ON BASE LOCI OF HIGHER FUNDAMENTAL FORMS OF TORIC VARIETIES

ANTONIO LAFACE AND LUCA UGAGLIA

Abstract. We study the base locus of the higher fundamental forms of a projective toric variety X at a general point. More precisely we consider the closure X of the image of a map $(\mathbb{C}^*)^k \to \mathbb{P}^n$, sending t to the vector of Laurent monomials with exponents $p_0, \dots, p_n \in \mathbb{Z}^k$. We prove that the m-th fundamental form of such an X at a general point has non empty base locus if and only if the points p_i lie on a suitable degree-m affine hypersurface.

We then restrict to the case in which the points p_i are all the lattice points of a lattice polytope and we give some applications of the above result. In particular we provide a classification for the second fundamental forms on toric surfaces, and we also give some new examples of weighted 3-dimensional projective spaces whose blowing up at a general point is not Mori dream.

Introduction

Let $X \subseteq \mathbb{P}^n$ be a projective variety and let $q \in X$ be a general point. Denote by $\pi : \tilde{X} \to X$ the blowing-up of X at q with exceptional divisor E. Given a hyperplane section H of X it is an open problem to provide necessary and sufficient conditions on the embedding $X \to \mathbb{P}^n$ in order for the linear system $|\pi^*H - mE|$ to be *special*, which means that its dimension is bigger than the expected one. The problem has been widely studied in case $m = 2$, see for instance [1, 6, 7] and the references therein, but it remains open even in this case. For higher values of m there are conjectures when X is the blowing up of $\mathbb{P}^2$ (Segre-Harbourne-Gimigliano-Hirschowitz Conjecture [12, 16, 19, 27]) and $\mathbb{P}^3$ (Laface-Ugaglia Conjecture [21, 22]) at points in very general position. These conjectures predict that a necessary condition for $|\pi^*H - mE|$ to be special is that it has positive dimensional base locus.

In this paper we investigate the above problem in case X is the closure of a monomial embedding $(\mathbb{C}^*)^k \to \mathbb{P}^n$, so that X is a not necessarily normal toric variety. The principal tool that we use is the restricted linear system

$$|\pi^*H - mE|_E,$$

which is also called the *m-th fundamental form* of X at q (see for instance [15, 20]). The m-th fundamental form turns out to be useful in two directions. On one hand, a base point for the m-th fundamental form is a base point for the system $|\pi^*H - mE|$ too. On the other hand, the dimension of the m-th form turns out to be related to the speciality of the system $|\pi^*H - (m+1)E|$ (see Proposition 1.2).

In order to state our results, let us fix a k-dimensional lattice $M \simeq \mathbb{Z}^k$ and a finite set of points $S = \{p_0, \dots, p_n\} \subseteq M$. It is possible to define a map $f : (\mathbb{C}^*)^k \to \mathbb{P}^n$

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which associates to t the vector of Laurent monomials with exponents $p_0, \dots, p_n$. The closure of the image of the above map is a k-dimensional projective toric variety $X(S) \subseteq \mathbb{P}^n$, and we denote by $\mathbf{1} \in X(S)$ the image of the neutral element of $(\mathbb{C}^*)^k$. The point $\mathbf{1}$ lies in the open torus orbit, and hence it is a general point of $X(S)$. An element $v \in N := \mathrm{Hom}(M, \mathbb{Z})$ defines a map $\mathbb{C}^* \to X$ by $t \mapsto f(t^v)$, whose derivative at $t = 1$ is a vector of $T_{\mathbf{1}}X$. This induces a linear map $N \otimes_{\mathbb{Z}} \mathbb{C} \to T_{\mathbf{1}}X$ which allows us to identify $\mathbb{P}(N \otimes_{\mathbb{Z}} \mathbb{C})$ with $\mathbb{P}(T_{\mathbf{1}}X) \simeq \mathbb{P}^{k-1}$ in our main theorem. In [25] it is shown that the m-th fundamental form at $\mathbf{1}$ is not the complete linear system if and only if the points $p_0, \dots, p_n$ lie on an affine hypersurface of degree m. Our main result shows that the m-th fundamental form at $\mathbf{1}$ has a base point if and only if the top degree part of the above affine hypersurface is a pure power. More precisely we have the following (see also Example 2.3 and 2.4 for the difference between our result and Perkinson's).

Theorem 1. *Given an integer $m \geq 2$ the following are equivalent:*

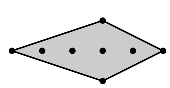
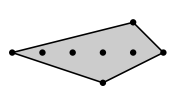
(1) *the m-th fundamental form at $\mathbf{1} \in X(S)$ has a base point $[v] \in \mathbb{P}(T_{\mathbf{1}}X)$;*
(2) *the points of S lie on an affine hypersurface of $M \otimes_{\mathbb{Z}} \mathbb{C}$ of equation*

$$(v \cdot x)^m + \text{lower degree terms} = 0.$$

We then restrict to the case of a toric variety associated to a polytope. Indeed, given a full-dimensional lattice polytope $\Delta \subseteq M \otimes_{\mathbb{Z}} \mathbb{Q}$, it is possible to define a polarized pair (X, H), where $X = X(\Delta)$ is the projective toric variety associated to the lattice points $\Delta \cap M$, while H is a very ample divisor of X. In what follows we will denote by $\pi : \tilde{X} \to X$ the blowing up of the toric variety X along the point $\mathbf{1}$ and by E the exceptional divisor. A first consequence of Theorem 1 is the following characterisation of projective toric surfaces whose second fundamental form at $\mathbf{1}$ is not full dimensional (see also Definition 1.8):

Proposition 2. *Let $\Delta \subseteq M \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}^2$ be a full dimensional lattice polytope such that $|\Delta \cap M| \geq 6$ and let (X, H) be the corresponding polarized pair. Then the following are equivalent:*

(1) *the second fundamental form of X at $\mathbf{1}$ is not full dimensional;*
(2) *the linear system $|\pi^* H - 3E|$ is special;*
(3) *Δ is either a Cayley polygon or it is equivalent, modulo $\mathrm{GL}(2, \mathbb{Z})$, to one of the following:*

Type		*Vertices*	
(i)		$(a,0)$, $(0,1)$, $(-b,0)$, $(0,-1)$,	*with* $a \geq 1$, $b \geq 0$ *and* $a + b \geq 3$
(ii)		$(a,0)$, $(0,1)$, $(-b,0)$, $(-1,-1)$,	*with* $a \geq 1$, $b \geq 0$ *and* $a + b \geq 3$

In particular the second fundamental form at $\mathbf{1}$ has non empty base locus if and only if Δ is Cayley.

Going back to the problem stated at the beginning of the introduction, an easy corollary of the above result is that if the linear system $|\pi^*H-3E|$ is special, then its base locus contains a curve (the strict transform of the closure of a one-parameter subgroup) intersecting E. We will show that if $m \geq 4$, this is no longer true, i.e. there are examples of special linear systems of the form $|\pi^*H - mE|$ whose base locus does not contain such a curve (see Example 3.4).

Finally, when $k \geq 2$, we make use of Theorem 1 in order to study stable base loci of divisors of the form $\pi^*H - mE$ on $\tilde{X}$. In particular we give a sufficient condition on Δ implying that π^*H-mE is not semiample (Corollary 3.2) and as an application we provide the following new list of 3-dimensional weighted projective spaces $\mathbb{P}(a_1,\dots,a_4)$, with $a_i \leq 30$, whose blowing up at $\mathbf{1}$ is not a Mori dream space.

