BEF Analogy for Concrete Box Girder Analysis of Bridges

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Summary

Box girder, due to its high torsional stiffness, is very appropriate for railway and highway long-medium span bridges. This type of cross section, subjected to transversely non-uniform loads, present warping and distortion phenomena. Accurate but time-consuming numerical procedures are available for determination of further strains and stresses caused by cross-section deformation. In this paper warping and distortion of box girders is evaluated through BEF analogy, by writing a 4^{th} order differential equation. The problem is solved for practical cases of box girders by considering internal diaphragms stiffness. Graphs are supplied to designers and main design parameters affecting cross section deformation are underlined. The proposed methodology is shown through the use of graphs by developing numerical examples on actual bridge girders.

Keywords: box-girder, bridges, warping, distortion, Winkler foundation, BEF analogy.

1. Introduction

Concrete box girders are common in beam bridges for their efficient cross section with high torsional stiffness. They can be built by cantilever method, by incremental launching or through gantries but in all these cases deck presents internal diaphragms in order to transversely stiffen the box section. These diaphragms are generally placed on piers and abutments, to drive the flow of forces in support regions. But they have also the function of reducing cross section deformability, so they can be placed also into the span, between two supports. In every cases each section of girder, resting between two stiffened sections by diaphragms, can be considered as a deformable frame in its plane. For current widths of served roads, the tendency is to have deck composed of unicellular hollow boxes; they are much higher for railway bridges and less for road bridges. Multi-cellular boxes are used in the case of very large decks or for particular purposes, because in this case formworks and casting are much more complicated.
Classical beam theory does not consider the deformation of box section but as a matter of fact cross sections do not remain stiff in their plane. In addition they present warping phenomena, i.e. deformation of slabs and webs in longitudinal direction (section does not maintain itself into the plane in the deformed configuration). When warping is prevented, normal stresses born in addition to those associated with bending, already found with the classical beam theory. Moreover for transversely eccentric loads, cross section distortion born. These phenomena have been studied by different authors [1-8]. Theory of non-uniform torsion faces the problem by maintaining the stiffness of section in its plane, with the addition of normal longitudinal stresses depending on prevented warping. A classical way to study box distortion, instead, is to consider girders as tubular frames composed of walls as membrane elements [8]; in this case, due to eccentricity of loads, slabs and webs do not maintain a rigid angle between themselves and a relative rotation between elements occurs, with the result of cross section distortion. A more complete theory that takes into account globally these phenomena is that of folded plates, in which box is considered and calculated as a deformable frame with symmetric and anti-symmetric loads. This approach is more

effective for concrete box girders in which joints between section walls have to be considered rigid. From this kind of analysis normal stresses due to pure bending can be found separately from those due to prevented warping and from effects induced by cross section distortion [2]. Folded plate analysis can be developed through the Beam on Elastic Foundation (BEF) Analogy, deriving a 4th order differential equation, to solve the problem of cross section deformability under anti-symmetric loads.
Wright et al. [9] developed the BEF analogy for analysis of box girders underlining torsional and distortional components for concentrated and distributed eccentric loads. They draw useful graphs of BEF solution. Stefanou et al. [10] discussed the influence of geometric parameters involved in the solution of BEF for box girders, giving tables and graphs for different positions of load. Even if these last papers have been presented many years ago, the followed approach has the aim to provide a useful tool for dimensioning and choosing geometrical and mechanical parameters in the first stage of design. The development of numerical solutions by FE models lead researchers and designers to forget these valuable approaches and today these contributions are difficult to be found in literature. Authors think instead that this kind of studies are fundamental in order to evaluate the influence of many parameters, avoiding time-consuming numerical simulations.
In this paper the solution of BEF analogy is given for a number of practical cases of box girders and reported in form of graphs. The target is to supply a useful tool for the designer, who needs a fast and simple procedure to take into account effects of concrete box deformability in the analysis of bridge structures. Even if the problem can be faced by the Finite Element Method, a 3-D FE model of a bridge is always very difficult to manage and it implies a big computational burden. A simplified but rigorous procedure can give instead useful indications to designers and immediate numerical results for practical cases in preliminary phases. FE analyses can be carried out only in the final stage of design. The followed procedure starts from the position of the problem, by considering eccentric loads on the deck and by writing the 4th order differential equation of deformable boxed concrete cross section [1]. The study is extended to a large range of unicellular boxes for beams with different restraint conditions. Results are shown through a number of graphs and the main parameters involved are explained, together with applications to real cases.

