3.1. Given an interval multimeasure $\Phi : \mathcal{I} \to C(X)$, a gauge δ and a set $E \subset [0,1]$, we define

$$Var(\Phi, \delta, E) := \sup \sum_{j=1}^{p} ||\Phi(I_j)||,$$

where the supremum is taken over all δ-fine partitions $\{(I_j, t_j)\}_{j=1}^{p}$ anchored on E.
Then we set

$$V_\Phi(E) := \inf \{Var(\Phi, \delta, E) : \delta \text{ gauge on } E\}.$$

V_Φ is called *the variational measure generated by* Φ.

It is clear that this definition coincides with the known definition of variational measure for interval X-valued measures and real-valued measures (see [4] and [20]).

Remark 3.3.1. If Φ is an interval multimeasure, then V_Φ coincides with the variational measure generated by the single valued map $R \circ \Phi$, where $R : CWK(X) \to \ell_\infty(B(X^*))$ is the *Rådstrom Embedding* defined, as well known, by $R(C) := s(\cdot, C)$, for every $C \in CWK(X)$.
In fact, for every $I \in \mathcal{I}$ we obtain:

$$\begin{aligned} ||R(\Phi(I))||_{l_\infty} &= ||s(\cdot, \Phi(I))||_{l_\infty} = \sup_{x^* \in B(X^*)} |s(x^*, \Phi(I))| \\ &= \sup_{x^* \in B(X^*)} |s(x^*, \Phi(I)) - s(x^*, \{0\})| = d_H(\Phi(I), \{0\}) = ||\Phi(I)||. \end{aligned}$$

Consequently, $Var(\Phi, \delta, E) = Var(R(\Phi), \delta, E)$ for any gauge δ and any set $E \subset [0,1]$, and $V_\Phi(E) = V_{R\circ\Phi}(E)$ for any set $E \subset [0,1]$.
Therefore, as in the X-valued case, V_Φ is a metric outer measure on $[0,1]$ (see [4]) and a measure over all Borel sets of $[0,1]$.

We say that the variational measure V_Φ is *σ-finite* if there exists a sequence of (pairwise disjoint) sets $(E_n)_{n=1}^{\infty}$ covering $[0,1]$ and such that $V_\Phi(E_n) < \infty$, for every $n \geq 1$. Moreover we say that V_Φ is *absolutely continuous* with respect to λ or briefly *λ-continuous* and we write $V_\Phi << \lambda$, if for every $E \in \mathcal{L}$

with $\lambda(E) = 0$ we have $V_\Phi(E) = 0$.
Taking into account that $V_\Phi = V_{R\circ\Phi}$ and using [4, Corollary 2.3], we have that every λ-continuous variational measure is also σ-finite.
Before to prove that the variational measure associated to a variational Henstock primitive is λ-continuous, we need some preliminary lemmas.
The first lemma is the multivalued version of Saks-Henstock lemma (see [29, Lemma 9.11] for the real valued case and [49, Lemma 3.4.1] for the Banach valued case).

Lemma 3.3.1 (Saks-Henstock Lemma). *Assume that $F : [0,1] \to C(X)$ is Henstock integrable. Given $\varepsilon > 0$, assume that a gauge δ on $[0,1]$ is such that*

$$d_H\left(\sum_{j=1}^{q} F(t_j)|I_j|, (H)\int_0^1 F\, d\lambda\right) < \varepsilon,$$

for every δ-fine Perron-partition $\{(I_j, t_j)\}_{j=1}^q$ of $[0,1]$.
Then if $\{(J_i, s_i)\}_{i=1}^p$ is an arbitrary δ-fine Perron-partition in $[0,1]$ we have

$$d_H\left(\sum_{i=1}^{p} F(s_i)|J_i|, \sum_{i=1}^{p}(H)\int_{J_i} F\, d\lambda\right) \leq \varepsilon.$$

Proof. Suppose that $\{(J_i, s_i)\}_{i=1}^p$ is a δ-fine Perron-partition in $[0,1]$. Then $[0,1] \setminus \bigcup_{i=1}^p \mathring{J}_i$ consists of a finite collection $\{M_k\}_{k=1}^m$ of non-overlapping intervals in $[0,1]$.
Fix $\alpha > 0$. For every $k = 1, \ldots, m$, there exists in M_k a gauge δ_k with $\delta_k \leq \delta$ and such that

$$d_H\left(\sum_{j=1}^{q_k} F(t_j^k)|I_j^k|, (H)\int_{M_K} F\, d\lambda\right) < \frac{\alpha}{m+1},$$

provided $\{(I_j^k, t_j^k)\}_{j=1}^{q_k}$ is a δ_k-fine Perron-partition of M_k.
The sum

$$W = \sum_{i=1}^{p} F(s_i)|J_i| + \sum_{k=1}^{m}\sum_{j=1}^{q_k} F(t_j^k)|I_j^k| \in C(X)$$

is an integral sum corresponding to a δ-fine Perron-partition of $[0,1]$. Consequently, we have

$$d_H\left(W, (H)\int_0^1 F\, d\lambda\right) \leq \varepsilon.$$

Hence

$$
\begin{aligned}
& d_H\left(\sum_{i=1}^{p} F(s_i)|J_i|, \sum_{i=1}^{p}(H)\int_{J_i} F\,d\lambda\right) \\
& = d_H\left(W, \sum_{i=1}^{p}(H)\int_{J_i} F\,d\lambda + \sum_{k=1}^{m}\sum_{j=1}^{q_k} F(t_j^k)|I_j^k|\right) \\
& \leq d_H\left(W, (H)\int_0^1 F\,d\lambda\right) + \\
& d_H\left((H)\int_0^1 F\,d\lambda, \sum_{i=1}^{p}(H)\int_{J_i} F\,d\lambda + \sum_{k=1}^{m}\sum_{j=1}^{q_k} F(t_j^k)|I_j^k|\right) \\
& < \varepsilon + d_H\left(\sum_{k=1}^{m}(H)\int_{M_k} F\,d\lambda, \sum_{k=1}^{m}\sum_{j=1}^{q_k} F(t_j^k)|I_j^k|\right) \\
& \leq \varepsilon + \sum_{k=1}^{m} d_H\left((H)\int_{M_k} F\,d\lambda, \sum_{j=1}^{q_k} F(t_j^k)|I_j^k|\right) < \varepsilon + \alpha.
\end{aligned}
$$

Since $\alpha > 0$ is arbitrary, we obtain

$$
d_H\left(\sum_{i=1}^{p} F(s_i)|J_i|, \sum_{i=1}^{p}(H)\int_{J_i} F\,d\lambda\right) \leq \varepsilon.
$$

□

Lemma 3.3.2. *Let $F : [0,1] \to C(X)$ be a variationally Henstock integrable multifunction and let $\Phi : \mathcal{I} \to C(X)$ be its variational Henstock primitive. Then the multifunction $G(t) := \Phi([0,t])$ is d_H-continuous on $[0,1]$.*

Proof. The continuity follows from Saks-Henstock Lemma 3.3.1 and the following inequality

$$
\begin{aligned}
& d_H(G(t), G(s)) = ||\Phi([s,t])|| \\
& \leq d_H(\Phi([s,t]), F(s)(t-s)) + ||F(s)|| \cdot |t-s|.
\end{aligned}
$$

□

Proposition 3.3.1. *Let $F : [0,1] \to C(X)$ be a variationally H-integrable multifunction and let $\Phi : \mathcal{I} \to C(X)$ be its variational Henstock primitive. Then $V_\Phi << \lambda$.*

Proof. By Lemma 3.3.2, the multifunction $G(t) := \Phi([0,t])$ is d_H-continuous on $[0,1]$. Assume that $\lambda(E) = 0$. If $E = \{0\}$ or $E = \{1\}$ then by continuity of G we have $V_\Phi(E) = 0$. So we may assume without losing generality that $E \subset (0,1)$.
For every positive integer n, let $E_n := \{t \in E : n-1 \leq ||F(t)|| < n\}$. The sets E_n are pairwise disjoint, $\bigcup_{n=1}^{\infty} E_n = E$ and $\lambda(E_n) = 0$ for every n.
Fix $\varepsilon > 0$ and, for every n, let $O_n \subset (0,1)$ be an open set such that $E_n \subseteq O_n$ and $\lambda(O_n) < \frac{\varepsilon}{n2^n}$.
By Lemma 3.3.1, there exists a gauge δ_0 on $[0,1]$ such that

$$\sum_{i=1}^{p} d_H(F(s_j)|J_i|, \Phi(J_i)) < \varepsilon,$$

for every δ_0-fine partition $\{(J_i, s_i)\}$ in $[0,1]$.
For every $t \in E_n$, take $\delta_n(t) > 0$ such that $(t - \delta_n(t), t + \delta_n(t)) \subseteq O_n$. Finally put $\delta(t) := \min\{\delta_0(t), \delta_n(t)\}$, $t \in E_n$.
In this way a gauge is defined in E. Let $\{(I_j, t_j)\}_{j=1}^{q}$ be an arbitrary δ-fine partition anchored on E. Note that since $t_j \in E_n$, then $I_j \subseteq O_n$. Therefore

$$\sum_{t_j \in E_n} |I_j| < \frac{\varepsilon}{n2^n}.$$

Then by Lemma 3.3.1,

$$\begin{aligned}\sum_{j=1}^{q} ||\Phi(I_j)|| &\leq \sum_{j=1}^{q} d_H(F(t_j)|I_j|, \Phi(I_j)) + \sum_{j=1}^{q} ||F(t_j)||\,|I_j| \\ &= \varepsilon + \sum_{n \geq 1} \sum_{t_j \in E_n} ||F(t_j)||\,|I_j| < 2\varepsilon.\end{aligned}$$