Proposition 3. *Let $X := \mathbb{P}(a_1,\dots,a_4)$ and let H be an ample divisor of degree $\operatorname{lcm}(a_1,\dots,a_4)$. If the vector of weights is in the following table then the divisor π^*H-mE is nef but not semiample. In particular the blowing up of X at $\mathbf{1}$ is not Mori dream.*

$[a_1,\dots,a_4]$	m
$[7,11,13,15]$	572
$[7,13,16,19]$	832
$[7,15,19,23]$	1140
$[7,17,22,27]$	1496
$[7,19,20,24]$	380
$[7,23,25,29]$	2300
$[9,10,13,17]$	702
$[9,13,16,23]$	1152
$[9,16,19,20]$	342
$[9,16,19,29]$	1710
$[9,17,23,28]$	2070
$[9,19,22,26]$	990
$[9,25,28,29]$	3024
$[10,11,16,19]$	480
$[10,11,17,23]$	1122
$[10,13,17,18]$	540
$[10,13,21,29]$	1638
$[10,17,19,21]$	1520
$[10,17,22,23]$	880
$[10,17,24,29]$	986
$[10,19,21,24]$	320
$[10,19,23,27]$	2300
$[10,19,23,28]$	1064
$[10,21,22,27]$	378
$[10,21,23,26]$	1196
$[10,21,26,29]$	1300
$[10,23,27,28]$	1400
$[11,12,13,17]$	816
$[11,13,23,28]$	1794
$[11,15,19,24]$	480
$[11,16,25,28]$	550

$[a_1,\dots,a_4]$	m
$[11,17,25,29]$	2550
$[11,18,20,21]$	280
$[11,19,24,26]$	1248
$[11,20,21,27]$	756
$[11,23,24,28]$	644
$[11,23,25,28]$	2800
$[12,13,16,19]$	304
$[12,13,17,22]$	663
$[12,17,19,23]$	1656
$[12,17,19,25]$	1800
$[12,17,20,23]$	460
$[12,17,25,26]$	1275
$[12,19,22,25]$	1100
$[12,19,25,28]$	700
$[12,23,25,29]$	2784
$[12,23,26,29]$	1508
$[13,14,15,22]$	616
$[13,14,17,25]$	1638
$[13,15,17,27]$	540
$[13,15,24,29]$	754
$[13,16,19,27]$	1728
$[13,17,23,29]$	2392
$[13,17,24,25]$	2550
$[13,18,22,29]$	1276
$[13,19,21,29]$	2436
$[13,20,21,29]$	2520
$[13,21,28,30]$	80
$[14,17,22,27]$	1232
$[14,17,23,24]$	1224
$[14,17,24,29]$	1392
$[14,19,23,30]$	1260

$[a_1,\dots,a_4]$	m
$[14,19,27,29]$	3192
$[16,17,19,22]$	969
$[16,18,19,29]$	1296
$[16,19,20,29]$	696
$[16,21,23,26]$	1449
$[16,22,25,27]$	1782
$[17,18,20,27]$	162
$[17,20,21,23]$	2520
$[17,20,26,27]$	1620
$[17,21,22,23]$	2772
$[17,21,22,29]$	3213
$[17,21,24,29]$	1218
$[17,23,25,26]$	3519
$[17,23,25,29]$	3450
$[17,23,26,30]$	2070
$[17,23,27,29]$	3726
$[17,25,27,29]$	4050
$[18,19,21,28]$	76
$[18,20,23,27]$	189
$[18,23,26,27]$	216
$[18,26,27,29]$	243
$[19,20,22,29]$	1740
$[19,22,24,25]$	1584
$[19,22,25,26]$	1672
$[19,23,24,25]$	3312
$[19,24,27,29]$	1296
$[19,24,29,30]$	696
$[19,25,26,27]$	3952
$[19,25,28,29]$	4275
$[22,25,27,28]$	2025
$[23,27,29,30]$	1827

In [14] and [18] there are examples of 3-dimensional weighted projective spaces whose blowing up at $\mathbf{1}$ is not Mori dream. We remark that there is no intersection between our list and the one of [18], since we consider only the cases in which no weight a_i belongs to the semigroup generated by the remaining ones. Concerning the list of [14], there is only one common case, namely $\mathbb{P}(17, 18, 20, 27)$ (see also Remark 3.10).

The paper is structured as follows. In Section 1 we first introduce higher fundamental forms on projective varieties, then we recall some definitions and facts about projective toric varieties and finally we specialize to the case of toric varieties associated to a lattice polytope Δ. In Section 2 we prove Theorem 1 and we present a couple of related examples. The last section deals with some applications of Theorem 1 to toric varieties associated to a polytope Δ. In particular we first prove a corollary which gives a condition on Δ implying that a suitable divisor on the blowing up of the toric variety $X(\Delta)$ is not semiample. Then we consider the dimension 2 case, proving Proposition 2 and some related results, and finally we restrict to weighted projective spaces, proving Proposition 3.

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1. Preliminaries

In this section we begin by recalling the definition of the m-th fundamental form of a projective variety, and then the definition of the projective toric variety $X \subseteq \mathbb{P}^n$ associated to a set $S = \{p_0, \dots, p_n\}$ of lattice points. Finally we restrict to the case in which S is the set of all lattice points in a lattice polytope Δ.

1.1. **Fundamental forms.** We recall the following definition (see [20, Definition 1.1]). Let $X \subseteq \mathbb{P}^n$ be a projective variety of dimension k, and let H be a hyperplane section of X. Given a point $q \in X$ denote by $\pi : \tilde{X} \to X$ the blowing-up of X at q, and by E the exceptional divisor. The *m-th fundamental form* of X at q is the linear system of degree m homogeneous polynomials of $\mathbb{P}^{k-1}$ defined by the image of the restriction map

$$\rho_m : H^0(\tilde{X}, \mathcal{O}(\pi^* H - mE)) \to H^0(E, \mathcal{O}(\pi^* H - mE)). \tag{1.1}$$

Recall also that $X \subseteq \mathbb{P}^n$ is *linearly normal* if the complete linear system $|H|$ has dimension n. For example a smooth rational quartic curve of $\mathbb{P}^n$ is linearly normal only if $n = 4$.

Remark 1.1. As we wrote in the introduction, the m-th fundamental form of X at q is related to the two divisors $\pi^* H - mE$ and $\pi^* H - (m+1)E$. Indeed, from the definition it follows immediately that if the m-th form has a base point, then the divisor $\pi^* H - mE$ on $\tilde{X}$ has a base point too. Moreover, concerning $\pi^* H - (m+1)E$ we have the following.

Proposition 1.2. *Assume that $X \subseteq \mathbb{P}^n$ is linearly normal and that the i-th fundamental form of X at q is full dimensional, for $2 \leq i \leq m-1$. Then the following are equivalent:*

1. *the m-th fundamental form of X at q is not full dimensional, i.e. ρ_m is not surjective;*
2. *the linear system $|\pi^* H - (m+1)E|$ does not have the expected dimension $n - \binom{m+2}{2}$.*

Proof. Denote by h_i the dimension of the vector space $H^0(\tilde{X}, \mathcal{O}(\pi^*H - iE))$. By hypothesis we have $h_{i+1} = h_i - \binom{i+k-1}{k-1}$ for any $0 \le i \le m-1$, so that $h_m = h_0 - \sum_{i=0}^{m-1} \binom{i+k-1}{k-1} = n+1-\binom{m-1+k}{k}$, where the last equality is due to the linear normality of X and to an elementary property of binomial coefficients. It follows that

$$h_{m+1} = n+1-\binom{m-1+k}{k} - \dim(\operatorname{im}(\rho_m)) = n+1-\binom{m+k}{k} + \operatorname{codim}(\operatorname{im}(\rho_m)),$$

which proves the statement. □

We are interested in describing the fundamental forms of a variety X in parametric form in the following sense. Let U be an open subset of $\mathbb{C}^k$ and let

$$f\colon U \to \mathbb{P}^n \qquad (u_1, \dots, u_k) \mapsto [f_0 : \cdots : f_n] \tag{1.2}$$

be a morphism whose image has dimension k as well. Denote by X the Zariski closure of the image and take a smooth point $p \in U$ such that its image $q = f(p) \in X$ is smooth as well.

Remark 1.3. In this setting we have the following isomorphism

$$H^0(\tilde{X}, \mathcal{O}(\pi^*H - mE)) \to H^0(X, \mathcal{O}(H) \otimes \mathcal{I}(q)^m) \qquad \sigma \mapsto \pi_*(\sigma),$$

whose inverse is $x \mapsto \pi^*(x)\sigma_E^{-m}$, where $\sigma_E \in H^0(\tilde{X}, E)$ is a non-zero section. The pullback of the sheaf $\mathcal{O}_X(H) \otimes \mathcal{I}(q)^m$ via f is isomorphic to $\mathcal{O}_U \otimes \mathcal{I}(p)^m$. Since U is quasiaffine the latter sheaf is generated by its global sections. Thus we have the isomorphism

$$H^0(X, \mathcal{O}(H) \otimes \mathcal{I}(q)^m) \to \langle f_0, \dots, f_n \rangle \cap I(p)^m \qquad x \mapsto f^*(x),$$

where we denote by $\langle f_0, \dots, f_n \rangle$ the complex vector space generated by the f_i. As a consequence the domain of ρ_m is isomorphic to the subspace of $\langle f_0, \dots, f_n \rangle$ consisting of elements whose partial derivatives of order up to $m-1$ vanish at p.

Given a multi-index $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{Z}_{\ge 0}^k$ let us denote by $\partial^\alpha g|_p$ the partial derivative of the function g, defined by α and evaluated at $p \in U$. We recall the following definition (see also [9, 25]).

