A SEQUENCE OF POSITIVE SOLUTIONS FOR SIXTH-ORDER ORDINARY NONLINEAR DIFFERENTIAL PROBLEMS

GABRIELE BONANNO, ROBERTO LIVREA

In honor of Jeff Webb, great master of higher order ordinary differential equations, on the occasion of seventy-fifth birthday. To Jeff with infinite admiration.

ABSTRACT. Infinitely many solutions for a nonlinear sixth-order differential equation are obtained. The variational methods are adopted and an oscillating behaviour on the nonlinear term is required, avoiding any symmetry assumption.

1. Introduction

Equations of the following type

$$\frac{\partial u}{\partial t} = -\frac{\partial^6 u}{\partial x^6} + A\frac{\partial^4 u}{\partial x^4} - B\frac{\partial^2 u}{\partial x^2} + Cu - \lambda h(t, x, u(t, x)) \tag{1.1}$$

arise when an interface between two phases is examined because they help to reveal a more detailed structure of the interface and a description of the behaviour of phase fronts in materials that are undergoing a transition between the liquid and solid state ([1, 7, 13]). Here we are concerned in periodic stationary solutions of (1.1). More precisely, we will give some multiple results for the following problem

$$\begin{cases} -u^{(vi)} + Au^{(iv)} - Bu'' + Cu = \lambda f(x, u) & x \in [0, 1] \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0 \end{cases} \tag{P_λ}$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, A, B and C are given real constants, while $\lambda > 0$. In [13], starting from the interest for the stationary solutions of a class of fourth-order equations, the so-called extended Fisher-Kolmogorov equation, a variational approach is proposed for obtaining existence and non existence of stationary periodic solutions, observing that the same arguments apply also to sixth-order equations. In [16, 6], taking advantage from a minimization theorem as well as Clark's theorem, the existence and the multiplicity of periodic solutions is investigated for a problem similar to (P_1), provided that A, B and C satisfy some suitable relations and the nonlinear term is a polynomial with a kind of symmetry. Again the variational methods have been exploited in [9] where two Brezis-Nirenberg linking theorems represent the main tool for assuring the existence of at least two or three periodic solutions for a sixth-order equation with super-quadratic nonlinearities, namely

$$\lim_{t\to+\infty} \frac{F(x, t)}{t^2} = +\infty, \quad \lim_{t\to 0} \frac{F(x, t)}{t^2} = 0$$

uniformly with respect to x, where $F(x, t) = \int_0^t f(x, s)\, ds$ for every $x \in [0, 1]$. We also cite [11, 21], where under suitable assumptions, in particular on the coefficients A, B, C, the existence of one or two positive solutions for problem (P_λ) is established by applying the theory of fixed point index in cones. Further nice results on higher-order differential equations are contained in [8, 17, 18, 19, 20], where non-local conditions have also been considered.

In this note we look at the existence of infinitely many solutions to problem (P_λ). In particular, under different assumptions on the parameters A, B and C and requiring a suitable oscillation on $f(x, \cdot)$ at infinity (see assumption ii) of Theorem 3.1), an unbounded sequence of classical solutions of (P_λ) is assured provided that λ belongs to a well determined interval. We explicitly stress that no symmetry conditions on the reaction term are involved. The variational structure of the problem is exploited and the solutions are obtained as local minima of the energy functional related to (P_λ). For this reason a crucial tool is a local minimum theorem proved in [2], see Theorem 2.1.

Key words and phrases. Sixth-order equations, Critical points, Infinitely many solutions.
2010 Mathematics Subject Classification: 34B15, 34B18, 35B38.

A further investigation is devoted to constant sign solutions of (P_λ). Whenever $f(x,0) \geq 0$ the classical solutions are assured to be nonnegative (see Theorem 3.2) and, in addition, in some particular cases, a maximum principle for sixth-order differential equations is pointed out for obtaining their positivity (see Remark 3.3).

As example, here is a consequence of our main results.

Theorem 1.1. *Let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function such that*

$$\liminf_{t\to+\infty} \frac{G(t)}{t^2} = 0, \qquad \limsup_{t\to+\infty} \frac{G(t)}{t^2} = +\infty,$$

where $G(t) = \int_0^t g(s)\, ds$ for every $t \in \mathbb{R}$, and fix $D \geq 0$.
Then, the problem

$$\begin{cases} -u^{(vi)} + 3Du^{(iv)} - 3D^2u^{''} + D^3u = g(u) & x \in [0,1] \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0 \end{cases} \tag{P}$$

admits an unbounded sequence of positive classical solutions.

Finally, when the oscillating behaviour is required at zero, instead that at infinity, a sequence of classical solutions that strong converges at zero is obtained (see Theorem 3.4).

In Section 2 we recall some useful preliminaries and detail the variational set pointing out the general strategy for obtaining classical solutions. The main results as well as their consequences and examples are contained in Section 3.

2. Basic notations and auxiliary results

Throughout the paper X denote the following Sobolev subspace of $H^3(0,1) \cap H^1_0(0,1)$

$$X = \{u \in H^3(0,1) \cap H^1_0(0,1) : \ u''(0) = u''(1) = 0\}$$

considered with the norm

$$\|u\| = \left(\|u'''\|_2^2 + \|u''\|_2^2 + \|u'\|_2^2 + \|u\|_2^2\right)^{1/2} \quad \forall\, u \in X, \tag{2.1}$$

where $\|\cdot\|_2$ denotes the usual norm in $L^2(0,1)$. It is well known that $\|\cdot\|$ is induced by the inner product

$$\langle u,v\rangle = \int_0^1 (u'''(x)v'''(x) + u''(x)v''(x) + u'(x)v'(x) + u(x)v(x))\, dx \quad \forall\, u,\ v \in X.$$

Now, arguing as in [13], we point out some useful Poincaré type inequalities.

Proposition 2.1. *For every $u \in X$, if $k = 1/\pi^2$, one has*

$$\|u^{(i)}\|_2^2 \leq k^{j-i}\|u^{(j)}\|_2^2, \quad i = 0,1,2,\ j = 1,2,3 \text{ with } i < j. \tag{2.2}$$

Proof. Let us consider all the possible situations.