Therefore $Var(\Phi, \delta, E) \leq 2\varepsilon$. Hence $V_\Phi(E) \leq 2\varepsilon$. □

Remark 3.3.2. At this point it is worth to observe that, if Φ is an *HKP*-primitive, the associated variation could be not λ-continuous, as the following example shows.
Let X be an infinite-dimensional Banach space and let $f : [0,1] \to X$ be a strongly measurable Pettis integrable function such that its Pettis integral

is nowhere differentiable in $[0,1]$ (such a function exists for every infinite-dimensional Banach space, see [19]). Let denote by ν the Pettis integral of f and define $\phi(I) := \nu(I)$, $I \in \mathcal{I}$. Then by [4, Corollary 4.2], V_ϕ is not λ-continuous.

3.4 Main Results

3.4.1 The $CK(X)$ case

We start by proving an extension result.

Proposition 3.4.1. *Let $\Phi : \mathcal{I} \to CK(X)$ be an interval multimeasure such that there exists a set $Q \in CK(X)$ with $\Phi(I) \subseteq |I|Q$ for every $I \in \mathcal{I}$.*
Then Φ can be extended to a multimeasure $M : \sigma(\mathcal{A}) \to CK(X)$ such that $M(B) \subseteq \lambda(B)Q$ for every $B \in \sigma(\mathcal{A})$.

Proof. We observe that for every $x^* \in X^*$, $s(x^*, \Phi)$ is a real-valued measure and

$$-s(-x^*, Q)|I| \leq s(x^*, \Phi(I)) \leq s(x^*, Q)|I|, \text{ for every } I \in \mathcal{I}.$$

Fix $x^* \in X^*$. Then $s(x^*, \Phi)$ can be extended to $\mathcal{A}$, the ring generated by $\mathcal{I}$. Hence for every $A \in \mathcal{A}$,

$$-s(-x^*, Q)\lambda(A) \leq s(x^*, \Phi(A)) \leq \lambda(A)s(x^*, Q).$$

Consequently,

$$|s(x^*, \Phi(A))| \leq |s(x^*, Q)|\lambda(A) + |s(-x^*, Q)|\lambda(A).$$

Since $A \mapsto \lambda(A)s(x^*, Q)$ is σ-additive on $\mathcal{A}$ and bounded, we get that $s(x^*, \Phi(\cdot))$ can be extended to a measure $\mu_{x^*} : \sigma(\mathcal{A}) \to \mathbb{R}$ [16, Theorem 7, p.116], where $\sigma(\mathcal{A})$ consists of all Borel subsets of $[0,1]$.
Now let $B \in \sigma(\mathcal{A})$ and consider a sequence $(A_n)_{n=1}^\infty$ of elements of $\mathcal{A}$ such that $\lambda(B \,\Delta\, A_n) \to 0$. We prove that $(\Phi(A_n))_{n=1}^\infty$ is a Cauchy sequence in $(CK(X), d_H)$.

In fact, for every natural numbers n, m, we have

$$\begin{aligned}
& d_H(\Phi(A_n), \Phi(A_m)) = \sup_{x^* \in B(X^*)} |s(x^*, \Phi(A_n)) - s(x^*, \Phi(A_m))| \\
& = \sup_{x^* \in B(X^*)} |s(x^*, \Phi(A_n \setminus A_m)) - s(x^*, \Phi(A_m \setminus A_n))| \\
& \leq \sup_{x^* \in B(X^*)} |s(x^*, \Phi(A_n \setminus A_m))| + \sup_{x^* \in B(X^*)} |s(x^*, \Phi(A_m \setminus A_n))| \\
& \leq 2 \sup_{x^* \in B(X^*)} |s(x^*, Q)| \lambda(A_n \setminus A_m) + 2 \sup_{x^* \in B(X^*)} |s(x^*, Q)| \lambda(A_m \setminus A_n) \\
& = k\lambda(A_n \setminus A_m) + k\lambda(A_m \setminus A_n) = k\lambda(A_n \,\Delta\, A_m),
\end{aligned}$$

where $k = 2||Q||$.
Since $\lambda(A_n \,\Delta\, A_m) \to 0$, also $d_H(\Phi(A_n), \Phi(A_m)) \to 0$. Since $(CK(X), d_H)$ is a complete metric space, we obtain that $(\Phi(A_n))_{n=1}^{\infty}$ is d_H-convergent to an element of $CK(X)$.
At this point let us define $M(B) := (d_H) \lim_n \Phi(A_n)$ for $B \in \sigma(\mathcal{A})$. The multifunction M is well defined. In fact, if $(A'_n)_{n=1}^{\infty} \subset \mathcal{A}$ is another sequence such that $\lambda(A'_n \,\Delta\, B) \to 0$, then also $\lambda(A'_n \,\Delta\, A_n) \to 0$. Consequently,

$$d_H(\Phi(A'_n), \Phi(A_n)) \leq k\lambda(A'_n \,\Delta\, A_n) \to 0.$$

Thus

$$(d_H) \lim_n \Phi_n(A'_n) = (d_H) \lim_n \Phi_n(A_n).$$

Moreover, M is $CK(X)$-valued and is an extension of Φ to $\sigma(\mathcal{A})$.
We claim that $s(x^*, M) = \mu_{x^*}$ for all $x^* \in X^*$. In fact, let fix $x^* \in X^*$. It follows from the definition of M that for every $B \in \sigma(\mathcal{A})$, one has $s(x^*, \Phi(A_n)) \to s(x^*, M(B))$, where $(A_n)_{n=1}^{\infty}$ is one of the above considerated sequence.
On the other hand,

$$\begin{aligned}
& |\mu_{x^*}(B) - s(x^*, \Phi(A_n))| = |\mu_{x^*}(B) - \mu_{x^*}(A_n)| \\
& = |\mu_{x^*}(B \setminus A_n) - \mu_{x^*}(A_n \setminus B)| \leq |\mu_{x^*}(B \setminus A_n)| + |\mu_{x^*}(A_n \setminus B)| \\
& \leq k\lambda(B \,\Delta\, A_n) \to 0,
\end{aligned}$$

for every $B \in \sigma(\mathcal{A})$.
Hence $s(x^*, M(B)) = \mu_{x^*}(B)$ for every $B \in \sigma(\mathcal{A})$.

Therefore for each $x^* \in X^*$, $s(x^*, M)$ is a measure. Since M is $CK(X)$-valued, by Theorem 1.6.1, M is a multimeasure.
Finally for each $B \in \sigma(\mathcal{A})$ and each $x^* \in X^*$

$$s(x^*, M(B)) = \mu_{x^*}(B) \leq s(x^*, Q)\lambda(B) = s(x^*, \lambda(B)Q).$$

Therefore $M(B) \subseteq \lambda(B)Q$ for each $B \in \sigma(\mathcal{A})$. □

The following result improves [8, Theorem 3.1], valid for dominated convex compact valued multimeasures that can be representated by Pettis integrable multifunctions. More precisely we show that a Pettis integrable density can be obtained even considering dominated interval multimeasures.

Theorem 3.4.1. *Let $\Phi : \mathcal{I} \to CK(X)$ be an interval multimeasure such that there exists a set $Q \in CK(X)$ with $\Phi(I) \subseteq |I|Q$ for every $I \in \mathcal{I}$. Then there exists a multifunction $F : [0,1] \to CK(X)$ Pettis integrable in $CK(X)$ such that:*

1. *for every finitely additive selection ϕ of Φ there exists a Pettis integrable selection f of F with $\phi(I) = (P)\int_I f\,d\lambda$ for all $I \in \mathcal{I}$;*

2. *$\Phi(I) = (P)\int_I F\,d\lambda$ for all $I \in \mathcal{I}$.*

Proof. By Proposition 3.4.1, Φ can be extended to a σ-additive multimeasure $M : \sigma(\mathcal{A}) \to CK(X)$ such that $M(B) \subseteq \lambda(B)Q$ for every $B \in \sigma(\mathcal{A})$. Therefore, by [8, Theorem 3.1], there exists a Pettis integrable multifunction $F : [0,1] \to CK(X)$ such that

1. for each countably additive selection m of M, there exists a Pettis integrable selection f of F such that $m(B) = (P)\int_B f\,d\lambda$, for each $B \in \sigma(\mathcal{A})$,

2. $M(B) = (P)\int_B F\,d\lambda$.

