Limit and shakedown analyses by Symmetric Boundary Element Method

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SUMMARY. A reformulation of the static approach to evaluate directly the shakedown and limit multipliers by using the Symmetric Boundary Element Method for multidomain type problems [1,2] is shown. The present formulation utilizes the self-equilibrium stress equation [3-5] connecting the stresses at the Gauss points of each substructure (bem-e) to plastic strains through a stiffness matrix (self stress matrix) involving all the bem-elements in the discretized system. The numerical method proposed is a direct approach because it permits to evaluate the multiplier directly as lower bound through the static approach. The analysis has been performed as a costrained optimization problem, solved through mathematical programming methods. In this approach the optimization problem has been rephrased in the canonic form of a Convex Optimization, in terms of discrete variables, and implemented by using Karnak.sbem code [6] coupled with the MatLab.

1 SHAKEDOWN ANALYSIS VIA SBEM AND CONVEX OPTIMIZATION

The multidomain Symmetric Boundary Element Method (SBEM), developed by some authors [1,2], is utilized to riformulate the static shakedown theorem [3,4], which represents a powerful tool for providing directly, by means of mathematical programming techniques, the safety condition of a structure. The proposed strategy uses the self-equilibrium stress equation [3,5] to define the self-equilibrium stress field employed in the classical shakedown approach. This equation connects stresses, computed at each bem-e Gauss point, to plastic strains through an influence matrix (self-stress matrix).

Then the shakedown multiplier is obtained as a constrained optimization problem within the canonical form of a Convex Optimization (CO) in terms of discrete variables.

1.1 *Self-equilibrium stress equation via multidomain SBEM*

The proposed strategy uses the stress equation [5], obtained by means of a displacement approach of the SBEM, to define the self-equilibrium stress field $\boldsymbol{\sigma}^p = \mathbf{Z}\mathbf{p}$. Indeed, the following equation:

$$\boldsymbol{\sigma} = \mathbf{Z}\mathbf{p} + \beta\hat{\boldsymbol{\sigma}}^e \tag{1}$$

provides the stress at the strain points of each bem-e as a function of the volumetric plastic strain $\mathbf{p}$ and of the external actions $\hat{\boldsymbol{\sigma}}^e$, the latter amplified by β . The matrix $\mathbf{Z}$, defined as the self-stress influence matrix of the assembled system, is a square matrix having $3m$x$3m$ dimensions, with m bem-elements, fully-populated, non-symmetric and semi-definite negative. The evaluation of this matrix only involves the knowledge of the material elastic characteristics and of the structure geometry within a discretization process.

The reader can refer to Zito et al. [5] for a more detailed discussion of the characteristics of this equation introduced for a multidomain SBEM problem.

1.2 *Shakedown analysis as a CO problem*

In order to evaluate the shakedown multiplier directly, the classic shakedown approach was rephrased by means of SBEM for multidomain type problems. In the hypothesis of a von Mises yield function, which is a convex quadratic function, the static theorem leads to a numerical optimization problem of a linear objective function subjected to linear and quadratic constraints. Therefore the analysis was developed by solving a constrained nonlinear optimization problem using known mathematical programming methods.

The present formulation couples a multidomain SBEM procedure with nonlinear optimization techniques through the introduction of the self-equilibrium stress field, defined in eq.(1).

According to the shakedown theorem, the safety condition for the structure is guaranteed by a stress state satisfying the yield condition, the latter rephrased in terms of discrete variables, i.e.:

$$F[\boldsymbol{\sigma}_i] \le 0 \tag{2}$$

with $i = 1....v$ the basic load and

$$\boldsymbol{\sigma}_i = \hat{\boldsymbol{\sigma}}_i^e + \boldsymbol{\sigma}^p \tag{3}$$

representing the total stress as the sum of the elastic stress vector $\hat{\boldsymbol{\sigma}}_i^e$, due to external actions, and the self-equilibrium stress vector $\boldsymbol{\sigma}^p$.