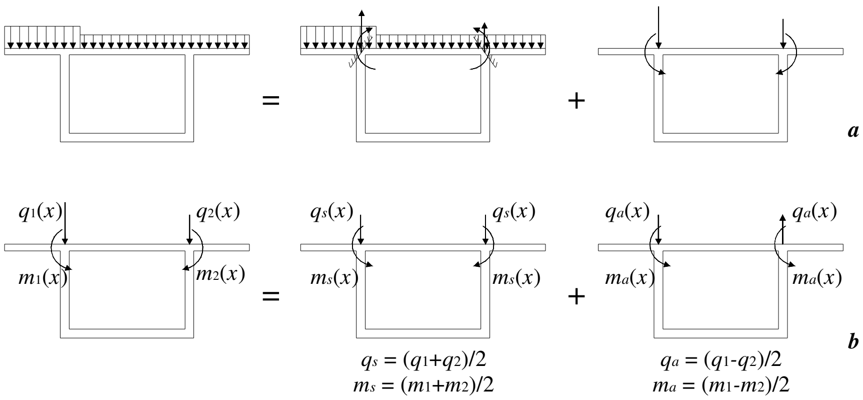


Figure 1. Symmetric and anti-symmetric loads on box girder

2. Position of the problem

2.1 Deformability of box cross section

Consider a unicellular concrete hollow box girder with the transverse section deformable in its plane. Generally high stiffness in plane is assured by internal diaphragms or very thick slabs and webs. Deformability of current cross sections depend on slabs and webs bending stiffness. Rigid joints connect box section concrete walls, allowing flexural deformations of each element as it occurs for frames in bending. Consider a transverse eccentric distribution of loads on the top slab (fig. 1*a*). It can be studied by fictitiously restraining joints: reactions can be found through the classical beam theory and re-applied with the opposite sign to the unrestrained joints, converting loads to concentrated nodal forces at the top of webs [1]. This scheme can be decomposed into two different schemes: one with symmetric loads and another with anti-symmetric loads (fig. 1*b*). Effects of symmetric loads can be studied by the classical beam theory giving mainly vertical displacements, longitudinal normal stresses into slabs and webs due to bending, tangential stresses due to shear. Anti-symmetric loads can be regarded as those causing torsion, warping and distortion. If section is considered stiff in its plane, anti-symmetric load causes only a rotation along the beam axis and the result is only torsion. It can be studied by Bredt theory but prevented warping has to be accounted by the non-uniform torsion theory. On the contrary, if section is considered deformable in its plane, an anti-symmetric load causes longitudinal and transverse deformations of slabs and webs; in this case transverse bending has to be considered in addition to prevented warping. Normal stresses increase with respect to those derived by pure bending. The final result is warping and distortion of cross section (fig. 2). It has been shown [1,3] that effects of non-uniform torsion for box sections, in terms of longitudinal stresses, are evident only in limited regions where the value of prevented warping is high, as it occurs near supports or for thick diaphragms. Along the beam instead warping can be disregarded while effects of distortion are much more significant.