Definition 1.4. The *matrix of m-jets* of f at $p \in U$, denoted by $J_m(f)|_p$, is the vertical join of the matrices

$$D_r(f)|_p := (\partial^\alpha f_i|_p : |\alpha| = r \text{ and } 0 \le i \le n),$$

for $r = 0, \dots, m$.

1.2. **Toric varieties.** We recall here some basic facts about projective toric varieties (see for instance [8, 10, 28]), and we introduce the notion of pseudonef cones for the blowing up of a toric variety at a general point.

In what follows M will be a rank k free abelian group, $N := \operatorname{Hom}(M, \mathbb{Z})$ its dual, and $S = \{p_0, \dots, p_n\} \subseteq M$ a finite subset whose differences generate M. For any $p \in M$ denote by $\chi^p(u) \in \mathbb{C}[u_1^{\pm 1}, \dots, u_k^{\pm 1}]$ the corresponding Laurent monomial. The closure $X(S)$ of the image of the morphism

$$f\colon (\mathbb{C}^*)^k \to \mathbb{P}^n \qquad u \mapsto [\chi^{p_0}(u) : \cdots : \chi^{p_n}(u)], \tag{1.3}$$

is the *projective toric variety* defined by S, and we denote by $\mathbf{1} \in X(S)$ the image of the neutral element $1 \in (\mathbb{C}^*)^k$. The toric variety $X := X(S)$ defined in this way is in general non normal. In what follows we will denote by $\pi : \tilde{X} \to X$ the blowing up of X at the point $\mathbf{1}$ (which lies in the open torus orbit) and by E the exceptional divisor of π or, with abuse of notation, its class.

Notation 1.5. Given an element $v \in N$, in what follows we denote by $C_v \subseteq X$ the Zariski closure of the image of the map $t \mapsto f(t^v)$, and by $\tilde{C}_v \subseteq \tilde{X}$ its strict transform. We denote by $A_i(X)$ (resp. $A_i(X)_{\mathbb{Q}}$) the $i-th$ (resp. rational) Chow group of X. Let $NE(X) \subseteq A_1(X)_{\mathbb{Q}}$ be the Mori cone of X, let $e \in \mathrm{NE}(\tilde{X})$ the class of a line contained in the exceptional divisor E and let $\Gamma_N \subseteq \mathrm{NE}(\tilde{X})$ be the subset consisting of the classes of the curves $\tilde{C}_v \subseteq \tilde{X}$, where $v \in N$. Finally we denote by $\pi_*\colon A_i(\tilde{X}) \to A_i(X)$ the homomorphism induced by pushforward of cycles [11, §1.4].

Definition 1.6. The *pseudonef cone* of $\tilde{X}$ is

$$\mathrm{PNef}(\tilde{X}) \coloneqq \mathrm{cone}(\pi_*^{-1}(\mathrm{NE}(X)) \cap E^\perp, e, \Gamma_N)^*,$$

and a *pseudonef class* is a class in $\mathrm{PNef}(\tilde{X})$.

Proposition 1.7. *We have the inclusion* $\mathrm{Nef}(\tilde{X}) \subseteq \mathrm{PNef}(\tilde{X})$.

Proof. It is enough to prove that the cone $\pi_*^{-1}(\mathrm{NE}(X)) \cap E^\perp$ is contained in the Mori cone $\mathrm{NE}(\tilde{X})$. In order to do that, observe that given a nef class $\tilde{D} \in \mathrm{Nef}(\tilde{X})$, we can write $\tilde{D} = \pi^* D - mE$, where $D \in \mathrm{Nef}(X)$ and $m \geq 0$. Therefore, for any γ in $\pi_*^{-1}(\mathrm{NE}(X)) \cap E^\perp$ we have

$$\gamma \cdot \tilde{D} = \gamma \cdot \pi^* D = \pi_* \gamma \cdot D \geq 0,$$

where the first equality follows from the fact that γ lies in $E^\perp$, and the second from the projection formula (see [23, pag. 17]). We conclude that γ has non negative intersection product with any nef class of $\tilde{X}$, and the claim follows. □

We remark that in general the other inclusion does not hold, since there are cases in which a pseudonef divisor is not nef, nor even effective, as we are going to see in Example 1.14. But in Proposition 3.8 we are going to prove that in some interesting cases we do have the equality $\mathrm{PNef}(\tilde{X}) = \mathrm{Nef}(\tilde{X})$.

1.3. **Toric varieties associated to a lattice polytope.** Most of the paper will deal with the case in which $S = \Delta \cap M$ is the set of all the lattice points of a full-dimensional lattice polytope $\Delta \subseteq M \otimes_{\mathbb{Z}} \mathbb{Q}$, and hence we recall here some definitions and observations.

Given a lattice polytope $\Delta \subseteq M \otimes_{\mathbb{Z}} \mathbb{Q}$, let us consider the set $\Delta \cap M$ of its lattice points and let us denote simply by $X(\Delta)$ the toric variety $X(\Delta \cap M)$. The polytope Δ defines indeed a polarized pair (X, H) consisting of the projective toric variety $X \coloneqq X(\Delta)$, together with a very ample divisor H of X (and in particular X is linearly normal). Let us recall the following definitions (see for instance [3, 24]).

Definition 1.8. Given a *lattice direction*, i.e. a non zero primitive vector $v \in N$, let us denote respectively by $\min\langle \Delta, v\rangle$ and $\max\langle \Delta, v\rangle$ the minimum and the maximum of $\langle m, v\rangle$ for $m \in \Delta$. The *lattice width of* Δ *in the direction* v can be defined as

$$\mathrm{lw}_v(\Delta) \coloneqq \max\langle \Delta, v\rangle - \min\langle \Delta, v\rangle.$$

The *lattice width of* Δ is defined as

$$\mathrm{lw}(\Delta) \coloneqq \min\{\mathrm{lw}_v(\Delta) : v \in N\},$$

and if $v \in N$ is such that $\mathrm{lw}_v(\Delta) = \mathrm{lw}(\Delta)$, we say that v is a *width direction* for Δ. The polytope Δ is called a *Cayley polytope* if $\mathrm{lw}(\Delta) = 1$.

Remark 1.9. Given a lattice polytope Δ, if $v \in N$ is a width direction, then Δ is bounded by the two hyperplanes $L_{\min}(\Delta, v) := \{\langle x, v\rangle = \min\langle \Delta, v\rangle\}$ and $L_{\max}(\Delta, v) := \{\langle x, v\rangle = \max\langle \Delta, v\rangle\}$ respectively. Therefore, all the lattice points of Δ lie on the union of $\mathrm{lw}(\Delta) + 1$ hyperplanes orthogonal to v. In particular, Δ is a Cayley polytope iff all its lattice points lie on two parallel hyperplanes at lattice distance 1.

Remark 1.10. Let (X, H) be the toric polarized pair defined by the lattice polytope Δ. Then $\mathrm{lw}_v(\Delta)$ is the degree of the curve C_v (see Notation 1.5).

Remark 1.11. Let Δ and (X, H) be as before and let us consider an integer $m \geq 2$. Proposition 1.2 implies that m is the smallest degree of an affine hypersurface passing through all the lattice points of Δ if and only if $|\pi^*H - iE|$ has the expected dimension for $0 \leq i \leq m$, while $|\pi^*H - (m+1)E|$ does not have the expected dimension. This is essentially the content of [25, Proposition 1.1], since $|\pi^*H - (m+1)E|$ corresponds to hyperplane sections of X containing the $(m+1)$-th osculating space to X at $\mathbf{1}$.

Proposition 1.12. *Let Δ and (X, H) be as above. Then the following are equivalent.*

(1) *$\pi^*H - mE$ is pseudonef;*
(2) *$0 \leq m \leq \mathrm{lw}(\Delta)$.*

Proof. We prove $(1) \Rightarrow (2)$. First of all, since $E\cdot e = -1$, we have that $(\pi^*H - mE)\cdot e = m \geq 0$. Therefore we have $(\pi^*H - mE)\cdot \tilde{C}_v \geq 0$, for any $\tilde{C}_v$ in Γ_N (see Notation 1.5). Thus by Remark 1.10 we deduce that $\mathrm{lw}_v(\Delta) - m \geq 0$ for any $v \in N$ and, by taking v to be a width direction we obtain the second inequality.