$j = 1$. In this case only $i = 0$ occurs and (2.2) reduces to the well known Poincaré inequality.

$j = 2$. The case $i = 1$ can be obtained observing that $\int_0^1 (u')^2 = -\int_0^1 uu''$. Hence, putting together the Hölder and the Poincaré inequalities one has

$$\|u'\|_2^2 \leq \|u\|_2\|u''\|_2 \leq k^{1/2}\|u'\|_2\|u''\|_2$$

from which directly follows (2.2).
For $i = 0$ condition (2.2) is derived putting together the Poincaré inequality with the case $i = 1$.

$j = 3$. The case $i = 2$ is directly the Poincaré inequality applied to $u'' \in H^1_0(0,1)$.
For $i = 1$, arguing as above one has

$$\|u'\|_2^2 \leq \|u\|_2\|u''\|_2 \leq k^{1/2}\|u'\|_2 k^{1/2}\|u'''\|_2 = k\|u'\|_2\|u'''\|_2,$$

where (2.2) for $i = 2$ has been also exploited. Hence, (2.2) is verified for $i = 1$.
Finally, for $i = 0$ the conclusion is achieved putting together the Poincaré inequality and using the case $i = 1$, indeed

$$\|u\|_2^2 \leq k\|u'\|_2^2 \leq k^3\|u'''\|_2^2.$$

□

Remark 2.1. The constants in (2.2) are the best ones as one can verify considering the function $\sin \pi x$ that realizes the equalities. Moreover, it is worth noting as follows. Indeed, we recall that, in general, one has

$$\|v\|_2^2 \leq 4k\|v'\|_2^2 \tag{2.3}$$

for all $v \in H^1([0,1])$ for which there is $c \in [0,1]$ such that $v(c) = 0$, and the equality for appropriate functions v also holds (see for instance [10, page 182]). So, if we apply the classical Poincaré inequality (2.3) to $v = u'$, then we obtain

$$\|u'\|_2^2 \leq 4k\|u''\|_2^2,$$

that, as shows (2.2), does not realize the best constant, on the contrary of (2.3). Clearly, this is due because in our case we have a greater regularity of u' (since $u \in X$).

We will introduce a convenient norm, equivalent to $\|\cdot\|$, that still makes X a Hilbert space. For this reason, for $A,\ B,\ C \in \mathbb{R}$ let us define the function $N : X \to \mathbb{R}$ by putting

$$N(u) = \|u'''\|_2^2 + A\|u''\|_2^2 + B\|u'\|_2^2 + C\|u\|_2^2$$

for every $u \in X$.

Now consider the following set of conditions according to the signs of the constants A, B and C:

$(H)_1$ $A \geq 0,\ B \geq 0,\ C \geq 0$;
$(H)_2$ $A \geq 0,\ B \geq 0,\ C < 0$ *and* $-Ak - Bk^2 - Ck^3 < 1$;
$(H)_3$ $A \geq 0,\ B < 0,\ C \geq 0$ *and* $-Ak - Bk^2 < 1$;
$(H)_4$ $A \geq 0,\ B < 0,\ C < 0$ *and* $-Ak - Bk^2 - Ck^3 < 1$;
$(H)_5$ $A < 0,\ B \geq 0,\ C \geq 0$ *and* $-Ak < 1$;
$(H)_6$ $A < 0,\ B \geq 0,\ C < 0$ *and* $\max\{-Ak,\ -Ak - Bk^2 - Ck^3\} < 1$;
$(H)_7$ $A < 0,\ B < 0,\ C \geq 0$ *and* $-Ak - Bk^2 < 1$;
$(H)_8$ $A < 0,\ B < 0,\ C < 0$ *and* $-Ak - Bk^2 - Ck^3 < 1$.

Moreover, fix $A, B, C \in \mathbb{R}$ and consider the following condition:

(H) $\max\{-Ak,\ -Ak - Bk^2,\ -Ak - Bk^2 - Ck^3\} < 1$.

We have the following result.

Proposition 2.2. *Condition (H) holds if and only if one of conditions $(H)_1 - (H)_8$ holds.*

Proof. Assume (H). Clearly, according to the signs of the constants A, B, C, one of conditions $(H)_1 - (H)_8$ is immediately verified. On the contrary, assuming one of conditions $(H)_1 - (H)_8$, then a direct computation shows that (H) is verified. As an example, assume at first $(H)_5$ and next $(H)_8$. In the first of such cases, since $B \geq 0$ and $C \geq 0$, one has $-Ak - Bk^2 \leq -Ak$ and $-Ak - Bk^2 - Ck^3 \leq -Ak$, for which $\max\{-Ak,\ -Ak - Bk^2,\ -Ak - Bk^2 - Ck^3\} \leq Ak < 1$, that is, (H) holds. In the other case, we have that the sum of three positive addends is less than 1, that is, $0 < -Ak - Bk^2 - Ck^3 < 1$. If, arguing by a contradiction, either $-Ak \geq 1$ or $-Ak - Bk^2 \geq 1$, then $-Ak - Bk^2 - Ck^3 \geq 1$ and this is absurd. So, $-Ak < 1$, $-Ak - Bk^2 < 1$ and $-Ak - Bk^2 - Ck^3 < 1$, for which (H) is satisfied. □

Proposition 2.3. *Assume (H). Then, there exits $m > 0$ such that*

$$N(u) \geq m\|u\|^2 \quad \forall\ u \in X. \tag{2.4}$$

Proof. Fix $u \in X$ and distinguish the different cases, taking Proposition 2.2 into account.

Assume $(H)_1$.
Then, in view of (2.2) one has

$$N(u) \geq \|u'''\|_2^2 \geq \frac{1}{4}\left(\|u'''\|_2^2 + \frac{1}{k}\|u''\|_2^2 + \frac{1}{k^2}\|u'\|_2^2 + \frac{1}{k^3}\|u\|_2^2\right) \geq \frac{1}{4}\|u\|^2 \tag{2.5}$$

and (2.4) holds with $m = \frac{1}{4}$.