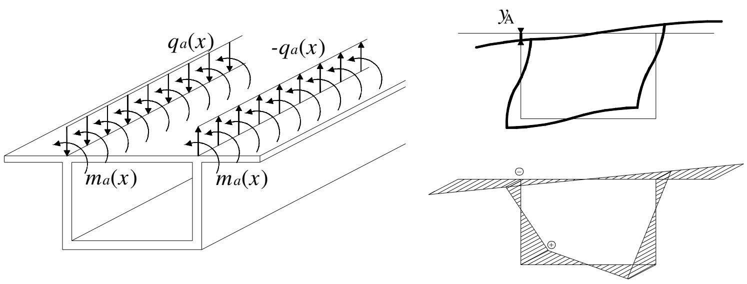


Figure 2. Cross section deformation and normal stresses due to anti-symmetric loads

In figure 3, forces into box girder walls are shown. Being q_a and m_a the distributed anti-symmetric nodal loads, stress resultants M_a, M_s and M_i give the transverse flexure of webs and slabs, while forces V_a, V_s and V_i give the resultants of tangential stress flow.

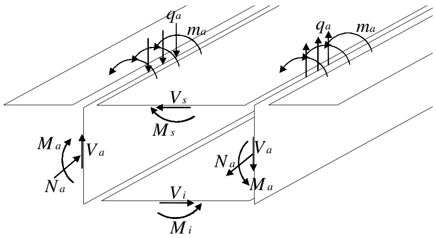


Figure 3. Forces into box girder walls under anti-symmetric loads

Figure 4 shows that anti-symmetric load that leads to a flow of tangential stresses (torsion) and to a couple of forces Q whose only effect is box section distortion [5]. Some authors choose the relative displacement between top-right and bottom-left joints, along the diagonal of hollow box, as the unknown parameter to study distortion of box sections. It is a good choice when torsional effects are separated from those due to distortion, by introducing fictitious stiff diagonals, in order to evaluate force Q. In this way torsion and prevented warping are separated from distortion.
Other authors [1, 4] do not make this distinction between torsion and distortion. In this last case, force Q is not found and the chosen displacement parameter is different. It is possible to write the differential equation in terms of the relative rotation between vertical axis of section and web axis [4] or in terms of the vertical displacement of upper web-flange joint $y_A(x)$ [1], but all these last approaches are equivalent. In the present paper the chosen parameter is displacement of upper web-flange joint $y_A(x)$, by following the procedure given by Calgaro and Virlogeux [1].

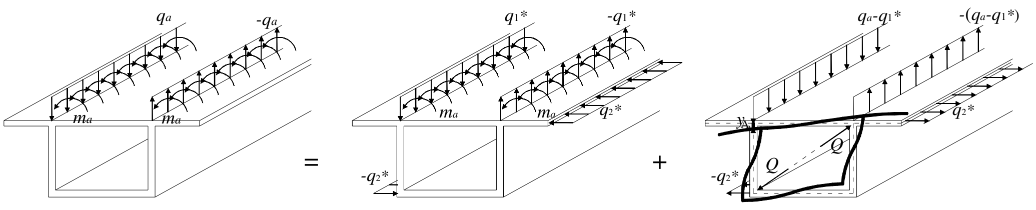


Figure 4. Decomposition of distortion forces.

2.2 BEF analogy

By applying anti-symmetric loads q_a and m_a and by considering the equilibrium and compatibility equations between elements of fig. 3, it is possible to solve the problem through the 4th order differential equation of Winkler's beam. Through symbols of figure 5 and with the positions [1]:

$$S_s = e_s b_s;\ S_i = e_i b_i;\ S_a = e_a h;\ I_a = \frac{e_a h^3}{12};\ \alpha_s = \frac{S_s}{S_a};\ \alpha_i = \frac{S_i}{S_a};\ \beta = \frac{b_i}{b_s};$$
$$I'_s = \frac{e_s^3}{12(1-\nu^2)};\ I'_i = \frac{e_i^3}{12(1-\nu^2)};\ I'_a = \frac{e_a^3}{12(1-\nu^2)};\ r_s = \frac{I'_a}{I'_s}\frac{b_i}{h};\ r_s = \frac{I'_a}{I'_i}\frac{b_i}{h}; \tag{1}$$
$$k_r = 3+2r_s+2r_i+r_i r_s;\ k_s = 3\beta^2+2\alpha_s+2\beta^2\alpha_i+\alpha_i\alpha_s; \rho = \frac{24EI'_a}{b_i h^2}\frac{6+r_s+r_i}{k_r};\ \mu = \frac{2r_s(r_i+3)}{k_r}.$$

