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Lower Semi-frames, Frames, and Metric Operators

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Abstract. This paper deals with the possibility of transforming a weakly measurable function in a Hilbert space into a continuous frame by a metric operator, i.e., a strictly positive self-adjoint operator. A necessary condition is that the domain of the analysis operator associated with the function be dense. The study is done also with the help of the generalized frame operator associated with a weakly measurable function, which has better properties than the usual frame operator. A special attention is given to lower semi-frames: indeed, if the domain of the analysis operator is dense, then a lower semi-frame can be transformed into a Parseval frame with a (special) metric operator.

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1. Introduction

In recent papers, one of us (RC) [21,22] has analyzed sesquilinear forms defined by sequences in Hilbert spaces and operators associated with them by means of representation theorems. In particular, he derived results about lower semi-frames and duality.

It turns out that most results from [21,22] can be extended to the continuous case and that is one of the aims of this present paper. The results are reported in Sect. 3, but we give here a brief summary. The continuous case involves a locally compact space (X, μ) with a Radon measure μ. A function $\phi : X \to \mathcal{H}, x \mapsto \phi_x$ is said to be *weakly measurable* if for every $f \in \mathcal{H}$ the function $x \mapsto \langle f|\phi_x\rangle$ is measurable. A weakly measurable function ϕ is said to be *μ-total* if $\langle f|\phi_x\rangle = 0$ for a.e. $x \in X$ implies that $f = 0$. A weakly measurable function ϕ is a *continuous frame* of $\mathcal{H}$ if there exist constants $0 < \mathsf{m} \leq \mathsf{M} < \infty$ (the frame bounds), such that:

$$\mathsf{m}\,\|f\|^2 \leq \int_X |\langle f|\phi_x\rangle|^2 \,\mathrm{d}\mu(x) \leq \mathsf{M}\,\|f\|^2, \quad \forall f \in \mathcal{H}. \tag{1.1}$$

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If a weakly measurable function ϕ satisfies:

$$\int_X |\langle f|\phi_x\rangle|^2 \,\mathrm{d}\mu(x) \leq \mathsf{M}\,\|f\|^2, \quad \forall\, f \in \mathcal{H},$$

then we say that ϕ is a *Bessel mapping* of $\mathcal{H}$. On the other hand, ϕ is a *lower semi-frame* of $\mathcal{H}$ if there exists a constant $\mathsf{m} > 0$, such that:

$$\mathsf{m}\,\|f\|^2 \leq \int_X |\langle f|\phi_x\rangle|^2 \,\mathrm{d}\mu(x), \quad \forall\, f \in \mathcal{H}. \tag{1.2}$$

Given a weakly measurable function ϕ, the operator $C_\phi : D(C_\phi) \subseteq \mathcal{H} \to L^2(X, d\mu)$ with domain

$$D(C_\phi) := \left\{ f \in \mathcal{H} : \int_X |\langle f|\phi_x\rangle|^2 \,\mathrm{d}\mu(x) < \infty \right\}$$

and $(C_\phi f)(x) = \langle f|\phi_x\rangle$, $f \in D(C_\phi)$, is called the *analysis* operator of ϕ. We can define the sesquilinear form:

$$\Omega_\phi(f, g) = \int_X \langle f|\phi_x\rangle\langle\phi_x|g\rangle d\mu(x), \quad f, g \in D(C_\phi)$$

and associate a positive self-adjoint operator T_ϕ in the space $\mathcal{H}_\phi$, the closure of $D(C_\phi)$ in $\mathcal{H}$, by Kato's representation theorem [26], which we call *generalized frame operator*. When ϕ is a lower semi-frame of $\mathcal{H}$, then the range of T_ϕ is $\mathcal{H}_\phi$ and the function $\psi : X \to \mathcal{H}$, defined by $\psi_x = \mathsf{T}_\phi^{-1} P_\phi \phi_x, x \in X$, where P_ϕ is the orthogonal projection onto $\mathcal{H}_\phi$, is a Bessel mapping and the reconstruction formula:

$$f = \int_X \langle f|\phi_x\rangle \psi_x \,\mathrm{d}\mu(x),$$

holds for every $f \in D(C_\phi)$ in a weak sense.

In this paper, we do not confine ourselves to extend results of [21,22], but actually we set two more goals. From one hand, given a lower semi-frame $\phi : X \to \mathcal{H}$ with $D(C_\phi)$ dense in $\mathcal{H}$, we consider general powers $\mathsf{T}_\phi^{-k}\phi$ with $k \geq 0$. These functions are Bessel mappings, frames or lower semi-frames in the space $\mathcal{H}(\mathsf{T}_\phi^m)$ (given by the domain of T_ϕ^m and the inner product $\langle \mathsf{T}_\phi^m \cdot | \mathsf{T}_\phi^m \cdot \rangle$) with $m \geq 0$ according to a simple relation between k and m (see Theorem 4.1).