Assume $(H)_2$.
Then, in view of (2.2)

$$N(u) \geq \|u'''\|_2^2 + A\|u''\|_2^2 + (B + Ck)\,\|u'\|_2^2.$$

Hence, if $B + Ck \geq 0$ we can argue as in (2.5) and (2.4) holds with $m = \frac{1}{4}$. Otherwise, again from (2.2)

$$N(u) \geq \|u'''\|_2^2 + (A + Bk + Ck^2)\|u''\|_2^2.$$

So, if $A+Bk+Ck^2 \geq 0$ we can argue as in (2.5) and conclude that (2.4) holds with $m = \frac{1}{4}$. Conversely, always from (2.2) one obtains

$$N(u) \geq (1 + Ak + Bk^2 + Ck^3)\|u'''\|_2^2$$

and assumption $(H)_2$, combined with the same above arguments, leads to (2.4) with $m = \frac{1+Ak+Bk^2+Ck^3}{4}$. Summarizing, (2.4) holds with $m = \min\left\{\frac{1}{4}, \frac{1+Ak+Bk^2+Ck^3}{4}\right\}$.

Assume $(H)_3$.
Then, in view of (2.2) one has

$$N(u) \geq \|u'''\|_2^2 + (A + Bk)\|u''\|_2^2.$$

If $A+Bk \geq 0$, following the reasoning as in (2.5) we conclude that (2.4) holds with $m = \frac{1}{4}$. Otherwise, (2.2) implies

$$N(u) \geq (1 + Ak + Bk^2)\|u'''\|_2^2.$$

and assumption $(H)_3$ implies that (2.4) holds with $m = \frac{1+Ak+Bk^2}{4}$.
Summarizing, (2.4) holds with $m = \min\left\{\frac{1}{4}, \frac{1+Ak+Bk^2}{4}\right\}$.

Assume $(H)_4$.
Then, from (2.2) one has

$$N(u) \geq \|u'''\|_2^2 + (A + Bk + Ck^2)\|u''\|_2^2.$$

If $A + Bk + Ck^2 \geq 0$ we conclude choosing $m = \frac{1}{4}$. Otherwise, with the same technique,

$$N(u) \geq (1 + Ak + Bk^2 + Ck^3)\|u'''\|_2^2$$

and we can complete also this case, pointing out that $m = \min\left\{\frac{1}{4}, \frac{1+Ak+Bk^2+Ck^3}{4}\right\}$.

Assume $(H)_5$.
Then, from (2.2) one has

$$N(u) \geq (1 + Ak)\|u'''\|_2^2$$

and (2.4) holds with $m = \frac{1+Ak}{4}$.

Assume $(H)_6$.
Then, from (2.2) one has

$$N(u) \geq (1 + Ak)\|u'''\| + (B + Ck)\|u'\|_2.$$

If $B + Ck \geq 0$ then assumption $(H)_6$ implies (2.4) with $m = \frac{1+Ak}{4}$. Otherwise,

$$N(u) \geq (1 + Ak + Bk^2 + Ck^3)\|u'''\|_2^2$$

we can conclude again but with $m = \frac{1+Ak+Bk^2+Ck^3}{4}$. Summarizing, in this case one has $m = \min\left\{\frac{1+Ak}{4}, \frac{1+Ak+Bk^2+Ck^3}{4}\right\}$.

Assume $(H)_7$.
Then, from (2.2) one has

$$N(u) \geq (1 + Ak + Bk^2)\|u'''\|_2^2$$

and (2.4) holds with $m = \frac{1+Ak+Bk^2}{4}$.

Assume $(H)_8$.
Then, from (2.2) one has

$$N(u) \geq (1 + Ak + Bk^2 + Ck^3)\|u'''\|_2^2$$

and (2.4) holds with $m = \frac{1+Ak+Bk^2+Ck^3}{4}$.

□

From the above considerations one can derive the following

Proposition 2.4. *Assume that* (H) *holds and put*

$$\|u\|_X = \sqrt{N(u)}, \quad \forall\, u \in X. \tag{2.6}$$

Then, $\|\cdot\|_X$ *is a norm equivalent to the usual one defined in (2.1) and* $(X, \|\cdot\|_X)$ *is a Hilbert space.*

Proof. The definition of N and Proposition 2.3 assure that $\|\cdot\|_X$ is the norm induced by the inner product

$$\langle\cdot,\cdot\rangle_X = \int_0^1 (u'''(x)v'''(x) + Au''(x)v''(x) + Bu'(x)v'(x) + Cu(x)v(x))\; dx \quad \forall\; u,\; v \in X.$$

It is simple to observe that there exists $M > 0$ such that

$$\|u\|_X^2 \le M\|u\|^2 \tag{2.7}$$

for every $u \in X$. Hence, the equivalence is an immediate consequence of (2.4) and (2.7) and the proof is complete. □

Clearly $(X, \|\cdot\|_X) \hookrightarrow (C^0(0,1), \|\cdot\|_\infty)$ and the embedding is compact. For a qualitative estimate of the constant of this embedding it is useful to introduce the following number

$$\delta = \begin{cases} 1 & \text{if } (H)_1 \text{ holds} \\ \min\{1, 1 + Ak + Bk^2 + Ck^3\} & \text{if } (H)_2 \text{ or } (H)_4 \text{ holds} \\ \min\{1, 1 + Ak + Bk^2\} & \text{if } (H)_3 \text{ holds} \\ 1 + Ak & \text{if } (H)_5 \text{ holds} \\ \min\{1 + Ak, 1 + Ak + Bk^2\} & \text{if } (H)_6 \text{ holds} \\ 1 + Ak + Bk^2 & \text{if } (H)_7 \text{ holds} \\ 1 + Ak + Bk^2 + Ck^3 & \text{if } (H)_8 \text{ holds.} \end{cases} \tag{2.8}$$

We explicitly observe that the proof of Proposition 2.3 shows in addition that

$$\|u\|_X^2 \ge \delta\|u'''\|_2^2 \tag{2.9}$$

for every $u \in X$, and $\delta = 4m$, where m is the number assured from the same Proposition 2.3.