We conclude that F satisfies the required properties. □

In the following result we prove that we get a Pettis density even if we weaken the hypothesis of previous theorem, assuming that the multimeasure is pointwise dominated.

Theorem 3.4.2. *Let $\Phi : \mathcal{I} \to CK(X)$ be an interval multimeasure. Assume that for all $t \in [0,1]$ there exist a set $Q_t \in CK(X)$ and $\delta_t > 0$ such that $\Phi(I) \subseteq Q_t|I|$, for every interval I containing t with $|I| < \delta_t$.*
Then there exists a Pettis integrable multifunction $F : [0,1] \to CK(X)$ in $CK(X)$ such that:

1. *for every selection ϕ of Φ there exists a Pettis integrable selection f of F such that $\phi(I) = (P)\int_I f\,d\lambda$ for all $I \in \mathcal{I}$;*
2. *$\Phi(I) = (P)\int_I F\,d\lambda$ for all $I \in \mathcal{I}$.*

Proof. Let us consider all the intervals of the form $(t - \delta_t, t + \delta_t)$, $t \in [0,1]$. Since $\{(t-\delta_t, t+\delta_t)\}_t$ is an open covering of $[0,1]$ and $[0,1]$ is compact, there exist $t_1, \dots, t_n$ such that $\bigcup_{i=1}^n (t_i - \delta_{t_i}, t_i + \delta_{t_i}) \supseteq [0,1]$.
Let $\{J_i : i \le m\}$, be the collection of non-overlapping closed intervals determined by the end-points of the intervals $(t_i - \delta_{t_i}, t_i + \delta_{t_i})$, $i \le n$ (in case 0 or 1 belongs to above intervals, we take 0 or 1 as end-points). Denote by Φ_i the restriction of Φ to J_i. Each of Φ_i satisfies the hypothesis of Theorem 3.4.1. Consequently, for each $i = 1, \dots, m$ there exists a Pettis integrable multifunction $F_i : J_i \to CK(X)$ which satisfies the thesis of Theorem 3.4.1. The multifunction $F = \sum_{i=1}^m F_i$ is still Pettis integrable in $CK(X)$ and clearly satisfies the required properties. □

Proposition 3.4.2. *Let $\Phi : \mathcal{I} \to CK(X)$ be an interval multimeasure such that $V_\Phi << \lambda$. Assume that there exists a sequence $(I_n)_{n=1}^\infty$ of non-overlapping intervals such that $\lambda([0,1] \setminus \bigcup_{n=1}^\infty I_n) = 0$ and for each natural number n there exists a compact set $Q_n \subset X$ with the property that $\Phi(I) \subseteq |I|Q_n$ for all subinterval I of I_n.*
Then Φ is the primitive of a $CK(X)$-valued multifunction HKP-integrable in $CK(X)$.

Proof. By Theorem 3.4.1, for each natural number n there exists a multifunction $G_n : I_n \to CK(X)$, Pettis integrable in $CK(X)$, such that

$$\Phi(I) = (P)\int_I G_n\,d\lambda, \text{ for each interval } I \subseteq I_n.$$

Let us consider now the multifunction $G : [0,1] \to CK(X)$ defined as

$$G(t) := \sum_{n=1}^{\infty} G_n(t).$$

Since $V_\Phi << \lambda$, we have also $V_{s(x^*,\Phi)} << \lambda$ for every $x^* \in X^*$. Therefore by [5, Theorem 3], for every $x^* \in X^*$ there exists $g_{x^*} \in HK([0,1])$ such that

$$s(x^*, \Phi(I)) = (HK) \int_I g_{x^*} \, d\lambda, \text{ for all } I \in \mathcal{I}.$$

Fix $x^* \in X^*$. For each n and each interval $I \subset I_n$ we have

$$s(x^*, \Phi(I)) = (HK) \int_I g_{x^*} \, d\lambda.$$

But for the same n and I we have also

$$s(x^*, \Phi(I)) = (HK) \int_I s(x^*, G_n) \, d\lambda.$$

Therefore we obtain $(HK) \int_I s(x^*, G_n) \, d\lambda = (HK) \int_I g_{x^*} \, d\lambda$ for each n and each interval $I \subset I_n$. It follows by [29, Theorem 9.12] that for every n, $s(x^*, G_n) = g_{x^*}$ almost everywhere on I_n (and the exceptional set depends only on x^*).
By the definition of G, we have that $s(x^*, G) = g_{x^*}$ almost everywhere on $[0,1]$ (and the exceptional set depends only on x^*). Therefore, by [29, Theorem 9.10], $s(x^*, G)$ is *HK*-integrable. Since x^* is arbitrary, then G is scalarly *HK*-integrable.
Finally, if $I \in \mathcal{I}$ and $x^* \in X^*$, we have

$$s(x^*, \Phi(I)) = (HK) \int_I g_{x^*} \, d\lambda = (HK) \int_I s(x^*, G) \, d\lambda.$$

We conclude that G is *HKP*-integrable in $CK(X)$ and that Φ is its *HKP*-primitive. □

In the particular case $X = \mathbb{R}$ we obtain the following result similar to that we have in case of X-valued functions (see [4, Theorem 3.6], and [5, Theorem 3]).

Theorem 3.4.3. *Let $\Phi : \mathcal{I} \to CK(\mathbb{R})$ be an interval multimeasure. Assume moreover that $V_\Phi << \lambda$. Then there exists an Henstock integrable multifunction $F : [0,1] \to CK(\mathbb{R})$ such that:*

1. *For every selection ϕ of Φ, there exists an HK-integrable selection f of F such that $\phi(I) = (HK)\int_I f\, d\lambda$ for every $I \in \mathcal{I}$.*
2. *$\Phi(I) = (HK)\int_I F\, d\lambda$ for every $I \in \mathcal{I}$.*

Proof. Since Φ is $CK(\mathbb{R})$-valued, $\Phi(I)$ is a closed bounded interval of the real line for all $I \in \mathcal{I}$.
Let us consider the real functions $\varphi, \psi : \mathcal{I} \to \mathbb{R}$ defined respectively by $\varphi(I) := \min \Phi(I)$ and $\psi(I) := \max \Phi(I)$.
Of course, φ and ψ are selections of Φ. Moreover, since by hypothesis $V_\Phi << \lambda$, we have $V_\varphi << \lambda$ and $V_\psi << \lambda$. So by [5, Theorem 3], φ and ψ are differentiable almost everywhere in $[0,1]$ and there exist $f, g \in \mathcal{HK}([0,1])$ such that $\varphi(I) = (HK)\int_I f\, d\lambda$ and $\psi(I) = (HK)\int_I g\, d\lambda$ for each $I \in \mathcal{I}$.
Moreover, $\varphi' = f$ and $\psi' = g$ a.e.
Since $\varphi \leq \psi$, we have $(HK)\int_I f\, d\lambda \leq (HK)\int_I g\, d\lambda$ for all $I \in \mathcal{I}$. Consequently $f \leq g$ a.e.
Now let consider the multifunction F defined by

$$F(t) := \begin{cases} [f(t), g(t)] & \text{if } f(t) \leq g(t) \\ \{0\} & \text{elsewhere.} \end{cases}$$

Clearly F is $CK(\mathbb{R})$-valued. Now we prove that F satisfies the required properties.

1. Let γ be a selection of Φ. Since by hypothesis $V_\Phi << \lambda$, also $V_\gamma << \lambda$. Therefore by [5, Theorem 3], γ is differentiable almost everywhere in $[0,1]$ and there exists $h \in \mathcal{HK}([0,1])$ such that $\gamma(I) = (HK)\int_I h\, d\lambda$. Moreover, $\gamma' = h$ a.e.
   Since $\varphi \leq \gamma \leq \psi$, then we get also that $f \leq h \leq g$ a.e. Consequently $h(t) \in F(t)$ for almost every $t \in [0,1]$. So, changing eventually the values in a negligible set, we have that h is a selection of F.

2. Since $f, g \in \mathcal{HK}([0,1])$, for each $\varepsilon > 0$, there exists a gauge δ on $[0,1]$ such that
$$\left|(HK)\int_0^1 f\,d\lambda - \sum_{j=1}^{p} f(t_j)|I_j|\right| < \varepsilon/2$$
and
$$\left|(HK)\int_0^1 g\,d\lambda - \sum_{j=1}^{p} g(t_j)|I_j|\right| < \varepsilon/2,$$
for every δ-fine Perron-partition $\{(I_j, t_j)\}_{j=1}^p$ of $[0,1]$.
If we put $u = (HK)\int_0^1 f\,d\lambda$ and $v = (HK)\int_0^1 g\,d\lambda$, then
$$\begin{aligned} & d_H\left(\Phi([0,1]), \sum_{j=1}^{p} F(t_j)|I_j|\right) \\ &= d_H\left([u,v], \left[\sum_{j=1}^{p} f(t_j)|I_j|, \sum_{j=1}^{p} g(t_j)|I_j|\right]\right) \\ &\le \left|u - \sum_{j=1}^{p} f(t_j)|I_j|\right| + \left|v - \sum_{j=1}^{p} g(t_j)|I_j|\right| < \varepsilon, \end{aligned}$$
for every δ-fine Perron-partition $\{(I_j, t_j)\}_{j=1}^p$ of $[0,1]$.
Therefore F is Henstock integrable and $(H)\int_0^1 F\,d\lambda = \Phi([0,1])$.
Finally, using Hausdorff distance we obtain that for every $I \in \mathcal{I}$,
$$\begin{aligned} & d_H\left(\Phi(I), (H)\int_I F\,d\lambda\right) \\ &\le \left|\varphi(I) - (HK)\int_I f\,d\lambda\right| + \left|\psi(I) - (HK)\int_I g\,d\lambda\right| = 0. \end{aligned}$$
Hence $\Phi(I) = (H)\int_I F\,d\lambda$ for every $I \in \mathcal{I}$ and the proof is over.