The classical static approach makes it possible to obtain the shakedown factor β_{sh} as the maximum of the shakedown factors β for which the structure does not fail:

$$\begin{cases} \beta_{\mathrm{sh}} = \max\limits_{(\beta,\boldsymbol{\sigma}^p)} \beta \\ \text{s.t.}: \\ F\,[\beta\hat{\boldsymbol{\sigma}}_i^e + \boldsymbol{\sigma}^p] \le 0 \end{cases} \tag{4}$$

Since the self-equilibrium stress vector $\boldsymbol{\sigma}^p$ is a function of the volumetric plastic strain vector $\mathbf{p}$, through the following relation:

$$\boldsymbol{\sigma}^p = \mathbf{Z}\,\mathbf{p} \tag{5}$$

the optimization problem can be written as follows:

$$\begin{cases} \beta_{\mathrm{sh}} = \max\limits_{(\beta,\mathbf{p})} \beta \\ \text{s.t.}: \\ \boldsymbol{F}[\beta\hat{\boldsymbol{\sigma}}_i^e + \mathbf{Z}\mathbf{p}] \le \mathbf{0} \end{cases} \tag{6}$$

or in explicit form:

$$\begin{cases} \beta_{sh} = \max\limits_{(\beta, \mathbf{p}_1, \cdots \mathbf{p}_m)} \beta \\ \text{s.t.}: \\ F_1[\beta \hat{\boldsymbol{\sigma}}^e_{i1} + \mathbf{Z}_{11}\mathbf{p}_1 \cdots + \mathbf{Z}_{1m}\mathbf{p}_m] \le 0 \\ \vdots \\ F_m[\beta \hat{\boldsymbol{\sigma}}^e_{im} + \mathbf{Z}_{m1}\mathbf{p}_1 \cdots + \mathbf{Z}_{mm}\mathbf{p}_m] \le 0 \end{cases} \tag{7}$$

where m is the bem-e number.

In the hypothesis of the von Mises yield law, the present approach allows one to write the problem through the optimization of an objective linear function subjected to quadratic constraints only:

$$\begin{cases} \beta_{sh} = \max\limits_{(\beta, \mathbf{p}_1, \ldots \mathbf{p}_m)} \beta \\ \text{s.t.}: \\ \frac{1}{2}(\beta \hat{\boldsymbol{\sigma}}^e_{i1} + \mathbf{Z}_{11}\mathbf{p}_1 \cdots + \mathbf{Z}_{1m}\mathbf{p}_m)^T \boldsymbol{M} (\beta \hat{\boldsymbol{\sigma}}^e_{i1} + \mathbf{Z}_{11}\mathbf{p}_1 \cdots + \mathbf{Z}_{1m}\mathbf{p}_m) - \boldsymbol{\sigma}^2_{1y} \le 0 \\ \vdots \\ \frac{1}{2}(\beta \hat{\boldsymbol{\sigma}}^e_{im} + \mathbf{Z}_{m1}\mathbf{p}_1 \cdots + \mathbf{Z}_{mm}\mathbf{p}_m)^T \boldsymbol{M} (\beta \hat{\boldsymbol{\sigma}}^e_{m1} + \mathbf{Z}_{m1}\mathbf{p}_1 \cdots + \mathbf{Z}_{mm}\mathbf{p}_m) - \boldsymbol{\sigma}^2_{my} \le 0 \end{cases} \tag{8}$$

where $\boldsymbol{M}$ is a constants matrix and σ_{iy} the uniaxial yield stress.

In order to solve the previous problem, the general form of a CO problem was rewritten as follows:

$$\begin{cases} \min\limits_{(\mathbf{y})} \mathbf{y} \\ \text{s.t.}: \\ \mathbf{y}^T \mathbf{B}\, \mathbf{y} \le 0 \end{cases} \tag{9}$$

where $\mathbf{B}$ is a symmetric positive matrix and $\mathbf{y}$ is the unknown quantity vector.