When $\phi : X \to \mathcal{H}$ is a μ-total weakly measurable function with $D(C_\phi)$ dense, then T_ϕ is in particular a metric operator, i.e., a strictly positive self-adjoint operator. Metric operators are a topic familiar in the theory of the so-called $\mathcal{PT}$-symmetric quantum mechanics [19,27]. In our previous works [10,11,14], we have analyzed thoroughly the structure generated by such a metric operator, bounded or unbounded, namely a lattice of Hilbert spaces.

As particular case of Theorem 4.1, if $\phi : X \to \mathcal{H}$ is a lower semi-frame with $D(C_\phi)$ dense, then $\mathsf{T}_\phi^{-1/2}\phi$ is a Parseval frame of $\mathcal{H}$. This inspired us to consider the following more general problem.

Question for which weakly measurable functions $\phi : X \to \mathcal{H}$, there exists a metric operator G on $\mathcal{H}$, such that $\phi_x \in D(G)$ for all $x \in X$ and $G\phi$ is a frame?

Partial answers to this problem are given in Theorem 6.1. In particular, necessary conditions are that $D(C_\phi)$ is dense, and that ϕ is μ-total if ϕ is in addition a Bessel mapping. In the discrete case, if $\phi : \mathbb{N} \to \mathcal{H}$ is a Schauder basis, then the problem has a positive solution (more precisely, one can again take $G = T_\phi^{-1/2}$ and $T_\phi^{-1/2}\phi$ is actually an orthonormal basis). A particular case of this question has been treated, in the discrete case and with a different approach, in [25].

The paper is organized as follows. After reviewing the conventional definitions about frames and semi-frames in Sect. 2, we introduce in Sect. 3 the generalized frame operator T_ϕ, whose properties are more convenient that those of the standard frame operator S_ϕ. In Sect. 4, we investigate the various (semi)-frames generated by a lower semi-frame. In Sect. 5, we review the lattice of Hilbert spaces generated by a metric operator. In Sect. 6, we face the question of transforming functions in frames. We conclude in Sect. 7 by several examples.

2. Preliminaries

Before proceeding, we list further definitions and conventions. The framework is a (separable) Hilbert space $\mathcal{H}$, with the inner product $\langle\cdot|\cdot\rangle$ linear in the first factor. $GL(\mathcal{H})$ denotes the set of all invertible bounded operators on $\mathcal{H}$ with bounded inverse. Throughout the paper, we will consider weakly measurable functions $\phi : X \to \mathcal{H}$, where (X, μ) is a locally compact space with a Radon measure μ.

Given a continuous frame ϕ, the analysis operator is defined and bounded on $\mathcal{H}$, i.e., $C_\phi : \mathcal{H} \to L^2(X, \mathrm{d}\mu)$[1] and the corresponding *synthesis operator* $C_\phi^* : L^2(X, \mathrm{d}\mu) \to \mathcal{H}$ is defined as (the integral being understood in the weak sense, as usual):

$$C_\phi^* \xi = \int_X \xi(x)\, \phi_x \;\mathrm{d}\mu(x), \quad \text{for} \quad \xi \in L^2(X, \mathrm{d}\mu). \tag{2.1}$$

Moreover, we set $S_\phi := C_\phi^* C_\phi$, which is self-adjoint. Then, it follows that:

$$\langle S_\phi f|g\rangle = \langle C_\phi^* C_\phi f|g\rangle = \langle C_\phi f|C_\phi g\rangle = \int_X \langle f|\phi_x\rangle\langle\phi_x|g\rangle \;\mathrm{d}\mu(x).$$

Thus, for continuous frames, S_ϕ and S_ϕ^{-1} are both bounded, that is, $S_\phi \in GL(\mathcal{H})$.

Following [6,7], we will say that a function ϕ is a *semi-frame* if it satisfies only one of the frame inequalities in (1.1). We already introduced the lower semi-frames (if ϕ satisfies (1.2), then it could still be a frame, and hence, we say that the lower semi-frame ϕ is *proper* if it is not a frame). Note that the lower frame inequality automatically implies that ϕ is μ-total. On the other hand, a weakly measurable function ϕ is an *upper semi-frame* if is μ-total, that is, $N(C_\phi) = \{0\}$, and there exists $\mathsf{M} < \infty$, such that:

[1] As usual, we identify a function ξ with its residue class in $L^2(X, \mathrm{d}\mu)$.

$$0 < \int_X |\langle f|\phi_x\rangle|^2 \, \mathrm{d}\mu(x) \le \mathsf{M} \, \|f\|^2, \quad \forall\, f \in \mathcal{H},\, f \neq 0. \tag{2.2}$$

Thus, an upper semi-frame is a total Bessel mapping [24]. Notice that this definition does not forbid ϕ to be a frame. Thus, we say that ϕ is a *proper* upper semi-frame if it is not a frame.