□

3.4.2 The $CWK(X)$ case

Now we are going to consider the more general case of $CWK(X)$-valued multifunctions. We need some preliminary results.

Proposition 3.4.3. *Let $g : [0,1] \to \mathbb{R}$ be a Henstock-Kurzweil integrable function such that $(HK)\int_I g\,d\lambda \geq 0$ for every $I \in \mathcal{I}$. Then $g \geq 0$ almost everywhere on $[0,1]$.*

Proof. By Theorem 1.5.1, the function $G(t) := (HK)\int_0^t g\,d\lambda$ is continuous, differentiable almost everywhere on $[0,1]$ and $G' = g$ almost everywhere on $[0,1]$.
Moreover, by hypothesis, G is monotone non-decreasing. Hence we obtain $G'(t) = g(t) \geq 0$ for almost every $t \in [0,1]$. □

Proposition 3.4.4. *Let $\Phi : \mathcal{I} \to CWK(X)$ be an interval multimeasure such that $V_\Phi << \lambda$. Assume that $s(x^*, \Phi(I)) \geq 0$ for every $x^* \in X^*$ and for every $I \in \mathcal{I}$. Then Φ can be extended to a σ-additive multimeasure $M : \mathcal{L} \to CWK(X)$ of σ-finite variation and with $M << \lambda$.*

Proof. Since $V_\Phi << \lambda$, we have also that $V_{s(x^*,\Phi)} << \lambda$ for each $x^* \in X^*$. By [5, Theorem 3], for every $x^* \in X^*$ there exists $g_{x^*} \in HK([0,1])$ such that

$$s(x^*, \Phi(I)) = (HK)\int_I g_{x^*}\,d\lambda, \text{ for every } I \in \mathcal{I}.$$

Since $s(x^*, \Phi) \geq 0$, it follows by Proposition 3.4.3 that $g_{x^*} \geq 0$ almost everywhere on $[0,1]$. By Theorem 1.5.2, g_{x^*} is Lebesgue integrable for every $x^* \in X^*$. Moreover, $V_{s(x^*,\Phi)}$ is a measure over all Borel sets of $[0,1]$. By [20, Theorem 2], $V_{s(x^*,\Phi)}(B) = \int_B g_{x^*}\,d\lambda$ for every $B \in \sigma(\mathcal{A})$.
Now let consider the family

$$\mathcal{B} := \left\{ B \in \sigma(\mathcal{A}) : \exists\, C_B \in CWK(X) | \,\forall\, x^* \in X^*,\, s(x^*, C_B) = \int_B g_{x^*}\,d\lambda \right\}.$$

We observe that $s(x^*, C_B) \leq \int_0^1 g_{x^*}\,d\lambda = s(x^*, \Phi([0,1]))$ for each $B \in \mathcal{B}$ and each $x^* \in X^*$. Hence $C_B \subseteq \Phi([0,1])$ for every $B \in \mathcal{B}$.
It is clear that $\mathcal{B}$ contains $\mathcal{A}$. We claim that $\mathcal{B}$ is a monotone class. In fact, let $(B_n)_{n=1}^\infty$ be a monotone increasing sequence of $\mathcal{B}$ and let $C_{B_n} \in CWK(X)$ such that $s(x^*, C_{B_n}) = \int_{B_n} g_{x^*}\,d\lambda$ for every $x^* \in X^*$. By the Monotone Convergence Theorem (see [29, Theorem 3.21]), $\lim_n \int_{B_n} g_{x^*}\,d\lambda = \int_{\bigcup_{n=1}^\infty B_n} g_{x^*}\,d\lambda$. Moreover, also $(C_{B_n})_{n=1}^\infty$ is a monotone increasing sequence. In fact, for every

n and every $x^* \in X^*$, $s(x^*, C_{B_n}) = \int_{B_n} g_{x^*}\, d\lambda \leq \int_{B_{n+1}} g_{x^*}\, d\lambda = s(x^*, C_{B_{n+1}})$. Hence $C_{B_n} \subseteq C_{B_{n+1}}$ for every n.
Consequently, $\lim_n s(x^*, C_{B_n}) = s(x^*, \bigcup_{n=1}^{\infty} C_{B_n}) = s(x^*, \overline{\bigcup_{n=1}^{\infty} C_{B_n}})$. In fact, first equality follows from the fact that $\lim_n s(x^*, C_{B_n}) = \sup_n s(x^*, C_{B_n}) = s(x^*, \bigcup_{n=1}^{\infty} C_{B_n})$, the second equality is a property of the support function. Since $\bigcup_{n=1}^{\infty} C_{B_n} \subseteq \Phi([0,1]) \in CWK(X)$, we have $\overline{\bigcup_{n=1}^{\infty} C_{B_n}} \in CWK(X)$. Hence $s(x^*, \overline{\bigcup_{n=1}^{\infty} C_{B_n}}) = \int_{\bigcup_{n=1}^{\infty} B_n} g_{x^*}\, d\lambda$ and therefore $\bigcup_{n=1}^{\infty} B_n \in \mathcal{B}$.
Let $(B_n)_{n=1}^{\infty}$ be a monotone decreasing sequence of $\mathcal{B}$ and let $C_{B_n} \in CWK(X)$ such that $s(x^*, C_{B_n}) = \int_{B_n} g_{x^*}\, d\lambda$ for every $x^* \in X^*$.
Clearly $\lim_n \int_{B_n} g_{x^*}\, d\lambda = \int_{\bigcap_{n=1}^{\infty} B_n} g_{x^*}\, d\lambda$. Moreover, also $(C_{B_n})_{n=1}^{\infty}$ is a monotone decreasing sequence.
Thus $\lim_n s(x^*, C_{B_n}) = s(x^*, \bigcap_{n=1}^{\infty} C_{B_n}) = s(x^*, \overline{\bigcap_{n=1}^{\infty} C_{B_n}})$.
Moreover, we observe that $\overline{\bigcap_{n=1}^{\infty} C_{B_n}} \in CWK(X)$, because $\bigcap_{n=1}^{\infty} C_{B_n} \subseteq \Phi([0,1]) \in CWK(X)$. Hence $s(x^*, \overline{\bigcap_{n=1}^{\infty} C_{B_n}}) = \int_{\bigcap_{n=1}^{\infty} B_n} g_{x^*}\, d\lambda$ for every $x^* \in X^*$. Therefore $\bigcap_{n=1}^{\infty} B_n \in \mathcal{B}$.
By the Monotone Class Theorem (see [50]), $\mathcal{B}$ contains the smallest σ-algebra containing $\mathcal{A}$. Hence $\mathcal{B} = \sigma(\mathcal{A})$.
Let define $M : \sigma(\mathcal{A}) \to CWK(X)$ as follows: $M(B) = C_B$, $B \in \sigma(\mathcal{A})$.
M is a multimeasure, because for every $x^* \in X^*$, $s(x^*, M(\cdot))$ is a Lebesgue integral.
Since M is $CWK(X)$-valued, by Theorem 1.6.1, M is a d_H-multimeasure (and a strong multimeasure).
We prove that $M << \lambda$. In fact, if $B \in \sigma(\mathcal{A})$ and $\lambda(B) = 0$, then for every $x^* \in X^*$, $s(x^*, M(B)) = \int_B g_{x^*}\, d\lambda = 0$. Consequently, $||M(B)|| = \sup_{x^* \in B(X^*)} |s(x^*, M(B))| = 0$, hence $M(B) = \{0\}$.
It remains to prove that M is of σ-finite variation. Since $V_\Phi << \lambda$, we have that V_Φ is σ-finite. Let $(B_n)_n \subseteq \sigma(\mathcal{A})$ be a partition of $[0,1]$ such that $V_\Phi(B_n) < +\infty$ for every n. Fix n and let $\{B_{n,1}, \ldots, B_{n,k}\} \subseteq \sigma(\mathcal{A})$ be a partition of B_n. Then for every $x^* \in B(X^*)$ and every $j = 1, \ldots, k$ we obtain $s(x^*, M(B_{n,j})) = V_{s(x^*,\Phi)}(B_{n,j}) \leq V_\Phi(B_{n,j})$. Hence for every $j = 1, \ldots, k$, $||M(B_{n,j})|| \leq V_\Phi(B_{n,j})$ and therefore $\sum_{j=1}^{k} ||M(B_{n,j})|| \leq V_\Phi(B_n)$. Finally, $|M|(B_n) \leq V_\Phi(B_n) < +\infty$.
Since $M << \lambda$, we can extend M to $\mathcal{L}$, because any measurable set is the

union of a Borel set and a set of zero Lebesgue measure. The proof is complete. □

Remark 3.4.1. The condition $s(x^*, \Phi(I)) \geq 0$ for every $x^* \in X^*$ and every $I \in \mathcal{I}$ implies that $0 \in \Phi(I)$ for every $I \in \mathcal{I}$.