Proposition 2.5. *Assume that* (H) *holds. One has*

$$\|u\|_\infty \le \frac{k}{2\sqrt{\delta}}\|u\|_X \tag{2.10}$$

for every $u \in X$*, where* δ *is given in* (2.8)*.*

Proof. It is well known that $H_0^1(0,1) \hookrightarrow C^0(0,1)$ and $\|u\|_\infty \le \frac{1}{2}\|u'\|_2$, thus, taking in mind (2.2),

$$\|u\|_\infty \le \frac{k}{2}\|u'''\|_2. \tag{2.11}$$

Moreover, from (2.9) one has

$$\|u'''\|_2 \le \frac{1}{\sqrt{\delta}}\|u\|_X$$

and (2.10) holds, in view of (2.11). □

In order to clarify the variational structure of problem (P_λ), we introduce the functionals $\Phi,\ \Psi :$ $X \to \mathbb{R}$ defined by putting

$$\Phi(u) = \frac{1}{2}\|u\|_X^2, \quad \Psi(u) = \int_0^1 F(x, u(x))\; dx \quad \forall\; u \in X, \tag{2.12}$$

where $F(x,t) = \int_0^t f(x,s)\; ds$ for every $(x,t) \in [0,1] \times \mathbb{R}$.
With standard arguments one can verify that Φ and Ψ are continuosly Gâteaux differentiable, being in particular

$$\Phi'(u)(v) = \int_0^1 (u'''(x)v'''(x) + Au''(x)v''(x) + Bu'(x)v'(x) + Cu(x)v(x))\; dx$$

and

$$\Psi'(u)(v) = \int_0^1 f(x,u(x))v(x)\ dx$$

for every $u,\ v \in X$.

Recall that a weak solution of problem (P_λ) is any $u \in X$ such that

$$\int_0^1 (u'''(x)v'''(x) + Au''(x)v''(x) + Bu'(x)v'(x) + Cu(x)v(x))\ dx = \lambda \int_0^1 f(x,u(x))v(x)\ dx \quad \forall v \in X. \tag{2.13}$$

Hence, the weak solutions of (P_λ) are exactly the critical points of the functional $\Phi - \lambda\Psi$.

Proposition 2.6. *Every weak solutions of (P_λ) is also a classical solution.*

Proof. Let $u \in X$ be a weak solution of (P_λ). Then, since

$$A\int_0^1 u''(x)v''(x)\ dx = -A\int_0^1 u'(x)v'''(x)\ dx$$

and

$$B\int_0^1 u'(x)v'(x)\ dx = -B\int_0^1 u''(x)v(x)\ dx,$$

one can observe that

$$\int_0^1 (u'''(x) - Au'(x))\, v'''(x)\ dx = \int_0^1 (Bu''(x) - Cu(x) + \lambda f(x,u(x)))\, v(x)\ dx$$

for every $v \in X$. Hence, $u''' - Au' \in H^3(0,1)$ and

$$(u''' - Au')''' = -Bu'' + Cu - \lambda f(x,u). \tag{2.14}$$

The continuity of f and the embedding $X \hookrightarrow C^2(0,1)$ imply that $u''' - Au' \in C^3(0,1)$. Thus, since

$$u''' = u''' - Au' + Au' \tag{2.15}$$

it is clear that $u \in C^4(0,1)$, namely $u' \in C^3(0,1)$ and (2.15) leads to $u \in C^6(0,1)$. From (2.14) one obtains

$$-u^{(vi)} + Au^{(iv)} - Bu'' + Cu = \lambda f(x,u). \tag{2.16}$$

At this point, integrating by parts (2.13) and exploiting (2.16) one has

$$\Big[-u^{(iv)}(x)v'(x)\Big]_0^1 = 0$$

for every $v \in X$, thus $u^{(iv)}(0) = u^{(iv)}(1) = 0$ and the proof is complete. □

The main tool in our approach is the following critical point theorem (see [2, Theorem 7.4])

Theorem 2.1. *Let X be a real Banach space and let $\Phi,\ \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functions with Φ bounded from below. Put*

$$\gamma = \liminf_{r\to+\infty} \varphi(r), \quad \chi = \liminf_{r\to(\inf_X \Phi)^+} \varphi(r),$$

where

$$\varphi(r) = \inf_{\Phi(v)<r} \frac{\sup_{\Phi(u)<r} \Psi(u) - \Psi(v)}{r - \Phi(v)} \quad (r > \inf_X \Phi).$$

(a) *If $\gamma < +\infty$ and for each $\lambda \in \left]0, \frac{1}{\gamma}\right[$ the function $I_\lambda = \Phi - \lambda\Psi$ satisfies $(PS)^{[r]}$-condition for all $r \in \mathbb{R}$ then, for each $\lambda \in \left]0, \frac{1}{\gamma}\right[$, the following alternative holds:*
either
  - (a$_1$) *I_λ possesses a global minimum,*
  or
  - (a$_2$) *there is a sequence $\{u_n\}$ of critical points (local minima) of I_λ such that $\lim_{n\to\infty} \Phi(u_n) = +\infty$.*

(b) *If $\chi < +\infty$ and for each $\lambda \in \left]0, \frac{1}{\chi}\right[$ the function $I_\lambda = \Phi - \lambda\Psi$ satisfies $(PS)^{[r]}$-condition for some $r > \inf_X \Phi$ then, for each $\lambda \in \left]0, \frac{1}{\chi}\right[$, the following alternative holds:*
either

(b_1) *there is a global minimum of* Φ *which is a local minimum of* I_λ,
or
(b_2) *there is a sequence* $\{u_n\}$ *of pairwise distinct critical points (local minima) of* I_λ *such that* $\lim_{n\to\infty}\Phi(u_n) = \inf_X \Phi$.

For the sake of completeness, we recall that for $r \in \mathbb{R}$, $I_\lambda = \Phi - \lambda\Psi$ is said to satisfy the $(PS)^{[r]}$-condition if any sequence $\{u_n\}$ such that

(α_1) $\{I_\lambda(u_n)\}$ is bounded,
(α_2) $\|I'_\lambda(u_n)\|_{X^*} \to 0$ as $n \to \infty$,
(α_3) $\Phi(u_n) < r \quad \forall\, n \in \mathbb{N}$

has a convergent subsequence.