We prove $(2) \Rightarrow (1)$. By hypothesis H is very ample, so that π^*H is nef and thus $\pi^*H - mE$ has non-negative intersection with any class in $E^\perp \cap \mathrm{NE}(\tilde{X})$. Moreover, by the same arguments given above, it follows that $\pi^*H - mE$ has non-negative intersection with e and all the curves $\tilde{C}_v$ in Γ_N. □

The proposition above gives a characterisation of divisors $\pi^*H - mE$, for $0 \leq m \leq \mathrm{lw}(\Delta)$. If m is bigger than $\mathrm{lw}(\Delta)$ we do not have such a characterisation, but we can prove that some implications hold. This is the content of the following:

Proposition 1.13. *Let (X, H) be the toric polarized pair defined by the lattice polytope Δ, and let $v \in N$ be a width direction for Δ. Given an integer $m \geq 2$ and the following statements:*

(1) *$m \geq \mathrm{lw}(\Delta) + 1$;*
(2) *the curve $\tilde{C}_v$ is contained in the base locus of $|\pi^*H - mE|$;*
(3) *the m-th fundamental form of X at $\mathbf{1}$ has a base point at $[v] \in \mathbb{P}(T_{X,\mathbf{1}})$;*
(4) *$h^1(\pi^*H - (m+1)E) > 0$;*

the implications (1)⇒(2)⇒(3)⇒(4) hold.

Proof. Let us suppose that (1) holds. If we denote as before by $\tilde{C}_v$ the strict transform of the rational curve C_v, we have that $(\pi^*H - mE)\cdot \tilde{C}_v = \mathrm{lw}(\Delta) - m \leq -1$, and (2) follows.

If (2) holds, then the tangent direction $[v]$ to the base curve at $\mathbf{1}$ is a base point for the m-th fundamental form at $\mathbf{1}$.

Finally, if (3) holds then the restriction map ρ_m in the following exact sequence

$$0 \longrightarrow H^0(\tilde{X}, \pi^*H - (m+1)E) \longrightarrow H^0(\tilde{X}, \pi^*H - mE) \xrightarrow{\rho_m} H^0(E, \pi^*H - mE_{|E})$$

is not surjective, which immediately implies (4). □

In the next section we will see (Remark 3.6) that there are some cases in which the conditions above are indeed equivalent, but in general only the given implications hold .

We conclude this section with an example of a pseudonef divisor which is not effective.

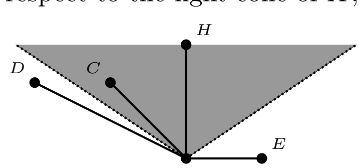
Example 1.14. Let Δ be the lattice triangle of vertices $(90,0),(0,117),(0,130)$ so that X is the weighted projective plane $\mathbb{P}(9,10,13)$, and $H = 1170A$, were A is the ample generator of the divisor class group of X. The one parameter subgroup of smallest degree of X is the one defined by the equation $x^4 - yz^2 = 0$ and it has degree 36. Thus the pseudonef cone is generated by the classe of the exceptional divisor E and the class of $D := \pi^*65A - 2E$, which has intersection product 0 with the strict transform of the above one parameter subgroup. To prove that D lies outside the effective cone it suffices to show that it has negative intersection product with some nef class C. Such a C is the strict transform of the irreducible curve defined by an element of a basis of the 4-th saturated power of the lattice ideal (3.1) (see [17, §5]). More explicitly

$$2x^{14}z - 3x^{11}y^4 - 5x^{10}yz^3 + 9x^7y^5z^2 + 3x^6y^2z^5 + x^4y^9z$$
$$-12x^3y^6z^4 + 4x^2y^3z^7 - xy^{13} - xz^{10} + 3y^{10}z^3 = 0.$$

To check that the curve has multiplicity 4 at the point $\mathbf{1} \in X$ observe that the invariant characters in Cox coordinates are generated by $u := \frac{x^4}{yz^2}$ and $v := \frac{z^3}{xy^3}$. Dividing the above polynomial by $y^{10}z^3$ we get a degree zero polynomial which can thus be written as a Laurent polynomial in the torus coordinates u, v. After performing the coordinate change and multiplying by v we get

$$2u^4v^3 - 5u^3v^3 - 3u^3v^2 + 3u^2v^3 + 9u^2v^2 - uv^4 + 4uv^3 - 12uv^2 + uv + 3v - 1 = 0.$$

Translating $(1,1)$ to the origin one sees that the multiplicity is 4. By the above discussion we have $C = \pi^*139A - 4E$, so that $C^2 = \frac{139^2}{1170} - 16 > 0$. Then C is nef, being the class of an irreducible curve with positive self-intersection, and we conclude observing that $C \cdot D = \frac{65\cdot 139}{1170} - 8 < 0$. In the following picture we show the position of D and C with respect to the light cone of $\tilde{X}$, which is the grey region.



2. Proof of Theorem 1

In this section we are going to prove Theorem 1, by following the idea of Perkinson [25], i.e. the study of suitable relations between left and right kernels of a slight modification of the m-th jet matrix.

Let us consider as in Subsection 1.2 any finite set $S = \{p_0, \ldots, p_n\} \subseteq M$ of lattice points, and the map $f\colon (\mathbb{C}^*)^k \to \mathbb{P}^n$ given by the Laurent monomials χ^p, for $p \in S$, whose image is the toric variety $X = X(S)$ (see (1.3)). Given an integer $m \geq 2$, for simplicity of notation we will set

$$J_m := J_m(f)|_1 \quad \text{and} \quad D_r := D_r(f)|_1,$$

for any $0 \leq r \leq m$ (see Definition 1.4). The columns of the matrices J_m and D_r are indexed by the points $p_0, \ldots, p_n$, while the rows of D_r (resp. J_m) are indexed by the partial derivatives ∂^α of order $|\alpha| = r$ (resp. $0 \leq |\alpha| \leq m$) in k variables. We fix the graded lexicographical order on these derivatives. Given $\alpha = (\alpha_1, \ldots, \alpha_k)$ we define the following polynomials of $\mathbb{C}[x_1, \ldots, x_k]$

$$P_\alpha := \prod_{i=1}^{k} x_i(x_i - 1)\cdots(x_i - \alpha_i + 1) \quad \text{and} \quad \mathrm{Lt}(P_\alpha) := \prod_{i=1}^{k} x_i^{\alpha_i},$$

where the product $x_i(x_i-1)\cdots(x_i-\alpha_i+1)$ is to be intended 1 if $\alpha_i = 0$. Observe that the $(i+1)$-th column of the matrix J_m is $P_\alpha(p_i)$, for $0 \leq |\alpha| \leq m$. If we denote by $\mathrm{Lt}(J_m)$ and $\mathrm{Lt}(D_m)$ the matrices of leading terms of J_m and D_m respectively, then by the definitions above the $(i+1)$-th column of $\mathrm{Lt}(J_m)$ consists of $\mathrm{Lt}(P_\alpha)(p_i)$, for $0 \leq |\alpha| \leq m$ (see also [5, 9, 25]).

Remark 2.1. The matrix $\mathrm{Lt}(J_m)$ is obtained from J_m by elementary row operations. In particular the two matrices have the same row span, the same rank, the same right kernel, and left kernels of the same dimension. An element $c = (c_p : p \in S)$ in the right kernel of J_m corresponds to the Laurent polynomial

$$R_c(u) := \sum_{p \in S} c_p \chi^p(u),$$

whose derivatives at $1 \in (\mathbb{C}^*)^k$ vanish up to order m. In other words, the zero locus of R_c is a hyperplane section of X which has multiplicity at least $m+1$ at $\mathbf{1} \in X$. On the other hand, since the $(i+1)$-th column of $\mathrm{Lt}(J_m)$ consists of all the monomials of degree up to m of $\mathbb{C}[x_1, \ldots, x_k]$ evaluated at p_i, an element $b = (b_\alpha : 0 \leq |\alpha| \leq m)$ in the left kernel of $\mathrm{Lt}(J_m)$ corresponds to the polynomial

$$L_b := \sum_{0 \leq |\alpha| \leq m} b_\alpha \, \mathrm{Lt}(P_\alpha)$$

of $\mathbb{C}[x_1, \ldots, x_k]$, of degree at most m, which vanishes at all the points of S.