If ϕ is a proper upper semi-frame, S_ϕ is bounded and S_ϕ^{-1} is unbounded, as follows immediately from (2.2). In the lower case, however, the definition of S_ϕ must be changed, since the domain $D(C_\phi)$ need not be dense, so that C_ϕ^* may not exist. Instead, following [6, Sec. 2], one defines the synthesis operator as:

$$D_\phi F = \int_X F(x)\, \phi_x \, \mathrm{d}\mu(x), \quad F \in L^2(X, \mathrm{d}\mu), \tag{2.3}$$

on the domain of all elements F for which the integral in (2.3) converges weakly in $\mathcal{H}$, and then, $S_\phi := D_\phi C_\phi$. With this definition, it is shown in [6, Sec. 2] that if ϕ is a proper lower semi-frame, then S_ϕ is unbounded and S_ϕ^{-1} is bounded. All these objects are studied in detail in our previous papers [6,7]. In particular, it is shown there that a natural notion of duality exists, namely, two measurable functions ϕ, ψ are dual to each other (the relation is symmetric) if one has:

$$\langle f|g\rangle = \int_X \langle f|\phi_x\rangle\langle\psi_x|g\rangle \, \mathrm{d}\mu(x), \quad \forall\, f, g \in \mathcal{H}. \tag{2.4}$$

This duality property extends to lower semi-frames and Bessel mappings, as shown in Proposition 3.2 below.

Consider the following sesquilinear form on the domain $\mathcal{D}_1 \times \mathcal{D}_2$:

$$\Omega_{\psi,\phi}(f,g) = \int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle \, \mathrm{d}\mu(x), \quad f \in \mathcal{D}_1, g \in \mathcal{D}_2. \tag{2.5}$$

If $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{H}$ and the form $\Omega_{\psi,\phi}$ is bounded on $\mathcal{H}\times\mathcal{H}$, that is, $|\Omega_{\psi,\phi}(f,g)| \le c\, \|f\|\, \|g\|$, for some $c > 0$, then the couple of weakly measurable functions (ψ, ϕ) is called a *reproducing pair* if the corresponding bounded operator $S_{\psi,\phi}$ given weakly by:

$$\langle S_{\psi,\phi} f|g\rangle = \int_X \langle f|\psi_x\rangle\langle\phi_x|g\rangle \, \mathrm{d}\mu(x), \quad \forall\, f, g \in \mathcal{H},$$

belongs to $GL(\mathcal{H})$. If $\psi = \phi$, we recover the notion of continuous frame.

Under certain conditions, boundedness of $\Omega_{\psi,\phi}$ is automatic, as shown in [21, Prop. 7].

Proposition 2.1. *If $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{H}$, X is locally compact and σ-compact, that is, $X = \bigcup_n K_n, K_n \subset K_{n+1}$, with K_j compact for every j, and $\sup_{x\in X}(\|\phi_x\|_{\mathcal{H}}\, \|\psi_x\|_{\mathcal{H}}) < \infty$, then the form $\Omega_{\psi,\phi}$ is bounded on $\mathcal{H} \times \mathcal{H}$.*

Proof. Define:

$$\Omega^n_{\psi,\phi}(f,g) := \int_{K_n} \langle f|\psi_x\rangle\langle\phi_x|g\rangle \, \mathrm{d}\mu(x), \quad f, g \in \mathcal{H}.$$

By assumption, there exists $c > 0$, such that:

$$|\Omega^n_{\psi,\phi}(f,g)| \le c\,\|f\|\,\|g\| \left|\int_{K_n} \mathrm{d}\mu(x)\right| < \infty.$$

Hence, there exists a bounded operator T_n, such that $\Omega^n_{\psi,\phi}(f,g) = \langle T_n f|g\rangle$. Applying the Banach–Steinhaus theorem to the functional $g \mapsto \Omega_{\psi,\phi}(f,g)$, one gets that the operator $T_{\psi,\phi}$ associated with $\Omega_{\psi,\phi}$ is defined on the whole of $\mathcal{H}$. Doing the same with T_n^*, as in [21, Prop. 7], we conclude that the form $\Omega_{\psi,\phi}$ is bounded on $\mathcal{H} \times \mathcal{H}$. □

The converse of Proposition 2.1 does not hold, as shown in the following example (which is based on [17, Example 2.5]), in the sense that boundedness of the form Ω does not imply the supremum condition.

Example 2.2. Let $h \in \mathcal{H}\backslash\{0\}$ and let $a : \mathbb{R} \to \mathbb{C}$ be such that $a \in L^2(\mathbb{R})\backslash L^\infty(\mathbb{R})$. Define $\phi_x = a(x)h$ for all $x \in \mathbb{R}$. Then, $\phi : \mathbb{R} \to \mathcal{H}$ is a weakly measurable function and a Bessel mapping, because:

$$\int_{\mathbb{R}} |\langle f|\phi_x\rangle|^2 dx \le \|a\|^2_{L^2(\mathbb{R})}\,\|h\|^2\|f\|^2, \quad f \in \mathcal{H}.$$

In conclusion, the sesquilinear form $\Omega_{\psi,\phi}$, where $\psi = \phi$, is defined and bounded on $\mathcal{H} \times \mathcal{H}$, but $\sup_{x\in X}(\|\phi_x\|_{\mathcal{H}}\,\|\psi_x\|_{\mathcal{H}}) = \infty$.