3. Main results

In this section we are going to present the announced multiplicity result. The following technical constant will be useful

$$\tau = 4\delta\pi^4\left(96\left(\frac{12}{5}\right)^5 + 4A\left(\frac{12}{5}\right)^4 + B\frac{1248}{175} + C\frac{493}{756}\right)^{-1} \tag{3.1}$$

where A, B and C are the real numbers involved in problem (P_λ) and such that (H) holds, while δ has been introduced in (2.8).

Remark 3.1. We wish to stress a useful estimate for τ. If we consider the function

$$w(x) = \begin{cases} v(x) & \text{if } x \in [0, 5/12[\\ 1 & \text{if } x \in [5/12, 7/12] \\ v(1-x) & \text{if } x \in]7/12, 1] \end{cases} \tag{3.2}$$

where $v(x) = \left(\frac{12}{5}\right)^4 x^4 - 2\left(\frac{12}{5}\right)^3 x^3 + \frac{24}{5}x$ for every $x \in [0, 5/12]$, a straightforward computation shows that $w \in X = H^3(0,1) \cap H^1_0(0,1)$ and, in particular

$$\|w\|_X^2 = \frac{4\delta\pi^4}{\tau}.$$

Recalling that (H) holds, the positivity of τ follows from the arguments presented in the previous section. Moreover, from (2.10), since $\|w\|_\infty = 1$, one can even conclude that

$$0 < \tau \le 1.$$

Here is the first main result.

Theorem 3.1. *Assume that*

i) $F(x,t) \ge 0$ *for every* $(x,t) \in ([0,5/12] \cup [7/12,1]) \times \mathbb{R}$,

ii) $\displaystyle \liminf_{t\to+\infty} \frac{\int_0^1 \max_{|s|\le t} F(x,s)\,dx}{t^2} < \tau \limsup_{t\to+\infty} \frac{\int_{5/12}^{7/12} F(x,t)\,dx}{t^2}.$

Then, for every

$$\lambda \in \Lambda = \left]\frac{2\delta\pi^4}{\tau}\frac{1}{\limsup_{t\to+\infty}\frac{\int_{5/12}^{7/12} F(x,t)\,dx}{t^2}}, \frac{2\delta\pi^4}{\liminf_{t\to+\infty}\frac{\int_0^1 \max_{|s|\le t} F(x,s)\,dx}{t^2}}\right[$$

the problem (P_λ) *admits an unbounded sequence of classical solutions.*

Proof. We wish to apply Theorem 2.1, case (a), with $X = H^3(0,1) \cap H^1_0(0,1)$ endowed with the norm $\|\cdot\|_X$ defined in (2.6), Φ and Ψ as in (2.12).
In the previous section we have already pointed out that $\Phi,\ \Psi \in C^1(X)$. It is simple to verify that Φ is bounded from below, coercive and its derivative is a homeomorphism. Moreover, the compactness of the embedding $X \hookrightarrow C^0(0,1)$ assures that Ψ' is a compact operator. Hence, we can conclude that, for every $\lambda > 0$ (indeed for every $\lambda \in \mathbb{R}$) the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies the $(PS)^{[r]}$-condition for

every $r \in \mathbb{R}$ (see [2, Remark 2.1]). Our aim is now to verify that $\gamma < +\infty$. Let us begin by observing that, in view of (2.10) one has

$$\{v \in X : \ \Phi(v) < r\} \subset \left\{ v \in C^0(0,1) : \ \|v\|_\infty \leq \frac{k}{\sqrt{\delta}} \sqrt{\frac{r}{2}} \right\}$$

for all $r > 0$. Let $\{t_n\}$ be in $\mathbb{R}^+$ such that $t_n \to +\infty$ and

$$\lim_{n\to\infty} \frac{\int_0^1 \max_{|s|\leq t_n} F(x,s)\ dx}{t_n^2} = \liminf_{t\to+\infty} \frac{\int_0^1 \max_{|s|\leq t} F(x,s)\ dx}{t^2}.$$

Put $r_n = 2\delta\pi^4 t_n^2$ for every $n \in \mathbb{N}$. Hence, one has

$$\begin{aligned} \varphi(r_n) &= \inf_{\Phi(v)<r_n} \frac{\sup_{\Phi(u)<r_n} \Psi(u) - \Psi(v)}{r - \Phi(v)} \\ &\leq \frac{\sup_{\Phi(u)<r_n} \Psi(u)}{r_n} \\ &\leq \frac{1}{2\delta\pi^4} \frac{\int_0^1 \max_{|s|\leq t_n} F(x,s)\ dx}{t_n^2}. \end{aligned}$$

Passing to the liminf in the previous inequality one obtains

$$\gamma \leq \liminf_{n\to\infty} \varphi(r_n) \leq \frac{1}{2\delta\pi^4} \liminf_{t\to+\infty} \frac{\int_0^1 \max_{|s|\leq t} F(x,s)\ dx}{t^2} < +\infty.$$

In particular, we have also verified that

$$\Lambda \subset \left] 0, \frac{1}{\gamma} \right[.$$

Fix now $\lambda \in \Lambda$ and let us check that I_λ is unbounded from below. We can explicitly observe that

$$\frac{1}{2\delta\pi^4} \liminf_{t\to+\infty} \frac{\int_0^1 \max_{|s|\leq t} F(x,s)\ dx}{t^2} < \frac{1}{\lambda} < \frac{\tau}{2\delta\pi^4} \limsup_{t\to+\infty} \frac{\int_{5/12}^{7/12} F(x,t)\ dx}{t^2}.$$

Pick $\eta > 0$ such that

$$\frac{1}{\lambda} < \eta < \frac{\tau}{2\delta\pi^4} \limsup_{t\to+\infty} \frac{\int_{5/12}^{7/12} F(x,t)\ dx}{t^2}$$

and consider a sequence $\{d_n\}$ in $\mathbb{R}^+$ such that $d_n \to +\infty$ and

$$\frac{\int_{5/12}^{7/12} F(x,d_n)}{d_n^2} > \eta \frac{2\delta\pi^4}{\tau}$$

for every $n \in \mathbb{N}$. If, for every $n \in \mathbb{N}$ we define

$$w_n(x) = d_n w(x),$$

where w has been defined in (3.2), it is clear that $0 \leq w_n(x) \leq d_n$ for every $x \in [0,1]$, $w_n \in X$ and, in particular

$$\|w_n\|_X^2 = \frac{4\delta\pi^4}{\tau} d_n^2.$$

Thus, also in view of i),

$$\begin{aligned} I_\lambda(w_n) &= \Phi(w_n) - \lambda\Psi(w_n) \\ &= \frac{2\delta\pi^4}{\tau} d_n^2 - \lambda \int_0^1 F(x, w_n(x))\ dx \\ &< \frac{2\delta\pi^4}{\tau}(1 - \lambda\eta) d_n^2. \end{aligned}$$

Namely, passing to the limit and taking in mind that $1 - \lambda\eta < 0$ one achieves that I_λ is unbounded from below.