Example 2.2. If we consider $S = \{(0,0), (1,0), (0,1), (3,1), (1,3), (6,3)\} \subseteq \mathbb{Z}^2$, and we fix $m = 2$, we have

$$J_2 = \left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 0 & 3 & 1 & 6 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ \hline 0 & 0 & 0 & 6 & 0 & 30 \\ 0 & 0 & 0 & 3 & 3 & 18 \\ 0 & 0 & 0 & 0 & 6 & 6 \end{array}\right) \begin{array}{l} \} D_0 \\ \left.\vphantom{\begin{array}{c}0\\0\end{array}}\right\} D_1 \\ \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\} D_2 \end{array} \qquad \mathrm{Lt}(J_2) = \left(\begin{array}{cccccc} 1 & 1 & 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 0 & 3 & 1 & 6 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ \hline 0 & 1 & 0 & 9 & 1 & 36 \\ 0 & 0 & 0 & 3 & 3 & 18 \\ 0 & 0 & 1 & 1 & 9 & 9 \end{array}\right) \begin{array}{l} \} \mathrm{Lt}(D_0) \\ \left.\vphantom{\begin{array}{c}0\\0\end{array}}\right\} \mathrm{Lt}(D_1) \\ \left.\vphantom{\begin{array}{c}0\\0\\0\end{array}}\right\} \mathrm{Lt}(D_2) \end{array}.$$

The right kernel of J_2 (which coincides with the right kernel of $\mathrm{Lt}(J_2)$) is generated by the vector $c = (10, -10, -5, 5, 1, -1)$, corresponding to the polynomial $R_c = 10 -$

$10u_1 - 5u_2 + 5u_1^3u_2 + u_1u_2^3 - u_1^6u_2^3$, vanishing at $(1,1)$ together with its derivatives up to order 2.

On the other hand, the left kernel of $\mathrm{Lt}(J_2)$ is generated by $b = (0,-1,-1,1,-2,1)$, corresponding to the degree 2 polynomial $L_b = -x_1 - x_2 + x_1^2 - 2x_1x_2 + x_2^2$, vanishing at all the points of S.

Proof of Theorem 1. Let $\mathfrak{m}_{\mathbf{1}}$ be the ideal sheaf of the point $\mathbf{1} \in X$, and let $T_{X,\mathbf{1}}$ and $T^*_{X,\mathbf{1}}$ be the tangent and cotangent spaces of X at $\mathbf{1}$ respectively. We fix $w_1, \ldots, w_k$ to be coordinates on $T_{X,\mathbf{1}}$, and we denote by rker the right kernel of a matrix. We then have a commutative diagram:

$$\begin{array}{ccc} H^0(\tilde{X}, \pi^*H - mE) & \xrightarrow{\quad \rho_m \quad} & H^0(E, \pi^*H - mE) \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{rker}(J_{m-1}) & \xrightarrow{g \mapsto \sum_{|\alpha|=m} w^\alpha \frac{m!}{\alpha_1! \cdots \alpha_k!} \partial^\alpha g|_1} & \mathrm{Sym}^m(T^*_{X,\mathbf{1}}) \end{array} \tag{2.1}$$

where the left vertical arrow is the isomorphism described in Remark 1.3, while the right vertical isomorphism follows from the usual identification $\mathcal{O}_E(-E) \simeq \mathfrak{m}_{\mathbf{1}}/\mathfrak{m}_{\mathbf{1}}^2$ and the fact that the latter quotient is isomorphic to $T^*_{X,\mathbf{1}}$. Let us fix the following notation:

$$\Phi_m(w) := \left(w^\alpha \frac{m!}{\alpha_1! \cdots \alpha_k!} \,:\, |\alpha| = m \right) \cdot D_m \in \mathbb{C}[w_1, \ldots, w_k]^{\binom{m+k}{k}},$$

so that $\Phi_m(w)$ is a vector of homogeneous polynomials of degree m in k variables, with $\binom{m+k}{k}$ entries. By (2.1), the m-th fundamental form of X at $\mathbf{1}$ corresponds to the linear system in $\mathbb{P}^{k-1}$ defined by the following degree m homogeneous polynomials of $\mathbb{C}[w_1, \ldots, w_k]$

$$\sum_{|\alpha|=m} w^\alpha \frac{m!}{\alpha_1! \cdots \alpha_k!} \partial^\alpha \Big(\sum_{p \in S} c_p \chi^p(u) \Big) = \Phi_m(w) \cdot c,$$

as c varies in $\mathrm{rker}(J_{m-1})$. Therefore the m-th fundamental form has a base point at $[v] \in \mathbb{P}^{k-1}$ if and only if $\Phi_m(v) \cdot c = 0$, for any $c \in \mathrm{rker}(J_{m-1})$ or equivalently if

$$\Phi_m(v) \in \mathrm{rker}(J_{m-1})^\perp = (\text{row span}(J_{m-1})^\perp)^\perp = \text{row span}(J_{m-1}). \tag{2.2}$$

If we define $\mathrm{Lt}(\Phi_m)(w)$ as the vector of polynomials obtained by replacing the matrix D_m with $\mathrm{Lt}(D_m)$ in the definition of $\Phi_m(w)$, we have that $\mathrm{Lt}(\Phi_m)(v) - \Phi_m(v)$ belongs to the row span of J_{m-1}. By (2.2) and Remark 2.1 we deduce that

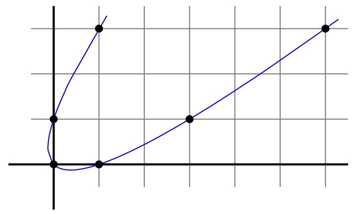
$$\mathrm{Lt}(\Phi_m)(v) \in \text{row span}(\mathrm{Lt}(J_{m-1})),$$

which gives a non trivial vector in the left kernel of $\mathrm{Lt}(J_m)$ (recall that $\mathrm{Lt}(J_m)$ is the vertical join of $\mathrm{Lt}(J_{m-1})$ and $\mathrm{Lt}(D_m)$). By the same Remark 2.1, this vector corresponds to a hypersurface of degree m containing the points $p \in S$. We conclude by observing that we can write $\mathrm{Lt}(\Phi_m)(v) = ((v \cdot p_i)^m \,:\, 0 \le i \le n)$, so that the defining polynomial for the hypersurface takes the form

$$(v \cdot x)^m + \text{lower degree terms},$$

which gives the statement. □

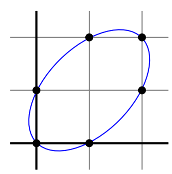
Example 2.3. Let us consider again the set $S \subseteq \mathbb{Z}^2$ of Example 2.2. The polynomial L_b we found before can be written as $(x_1 - x_2)^2 - x_1 - x_2 = (v \cdot x)^2 - x_1 - x_2$, where $v = (1,-1)$. This is the equation of a parabola passing through the points of S.



Indeed, the second fundamental form of the surface $X(S) \subseteq \mathbb{P}^5$ at $\mathbf{1}$ is the 1 dimensional linear system generated by $w_1(w_1+w_2)$ and $w_2(w_1+w_2)$, so that it has the base point $[v] = (1:-1) \in \mathbb{P}^1$.

We are now going to present another example of toric surface in order to stress the difference between Perkinson's result [25] and Theorem 1. This is the well known Togliatti surface (see also [29] and [25, Example 2.4]).

Example 2.4. If we now consider $S = \{(0,0),(1,0),(0,1),(2,1),(1,2),(2,2)\} \subseteq \mathbb{Z}^2$, and $m = 2$, we have that the left kernel of $\mathrm{Lt}(J_2)$ is generated by $(0,1,1,-1,1,-1)$, corresponding to the conic $V(x_1+x_2-x_1^2+x_1x_2-x_2^2)$ containing all the points of S.



In this case the second fundamental form is the 1 dimensional linear system generated by $w_1^2 - w_2^2$ and $2w_1w_2 + w_2^2$. Indeed, according to Perkinson's result, the second form is not full dimensional because the points of S lie on a conic, but since the conic is not a parabola, according to Theorem 1 the form has no base point.

3. Applications

From now on we are going to apply the above results to projective toric varieties $X := X(\Delta)$, associated to a full-dimensional lattice polytope $\Delta \subseteq M \otimes_{\mathbb{Z}} \mathbb{Q}$ (see Subsection 1.3).

In particular, we first prove a corollary of Theorem 1, which allows to construct some interesting non semiample divisors on $\tilde{X}$, then we restrict to the case of toric surfaces and finally we consider weighted projective spaces.

Remark 3.1. Let us fix a lattice polytope $\Delta \subseteq M \otimes \mathbb{Q}$ and let us consider $m := \mathrm{lw}(\Delta)+1$. Observe that if $v \in N$ is a width direction, then all the lattice points of Δ lie on m parallel hyperplanes of equation $v \cdot x - \alpha = 0$, with $\alpha \in \{\min\langle\Delta, v\rangle, \dots, \max\langle\Delta, v\rangle\}$. The product of these hyperplanes gives a hypersurface whose homogeneous part of higher degree is $(v \cdot x)^m$, so that, by Theorem 1, $[v]$ is a base point for the m-th fundamental form of $X(\Delta)$ at $\mathbf{1}$ (if the form is not empty), and hence it is also a base point for $\pi^*H - mE$. Moreover, a similar argument shows that any positive multiple of $\pi^*H - mE$ has the same point $[v]$ in its base locus, so that the divisor $\pi^*H - mE$ is not semiample.