3. The Generalized Frame Operator T_ϕ

In the previous section, we defined the *frame operator* S_ϕ for a lower semi-frame. However, this operator lacks good properties, in general (for instance S_ϕ need not be self-adjoint like in the case of an upper semi-frame, even if S_ϕ is non-negative). In this section, we are going to construct a new operator associated with ϕ which plays the rôle of S_ϕ for lower semi-frames. We show its main properties, in particular, concerning the definition of a Bessel dual mapping in a natural way.

We note that if ϕ is a proper lower semi-frame, the r.h.s. of (1.2) actually diverges for some f. As already said, the domain $D(C_\phi)$ need not be dense in $\mathcal{H}$. It is useful to work with the Hilbert space $\mathcal{H}_\phi$ made of the closure of $D(C_\phi)$ endowed with the topology of $\mathcal{H}$.

The analysis operator C_ϕ is closed [6, Lemma 2.1]. Therefore, the sesquilinear form:

$$\Omega_\phi(f,g) = \langle C_\phi f|C_\phi g\rangle = \int_X \langle f|\phi_x\rangle\langle\phi_x|g\rangle d\mu(x), \quad f,g \in D(C_\phi)$$

is non-negative and closed. By Kato's second representation theorem [26, Theorem 2.23], there exists an operator $\mathsf{T}_\phi : D(\mathsf{T}_\phi) \subset \mathcal{H}_\phi \to \mathcal{H}_\phi$, with $D(\mathsf{T}_\phi) \subset D(C_\phi)$, such that[2]

- $\Omega_\phi(f,g) = \langle \mathsf{T}_\phi f|g\rangle$ for all $f \in D(\mathsf{T}_\phi)$ and $g \in D(C_\phi)$;

[2] We use this sans serif font to avoid confusion with the generalized synthesis operator T_ϕ introduced in our papers about reproducing pairs [12,13,15].

- if $T : D(T) \subset \mathcal{H}_\phi \to \mathcal{H}_\phi$ is such that $\Omega_\phi(f,g) = \langle Tf|g\rangle$ for all $f \in D(T)$ and $g \in D(C_\phi)$, then $T \subset \mathsf{T}_\phi$;
- T_ϕ is non-negative and self-adjoint in $\mathcal{H}_\phi$;
- $\Omega_\phi(f,g) = \langle \mathsf{T}_\phi^{1/2} f|\mathsf{T}_\phi^{1/2} g\rangle$, for all $f, g \in D(C_\phi)$.

We call T_ϕ the *generalized frame operator* of ϕ. The motivation behind this name is that when ϕ is a continuous frame, then $\mathsf{T}_\phi = S_\phi$. The generalized frame operator has been studied in [21,22] in the discrete case and a preliminary extension to the continuous setting has been given in [18].

If $D(C_\phi)$ is dense, then, of course, $\mathcal{H}_\phi = \mathcal{H}$ (i.e., T_ϕ is a non-negative self-adjoint operator on $\mathcal{H}$) and $S_\phi \subset \mathsf{T}_\phi$. In [22], it was proved that in the discrete setting ($X = \mathbb{N}$) we may have a strict inclusion $S_\phi \subsetneq \mathsf{T}_\phi$ (recall that in our paper S_ϕ is weakly defined; therefore, in the discrete setting, it corresponds to the operator W_ϕ in [22]).

Since $\Omega_\phi(f,g) = \langle C_\phi f|C_\phi g\rangle$, we have $\mathsf{T}_\phi := |C_\phi|^2 = C_\phi^\times C_\phi$, where $C_\phi^\times$ and $|C_\phi|$ are the adjoint and the modulus of C_ϕ when we think of it as an operator $C_\phi : \mathcal{H}_\phi \to L^2(X,\mu)$.

The following characterization can be proved as in [21].

Proposition 3.1. *Let ϕ be a weakly measurable function and $\mathsf{m} > 0$. The following statements are equivalent.*

(i) *ϕ is a lower semi-frame of $\mathcal{H}$ with lower bound m;*

(ii) *Ω_ϕ is bounded from below by m, that is:*

$$\Omega_\phi(f,f) \geq \mathsf{m}\|f\|^2, \quad \forall f \in D(C_\phi);$$

(iii) *C_ϕ is bounded from below by $\sqrt{\mathsf{m}}$, that is:*

$$\|C_\phi f\| \geq \sqrt{\mathsf{m}}\|f\|, \quad \forall f \in D(C_\phi);$$

(iv) *T_ϕ is bounded from below by m, that is:*

$$\|\mathsf{T}_\phi f\| \geq \mathsf{m}\|f\|, \quad \forall f \in D(\mathsf{T}_\phi);$$

(v) *T_ϕ is invertible and $\mathsf{T}_\phi^{-1} \in \mathcal{B}(\mathcal{H}_\phi)$ with $\|\mathsf{T}_\phi^{-1}\| \leq \mathsf{m}$.*

Proof. (i) $\iff$ (ii) and (i) $\iff$ (iii) By definition:

(ii) $\implies$ (iv) Let $f \in D(\mathsf{T}_\phi)$. Then, $\|\mathsf{T}_\phi f\|\|f\| \geq \langle \mathsf{T}_\phi f|f\rangle \geq \mathsf{m}\|f\|^2$. Thus $\|\mathsf{T}_\phi f\| \geq \mathsf{m}\|f\|$.