We are now in the position to apply Theorem 2.1, case (a), and obtain a sequence $\{u_n\}$ in X of critical points (local minima) of I_λ such that $\|u_n\|_X \to +\infty$. Taking in mind that the critical points of I_λ are classical solutions of (P_λ), see Proposition 2.6, we have completed the proof. $\square$

The following remark will be useful in order to obtain a sign condition on the solutions of (P_λ).

Remark 3.2. Recall that if $\mu \geq 0$ and $w \in H^2(0,T)$ is such that

$$\begin{cases} -w'' + \mu w \geq 0 & x \in [0,1] \\ w(0) = w(1) = 0 \end{cases}$$

then $w(x) \geq 0$ for every $x \in [0,1]$ (see also [4, Théorème VIII.17]).

Remark 3.3. We wish to point out that if $u \in C^6(0,1)$ is a nonnegative and nontrivial function such that

$$\begin{cases} -u^{(vi)} + Au^{(iv)} - Bu'' + Cu \geq 0 & x \in [0,1] \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0 \end{cases}$$

and there exist three nonnegative numbers $X,\ Y$ and Z such that

$$\begin{cases} X + Y + Z = A \\ XY + XZ + YZ = B \\ XYZ = C \end{cases} \tag{3.3}$$

then $u(x) > 0$ for every $x \in (0,1)$. To justify this, we can observe that for the following linear differential operators

$$L_1(w) = -w'' + Xw, \quad L_2(w) = -w'' + Yw, \quad L_3(w) = -w'' + Zw$$

is possible to apply the strong maximum principle (see [14]). Hence, in particular, since in view of (3.3)

$$L_1(L_2(L_3(u))) = -u^{(vi)} + (X+Y+Z)u^{(iv)} - (XY+XZ+YZ)u'' + XYZu = -u^{(vi)} + Au^{(iv)} - Bu'' + Cu,$$

one has, using several times Remark 3.2,

$$L_1(L_2(L_3(u))) \geq 0 \ \Rightarrow \ L_2(L_3(u)) \geq 0 \ \Rightarrow \ L_3(u) \geq 0 \ \Rightarrow \ u \geq 0 \quad \text{in } [0,1].$$

Finally, from [14, Theorem 3] one can conclude that $u(x) > 0$ for every $x \in (0,1)$.
We remand to [12] for further considerations on maximum principle for high-order differential equations.

Example 3.1. If $u \in C^6(0,1)$ is such that

$$\begin{cases} -u^{(vi)} + 3u^{(iv)} - 3u'' + u \geq 0 & x \in [0,1] \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0, \end{cases}$$

then $u > 0$ in $(0,1)$. It suffices to take $X = Y = Z = 1$ in (3.3), so that $A = B = 3$ and $C = 1$.

Example 3.2. If $C \leq 0$ and $A,\ B \geq 0$ are such that $A^2 - 4B \geq 0$ then every nonnegative and nontrivial $u \in C^6(0,1)$ such that

$$\begin{cases} -u^{(vi)} + Au^{(iv)} - Bu'' + Cu \geq 0 & x \in [0,1] \\ u(0) = u(1) = u''(0) = u''(1) = u^{iv}(0) = u^{iv}(1) = 0, \end{cases}$$

is positive in $(0,1)$. Indeed, observing that from $C \leq 0$ and $u \geq 0$ one has

$$-u^{(vi)} + Au^{(iv)} - Bu'' \geq 0$$

we can recall Remark 3.3, in (3.3) consider $X = \frac{A+\sqrt{A^2-4B}}{2}$, $Y = \frac{A-\sqrt{A^2-4B}}{2}$, $Z = 0$ and conclude that $u > 0$ in $(0,1)$.

In the following we say that $A,\ B,\ C$ satisfy the (H_+) condition if

(H_+) there exist nonnegative numbers $X,\ Y$ and Z such that (3.3) holds.

The existence of constant sign solutions can be pointed out, provided

$$f(x,t) \geq 0 \qquad \forall (x,t) \in [0,1] \times [0,+\infty[. \tag{3.4}$$

In particular the following result holds.

Theorem 3.2. *Assume that assumption (3.4) and (H_+) hold and*

ii') $\displaystyle \liminf_{t\to+\infty} \frac{\int_0^1 F(x,t)\,dx}{t^2} < \tau \limsup_{t\to+\infty} \frac{\int_{5/12}^{7/12} F(x,t)\,dx}{t^2}.$

Then, for every

$$\lambda \in \tilde{\Lambda} = \left] \frac{2\delta\pi^4}{\tau} \frac{1}{\limsup_{t\to+\infty} \frac{\int_{5/12}^{7/12} F(x,t)\ dx}{t^2}}, \frac{2\delta\pi^4}{\liminf_{t\to+\infty} \frac{\int_0^1 F(x,t)\ dx}{t^2}} \right[$$

the problem (P_λ) admits an unbounded sequence of positive classical solutions.