We remark that the above conclusion easily follows either from Proposition 1.13 or Proposition 1.12, since $\pi^*H - mE$ has negative intersection with the curve $\tilde{C}_v$, so that it is not pseudonef (and hence it is not nef nor semiample) and in particular the tangent direction $[v]$ lies in the stable base locus of $\pi^*H - mE$.

On the other hand, if we take $m := \mathrm{lw}(\Delta)$, by Proposition 1.12 we have that $\pi^*H - mE$ is pseudonef, but we do not know whether it is nef (or semiample). Our next goal is to give a sufficient condition on Δ, based on Theorem 1, which guarantees that $\pi^*H - mE$ is not semiample.

Corollary 3.2. *Let $v \in N$ be a width direction for Δ, and suppose that the following conditions hold:*

(1) *the hyperplanes $L_{\min}\langle\Delta, v\rangle$ and $L_{\max}\langle\Delta, v\rangle$ intersect Δ exactly in two vertices, $p_{\min}$ and $p_{\max}$ respectively;*
(2) *the linear span Λ of the lattice points on $\{\langle x, v\rangle = \min\langle\Delta, v\rangle + 1\} \cap \Delta$ has codimension at least 2;*
(3) *the line through the two vertices $p_{\min}$ and $p_{\max}$ does not intersect Λ.*

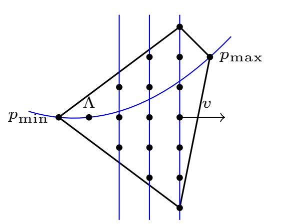
Then $\pi^*H - \mathrm{lw}(\Delta)E$ is not semiample.

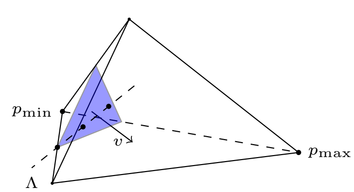
Proof. By translating Δ we can suppose that Λ passes through the origin and (by taking $-v$ instead of v if necessary) that $\min\langle\Delta, v\rangle = -1$ and $\max\langle\Delta, v\rangle = m - 1$, where we set $m := \mathrm{lw}(\Delta)$ for simplicity of notation.

By (2) and (3) there exist two non-associated homogeneous polynomials f, g of degree one, such that $\{p_{\min}\} \cup \Lambda \subseteq V(f)$ and $\{p_{\max}\} \cup \Lambda \subseteq V(g)$. Moreover, by (3) and the fact that v is constant along Λ, we can choose f and g in such a way that the hyperplane orthogonal to v and containing Λ has equation $f + g = 0$. Therefore the lattice points of $\Delta \cap \Lambda$ together with $p_{\min}$ and $p_{\max}$ lie on the affine quadric of equation

$$(f + g)^2 - f(p_{\max})f - g(p_{\min})g = 0.$$

We conclude that all the lattice points of Δ lie on the union of this quadric and the $m - 2$ hyperplanes defined by $f + g - \alpha$, for $\alpha = 1, \ldots, m - 2$. This union is a degree m affine hypersurface whose homogeneous part of maximal degree is the power $(f + g)^m = (v \cdot x)^m$. We illustrate the above results with the following two pictures.

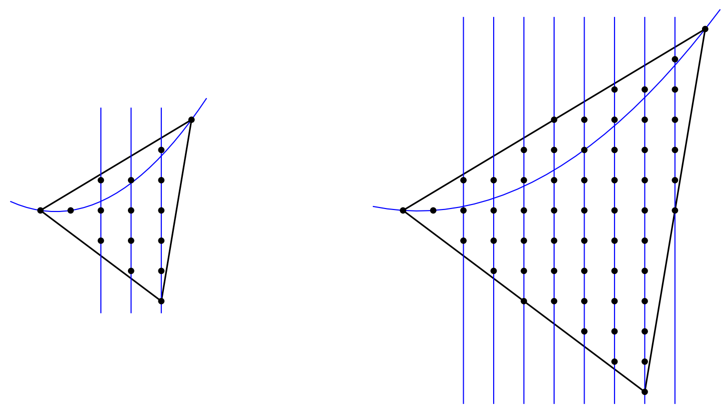




By Theorem 1, the m-th fundamental form of $X(\Delta)$ at $\mathbf{1}$ has a base point corresponding to $[v] \in \mathbb{P}^{k-1}$, and by Remark 1.1 we deduce that $\pi^*H - mE$ has a base point too.

We now claim that the hypotheses are indeed satisfied by any positive multiple $r\Delta$. This is immediately clear for (1). Concerning (2), observe that the cone $\mathbb{R}_{>0} \cdot (\Delta - p_{\min})$ at $p_{\min}$ coincides with the cone $\mathbb{R}_{>0} \cdot (r\Delta - rp_{\min})$ at $rp_{\min}$. In particular the dimension of the linear span of lattice points in $\{\langle x, v\rangle = \min\langle r\Delta, v\rangle + 1\} \cap r\Delta$

is equal to the dimension of Λ. We illustrate this in the following picture (the corresponding toric variety is the fake projective plane obtained by quotienting $\mathbb{P}^2$ by the action of $\mathbb{Z}/27\mathbb{Z}$ defined by $\varepsilon \cdot [x_0 : x_1 : x_2] = [x_0 : \varepsilon^{20} x_1 : \varepsilon x_2]$).



Δ, $m = 5$ $\quad$ 2Δ, $2m = 10$

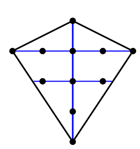
Finally, by the hyperplane separation theorem, the line through $p_{\min}$ and $p_{\max}$ is separated from Λ by a hyperplane and this property is preserved by dilations, which proves (3). We conclude that any positive multiple of $\pi^* H - mE$ has the same base point, and the statement follows. $\square$

3.1. **Toric surfaces.** Let us focus now on the case $k = 2$, so that $\Delta \subseteq M \otimes \mathbb{Q}$ is a lattice polygon and $X(\Delta)$ is a projective toric surface. We first present some nice consequences of the previous results and of Proposition 2, and then we give a proof for the latter.

Remark 3.3. Let us consider a lattice polygon Δ such that $|\Delta \cap M| \geq 6$, and the corresponding toric pair (X, H). If the linear system $|\pi^* H - 3E|$ is special, by Proposition 2 we have that $\mathrm{lw}(\Delta) \leq 2$, and in particular Proposition 1.13 implies that the base locus of $|\pi^* H - 3E|$ contains a curve intersecting E.

We are now going to give a counterexample showing that the above result is no longer true for systems of the form $|\pi^* H - mE|$, when $m \geq 4$.

Example 3.4. Let $k \geq 2$ be an integer and let us consider the polygon $\Delta \subseteq M \otimes_{\mathbb{Z}} \mathbb{Q}$ with vertices $(0, -k-1), (k, 0), (-k, 0), (0, k-1)$. The lattice width of Δ is $2k$, and there exist curves of degree $2k - 1$, but not smaller, passing through all its lattice points (the following is the picture for $k = 2$).



Therefore, if we consider the corresponding toric pair (X, H), by Remark 1.11 we have that $h^1(\pi^* H - (2k)E) > 0$. We now claim that the base locus of $|\pi^* H - (2k)E|_E$ (equivalently, the base locus of the $2k$-th fundamental form of X at $\mathbf{1}$) is empty. Indeed, if it were not the case, by Theorem 1 there would exist a curve of degree

$2k$ passing through the lattice points of $\Delta \cap M$ and having an inflection point of order $2k$ at infinity. But since there are $2k+1$ lattice points of Δ on each of the two axes, any curve of degree $2k$ passing through $\Delta \cap M$ must contain the factor xy, so that it can not have an inflection of maximal order at infinity, which proves the claim. Observe that on one hand this implies that the system $|\pi^*H - (2k)E|$ is not empty (and hence it is special), and on the other hand that the base locus of $|\pi^*H - (2k)E|$ does not contain any curve intersecting E (when k is small, it is possible to prove, by a heavier calculation, that the above base locus is indeed empty).

We also remark that even if the toric surfaces corresponding to the above polytopes are singular, it is possible to resolve their singularities in order to obtain smooth examples.