the differential equation of BEF analogy can be written:

$$y_A^{IV}(x) + \frac{4h}{b_i}\rho\frac{\alpha_s+\alpha_i\beta^2+6\beta^2}{2k_s EI_a} y_A(x) = \frac{\alpha_s+\alpha_i\beta^2+6\beta^2}{2k_s EI_a}\left(q_a + \frac{\mu}{b_i}m_a\right) \tag{2}$$

in which E is the elastic modulus of concrete section and ν is the Poisson coefficient.

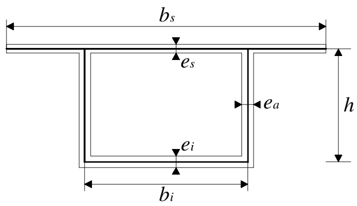


Figure 5. Geometric characteristics of box section

By the following positions:

$$k_w = \frac{4h}{b_i}\rho;\ q_w = q_a + \frac{\mu}{b_i}m_a;\ I_w = I_a\frac{2k_s}{\alpha_s+\alpha_i\beta^2+6\beta^2} \tag{3a,b,c}$$

in which ρ is the parameter related to cross section rigidity, equation (2) can be written as:

$$y_A^{IV}(x) + \frac{k_w}{EI_w} y_A(x) = \frac{q_w}{EI_w} \tag{4}$$

Equation (4) represents the 4th order differential equation of a BEF characterized by the Winkler soil constant k_w, by the beam stiffness EI_w and by the distributed load q_w. In this way the BEF analogy is established: by solving equation (4) related to an equivalent fictitious Winkler beam, equation (2) is solved. Solution of BEF is given by one of literature methods [11-13]. Bending moment in webs is:

$$M_a(x) = M(x)\frac{I_a}{I_w} = M(x)\frac{\alpha_s+\alpha_i\beta^2+6\beta^2}{2k_s} \tag{5}$$

from which normal stresses at the top and bottom of webs are:

$$\sigma_{\sup}(x) = -\frac{M(x)}{I_w}\frac{\beta^2(\alpha_i+3)}{\alpha_s+\alpha_i\beta^2+6\beta^2}h;\ \sigma_{\inf}(x) = \frac{M(x)}{I_w}\frac{(\alpha_s+3\beta^2)}{\alpha_s+\alpha_i\beta^2+6\beta^2}h \tag{6}$$

from which longitudinal normal stress diagram of box section is known (fig. 6*a*).
Transverse bending moments, depending on $y_A(x)$, are:

$$m_s(x) = -h\rho\frac{3+r_s}{6+r_s+r_i}y_A(x);\ m_i(x) = h\rho\frac{3+r_i}{6+r_s+r_i}y_A(x) \tag{7}$$

that have to be added to the local effect of nodal moment $m_a(x)$ (fig. 6*b*).
From eqn. (7) it is evident that BEF displacements $y_A(x)$ are directly related to transverse moments into the section, while bending moment $M(x)$ in each section is directly related to longitudinal stresses at top and bottom. The parameter related to the characteristic length $\lambda = 2\pi/\alpha$ of BEF is:

$$\alpha L = \sqrt[4]{\frac{k_w}{4EI_w}}L \tag{8}$$

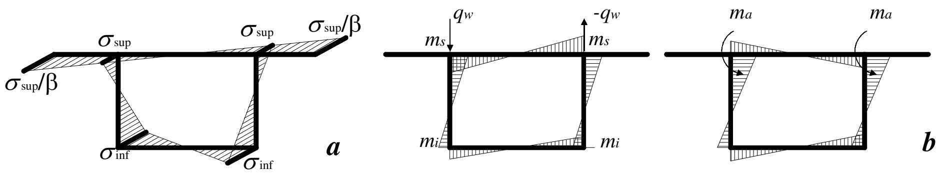


Figure 6. Normal stresses and transverse bending moments in the section

Another consideration can be done about transverse diaphragms. When there are no diaphragms, beam lies on Winkler soil, free at the ends, i.e. no external restraints have to be considered. When diaphragms are infinitely stiff in their plane, instead, cross section deformation is not allowed in those points. The related BEF lies on Winkler soil with external rigid supports. This is the case of a diaphragm that closes the hollow section or with only a little manhole. When diaphragms are very thick or in presence of counterweight at the ends of girder, warping is prevented and BEF has to be considered embedded. When diaphragm is composed of slabs and webs with increased thickness, warping and distortion are not totally prevented in those sections; so analogous BEF lies on Winkler soil with concentrated external elastic restraints. In this study deformable diaphragms have been considered through the ratio $\gamma=\rho_d/\rho$ between diaphragm stiffness value ρ_d, given by relations (1) re-written with the thickness of slabs and webs of diaphragm zone, and current box section stiffness value ρ. Value $\gamma = 1$ means the girder has not diaphragms along its axis, while $\gamma \to\infty$ means infinitely stiff diaphragms. Stiffness of the concentrated elastic restraint, for an intermediate situation, is given by the product $4\rho_d h\, t_d/b_i$, in which t_d is diaphragm thickness. From BEF solution support reactions can be found, i.e. shear forces useful to calculate reinforcement into diaphragms.

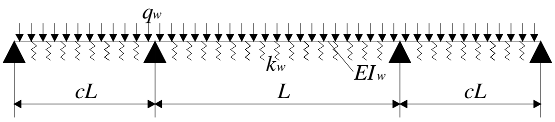


Figure 7. Geometry of a three-span BEF.

3. Solution of BEF analogy for concrete box girders

3.1 Construction of graphs

On the base of previous equations, solution of BEF analogy is given in this section for the case of a BEF with span length L. Calculus has been repeated for different values of parameters αL and $q_{w,ref}/k_w$, with $q_{w,ref}$ = 10 KN/m. Solution of 4th order differential equation is shown through values of displacement $y_A(x)$ and bending moment ratio $M(x)/k_wL^2$, put in semi-logarithmic graphs. Variation range of chosen parameters is related to real box girder sections and typical lengths of bridge spans. Entire beam is uniformly loaded by $q_{w,ref}$. Values of bending moment $M(x)$ are significant in the midspan section and at the ends of embedded BEF.

Three-span continuous beams with central span length L and lateral spans cL have been analysed by varying ratio c between span lengths (fig. 7). It can be seen that the influence of c is very little significant for purposes of this study. Moreover when $c \to 0$ three-span beam degenerates to an embedded simple span, while for $c >> 1$ the beam degenerates into a simple supported span. As a consequence, each span between subsequent diaphragms can be studied with one of the single-span schemes of fig. 8, depending them only on the boundary conditions. In figure 8 beam *i*) refers to a span without diaphragms, beam *ii*) refers to a span between two stiff diaphragms preventing distortion, while beam *iii*) refers to a span between diaphragms preventing torsion warping and distortion. In the first case, for given values of q_w, k_w and EI_w, displacements of BEF are equal in every sections and no bending moment appears. Diaphragm in the real girder can be placed over supports or at the centre of spans. When diaphragms are put in intermediate sections between two supports, the span length influenced by distortion is reduced to that between two successive diaphragms. It is so possible to reduce these effects in long spans by placing diaphragms in intermediate sections between piers. So length L of BEF is that between two diaphragms and if they are over supports, L coincides with the entire girder span length. In this way transverse moments in webs can be reduced by adding diaphragms along the beam. On the other hand, in diaphragm sections warping is prevented and longitudinal normal stresses become significant.

Designer has then to take decisions regarding the presence, number and position of internal diaphragms, the thickness of slabs and webs and the span lengths, in the conceptual design stage.