Theorem 3.4.4. *Assume that X is a Banach space with the RNP and let $\Phi : \mathcal{I} \to CWK(X)$ be an interval multimeasure such that $V_\Phi << \lambda$. Then Φ admits a CBC(X)-valued density F which is HKP-integrable in CWK(X).*

Proof. Let us consider first the particular case when $s(x^*, \Phi) \geq 0$ for every $x^* \in X^*$. By Proposition 3.4.4, Φ can be extended to a σ-additive multimeasure $M : \mathcal{L} \to CWK(X)$ such that M is of σ-finite variation and $M << \lambda$.
Let $(A_n)_{n=1}^{\infty}$ be a sequence of pairwise disjoint sets of $\mathcal{L}$ such that $\bigcup_{n=1}^{\infty} A_n = [0,1]$ and $|M|(A_n) < +\infty$ for all n. Let us denote by M_n the restriction of M to all measurable subsets of A_n. Each M_n is a $CWK(X)$-valued (hence $CBC(X)$-valued) multimeasure of finite variation. Moreover, since $M << \lambda$, also $M_n << \lambda$, for all n.
Since X has the RNP, by [8, Theorem 4.1], we have that for all n, M_n has a density $F_n : A_n \to CBC(X)$ which is Pettis integrable in $CBC(X)$.
Now let us define the multifunction $F : [0,1] \to CBC(X)$ as follows:

$$F(t) := F_n(t), \text{ if } t \in A_n.$$

We check that F is scalarly integrable. Let us fix $x^* \in X^*$. Since M is $CWK(X)$-valued, for all $x^* \in X^*$ $s(x^*, M)$ is a positive (by construction) real-valued measure absolutely continuous with respect to λ. Therefore by the classic Radon-Nikodým Theorem [16, Theorem 5, p.163], there exists $h_{x^*} \in L^1([0,1])$ such that

$$s(x^*, M(A)) = \int_A h_{x^*}\, d\lambda, \text{ for every } A \in \mathcal{L}.$$

Moreover, for each n, F_n is a Pettis integrable density of M_n, hence

$$s(x^*, M_n(A)) = \int_A s(x^*, F_n)\, d\lambda, \text{ for every } A \in \mathcal{L}, A \subseteq A_n.$$

It follows that for every n, $s(x^*, F_n) = h_{x^*}$ almost everywhere on A_n (and the exceptional set depends only on x^*).
By the definition of F, we have also that $s(x^*, F) = h_{x^*}$ (and the exceptional set depends only on x^*). Therefore $s(x^*, F)$ is integrable. Since x^* is arbitrary, then F is scalarly integrable.
Finally we observe that for every $A \in \mathcal{L}$ and every $x^* \in X^*$,

$$s(x^*, M(A)) = \int_A h_{x^*} \, d\lambda = \int_A s(x^*, F) \, d\lambda.$$

Therefore F is a Pettis integrable (in $CWK(X)$) density of M. In particular,

$$\Phi(I) = (P) \int_I F \, d\lambda, \text{ for every } I \in \mathcal{I}.$$

In the general case, let ϕ be a finitely additive selection of Φ (existing by Proposition 3.2.1) and let consider $\Psi := \Phi - \phi$. It is clear that $s(x^*, \Psi) \geq 0$ for every $x^* \in X^*$. We have also that $V_\Psi << \lambda$, since $V_\Phi << \lambda$ and $V_\phi << \lambda$. Consequently, Ψ has a density $G : [0,1] \to CBC(X)$ Pettis integrable in $CWK(X)$. By [4, Theorem 3.6], ϕ has a variationally Henstock integrable (and then Henstock integrable) density $f : [0,1] \to X$.
Now let consider the multifunction $F := G + f$. Clearly F is $CBC(X)$-valued. Moreover, $s(x^*, F) = s(x^*, G) + \langle x^*, f \rangle$, for every $x^* \in X^*$. Since each $s(x^*, G)$ is Lebesgue integrable and each $\langle x^*, f \rangle$ is HK-integrable, also $s(x^*, F)$ is HK-integrable. Hence F is scalarly HK-integrable.
Finally for every $x^* \in X^*$ and for every $I \in \mathcal{I}$ we have

$$\begin{aligned} & s(x^*, \Phi(I)) = s(x^*, \Psi(I)) + \langle x^*, \phi(I) \rangle \\ & = \int_I s(x^*, G) \, d\lambda + (HK) \int_I \langle x^*, f \rangle \, d\lambda = (HK) \int_I s(x^*, F) \, d\lambda. \end{aligned}$$

We conclude that F is HKP-integrable in $CBC(X)$ and

$$\Phi(I) = (HKP) \int_I F \, d\lambda, \text{ for every } I \in \mathcal{I}$$

. □

Remark 3.4.2. In general, under the hypothesis of Theorem 3.4.4, the density of Φ is only $CBC(X)$ and not $CWK(X)$ valued, as the following example

shows (see [14, Exemple 2]).
Let X be the space ℓ^1 and let $(e_n)_{n\geq 0}$ be the canonical base of ℓ^1. Let $(\alpha_n^k)_{n,k\geq 0}$ be a sequence of real numbers such that

$$\sum_{n\geq 0} |\alpha_n^k| = 1 \text{ for every } k \geq 0 \text{ and } \sum_{k\geq 0} \left(\sum_{n\geq 0} |\alpha_n^k|^2 \right)^{\frac{1}{2}} < +\infty.$$

Let $(r_n)_{n\geq 0}$ be the sequence of the Rademacher functions. For $k \geq 0$ and $t \in [0,1]$, set $\sigma_k(t) := (\alpha_n^k r_n(t))_{n\geq 0} \in \ell^1$.
Now let define the multifunction $F(t) := \overline{co}\{\sigma_k(t) : k \geq 0\}$, $t \in [0,1]$. Then, F is with values in $CBC(\ell^1)$ and Pettis integrable in $CWK(\ell^1)$, but $F(t) \notin CWK(\ell^1)$ almost everywhere.

Remark 3.4.3. Since on the real line $CBC(\mathbb{R}) = CK(\mathbb{R}) = CWK(\mathbb{R})$ and the Henstock integrability coincides with the *HKP*-integrability, we have that Theorem 3.4.3 is included in Theorem 3.4.4. Nevertheless we prefered to give its proof, since it uses properties of the real line.

In [51] it has been proved the following result.

Theorem 3.4.5. *Let X be a separable Banach space with the RNP. Assume that also X^* has the RNP. Let M be a $CWK(X)$-valued multimeasure of σ-finite variation and such that $M << \lambda$. Then M admits a unique density $F : [0,1] \to CWK(X)$ which is Pettis integrable in $CWK(X)$.*

Under the same assumptions of Theorem 3.4.5 we can obtain the following result.

Theorem 3.4.6. *Let X be a separable Banach space with the RNP. Assume that also X^* has the RNP. Let $\Phi : \mathcal{I} \to CWK(X)$ be an interval multimeasure such that $V_\Phi << \lambda$. Then Φ admits a CWK(X)-valued density F which is HKP-integrable in CWK(X).*

Proof. First let us consider the particular case when $s(x^*, \Phi) \geq 0$ for every $x^* \in X^*$. By Proposition 3.4.4, Φ can be extended to a σ-additive multimeasure $M : \mathcal{L} \to CWK(X)$ such that M is of σ-finite variation and $M << \lambda$. By hypothesis, X is separable, has the RNP and also its dual X^* has the

RNP. Therefore by Theorem 3.4.5, M has a density $F : [0,1] \to CWK(X)$ which is Pettis integrable in $CWK(X)$. Consequently, we have

$$\Phi(I) = (P)\int_I F\, d\lambda, \text{ for every } I \in \mathcal{I}.$$

In the general case, let ϕ be a finitely additive selection of Φ and let consider $\Psi := \Phi - \phi$. It is clear that $s(x^*, \Psi) \geq 0$ for every $x^* \in X^*$. We have also that $V_\Psi << \lambda$, since $V_\Phi << \lambda$ and $V_\phi << \lambda$. Consequently, Ψ has a density $G : [0,1] \to CWK(X)$ Pettis integrable in $CWK(X)$. By [4, Theorem 3.6], ϕ has a variationally Henstock (then a Henstock) integrable density $f : [0,1] \to X$.