Remark 3.5. There exist other examples of toric surfaces such that $|\pi^*H - mE|$ is special but it contains no curve in its base locus, such as the Togliatti surface (see Example 2.4). But the toric surfaces appearing in Example 3.4 have the additional property of being linearly normal, since they correspond to all the lattice points of a lattice polygon (see for instance [8, Chapter 2]).

Remark 3.6. Another direct consequence of Proposition 2 is that, when $k = m = 2$ the first three conditions of Proposition 1.13 are equivalent. Moreover, if X is smooth, they are also equivalent to the fourth one, $h^1(\pi^*H - 3E) > 0$. Indeed, the homomorphism ρ_1 defined in (1.1) is surjective since the hyperplanes of $\mathbb{P}^n$ through $\mathbf{1}$ do not have a fixed direction. Hence $h^1(\pi^*H - 2E) = 0$, so that the hypothesis $h^1(\pi^*H - 3E) > 0$ implies that the map ρ_2 is not surjective. In particular the second fundamental form is not full dimensional. By Proposition 2 and the fact that Δ is smooth, the latter must be a Cayley polytope.

If $m = 2$ and $k \geq 3$, condition (3) of Proposition 1.13 implies (by Theorem 1) that the lattice points of Δ lie on an affine paraboloid $V(l_1^2 - l_2)$, with l_1 and l_2 linear forms. In particular Δ can not have any lattice point in its interior by the convexity of the paraboloid, i.e. Δ is a so called *hollow polytope*. For $k = 3$ there exists a complete classification of hollow polytopes (see [2, 26]), and looking at the list we can again conclude that, under our hypotheses, it must be $\mathrm{lw}(\Delta) = 1$, so that (3) $\Rightarrow$ (1) (and hence the first three conditions are equivalent). For $k > 3$, we believe that the implication still holds true, but since in this case there is no complete classification of hollow polytopes, so far we were not able to prove the result.

Finally, for bigger values of m, the implication (3) $\Rightarrow$ (1) of Proposition 1.13 is no longer true in general (even in dimension $k = 2$). For instance, given any polygon Δ satisfying the hypotheses of Corollary 3.2, we have seen that the m-th fundamental form of $X(\Delta)$ has a base point, but $\mathrm{lw}(\Delta) = m$.

Proof of Proposition 2. The equivalence of (1) and (2) follows from Proposition 1.2. Let us prove (2) $\Rightarrow$ (3). By Remark 1.11, the lattice points of Δ lie on a conic, which by Lemma 3.7 below is the union of two lines, say L_1 and L_2. If they are parallel, they must be at lattice distance 1, so that Δ is a Cayley polygon. Let us consider then the case in which L_1 and L_2 meet in a point q. We set $r_i := |\Delta \cap L_i \cap M|$, for $i = 1, 2$, and we suppose that $r_1 \geq r_2$ (so that in particular $r_1 \geq 3$). We also assume that L_1 coincides with the x-axis and we distinguish the following two cases.

a) $r_1 \geq 4$. We claim that in this case any point $p = (x_p, y_p) \in \Delta \cap (L_2 \smallsetminus L_1)$ satisfies $|y_p| \leq 1$. Let us suppose on the contrary that $|y_p| \geq 2$ and let us consider the intersection of the line $y = \pm 1$ (depending on the sign of y_p) with the triangle generated by p and $\Delta \cap L_1$. If $|y_p| \geq 3$ or $r_1 \geq 5$, the length of this segment is at least 2, while if $|y_p| = 2$ and $r_1 = 4$, the length is $3/2$, but one of its endpoints is a lattice point, so that in both cases we have at least two lattice points on this segment. This is a contradiction and proves the claim. In particular there are at most 2 lattice points on $\Delta \cap (L_2 \smallsetminus L_1)$. If there is only one point, Δ turns out to be a Cayley triangle. If there are 2 points, Δ is equivalent to a polygon of type (i) or (ii), depending on whether q is a lattice point or not.

b) $r_1 = 3$. In this case we must have $r_2 = 3$ and the intersection point q is not a lattice point. We are going to show that it not possible. First of all observe that on one of the two half-lines determined by q on L_1 (resp. L_2) there are at least 2 lattice points, say $(a, 0)$ and $(a+1, 0)$ (resp. p_1 and p_2). Moreover the triangles with vertices $(a, 0)$, $(a+1, 0)$ and p_i for $i = 1, 2$, contain no lattice point but the vertices, which implies that they both have area $1/2$. Therefore p_1 and p_2 lie on one of the two lines $y = \pm 1$, contradicting the fact that L_1 and L_2 intersect. This concludes the proof of $(2) \Rightarrow (3)$.

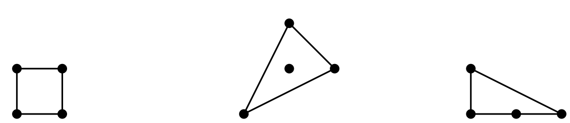
In order to prove $(3) \Rightarrow (1)$ observe that if Δ is either Cayley or of type (i) or (ii), its width satisfies the inequality $\mathrm{lw}(\Delta) \leq 2$ and hence we conclude by means of Proposition 1.13.

Finally, concerning the last assertion, by Theorem 1 the second fundamental form at $\mathbf{1}$ has a base point if and only if the lattice points of $\Delta \cap M$ lie on a parabola. Since in this case the parabola is degenerate, it must be the union of two parallel lines at lattice distance 1, which is equivalent to say that Δ is Cayley. $\square$

We conclude this subsection with the the following lemma that we used in the proof of Proposition 2 (we believe that its content is well known, but we give a proof anyway since we could not find any explicit reference to this result).

Lemma 3.7. *Let $\Delta \subseteq M \otimes_{\mathbb{Z}} \mathbb{Q}$ be a lattice polygon such that $|\Delta \cap M| \geq 5$. Then there are at least 3 lattice points of Δ lying on a line.*

Proof. We can reduce to the case $|\Delta \cap M| = 5$. In this case, removing one of the vertices and taking the convex hull of the remaining 4 lattice points we obtain a lattice polygon Δ' with 4 lattice points and we claim that it is equivalent to one of the following.



(a) (b) (c)

Indeed, if Δ' is a quadrilateral then by Pick's Theorem its area is 1, so that it is equivalent to the unitary square. If Δ' is a triangle with one internal lattice point, joining this point with any pair of vertices we obtain a triangle of area $1/2$. Then Δ' is equivalent to the triangle (b). Finally, if Δ' is a triangle without internal lattice points, its area is 1. Acting with $\mathrm{GL}(2, \mathbb{Z})$ we can suppose that the edge having one lattice point in its relative interior lies on the x-axis, which means that the height of the triangle is 1, so that it is equivalent to (c). This proves the claim.

We conclude by observing that if we add one lattice point to any of the polygons above and we take the convex hull, we obtain at least three lattice points on a line. □

3.2. **Weighted projective spaces.** In this last subsection we are going to prove Proposition 3. From now on we restrict our study to weighted projective spaces $\mathbb{P}(a_1, \ldots, a_{k+1})$, where $w = [a_1, \ldots, a_{k+1}] \in (\mathbb{Z}_{>0})^{k+1}$ is the vector of weights. Let us denote by A a generator for the divisor class group of X (which is torsion-free of rank one). In what follows we recall how to construct a (non unique) lattice simplex $\Delta \subseteq M \otimes_{\mathbb{Z}} \mathbb{Q}$, such that the toric pair (X, H) associated to Δ consists of $X = \mathbb{P}(a_1, \ldots, a_{k+1})$ and $H := dA$, where $d := \mathrm{lcm}(w)$. Let us consider the following exact sequence

$$0 \longrightarrow M \xrightarrow{P^*} \tilde{M} \xrightarrow{\cdot w} \mathbb{Z} \longrightarrow 0,$$

where P^* is the kernel matrix of the product by w, so that M is included in $\tilde{M} \simeq \mathbb{Z}^{k+1}$ as the orthogonal to w. We can define the simplex

$$\tilde{\Delta} := \{u \in \tilde{M} \otimes_{\mathbb{Z}} \mathbb{Q} : u \cdot w = \mathrm{lcm}(w) \,\text{and}\, u_i \geq 0\},$$

i.e. the intersection of the non negative ortant of $\tilde{M}$ with the affine hyperplane defined by $u \cdot w = d$. Observe that if we denote by $e_1, \ldots, e_{n+1}$ the canonical basis of $\tilde{M}$, then the vertices of $\tilde{\Delta}$ are $\delta_i e_i$, where $\delta_i := d/a_i$. If we now translate $\tilde{\Delta}$ moving one of its vertices (for instance the first one) to the origin and we pull it back to M via P^*, we obtain the simplex

$$\Delta := (P^*)^{-1}(\mathrm{chull}(\delta_i e_i - \delta_1 e_1 : i \in \{1, \ldots, n+1\})).$$

We can now prove the following sufficient condition for the nefness of the divisor $\pi^* dA - \mathrm{lw}(\Delta) E$ on the blowing up of $\mathbb{P}(w)$ at the point $\mathbf{1}$.