(iv) $\implies$ (ii) For $f \in D(\mathsf{T}_\phi)$, we have $\Omega_\phi(f,f) = \langle \mathsf{T}_\phi^{1/2} f|\mathsf{T}_\phi^{1/2} f\rangle \geq \mathsf{m}\|f\|^2$ by hypothesis. The inequality now extends to every $f \in D(C_\phi)$ noting that $D(\mathsf{T}_\phi)$ is a core of Ω_ϕ (see [26, Theorem 2.1]).

(iv) $\iff$ (v) This is clear, since T_ϕ is self-adjoint. □

Now, assume that ϕ is a lower semi-frame of $\mathcal{H}$. By [6, Proposition 2.6], there exists a Bessel mapping $\psi : X \to \mathcal{H}$ dual to ϕ, i.e., for some $\mathsf{M} > 0$:

$$\int_X |\langle f|\psi_x\rangle|^2 \, \mathrm{d}\mu(x) \leq \mathsf{M}\, \|f\|^2, \quad \forall\, f \in \mathcal{H},$$

and

$$f = \int_X \langle f|\phi_x\rangle \psi_x \, \mathrm{d}\mu(x), \quad \text{for all } f \in D(C_\phi) \text{ in weak sense.} \tag{3.1}$$

A proof may be found in [6, Lemma 2.5 and Proposition 2.6]. Note that the proof given there is incomplete, in the sense, there is no guarantee that ψ is total, see Item (4) in Proposition 3.2 below. Here, we give a different proof involving the operator T_ϕ using the same argument as in [22, Theorem 4.1].

Proposition 3.2. *Let ϕ be a lower semi-frame of $\mathcal{H}$. Let ψ be the function defined by $\psi = \mathsf{T}_\phi^{-1} P_\phi \phi$, i.e., $\psi_x = \mathsf{T}_\phi^{-1} P_\phi \phi_x$ for $x \in X$, where $P_\phi : \mathcal{H} \to \mathcal{H}_\phi$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_\phi$. The following statements hold.*

(1) *ψ is a weakly measurable function;*
(2) *ψ a Bessel mapping of $\mathcal{H}$;*
(3) *ψ is dual to ϕ, i.e., satisfies (3.1).*
(4) *If $D(C_\phi)$ is dense, then ψ is an upper semi-frame of $\mathcal{H}$.*

Proof. We recall that T_ϕ is invertible with $\mathsf{T}_\phi^{-1} \in \mathcal{B}(\mathcal{H}_\phi)$.

(1) This is immediate.
(2) Let $f \in \mathcal{H}$. Then:

$$\begin{aligned}\int_X |\langle f|\mathsf{T}_\phi^{-1} P_\phi \phi_x\rangle|^2 \,\mathrm{d}\mu(x) &= \int_X |\langle P_\phi f|\mathsf{T}_\phi^{-1} P_\phi \phi_x\rangle|^2 \,\mathrm{d}\mu(x)\\ &= \int_X |\langle \mathsf{T}_\phi^{-1} P_\phi f|\phi_x\rangle|^2 \,\mathrm{d}\mu(x)\\ &= \|\mathsf{T}_\phi^{1/2}\mathsf{T}_\phi^{-1} P_\phi f\|^2\\ &= \|\mathsf{T}_\phi^{-1/2} P_\phi f\|^2 \le \|\mathsf{T}_\phi^{-1/2}\|^2\|f\|^2. \end{aligned} \tag{3.2}$$

(3) Let $f \in D(C_\phi)$ and $h \in \mathcal{H}$. Then:

$$\begin{aligned}\langle f|h\rangle = \langle f|P_\phi h\rangle = \langle f|\mathsf{T}_\phi \mathsf{T}_\phi^{-1} P_\phi h\rangle &= \int_X \langle f|\phi_x\rangle\langle \phi_x|\mathsf{T}_\phi^{-1} P_\phi h\rangle \,\mathrm{d}\mu(x)\\ = \int_X \langle f|\phi_x\rangle\langle P_\phi\phi_x|\mathsf{T}_\phi^{-1} P_\phi h\rangle \,\mathrm{d}\mu(x) &= \int_X \langle f|\phi_x\rangle\langle \mathsf{T}_\phi^{-1} P_\phi \phi_x|h\rangle \,\mathrm{d}\mu(x).\end{aligned}$$