Now let consider the multifunction $F := G + f$. Clearly F is $CWK(X)$-valued. Moreover, it is easy to check that $s(x^*, F) = s(x^*, G) + \langle x^*, f\rangle$, for every $x^* \in X^*$. Since each $s(x^*, G)$ is Lebesgue integrable and each $\langle x^*, f\rangle$ is HK-integrable, also $s(x^*, F)$ is HK-integrable. Hence F is scalarly HK-integrable.

Finally for every $x^* \in X^*$ we have

$$\begin{aligned} &s(x^*, \Phi(I)) = s(x^*, \Psi(I)) + \langle x^*, \phi(I)\rangle \\ &= \int_I s(x^*, G)\, d\lambda + (HK)\int_I \langle x^*, f\rangle\, d\lambda = (HK)\int_I s(x^*, F)\, d\lambda, \end{aligned}$$

for every $I \in \mathcal{I}$.

We conclude that F is HKP-integrable in $CWK(X)$ and

$$\Phi(I) = (HKP)\int_I F\, d\lambda, \text{ for every } I \in \mathcal{I}$$

. □

CHAPTER 4

HENSTOCK INTEGRABILITY OF HUKUHARA DIFFERENTIAL

4.1 Introduction.

There have been several attempts to develop a differential calculus for multifunctions. Unfortunately, none of them produced a completely satisfactory theory and each one is useful and effective only within a particular class of problems.
The most popular of these approaches are due by T. F. Bridgland [7], F. S. De Blasi [17], M. Martelli and A. Vignoli [37] and M. Hukuhara [33]. They were motivated essentially by the perturbation theory of differential inclusions and the theory of set differential equations which generalizes the theory of ordinary differential equations.
In this chapter, we use the definition of differentiability introduced by M. Hukuhara, as it is well suited for our purposes. This notion is very useful and plays a fundamental role in the theory of set differential equations (see for instance [25, 26, 27]).
In particular, we prove for the multivalued case some results valid for vector-valued functions. More precisely we show the almost everywhere Hukuhara

differentiability for a variational Henstock primitive (see Theorem 4.2.1) and the variational Henstock integrability of a Hukuhara derivative (see Theorem 4.2.2).
A characterization of the variationally Henstock primitives is also given (see Theorem 4.2.4). As an application of the Hukuhara differentiability, we prove that all the scalarly measurable selections of a variationally Henstock integrable multifunction are variationally Henstock integrable (see Theorem 4.3.1).
We end the chapter showing that Theorem 4.3.1 holds for $CK(X)$-valued variationally Henstock integrable multifunctions, but it fails to be true for $CWK(X)$-valued multifunctions (see Example 4.3.1).

4.2 The Hukuhara derivative

We start with some definitions.

Definition 4.2.1. Let $A, B \in C(X)$. The set $C \in C(X)$ is said to be the *Hukuhara difference* or simply *H-difference* of A and B if $A = B + C$. We denote it by $A \ominus B$.

Remark 4.2.1. If $C \in C(X)$ is the H-difference of $A, B \in C(X)$, then C is uniquely determined [43, Lemma 2]. Moreover, if there exist $A \ominus B$ and $B \ominus C$, then also $A \ominus C$ exists and $A \ominus C = (A \ominus B) + (B \ominus C)$.
In fact, if $A = B + K_1$ for some K_1 and $B = C + K_2$ for some K_2, then $A = C + (K_1 + K_2)$. But $K_1 = A \ominus B$ and $K_2 = B \ominus C$. Therefore $A \ominus C = (A \ominus B) + (B \ominus C)$.

Definition 4.2.2. Let $F : [0, 1] \to C(X)$ be a multifunction. We say that F *satisfies condition (H)* on $[0, 1]$, if for every $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$, there exists the H-difference $F(t_2) \ominus F(t_1)$.

Definition 4.2.3. Let $F : [0, 1] \to C(X)$ be a multifunction satisfying condition (H). We say that F admits a *Hukuhara differential* (or simply *H-differential*) at $t_0 \in (0, 1)$ if there exists a set $F'(t_0) \in C(X)$ such that the

limits
$$\lim_{h\to 0^+} \frac{F(t_0+h)\ominus F(t_0)}{h}$$
and
$$\lim_{h\to 0^+} \frac{F(t_0)\ominus F(t_0-h)}{h}$$
exist (with respect to the Hausdorff distance d_H) and are equal to $F'(t_0)$. We call $F'(t_0)$ the *H-derivative* of F in t_0.

Remark 4.2.2. A $C(X)$-valued multifunction F is H-differentiable $t_0 \in [0,1]$ with H-derivative $F'(t_0)$, if and only if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for any interval $[u,v]$ satisfying $t_0 \in [u,v] \subset (t_0-\delta, t_0+\delta)$, we have $d_H(\frac{F(v)\ominus F(u)}{v-u}, F'(t_0)) < \varepsilon$.

Proof. We observe that if $u \leq t_0 \leq v$, then
$$d_H\left(\frac{F(v)\ominus F(u)}{v-u}, F'(t_0)\right) = \frac{d_H(F(v)\ominus F(u), F'(t_0)(v-u))}{v-u}$$
$$= \frac{d_H((F(v)\ominus F(t_0)) + (F(t_0)\ominus F(u)), F'(t_0)((v-t_0)+(t_0-u)))}{v-u}$$
$$\leq \frac{d_H(F(v)\ominus F(t_0), F'(t_0)(v-t_0))}{v-t_0} + \frac{d_H(F(t_0)\ominus F(u), F'(t_0)(t_0-u))}{t_0-u}$$
$$= \frac{v-t_0}{v-u}\cdot d_H\left(\frac{F(v)\ominus F(t_0)}{v-t_0}, F'(t_0)\right) + \frac{t_0-u}{v-u}\cdot d_H\left(\frac{F(t_0)\ominus F(u)}{t_0-u}, F'(t_0)\right).$$
Hence if F admits H-differential at t_0, then
$$\lim_{v\to t_0^+} d_H\left(\frac{F(v)\ominus F(t_0)}{v-t_0}, F'(t_0)\right) = 0$$
and
$$\lim_{u\to t_0^-} d_H\left(\frac{F(t_0)\ominus F(u)}{t_0-u}, F'(t_0)\right) = 0.$$
Taking into account that $0 \leq \frac{v-t_0}{v-u} \leq 1$ and $0 \leq \frac{t_0-u}{v-u} \leq 1$, we obtain that
$$d_H\left(\frac{F(v)\ominus F(u)}{v-u}, F'(t_0)\right) \to 0$$
whenever $v \to t_0^+$ and $u \to t_0^-$.
The converse is obvious. $\square$

Proposition 4.2.1. *Let $\Phi : \mathcal{I} \to C(X)$ be finitely additive and let G be the $C(X)$-valued multifunction defined as $G(t) := \Phi([0,t])$. Then G satisfies condition (H) on $[0,1]$ and $\Phi([a,b]) = G(b) \ominus G(a)$.*

Proof. Let $[a,b]$ be a subinterval of $[0,1]$. Since Φ is finitely additive we have $\Phi([0,b]) = \Phi([0,a]) + \Phi([a,b])$.
Therefore $G(b) \ominus G(a) = \Phi([0,b]) \ominus \Phi([0,a]) = \Phi([a,b])$. □

The following result concerning the H-diffentiability of the primitive is a generalization of [49, Theorem 7.4.2], valid for X-valued functions.

Theorem 4.2.1. *Let $F : [0,1] \to C(X)$ be a variationally Henstock integrable multifunction, let Φ be its primitive and let G be the C(X)-valued multifunction defined as $G(t) := \Phi([0,t])$.*
Then G satisfies condition (H) on $[0,1]$, is H-differentiable almost everywhere and $G'(t) = F(t)$ almost everywhere in $[0,1]$.