Proof. Put

$$f^+(x,t) = \begin{cases} f(x,t) & \text{if } (x,t) \in [0,1] \times [0,+\infty[\\ \\ f(x,0) & \text{if } (x,t) \in [0,1] \times]-\infty, 0[\end{cases}$$

and $F^+(x,t) = \int_0^t f^+(x,s)\ ds$. Clearly $f^+_{|[0,1]\times[0,+\infty[} = f_{|[0,1]\times[0,+\infty[}$ as well as $F^+_{|[0,1]\times[0,+\infty[} = F_{|[0,1]\times[0,+\infty[}$. Hence, in view of ii'),

$$\begin{aligned} \liminf_{t\to+\infty} \frac{\int_0^1 \max_{|s|\le t} F^+(x,s)\ dx}{t^2} &= \liminf_{t\to+\infty} \frac{\int_0^1 \max_{0\le s\le t} F(x,s)\ dx}{t^2} \\ &= \liminf_{t\to+\infty} \frac{\int_0^1 F(x,t)\ dx}{t^2} \\ &< \tau \limsup_{t\to+\infty} \frac{\int_{5/12}^{7/12} F(x,t)\ dx}{t^2} \\ &= \tau \limsup_{t\to+\infty} \frac{\int_{5/12}^{7/12} F^+(x,t)\ dx}{t^2}. \end{aligned}$$

Thus, we can apply Theorem 3.1 to f_+ and F_+ and assure that for every $\lambda \in \tilde{\Lambda}$, the problem

$$\begin{cases} -u^{(vi)} + Au^{(iv)} - Bu'' + Cu = \lambda f^+(x,u) & x \in [0,1] \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0 \end{cases} \tag{3.5}$$

admits an unbounded sequence of classical solutions.
We claim that

Every solution of (3.5) is a nonnegative solution of (P_λ).

Indeed, if u solves (3.5), since $f^+(x,t) \ge 0$ for every $(x,t) \in [0,1] \times \mathbb{R}$, we can recall Remark 3.3 and deduce that u is positive. Hence, $f^+(x,u(x)) = f(x,u(x))$ for every $x \in [0,1]$ and u solves (P_λ). The claim is now verified and the proof is completed. $\square$

We now present an autonomous version of the previous result.

Theorem 3.3. *Suppose (H_+) holds and assume that g is a nonnegative continuous function such that*

$$\liminf_{t\to+\infty} \frac{G(t)}{t^2} < \frac{\tau}{6} \limsup_{t\to+\infty} \frac{G(t)}{t^2}, \tag{3.6}$$

where $G(t) = \int_0^t g(s)\ ds$ for every $t \in \mathbb{R}$.
Then, for every $\lambda \in \left] \frac{12\delta\pi^4}{\tau} \frac{1}{\limsup_{t\to+\infty} \frac{G(t)}{t^2}}, \frac{2\delta\pi^4}{\liminf_{t\to+\infty} \frac{G(t)}{t^2}} \right[$ the problem

$$\begin{cases} -u^{(vi)} + Au^{(iv)} - Bu'' + Cu = \lambda g(u) & x \in [0,1] \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0 \end{cases} \tag{$\tilde{P}_\lambda$}$$

admits an unbounded sequence of classical postive solutions.

Proof. Apply Theorem 3.2 to $f(x,t) = g(t)$ for all $(x,t) \in [0,1] \times \mathbb{R}$ and observe that

$$F(x,t) = G(t), \quad \int_{5/12}^{7/12} F(x,t)\ dx = \frac{1}{6} G(t).$$

$\square$

Example 3.3. Fix A, B and C (as usual such that (H_+) holds) let τ be the number defined in (3.1), pick $\rho > \frac{6-\tau}{\tau}$ and consider the continuous function $g : \mathbb{R} \to \mathbb{R}$ defined by putting

$$g(t) = \begin{cases} 2t\left[1 + \rho\sin^2(\ln(\rho^2 + \ln^2 t)) + \sin(2\ln(\rho^2 + \ln^2 t))\frac{\rho \ln t}{\rho^2 + \ln^2 t}\right] & \text{if } t > 0 \\ \\ 0 & \text{if } t \leq 0. \end{cases}$$

Then, for every $\lambda \in \left]\frac{12\delta\pi^4}{\tau(1+\rho)}, 2\delta\pi^4\right[$ the problem

$$\begin{cases} -u^{(vi)} + Au^{(iv)} - Bu'' + Cu = \lambda g(u) & x \in [0,1] \\ u(0) = u(1) = u''(0) = u''(1) = u^{(iv)}(0) = u^{(iv)}(1) = 0 \end{cases}$$

admits an unbounded sequence of classical positive solutions.
Indeed, a direct computation shows that $0 < \tau \leq 1$. Hence, $\rho > 0$ and exploiting the boundedness of the function $t \to \sin(2\ln(\rho^2 + \ln^2 t))\frac{\rho \ln t}{\rho^2 + \ln^2 t}$ one has

$$g(t) \geq 2t\left[\frac{1}{2} + \rho\sin^2(\ln(\rho^2 + \ln^2 t))\right] > 0$$

for every $t > 0$. Moreover,

$$G(t) = \begin{cases} t^2(1 + \rho\sin^2(\ln(\rho^2 + \ln^2 t))) & \text{if } t > 0 \\ \\ 0 & \text{if } t \leq 0 \end{cases}$$

Hence, if we put $a_n = e^{\sqrt{e^{n\pi} - \rho^2}}$ and $b_n = e^{\sqrt{e^{(2n+1)\pi/2} - \rho^2}}$ for every $n \in \mathbb{N}$ with $n > (2/\pi)\ln\rho$, one has

$$\begin{aligned} \liminf_{t\to+\infty} \frac{G(t)}{t^2} &\leq \lim_{n\to+\infty} \frac{G(a_n)}{a_n^2} = 1 \\ &< \frac{\tau}{6}(1+\rho) \\ &= \frac{\tau}{6}\lim_{n\to+\infty} \frac{G(b_n)}{b_n^2} \\ &\leq \frac{\tau}{6}\limsup_{t\to+\infty} \frac{G(t)}{t^2} \end{aligned}$$

A this point we can apply Theorem 3.3 observing that

$$\left]\frac{12\delta\pi^4}{\tau(1+\rho)}, 2\delta\pi^4\right[\subseteq \left]\frac{12\delta\pi^4}{\tau}\frac{1}{\limsup_{t\to+\infty}\frac{G(t)}{t^2}}, \frac{2\delta\pi^4}{\liminf_{t\to+\infty}\frac{G(t)}{t^2}}\right[.$$

We can directly derive the proof of Theorem 1.1 from Theorem 3.3.