(4) This is immediate from (3.1). □

Remark 3.3. Part (3) of the above proposition comes from general properties of semi-bounded operators. If $\mathcal{H}, \mathcal{K}$ are Hilbert spaces, and $C : D(C) \subset \mathcal{H} \to \mathcal{K}$ is a closed operator, such that $D(C)$ is dense and $\|Cf\| \geq \gamma\|f\|$, for every $f \in D(C)$, then by Kato's theorem, there exists $T \geq 0$, with the properties described at the beginning of Sect. 3, such that $\langle Cf|Cg\rangle = \langle T^{1/2}f|T^{1/2}g\rangle$ for every $f, g \in D(C)$; as before T is invertible with bounded inverse. On the other hand:

$$|\langle f|g\rangle| \leq \frac{1}{\gamma^2}\|f\|_C\|g\|_C, \quad \forall f, g \in D(C), \text{ where } \|f\|_C := \|Cf\|\,.$$

Hence, there exists an operator X, bounded in $\mathcal{H}(C) := D(C)[\|\cdot\|_C]$, such that:

$$\langle f|g\rangle = \langle Cf|CXg\rangle, \quad \forall f, g \in D(C).$$

Then:

$$\langle Cf|CXg\rangle = \langle T^{1/2}f|T^{1/2}Xg\rangle = \langle f|g\rangle, \quad \forall f, g \in D(C).$$

This implies that $T^{1/2}Xg \in D(T^{1/2}) = D(C)$, for every $g \in D(C)$ and $TXg = g$, $\forall g \in D(C)$. Thus, $X \subset T^{-1}$. In conclusion, $\langle f|g\rangle = \langle Cf|CT^{-1}g\rangle$, for all $f, g \in D(C)$. Therefore, if $C = C_\phi$ is densely defined, the *dual* is found. If $D(C_\phi)$ is not dense, we can proceed with the projection P_ϕ as before.

Finally, as mentioned in [22] for the discrete case, calculations similar to (3.2) show the next result.

Proposition 3.4. *Let ϕ be a lower semi-frame of $\mathcal{H}$. Then, the function $\psi = \mathsf{T}_\phi^{-1/2} P_\phi \phi$ is a Parseval frame for $\mathcal{H}_\phi$.*

Thus, following the standard terminology in frame theory, we can call $\mathsf{T}_\phi^{-1} P_\phi \phi$ and $\mathsf{T}_\phi^{-1/2} P_\phi \phi$ the *canonical dual Bessel mapping* and the *canonical tight frame* of ϕ.

Now, we exploit Proposition 5 of [21] and the discussion after it.

Proposition 3.5. *Let ϕ be a weakly measurable function of $\mathcal{H}$. Then, ϕ is a lower semi-frame of $\mathcal{H}$ if and only if there exists an inner product $\langle\cdot|\cdot\rangle_+$ inducing a norm $\|\cdot\|_+$ on $D(C_\phi)$ for which $D(C_\phi)[\|\cdot\|_+]$ is complete, continuously embedded into $\mathcal{H}$, and for some $\alpha, \mathsf{m}, \mathsf{M} > 0$, one has:*

$$\alpha \|f\| \leq \|f\|_+ \ \textit{and}$$
$$\mathsf{m}\, \|f\|_+^2 \leq \int_X |\langle f|\phi_x\rangle|^2 \, \mathrm{d}\mu(x) \leq \mathsf{M}\, \|f\|_+^2, \quad \forall \in D(C_\phi).$$

Proof. It is sufficient to take $\|f\|_+^2 = \|f\|_{C_\phi}^2 = \int_X |\langle f|\phi_x\rangle|^2 \, \mathrm{d}\mu(x)$, for $f \in D(C_\phi)$. □

Let ϕ be a lower semi-frame in the Hilbert space $\mathcal{H}$, with domain $D(C_\phi)$, assumed to be dense. For every $x \in X$, the map $f \mapsto \langle f|\phi_x\rangle$ is a bounded linear functional on the Hilbert space $\mathcal{H}(C_\phi) := D(C_\phi)[\|\cdot\|_+]$. By the Riesz Lemma, there exists an element $\chi_x^\phi \in D(C_\phi)$, such that:

$$\langle f|\phi_x\rangle = \langle f|\chi_x^\phi\rangle_+ \ \ \forall f \in D(C_\phi). \tag{3.3}$$

By Proposition 3.5, χ^ϕ is a frame.

We can explicitly determine the element χ^ϕ when $\|\cdot\|_+$ is the norm $\|f\|_{1/2}^2 = \|\mathsf{T}_\phi^{1/2} f^2\|$. Notice that, by Proposition 3.1(iv), this norm is equivalent to the graph norm of $\mathsf{T}_\phi^{1/2}$.

Then, we have $\langle f|\phi_x\rangle = \langle f|\chi_x^\phi\rangle_{1/2} = \langle f|\mathsf{T}_\phi \chi_x^\phi\rangle$ for all $f \in D(C_\phi)$. Thus $\chi_x^\phi = \mathsf{T}_\phi^{-1}\phi_x$ for all $x \in X$, i.e., χ^ϕ is the canonical dual Bessel mapping of ϕ.