Proof. Since Φ is finitely additive, by Proposition 4.2.1 G satisfies condition (H) on $[0,1]$.
Now let us fix $\varepsilon > 0$. Since F is variationally Henstock integrable, there exists a gauge δ on $[0,1]$ such that

$$\sum_{i=1}^{p} d_H(F(t_i)(a_i - a_{i-1}), \Phi([a_{i-1}, a_i])) < \varepsilon,$$

for every δ-fine Perron-partition $\{([a_{i-1}, a_i], t_i)\}_{i=1}^{p}$ of $[0,1]$.
In particular, for every δ-fine Perron-partition $\{([u_j, v_j], t_j)\}_{j=1}^{p}$ in $[0,1]$ one has

$$\sum_{j=1}^{p} d_H(F(t_j)(v_j - u_j), \Phi([u_j, v_j])) < \varepsilon.$$

Let N be the set of points $t \in [0,1]$ such that $G'(t)$ does not exist or, if it does, is not equal to $F(t)$. We prove that $\lambda(N) = 0$.
If $t \in N$, there exists a $\eta(t) > 0$ such that for every $\delta(t) > 0$, there exists an interval I satisfying $t \in I \subset (t - \delta(t), t + \delta(t))$ such that

$$d_H(F(t)(v-u), \Phi([u,v])) \geq \eta(t)(v-u), \text{ where } I = [u,v].$$

Let $N_k := \{t \in N : \eta(t) \geq \frac{1}{k}\}$, then $N = \bigcup_{k=1}^{\infty} N_k$. Fixed k, then the above family of closed intervals covers N_k in the Vitali sense. Applying the Vitali Covering Lemma (see [29, Lemma 4.6]), we can find $\{[u_j, v_j]\}_{j=1}^{m}$ such that $\lambda^*(N_k \setminus \bigcup_{j=1}^{m}[u_j, v_j]) < \varepsilon$. It follows that $\lambda^*(N_k) < \sum_{j=1}^{m}(v_j - u_j) + \varepsilon$. Therefore

$$\lambda^*(N_k) < \sum_{j=1}^{m}(v_j - u_j) + \varepsilon \leq \sum_{j=1}^{m} \frac{d_H(F(t_j)(v_j - u_j), G(v_j) \ominus G(u_j))}{\eta(t_j)} + \varepsilon$$
$$\leq k \sum_{j=1}^{m} d_H(F(t_j)(v_j - u_j), G(v_j) \ominus G(u_j)) + \varepsilon < \varepsilon(k+1).$$

Since $\varepsilon > 0$ is arbitrary, we have $\lambda^*(N_k) = \lambda(N_k) = 0$ for every k. Therefore $\lambda(N) = 0$. □

Proposition 4.2.2. *Let $F : [0,1] \to C(X)$ be a multifunction satisfying condition (H) on $[0,1]$. Then the interval multifunction defined as $\Phi([a,b]) := F(b) \ominus F(a)$ is finitely additive.*

Proof. Let consider $a, b, c \in [0,1]$ such that $a < c < b$. Then $F(b) = F(c) + \Phi([c,b])$ and $F(c) = F(a) + \Phi([a,c])$. Consequently, $F(b) = F(a) + (\Phi([a,c]) + \Phi([c,b]))$. It follows that $\Phi([a,b]) := F(b) \ominus F(a) = \Phi([a,c]) + \Phi([c,b])$. □

The following result is well known for the case of X-valued functions [49, Theorem 7.3.10]. It states that every H-derivative on $[0,1]$ is variationally Henstock integrable.

Theorem 4.2.2. *Let $F : [0,1] \to C(X)$ be a multifunction that satisfies condition (H) on $[0,1]$ and let assume that F admits H-differential at each point of $[0,1]$. Then the H-derivative F' is variationally Henstock integrable and*

$$F(b) \ominus F(a) = (vH) \int_a^b F' \, d\lambda, \text{ for every } [a,b] \subseteq [0,1].$$

Proof. Since F satisfies condition (H) on $[0,1]$ and admits H-differential at each point of $[0,1]$, for every $t \in [0,1]$ and every $\varepsilon > 0$, there exists a $\delta(t) > 0$ such that for every interval I satisfying $t \in I \subset (t - \delta(t), t + \delta(t))$ we have $d_H(F'(t)(v-u), \Phi([u,v])) < \varepsilon(v-u)$, where $\Phi([u,v]) = F(v) \ominus F(u)$ and

$I = [u, v]$.
Hence for any δ-fine Perron-partition $\{([u_{i-1}, u_i], t_i)\}_{i=1}^{p}$ of $[0, 1]$, by the H-differentiability of F at t_i, $i = 0, \ldots, p$ we have

$$\sum_{i=1}^{p} d_H(F'(t_i)(u_i - u_{i-1}), F(u_i) \ominus F(u_{i-1})) < \varepsilon.$$

Therefore F' is variationally Henstock integrable and

$$F(b) \ominus F(a) = (vH) \int_a^b F' \, d\lambda, \text{ for every } [a, b] \subseteq [0, 1].$$

□

The following result is a further generalization of Theorem 4.2.2. The variationally Henstock integrability of the H-derivative can be obtained even if the multifunction is H-differentiable in a subset of $[0, 1]$ whose complementar is negligible with respect to the variational measure generated by the primitive.

Theorem 4.2.3. *Let $F : [0, 1] \to C(X)$ be a multifunction that satisfies condition (H) on $[0, 1]$. Assume that there exists a set $A \in \mathcal{L}$ with the property that F is H-differentiable at each point of A and such that $V_\Phi(A^c) = 0$, where $\Phi([a, b]) = F(b) \ominus F(a)$.*
Then the multifunction $G : [0, 1] \to C(X)$ defined as

$$G(t) = \begin{cases} F'(t) & \text{if } t \in A \\ \{0\} & \text{if } t \in A^c \end{cases}$$

is variationally Henstock integrable and Φ is its variational Henstock primitive.

Proof. By Proposition 4.2.2, Φ is finitely additive. Now fix $\varepsilon > 0$.
If $t \in A$, define $\delta(t) > 0$ such that

$$d_H(\Phi(I), F'(t)|I|) < \varepsilon |I|, \tag{4.1}$$

for every interval $I \in \mathcal{I}$ such that $t \in I \subset (t - \delta(t), t + \delta(t))$.
Moreover, since $V_\Phi(A^c) = 0$, there exists a gauge $\overline{\delta}$ on A^c such that

$$\sum_{i=1}^{s} ||\Phi(J_i)|| < \varepsilon, \tag{4.2}$$

for every $\overline{\delta}$-fine partition $\{(J_i, t_i)\}_{i=1}^{s}$ anchored on A^c.
So we set $\delta(t) := \overline{\delta}(t)$, for every $t \in A^c$.
Now let $\{(I_j, t_j)\}_{j=1}^{p}$ be a δ-fine Perron-partition of $[0, 1]$. Then, by (4.1) and (4.2), we have

$$\begin{aligned}\sum_{j=1}^{p} d_H(G(t_j)|I_j|, \Phi(I_j)) &= \sum_{t_j \in A} d_H(F'(t_j)|I_j|, \Phi(I_j)) + \sum_{t_j \in A^c} ||\Phi(I_j)|| \\ &< \varepsilon + \varepsilon = 2\varepsilon.\end{aligned}$$

Thus G is variationally Henstock integrable and Φ is its variational Henstock primitive. □

At this point we can characterize the interval multifunctions that are variational Henstock primitives.

Theorem 4.2.4. *Let $\Phi : \mathcal{I} \to C(X)$ be an interval multifunction. The following statements are equivalent.*

1. Φ *is a variational Henstock primitive.*

2. $V_\Phi << \lambda$ *and the multifunction $G(t) := \Phi([0, t])$ satisfies condition (H) on $[0, 1]$ and is H-differentiable a.e.*

Proof.

(1. $\Rightarrow$ 2.) It follows from Proposition 3.3.1 and Theorem 4.2.1.

(2. $\Rightarrow$ 1.) Let denote by A the set of all points $t \in [0, 1]$ at which G is H-differentiable. By hypothesis, $\lambda(A^c) = 0$. Moreover, $V_\Phi << \lambda$. Therefore $V_\Phi(A^c) = 0$.
Let define the multifunction F by

$$F(t) = \begin{cases} G'(t) & \text{if } t \in A \\ \{0\} & \text{if } t \in A^c \end{cases}.$$

By Theorem 4.2.3, F is variationally Henstock integrable and Φ is its primitive. So we conclude that Φ is a variational Henstock primitive.

□

Definition 4.2.4. Let $F : [0,1] \to C(X)$ be a multifunction. We say that F is *scalarly H-differentiable* at $t_0 \in [0,1]$ if there exists a set $F_s'(t_0) \in C(X)$ with the following property:
for every $x^* \in X^*$ and every $\varepsilon > 0$, there exists a $\delta_{x^*,\varepsilon} > 0$ such that for any interval $[u,v]$ satisfying $t_0 \in [u,v] \subset (t_0 - \delta_{x^*,\varepsilon}, t_0 + \delta_{x^*,\varepsilon})$, we have

$$\left| \frac{s(x^*, F(v)) - s(x^*, F(u))}{v-u} - s(x^*, F_s'(t_0)) \right| < \varepsilon.$$

We call $F_s'(t_0)$ the *scalar H-derivative* of F at t_0.