The canonical form (9) is obtained by collecting in the $\mathbf{B}$ matrix the constant terms of eq.(8), i.e. for the j-th bem-e:

$$F_{ij} = \underbrace{\left|\beta \;\; \mathbf{p}^T_1 \cdots \mathbf{p}^T_m\right|}_{\mathbf{y}^T} \underbrace{\left|\hat{\boldsymbol{\sigma}}^e_{ij} \;\; \mathbf{Z}_{j1} \cdots \mathbf{Z}_{jm}\right|^T \frac{\boldsymbol{M}}{2\sigma^2_y} \left|\hat{\boldsymbol{\sigma}}^e_{ij} \;\; \mathbf{Z}_{j1} \cdots \mathbf{Z}_{jm}\right|}_{\mathbf{B}_{ij}} \underbrace{\left|\beta \;\; \mathbf{p}^T_1 \cdots \mathbf{p}^T_m\right|^T}_{\mathbf{y}} - 1 \le 0 \tag{10}$$

and in compact form:

$$F_{ij} = \mathbf{y}^T \mathbf{B}_{ij}\, \mathbf{y} - 1 \le 0 \tag{11}$$

The shakedown problem can be rewritten as follows:

$$
\begin{cases}
\underset{(\mathbf{y})}{min} \ \ \mathbf{c}^T \mathbf{y} \\
\text{s.t.:} \\
\mathbf{y}^T \mathbf{B}_1 \mathbf{y} - 1 \le 0 \\
\vdots \qquad\qquad\qquad\qquad k = m*v \\
\mathbf{y}^T \mathbf{B}_k \mathbf{y} - 1 \le 0
\end{cases}
\tag{12}
$$

where the vector $\mathbf{c}^T = \begin{vmatrix} -1 & 0 & \cdots & 0 \end{vmatrix}$ has been introduced.

Problem (6), in the form (12), was implemented by coupling the Karnak.sbem code [6] with a Matlab 7.6.0 optimization toolbox.

In this procedure, using multidomain SBEM, it was also possible to reduce the size of the problem. Indeed, since this method introduces a domain discretization exclusively in the zones of potential store of the plastic strains, the remaining part of the structure can be considered as made up of elastic macroelements, and therefore governed by few boundary variables. This aspect makes the strategy proposed computationally advantageous.

2 CONCLUSIONS

A new strategy utilizing the Multidomain SBEM for rapidly performing shakedown analysis as a convex optimization problem has been shown. The present multidomain approach, called displacement method [1], makes it possible to consider step-wise physically non-homogeneous materials and obtains a self-equilibrium stress equation regarding all the bem-elements of the structure. It provides a nonlinear optimization problem solved as a CO problem. Furthermore, the strategy makes it possible to introduce a domain discretization exclusively in the zones involved by plastic strain storage, leaving the rest of the structure as elastic macroelements, thus governed by few boundary variables. It limits considerably the number of variables in the nonlinear analysis and makes the proposed strategy extremely advantageous. The procedure is implemented within the Karnak.sbem code, coupled with optimization toolbox Matlab 7.6.0.

References

[1] Panzeca, T., Cucco, F., Terravecchia, S., "Symmetric boundary element method versus Finite element method". *Comp. Meth. Appl. Mech. Engng.* **191**, 3347-3367 (2002).

[2] Perez-Gavilan, J.J. and Aliabadi, M.H., "Symmetric Galerkin BEM for Multi-connected bodies". *Commun. Numer. Meth. Engng*. **17**, 761-770 (2001).

[3] Panzeca, T., "Shakedown and limit analysis by the boundary integral equation method". *Eur. J. Mech., A/Solids*. **11**, 685-699 (1992).

[4] Zhang, X., Liu, Y., Zhao, Y., Cen, Z.,. "Lower bound limit analysis by the symmetric Galerkin boundary element method and Complex method". *Meth. Appl. Mech. Engng.*. **191**, 1967-1982 (2002).

[5] Zito, L., Cucco, F., Parlavecchio, E., Panzeca, T., "Incremental elastoplastic analysis for active macro-zones". *Int. J. Num. Meth. Engng*. In press. doi: 10.1002/nme.4319.

[6] Cucco, F., Panzeca, T., Terravecchia, S., "The program Karnak.sbem Release 2.1", Palermo University. (2002).