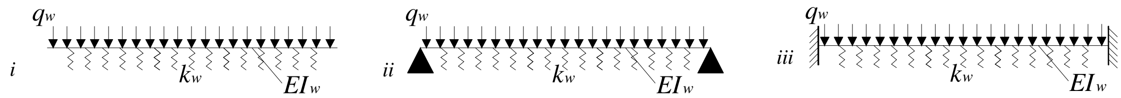


Figure 8. Simple span schemes of BEF between two diaphragms.

In this study the parameter of diaphragms stiffness γ varies from 100 (deformable) to 10000 (infinitely rigid). Figures 11-15 show graphs of y_A and M/k_wL^2 in midspan section of simply supported span and at the end of embedded span. Designer can apply the following procedure:

1) values of k_w, αL and γ have to be calculated for the beam, by eqns. (1), (3) and (9);
2) graphs have to be entered with these values (for a fixed value of diaphragm stiffness γ), in order to found y_A and M/k_wL^2 in the desired section;
3) values of normal stresses and transverse bending moments are then found by eqns. (6), (7).

Graphs for midspan are useful in order to evaluate maximum transverse moments. Graphs for end section of embedded span are useful in order to evaluate maximum values of stress due to prevented warping. Length L is always the distance between two subsequent diaphragms. Influence of diaphragms has been addressed by Sana et al. in [14].

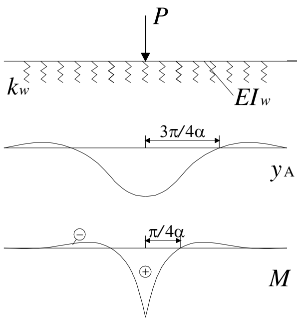


Figure 9. Displacements and bending moments in BEF with concentrated load.

Figure 9 shows the effect of concentrated forces in BEF. It can be seen that the effect of concentrated loads, in terms of transverse moments, disappears at a distance of $3\lambda/8$, with λ the characteristic BEF length. Effect of concentrated loads in terms of longitudinal stresses disappear instead at a distance $\lambda/8$. Functions of displacements and moments [11] are:

$$y_A(x)=\frac{P\alpha}{2k_w}e^{-\alpha x}\left(\cos\alpha x+\sin\alpha x\right);\quad M(x)=\frac{P}{4\alpha}e^{-\alpha x}\left(\cos\alpha x-\sin\alpha x\right) \tag{10}$$

3.2 Discussion and numerical application

By the observation of graphs, it can be seen that effect of diaphragms is important only when ratio αL is small, while this effect becomes less and less fundamental for higher values αL. In the same way a higher value of γ is remarkable only for low values of αL, by contributing to limit the magnitude of displacement y_A in a drastic way. When the characteristic BEF length λ is exceeded, value of y_A becomes practically constant, depending only on $q_{w,ref}/k_w$ and it doesn't result influenced by the presence and by diaphragms stiffness. Besides, for increasing value of αL, bending moment M decreases, both in midspan and near diaphragms, independently from the boundary conditions.
Two numerical examples are considered here, in order to make comparisons with literature data.
First application is carried out on a simply supported span with length L = 30 m [4]; b_s =9,00 m, b_i = 6,00 m, h= 1,50 m, e_a = 0,35 m, e_i = e_s = 0,25 m, q_w = 50 kN/m (uniformly distributed only on the central span). Solution is found in midspan section. From relations (1), (3) and (9) we have: k_w = 31,822 MN/m^2, αL = 4,40, $q_{w,ref}/k_w$ = 0,000314 m. From graph of figure 11, with γ = 10000, the value of y_A is found: y_A = 0,035 cm. From graph of fig. 12, with γ = 10000, the value of M is found: M/k_wL^2 = 0,00017 cm. By relating these values (found with $q_{w,ref}$ = 10 kN/m) to the real value of q_w, results in terms of stresses at top and bottom of webs are: σ_{sup} = -0,20 MPa; σ_{inf} = 0,42 MPa, while transverse moments are: m_s = m_i = 0,0417 MNm/m.
Second numerical application consists of an embedded beam[1]. It has L = 50 m, b_s =11,00 m, b_i = 5,50 m, h= 3,00 m, e_a = e_i = e_s = 0,30 m, q_w=1MN/m. Solution is found in midspan section.