Theorem 4.2.5. *Let $F : [0,1] \to C(X)$ be a multifunction that satisfies condition (H) on $[0,1]$ and is scalarly H-differentiable at each point of $[0,1]$. Then the scalar H-derivative F_s' is HKP-integrable in C(X) and*

$$F(b) \ominus F(a) = (HKP) \int_a^b F_s' \, d\lambda, \ \textit{for every } [a,b] \subseteq [0,1].$$

Proof. Fix $\varepsilon > 0$. Since F is scalarly H-differentiable on $[0,1]$, for every $t \in [0,1]$ and every $\varepsilon > 0$ there exists $\delta_{x^*,\varepsilon}(t) > 0$ such that for every interval I satisfying $t \in I \subset (\delta_{x^*,\varepsilon}(t), t + \delta_{x^*,\varepsilon}(t))$ we have

$$|s(x^*, F_s'(t))(v-u) - s(x^*, \Phi([u,v]))| < \varepsilon(v-u),$$

where $\Phi([u,v]) = F(v) \ominus F(u)$ and $I = [u,v]$. $\delta_{x^*,\varepsilon}$ is clearly a gauge on $[0,1]$. Moreover, for any $\delta_{x^*,\varepsilon}$-fine Perron-partition $\{([u_{i-1}, u_i], t_i)\}_{i=1}^p$ of $[0,1]$, by the scalar H-differentiability of F at t_i, $i = 0, \ldots, p$ we have

$$\sum_{i=1}^p |s(x^*, F_s'(t_i))(u_i - u_{i-1}) - s(x^*, F(u_i) \ominus F(u_{i-1}))| < \varepsilon.$$

Therefore $s(x^*, F_s')$ is HK-integrable for every $x^* \in X^*$.
Moreover, if $[a,b]$ is a subinterval of $[0,1]$, then

$$s(x^*, F(b) \ominus F(a)) = (HK) \int_a^b s(x^*, F_s') \, d\lambda, \ \text{for every } [a,b] \subseteq [0,1].$$

Hence F_s' is HKP-integrable in $C(X)$ and

$$F(b) \ominus F(a) = (HKP) \int_a^b F_s' \, d\lambda, \ \text{for every } [a,b] \subseteq [0,1].$$

□

4.3 Applications

We start with two lemmas.

Lemma 4.3.1. *Let $\Gamma : [0,1] \to CK(X)$ be a multifunction variationally Henstock integrable. Then Γ is Bochner measurable.*

Proof. Applying Theorem 4.2.1, the $CK(X)$-valued multifunction defined as $G(t) := \Phi([0,t])$, where Φ is the variational Henstock primitive of Γ, has condition (H), is H-differentiable almost everywhere and $G'(t) = \Gamma(t)$ for almost every $t \in [0,1]$.
For every positive integer n let define

$$\Gamma_n(t) := \sum_{k=0}^{2^n-1} \frac{G(\frac{k+1}{2^n}) \ominus G(\frac{k}{2^n})}{\frac{1}{2^n}} \chi_{[\frac{k}{2^n}, \frac{k+1}{2^n}]}.$$

By definition, every Γ_n is a $CK(X)$-valued step multifunction.
If $t_0 \in [0,1]$ is a point such that $G'(t_0) = \Gamma(t_0)$ and t_0 is not a dyadic point, then we have

$$\lim_n \Gamma_n(t_0) = \lim_n \frac{G(\frac{k_0+1}{2^{n_0}}) \ominus G(\frac{k_0}{2^{n_0}})}{\frac{1}{2^{n_0}}} \chi_{[\frac{k_0}{2^{n_0}}, \frac{k_0+1}{2^{n_0}}]} = G'(t_0) = \Gamma(t_0).$$

Therefore $\Gamma_n \to \Gamma$ almost everywhere.

□

Lemma 4.3.2. *Let $\Gamma : [0,1] \to CK(X)$ be a Bochner measurable multifunction. Then the range of Γ is essentially separable, i.e. there exists a measurable set $N \subseteq [0,1]$ with $\lambda(N) = 0$ such that $\Gamma([0,1] \setminus N) = \bigcup_{t \in [0,1]\setminus N} \Gamma(t)$ is a separable subset of X.*

Proof. Let $(\Gamma_n)_{n=1}^{\infty}$ be a sequence of $CK(X)$-valued step multifunctions such that $\Gamma_n \to \Gamma$ almost everywhere. Fix n, we have that $\bigcup_{t\in[0,1]} \Gamma_n(t)$ is a finite union of compact convex sets. Hence $\bigcup_{t\in[0,1]} \Gamma_n(t)$ is separable, because every compact set is separable and every finite union of separable sets is separable. Consequently $\bigcup_{n=1}^{\infty} \bigcup_{t\in[0,1]} \Gamma_n(t)$ is separable. Since $\Gamma_n \to \Gamma$ almost everywhere, we have that $\bigcup_{n=1}^{\infty} \bigcup_{t\in[0,1]} \Gamma_n(t)$ is dense in $\bigcup_{t\in[0,1]\setminus N} \Gamma(t)$ for some $N \subset [0,1]$ with $\lambda(N) = 0$. Indeed, let N be the set of points of $[0,1]$

such that $\Gamma_n(t) \to \Gamma(t)$, for every $t \in [0,1] \setminus N$. N is a set of zero measure. Now let $x_0 \in \bigcup_{t\in[0,1]\setminus N} \Gamma(t)$. $x_0 \in \Gamma(t_0)$ for some $t_0 \in [0,1] \setminus N$. Fix $\varepsilon > 0$ and let n_0 be sufficiently large such that $d_H(\Gamma_{n_0}(t_0), \Gamma(t_0)) < \varepsilon$. Then also $d(x_0, \Gamma_{n_0}(t_0)) < \varepsilon$. Hence $||x_0 - x_{n_0}|| < \varepsilon$ for some $x_{n_0} \in \Gamma_{n_0}(t_0)$.
We conclude that $\bigcup_{t\in[0,1]\setminus N} \Gamma(t)$ is a separable subset of X. □

Now we are going to prove the main result of this chapter. It is known that every measurable selection of a $CK(X)$ or $CWK(X)$ valued Pettis integable multifunction is Pettis integrable (see [24] for the separable case and [9] for the general case).
Similarly, every measurable selection of a $CK(X)$ or $CWK(X)$ valued *HKP*-integrable multifunction is *HKP*-integrable (see [21] for the separable and [23] for the general case).
Our purpose is to obtain a similar result for variationally Henstock integrable multifunctions taking values in $CK(X)$. Here the separability of X is dropped but we use the hypothesis that X has the RNP.

Theorem 4.3.1. *Assume that X is a Banach space with the Radon-Nikodým property and let $\Gamma : [0,1] \to CK(X)$ be a variationally Henstock integrable multifunction. Then every scalarly measurable selection of Γ is variationally Henstock integrable.*

Proof. Let $\gamma : [0,1] \to X$ be a scalarly measurable selection of Γ. By Lemma 4.3.1, Γ is Bochner measurable and by Lemma 4.3.2, the range of Γ is essentially separable. Applying the Pettis measurability Theorem we get the strong measurability of γ.
Since Γ is variationally Henstock integrable, it is also *HKP*-integrable. Therefore by Theorem 1.5.3, γ is *HKP*-integrable.
Moreover, if we denote by Φ the variational Henstock primitive of Γ, then by Proposition 3.3.1, we have $V_\Phi << \lambda$. Hence also $V_\phi << \lambda$, where ϕ is the variational Henstock primitive of γ.
Now by hypothesis, X has the RNP. Therefore, by [4, Theorem 3.6], ϕ is differentiable a.e. in $[0,1]$, ϕ' is variationally Henstock integrable and $\phi(I) = (vH) \int_I \phi' \, d\lambda$, for every $I \in \mathcal{I}$.

Hence for every $I \in \mathcal{I}$,

$$(HKP) \int_I \gamma \, d\lambda = (vH) \int_I \phi' \, d\lambda = (HKP) \int_I \phi' \, d\lambda.$$

It follows that γ and ϕ' are scalarly equivalent. But γ and ϕ' are also strongly measurable. Therefore by [18, Corollary 2.2.7], $\gamma = \phi'$ a.e. and we conclude that γ is variationally Henstock integrable. $\square$

Theorem 4.3.1 is false if $CK(X)$ is replaced by $CWK(X)$, as the following example shows.

Example 4.3.1. Let $X = \ell_2([0,1])$. X is a Hilbert space hence it has the RNP. Moreover, X is not separable.

The unit ball $B(X)$ is a convex weakly compact set of X but it is not norm-compact. Let define the constant multifunction $\Gamma : [0,1] \to CWK(X)$ by $\Gamma(t) := B(X)$, $t \in [0,1]$.

Clearly Γ is variationally Henstock integrable and its variational Henstock primitive is $\Phi(I) := B(X)|I|$, $I \in \mathcal{I}$.

Now let consider an orthonormal basis $(e_t)_{t\in[0,1]}$ of X and let define the function $\gamma : [0,1] \to X$, by $\gamma(t) := e_t$, $t \in [0,1]$.

Since $||e_t|| = 1$ for every $t \in [0,1]$, we have that γ is a selection of Γ. Moreover, γ is scalarly measurable, Pettis integrable and $(P) \int_I \gamma \, d\lambda = 0$, for every $I \in \mathcal{I}$.

If $\{(I_j, t_j)\}_{j=1}^p$ is an arbitrary Perron-partition of $[0,1]$, then

$$\sum_{j=1}^p ||\gamma(t_j)|I_j| - (P) \int_{I_j} \gamma \, d\lambda|| = \sum_{j=1}^p ||e_{t_j}|| \cdot |I_j| = \sum_{j=1}^p |I_j| = 1.$$

We conclude that γ is a scalarly measurable but not variationally Henstock integrable selection of Γ. $\square$

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