Following the notation of [8], denote by $\mathcal{H}(\mathsf{T}_\phi^{1/2})$ the Hilbert space $D(\mathsf{T}_\phi^{1/2})$ with the norm $\|\cdot\|_{1/2}$. In the same way, denote by $\mathcal{H}(C_\phi)$ the Hilbert space $D(C_\phi)$ with inner product $\langle C_\phi \cdot|C_\phi \cdot\rangle$. Hence, we have proved the following result.

Proposition 3.6. *Let ϕ be a lower semi-frame of $\mathcal{H}$ with $D(C_\phi)$ dense. Then, the canonical dual Bessel mapping of ϕ is a tight frame for the Hilbert space $\mathcal{H}(C_\phi) = \mathcal{H}(\mathsf{T}_\phi^{1/2})$.*

We can proceed in the converse direction, i.e., starting with a frame $\chi \in D(C_\phi)$, does there exist a lower semi-frame η of $\mathcal{H}$, such that χ is the frame χ^η constructed from η in the way described above? The answer is formulated in the following.

Proposition 3.7. [21, Prop. 6] *Let χ be a frame of $\mathcal{H}(C_\phi) = \mathcal{H}(\mathsf{T}_\phi^{1/2})$. Then:*

(i) *There exists a lower semi-frame η of $\mathcal{H}$, such that $\chi = \chi^\eta$ if, and only if, $\chi \in D(\mathsf{T}_\phi)$.*

(ii) *If $\chi = \chi^\eta$ for some lower semi-frame η of $\mathcal{H}$, then $\eta = \mathsf{T}_\phi \chi$.*

4. The Functions Generated by a Lower Semi-frame

Throughout this section, we continue to consider a lower semi-frame ϕ in $\mathcal{H}$, with $D(C_\phi)$ dense.

Since T_ϕ^{-1} is defined on $\mathcal{H}$, we can actually apply different powers of T_ϕ^{-1} on ϕ and get the functions $\mathsf{T}_\phi^{-k}\phi$, $k \in [0, \infty)$. Hence, we can ask for the properties of $\mathsf{T}_\phi^{-k}\phi$. Of course, the answer depends on the Hilbert space where $\mathsf{T}_\phi^{-k}\phi$ is considered. For instance, for $k = 0$, we have a lower semi-frame of $\mathcal{H}$ and, as seen in the previous section, for $k = \frac{1}{2}$, we have a frame for $\mathcal{H}$, while for $k = 1$, we have a frame for $\mathcal{H}(\mathsf{T}_\phi^{1/2})$. When we have powers of an unbounded, closed, densely defined operator, then we can consider *scales* and *lattices* of Hilbert spaces, which we will consider in more detail in Sect. 5. For a while, let us simply denote by $\mathcal{H}(\mathsf{T}_\phi^m)$, $m \geq 0$, the domain of T_ϕ^m considered as a Hilbert space with norm $\|f\|_m = \|\mathsf{T}_\phi^m f\|$, $f \in D(\mathsf{T}_\phi^m)$. Then, if $m > n \geq 0$, we have $\mathcal{H}(\mathsf{T}_\phi^m) \subset \mathcal{H}(\mathsf{T}_\phi^n) \subset \mathcal{H}$. The function $\mathsf{T}_\phi^{-k}\phi$ is not always a well-defined function of $\mathcal{H}(\mathsf{T}_\phi^m)$; a sufficient condition for $\mathsf{T}_\phi^{-k}\phi \in \mathcal{H}(\mathsf{T}_\phi^m)$ is that $k \geq m$.

Having at our disposal the notion of scale of Hilbert spaces, we now come back to a lower semi-frame ϕ with $D(C_\phi)$ dense and to the functions $\mathsf{T}_\phi^{-k}\phi$ with $k \geq 0$. For simplicity of notation, we write in a compact way the inner product of $\mathcal{H}(\mathsf{T}_\phi^m)$ in the following way $\langle f|g\rangle_m := \langle f|g\rangle_{\mathsf{T}_\phi^m} = \langle \mathsf{T}_\phi^m f|\mathsf{T}_\phi^m g\rangle$.

Theorem 4.1. *Let ϕ be a lower semi-frame of $\mathcal{H}$ with $D(C_\phi)$ dense and $m, k \in [0, \infty)$, $k \geq m$. Then, the following statements hold.*

(i) *$\mathsf{T}_\phi^{-k}\phi$ is a Bessel mapping of $\mathcal{H}(\mathsf{T}_\phi^m)$ if and only if $k \geq m + \frac{1}{2}$;*

(ii) *$\mathsf{T}_\phi^{-k}\phi$ is a lower semi-frame of $\mathcal{H}(\mathsf{T}_\phi^m)$ if and only if $m \leq k \leq m + \frac{1}{2}$;*

