A novel procedure for the solution of heterogeneous anisotropic transport problems. Part 1: the diffusion problem

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Anisotropic problems arise in various areas of science and engineering, for example groundwater flow simulations, pretroleum reservoir simulations, transport problems, …
The present work is articulated in two companions papers: in the present one, a numerical solution of the 2D heterogeneous anisotropic diffusion problem is proposed. Tensor diffusion matrix is assumed symmetric and positive definite.
The governing diffusion PDE in the *u* unknown variable is discretized over a generally unstructured triangular Delaunay mesh and a flux spatial discretization similar to the Galerkin Finite Element scheme is adopted.
A simple mesh adjustment is suggested, that attains the Delaunay condition for all the triangle sides without changing the original nodes location and also maintains the internal boundaries. Analogously to the standard Galerkin FE scheme, where the local control volume is the Voronoi polygon and the fluxes between two adjacent control volumes are orthogonal to triangle sides, in the proposed procedure the control volume relative to node *i* is defined by linking the "modified" circumcentres of the triangles sharing node *i* with the midpoints of the triangle edges. Let c_T be the modified circumcentre of triangular element *T*. Call $P_{i,ip}$ the midpoint of edge $\overrightarrow{i,ip}$ following in counterclockwise direction node *i* of element *T*. The "modified" circumcentre is computed in order to set to zero the flux through the direction going from c_T to $P_{i,ip}$, due to the component of the *u* gradient normal to edge $\overrightarrow{i,ip}$, for all the element edges. Diffusive flux between nodes *i* and *ip* is proportional to the vectorial product between two vectors: the first one is given by the matricial product between the diffusion tensor and side $\overrightarrow{i,ip}$, the second on is the vector linking c_T with midpoint of side $\overrightarrow{i,ip}$.
Starting from the previous diffusive flux discretization, a mapping procedure is defined along with a transformation matrix. According to this mapping procedure, the physical anisotropic problem can be regarded as an isotropic one in the computational space, where the physical triangular mesh is transformed into a computational one. It is shown that: 1) if the Delaunay property holds in the computational space, the system matrix of the physical problem is an M-matrix ; 2) if the Delaunay property does not hold for the computational mesh, *M*-property of the matrix system of in the physical problem is lost. In the second case, it is also shown that an edge swap in the computational space leads to an *M*-matrix system of the physical problem. Internal fixed boundaries are easily handled by the proposed algorithm.
The main advantages of the proposed procedure are: 1) the algorithm acts directly on the physical mesh, without dealing with the computational space; 2) the number of nodes along with their spatial co-ordinates are not changed; 3) the method is locally and globally conservative; 4) it is not computationally expensive; 5) it can be easily included in time-dependent problems.
The important requirement for the application of the described methodology is that the Delaunay property holds for the triangular mesh in the physical space.
The scheme is tested using several literature tests. For some of the presented tests, analytical solutions are known. Comparison with other literature procedures is carried out, as well as investigation of the convergence order and of the computational costs.