Remembering that graphs have been drawn with the reference load value $q_{w,ref}$ = 10 kN/m, from relations (1), (3) and (8) we have: k_w = 70,0 MN/m^2, αL = 5,0, $q_{w,ref}/k_w$ = 0,000143 m. From graph of figure 13, with γ = 10000 (infinitely rigid diaphragms) the value of y_A is found: y_A = 0,014 cm. From graph of figure 14, with γ = 10000, the value of M is found: M/k_wL^2 = 0,00006 cm. Being q_w = 1 MN/m, results in terms of stresses at top and bottom of webs are: σ_{sup} = -1,64 MPa; σ_{inf} = 6,00 MPa, while transverse moments are: $m_s = m_i$ = 0,722 MNm/m

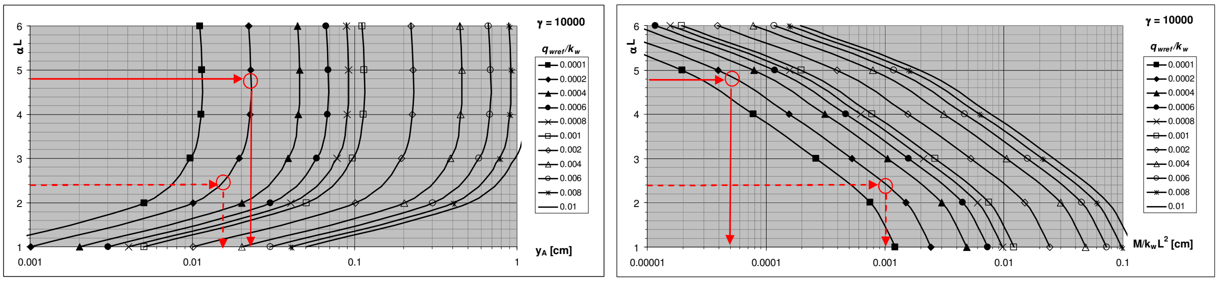


Figure 10. Examples of paths for different values of αL.

Let see now the influence of internal diaphragms in the middle of span (fig. 10). If cross section of girder is maintained constant and one or more stiff diaphragms are put in intermediate sections, length L between two subsequent diaphragms decreases as well as value αL, but the curve $q_{w,ref}/k_w$ of graphs is the same. In this way, by reducing αL, displacement y_A in midspan section reduces too. It means that by adding diaphragms into the beam, values of transverse moments reduce. Alternatively designer can choose to modify cross section rigidity by varying web and slab thickness and by maintaining diaphragms spacing. In this case both values αL and $q_{w,ref}/k_w$ change.

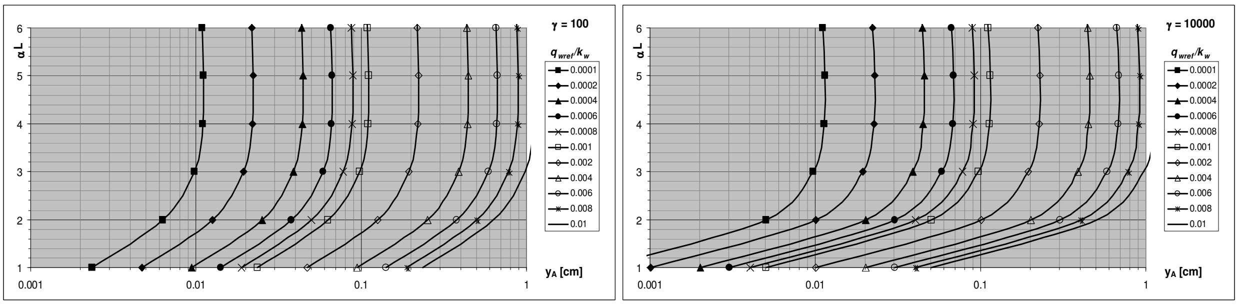


Figure 11. Graphs of y_A in midspan section. Simple supported beam.