Proposition 3.8. *Let $v \in N$ be a width direction such that the corresponding one parameter subgroup is the intersection of hypersurfaces of degree smaller than $d/\mathrm{lw}(\Delta)$. Then $\pi^* dA - \mathrm{lw}(\Delta) E$ is nef or equivalently* $\mathrm{Nef}(\tilde{X}) = \mathrm{PNef}(\tilde{X})$.

Proof. For simplicity of notation let us set $\tilde{D} = \pi^* dA - \mathrm{lw}(\Delta) E$, and let us consider the curve $C_v \subseteq X$ (see Notation 1.5). By Remark 1.10 we have that $\tilde{C}_v \cdot \tilde{D} = 0$, where $\tilde{C}_v$ is the strict transform of C_v. Moreover, since the divisor class group of $\tilde{X}$ has rank two, the nef cone and its dual, the Mori cone, are two dimensional, so that in order to prove that $\tilde{D}$ is nef it is enough to show that $\tilde{C}_v$ generates an extremal ray of the Mori cone.

Let n be a positive integer such that $n\tilde{C}_v \equiv \tilde{C}_1 + \tilde{C}_2$, with $\tilde{C}_1$ and $\tilde{C}_2$ effective curves of $\tilde{X}$. Let $C_v = D_1 \cap \cdots \cap D_r$ and let $\tilde{D}_j \equiv \alpha_j \pi^* A - E$ be the strict transform of D_j. By hypothesis $\alpha_j < d/\mathrm{lw}(\Delta)$ for any $1 \leq j \leq r$. We claim that $\tilde{C}_i \cdot \tilde{D} \geq 0$, for $i = 1$ and 2. Let us suppose by contradiction that $\tilde{C}_1 \cdot \tilde{D} < 0$. Then at least one irreducible component $\tilde{\Gamma}$ of $\tilde{C}_1$ would intersect negatively $\tilde{D}$ and thus

$$\tilde{\Gamma} \cdot \tilde{D}_j = \tilde{\Gamma} \cdot (\alpha_j \pi^* A - E) < \tilde{\Gamma} \cdot (d/\mathrm{lw}(\Delta) \pi^* A - E) = \frac{1}{\mathrm{lw}(\Delta)} \tilde{\Gamma} \cdot \tilde{D} < 0$$

for any $j = 1, \ldots, r$. In particular $\tilde{\Gamma}$ would be contained in the intersection of all the $\tilde{D}_j$, so that $\tilde{\Gamma} = \tilde{C}_v$, because $\tilde{C}_v$ is irreducible. But this contradicts the equality

$\tilde{C}_v \cdot \tilde{D} = 0$ and proves the claim. Using the equalities

$$0 = n\tilde{C}_v \cdot \tilde{D} = (\tilde{C}_1 + \tilde{C}_2) \cdot \tilde{D}$$

and the claim, we conclude that $\tilde{C}_i \cdot \tilde{D} = 0$ for each $i = 1, 2$, which implies that the classes of $\tilde{C}_1$ and $\tilde{C}_2$ are proportional to that of $\tilde{C}_v$. Therefore $\tilde{D} = \pi^* dA - \mathrm{lw}(\Delta)E$ is nef. Moreover, since the Picard group of $\tilde{X}$ has rank 2, by Proposition 1.12 we conclude that $\mathrm{Nef}(\tilde{X}) = \mathrm{PNef}(\tilde{X})$. □

Proof of Proposition 3. The proof is a case by case analysis performed with the Computer Algebra package Magma [4] and we explain it in the next lines. Given any $w = [a_1, \ldots, a_4]$ appearing in the list, we construct Δ as explained at the beginning of the section. Let $m := \mathrm{lw}(\Delta)$ be the lattice width of Δ and let $v \in N$ be a width direction for Δ. In each case we verify that the hypotheses of Corollary 3.2 are satisfied for the pair (Δ, v), so that $\pi^* dA - mE$ is not semiample.

In order to conclude the proof it is enough to prove that $\pi^* dA - mE$ is nef by means of Proposition 3.8. Consider the kernel I of the map

$$\mathbb{C}[x_1, \ldots, x_4] \to \mathbb{C}[t], \quad x_i \mapsto t^{a_i}, \text{ for } i = 1 \ldots, 4, \tag{3.1}$$

which is an ideal generated by binomials and it is called *lattice ideal*, according to [17, §5]. Let $(f_1, \ldots, f_r)$ be a minimal basis for I, ordered by increasing w-degree, and let d_i be the w-degree of f_j, for any $j = 1, \ldots, r$. We verified that in all but one case the following holds

$$\deg V(f_1, f_2) = m,$$

where the degree is calculated with respect to the very ample class dA, i.e. it is given by $d_1 A \cdot d_2 A \cdot dA = d_1 d_2 d/(a_1 a_2 a_3 a_4)$. This immediately implies that the complete intersection $V(f_1, f_2)$ is irreducible, since otherwise one of its prime components would be a one-parameter subgroup of degree smaller than the width m of Δ, a contradiction. In all these cases one verifies that $d_i < d/\mathrm{lw}(\Delta) = \mathrm{lcm}(w)/m$, for $i = 1, 2$, so that $\pi^* dA - mE$ is nef by Proposition 3.8. Finally in the remaining case, namely $[23, 27, 29, 30]$, the curve $V(f_1, f_2, f_3)$ is irreducible, where $f_1 = x_1 x_3^2 - x_2^3$, $f_2 = x_1 x_4^2 - x_2^2 x_3$ and $f_3 = x_2 x_4^2 - x_3^3$, so that $d_1 = 81$, $d_2 = 83$ and $d_3 = 87$. Also in this case we have that $d_i < d/\mathrm{lw}(\Delta) = \mathrm{lcm}(w)/m = 690/7$, for any $i = 1, 2, 3$, and we conclude again by means of Proposition 3.8. □

Remark 3.9. Even if in all the cases studied in the preceding proof the width of Δ is realised by the binomials of lowest degree of the lattice ideal I, in general this is not true. For example let us consider the vector of weights $w = [4, 5, 6, 7]$. In this case $d := \mathrm{lcm}(w) = 420$ and the inclusion of M in $\tilde{M}$ is given by

$$P^* = \begin{pmatrix} 1 & 0 & 4 & -4 \\ 0 & 1 & 5 & -5 \\ 0 & 0 & 7 & -6 \end{pmatrix}.$$

Therefore the vertices of Δ are $(0,0,0)$, $(-105, 84, 0)$, $(-105, 0, 70)$ and $(-105, 0, 60)$, and $\mathrm{lw}(\Delta) = \mathrm{lw}_v(\Delta) = 42$, where $v = (2, 2, 3)$. A minimal basis for the lattice ideal is $(f_1, \ldots, f_6) = (x_1 x_3 - x_2^2,\ x_1 x_4 - x_2 x_3,\ x_2 x_4 - x_3^2,\ x_1^3 - x_3^2,\ x_1^2 x_2 - x_3 x_4,\ x_1 x_2^2 - x_4^2)$. The dual of the inclusion $M \to \tilde{M}$ is the map $\tilde{N} \to N$ given by the transpose of P^*. A preimage of v in $\tilde{N}$ is the vector $\tilde{v} = (2, 2, 3, 3)$, defining a one parameter subgroup whose closure is a prime component of $V(f_2, f_4)$, but not of $V(f_1, f_2)$.

Remark 3.10. When $k = 2$, the second condition of Corollary 3.2 states that there is only one lattice point on the line at lattice distance 1 from one of the two vertices p_{min} and p_{max} of the triangle. Therefore Corollary 3.2 turns out to be equivalent to [13, Theorem 1.5], in case $n = 1$ (with the notation of [13]), so that we could not find any new example in the class of weighted projective planes. Anyway, our technique is different from the one used by the authors of the cited paper. Their generalisation to dimension 3 (see [14, Theorem 2.11]) gives rise to the list appearing in [14, Table 1], where there is only one example with $a_i \leq 30$ for any i, namely $\mathbb{P}(17, 18, 20, 27)$.

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Departamento de Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile
E-mail address: alaface@udec.cl

Dipartimento di Matematica e Informatica, Università degli studi di Palermo, Via Archirafi 34, 90123 Palermo, Italy
E-mail address: luca.ugaglia@unipa.it