Proof of Theorem 1.1. Apply Theorem 3.3, with $A = 3D$; $B = 3D^2$; $C = D^3$, and exploit Remark 3.3, by choosing $X = Y = Z = D$.

We conclude the present note pointing out that, adapting the previous arguments, one can exploit case (b) of Theorem 2.1 in order to prove the existence of arbitrary small solutions of problem (P_λ).

Theorem 3.4. *Assume that*

j) *there exists $r > 0$ such that $F(x,t) \geq 0$ for every $(x,t) \in ([0, 5/12] \cup [7/12, 1]) \times [0, r]$,*

jj) $\liminf_{t\to 0^+} \frac{\int_0^1 \max_{|s|\leq t} F(x,s)\, dx}{t^2} < \tau \limsup_{t\to 0^+} \frac{\int_{5/12}^{7/12} F(x,t)\, dx}{t^2}.$

Then, for every

$$\lambda \in \Gamma = \left]\frac{2\delta\pi^4}{\tau}\frac{1}{\limsup_{t\to 0^+}\frac{\int_{5/12}^{7/12} F(x,t)\,dx}{t^2}}, \frac{2\delta\pi^4}{\liminf_{t\to 0^+}\frac{\int_0^1 \max_{|s|\leq t} F(x,s)\,dx}{t^2}}\right[$$

the problem (P_λ) *admits a sequence of pairwise distinct nontrivial classical solutions, which strongly converges to* 0 *in* X.

Clearly, starting from Theorem 3.4 and arguing as above, further results dealing with the existence of arbitrary small (positive) classical solutions could be furnished.

Acknowledgments. The authors are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM). The paper is partially supported by PRIN 2017 – Progetti di Ricerca di rilevante Interesse Nazionale, "Nonlinear Differential Problems via Variational, Topological and Set-valued Methods" (2017AYM8XW).

References

[1] P. Bates, P. Fife, R. Gardner and C.K.R.J. Jones, *The existence of traveling wave solutions of a generalized phase-field model*, SIAM J. Math. Anal. **28** (1997), 60–93.
[2] G. Bonanno, *A critical point theorem via the Ekeland variational principle*, Nonlinear Anal. **75** (2012), 2992–3007
[3] G. Bonanno and B. Di Bella, *Infinitely many solutions for a fourth-order elastic beam equation*, Nonlinear Differ. Equ. Appl. **18** (2011), 357–368.
[4] H. Brezis, *Analyse fonctionnelle, theorie et applications*, Masson, Paris, 1987.
[5] G. Caginalp and P. Fife, *Higher order phase field models and detailed anisotropy*, Phys. Rev. B **34** (1986), 4940–4943
[6] J. Chaparova, L. A. Peletier and S. Tersian, *Existence and nonexistence of nontrivial solutions of semilinear fourth- and sixth-order differential equations*, Adv. Differential Equations **8** (2003), no. 10, 1237–1258
[7] R. A. Gardner and C. K. R. T.Jones, *Traveling waves of a perturbed diffusion equation arising in a phase field model*, Indiana Univ. Math. J. **39** (1990), 1197-1222
[8] J. R. Graef and J. R. L. Webb, *Third order boundary value problems with nonlocal boundary conditions*, Nonlinear Anal. **71** (2009), no. 5–6, 1542–1551
[9] T. Gyulov, G. Morosanu and S. Tersian, *Existence for a semilinear sixth-order ODE*, J. Math. Anal. Appl. **321** (2006), 86–98
[10] G.H. Hardy, J.E. Littlewood and G Pólya, Inequalities, Cambridge University Press, London, 1934.
[11] W. Li, *The existence and multiplicity of positive solutions of nonlinear sixth-order boundary value problem with three variable coefficients*, Bound. Value Probl. **2012** (2012), no. 22, 1-16.
[12] A. Mohammed and G. Porru, *Maximum priciples for ordinary differential inequalities of fourth and sixth order*, J. Math. Anal. Appl. **146** (1990), 408-419.
[13] L.A. Peletier, W.C. Troy and Van der Vorst, *Stationary solutions of a fourth-order nonlinear diffusion equation*, Differential Equations **31**, No.2 (1995), 301–314.
[14] M.H. Protter and H.F. Weinberger, Maximum principles in differential equations, Springer-Verlag, New York, 1984.
[15] S. Tersian and J. Chaparova, *Periodic and homoclinic solutions of extended Fisher-Kolmogorov equations*, J. Math. Anal. Appl. **260** (2001), no. 2, 490-506
[16] S. Tersian and J. Chaparova, *Periodic and homoclinic solutions of some semilinear sixth-order differential equations*, J. Math. Anal. Appl. **272** (2002), no. 1, 223-239
[17] J. R. L. Webb, *Positive solutions of some higher order nonlocal boundary value problems*, Electron. J. Qual. Theory Differ. Equ. 2009, Special Edition I, No. 29, 15 pp.
[18] J. R. L. Webb and G. Infante, *Semi-positone nonlocal boundary value problems of arbitrary order*, Commun. Pure Appl. Anal. **9** (2010), no. 2, 563–581
[19] J.R.L. Webb, G. Infante and D. Franco, *Positive solutions of nonlinear fourth-order boundary-value problems with local and non-local boundary conditions*, Proc. Roy. Soc. Edinburgh Sect. A **138** (2008), no. 2, 427–446
[20] J. R. L. Webb and M. Zima, *Multiple positive solutions of resonant and non-resonant non-local fourth-order boundary value problems*, Glasg. Math. J. **54** (2012), no. 1, 225–240
[21] L. Zhang and Y. An, *Existence and multiplicity of positive solutions of a boundary-value problem for sixth-order ODE with three parameters*, Bound. Value Probl. **2010** (2010), 1-13.

(G. Bonanno) Department of Engineering, University of Messina, c/da di Dio (s. Agata), 98166 Italy.
Email address: `bonanno@unime.it`

(R. Livrea) Department of Mathematics and Computer Science, University of Palermo, Via Archirafi, 34, 90123 Palermo, Italy.
Email address: `roberto.livrea@unipa.it`