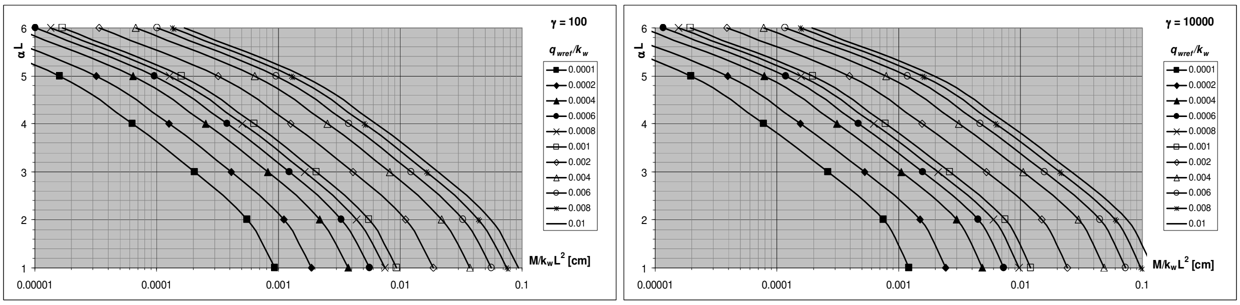
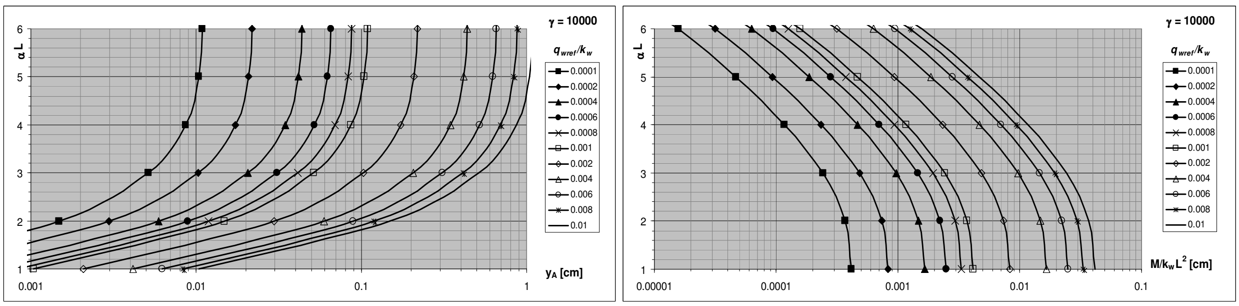


Figure 12. Graphs of M in midspan section. Simple supported beam.



Figures 13 and 14. Graph of y_A and M in midspan section. Embedded beam.

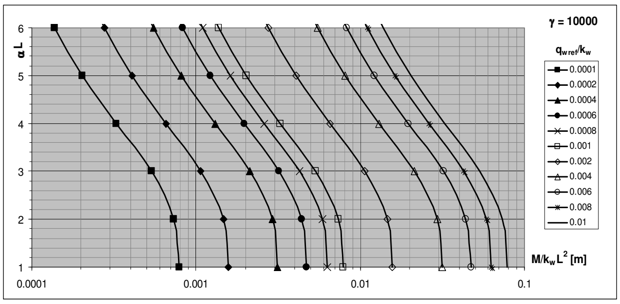


Figure 15. Graph of M at the end section of embedded beam.

4. Conclusions

In this study deformation of concrete box girders has been addressed in order to apply the Beam on Elastic Foundation Analogy and to find longitudinal stresses and transverse bending moments due to single cell box warping and distortion. The solution has been shown for a large range of practical cases through general graphs, taking into account rigidity of box cross section and diaphragms stiffness in beams with different restraints. A useful tool for designers has been supplied, avoiding time-consuming and complicated finite element analyses of bridges. Influence of main parameters involved has been discussed and numerical examples have been presented.

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