(iii) *$\mathsf{T}_\phi^{-k}\phi$ is a frame of $\mathcal{H}(\mathsf{T}_\phi^m)$ if and only if $\mathsf{T}_\phi^{-k}\phi$ is a Parseval frame of $\mathcal{H}(\mathsf{T}_\phi^m)$, if and only if $k = m + \frac{1}{2}$.*

Proof. We have for $f \in \mathcal{H}(\mathsf{T}_\phi^m)$:

$$\begin{aligned}\int_X |\langle f|\mathsf{T}_\phi^{-k}\phi_x\rangle|_m^2 \,\mathrm{d}\mu(x) &= \int_X |\langle \mathsf{T}_\phi^m f|\mathsf{T}_\phi^m \mathsf{T}_\phi^{-k}\phi_x\rangle|^2 \,\mathrm{d}\mu(x)\\ &= \int_X |\langle \mathsf{T}_\phi^{2m-k} f|\phi_x\rangle|^2 \,\mathrm{d}\mu(x)\\ &= \|\mathsf{T}_\phi^{1/2}\mathsf{T}_\phi^{2m-k} f\|^2 = \|\mathsf{T}_\phi^{2m-k+1/2} f\|^2. \quad (4.1)\end{aligned}$$

Hence, taking into account that $0 \in \rho(\mathsf{T}_\phi)$ and that $\|f\|^2_{\mathsf{T}_\phi^m} = \|\mathsf{T}_\phi^m f\|^2$, (i) and (ii) follow by comparing $2m-k+\frac{1}{2}$ and m. We give the details for (i) as example.

Assume that $\mathsf{T}_\phi^{-k}\phi$ is a Bessel mapping of $\mathcal{H}(\mathsf{T}_\phi^m)$, then there exists $B > 0$, such that for all $f \in \mathcal{D}(\mathsf{T}_\phi^m)$, we have $\|\mathsf{T}_\phi^{2m-k+1/2} f\|^2 \leq B\|\mathsf{T}_\phi^m f\|^2$. Rewriting the inequality as $\|\mathsf{T}_\phi^{m-k+\frac{1}{2}}\mathsf{T}_\phi^m f\|^2 \leq B\|\mathsf{T}_\phi^m f\|^2$, we find that $\mathsf{T}_\phi^{m-k+\frac{1}{2}}$ is bounded, i.e., $m-k+\frac{1}{2} \leq 0$. The other implication trivially holds by the same estimate.

Finally, combining the two cases above, we obtain that $\mathsf{T}_\phi^{-k}\phi$ is a frame of $\mathcal{H}(\mathsf{T}_\phi^m)$ if and only if $k = m+\frac{1}{2}$. Moreover, in this case, $\mathsf{T}_\phi^{-k}\phi$ is actually a Parseval frame, as one can see in (4.1). □

In particular, we recover some cases we discussed in the previous section, namely $(k,m) = (1,\frac{1}{2})$ and $(k,m) = (1,0)$ (corresponding to the canonical Bessel mapping of ϕ) and $(k,m) = (\frac{1}{2},0)$ (corresponding to the canonical tight frame of ϕ).

It is possible to generalize Theorem 4.1 by considering more general functions than powers of T_ϕ. For instance we could take the set Σ of real valued functions g defined on the spectrum $\sigma(\mathsf{T}_\phi)$, which are measurable with respect to the spectral measure of T_ϕ and such that g and $\widetilde{\mathsf{g}} := 1/\mathsf{g}$ are bounded on compact subsets of $\sigma(\mathsf{T}_\phi)$. For every $\mathsf{g} \in \Sigma$, we denote by $\mathcal{H}_\mathsf{g}$ the Hilbert space completion of $D(\mathsf{g}(\mathsf{T}_\phi))$ with respect to the norm $\|f\|_\mathsf{g} = \|\mathsf{g}(\mathsf{T}_\phi)f\|$, $f \in D(\mathsf{g}(\mathsf{T}_\phi))$. As shown in [2, Sec. 10.4], we get an LHS if the order is defined by $\mathsf{h} \preceq \mathsf{g} \iff \exists\, \gamma > 0$, such that $\mathsf{h} \leq \gamma\mathsf{g}$. We put $\mathsf{i}(t) := t$, $t \in \sigma(\mathsf{T}_\phi)$. Then, we get the following:

Theorem 4.2. *Let ϕ be a lower semi-frame of $\mathcal{H}$ with $\mathcal{D}(C_\phi)$ dense and g, h non-negative functions from Σ, with $\mathsf{g} \succeq \mathsf{h}$. Suppose that $\widetilde{\mathsf{g}}$ and $\mathsf{h}\widetilde{\mathsf{g}}$ are bounded functions. Then, the following statements hold.*

- (i) *$\widetilde{\mathsf{g}}(\mathsf{T}_\phi)\phi$ is a Bessel mapping of $\mathcal{H}_\mathsf{h}$ if and only if $\mathsf{i}^{1/2}\mathsf{h} \preceq \mathsf{g}$;*
- (ii) *$\widetilde{\mathsf{g}}(\mathsf{T}_\phi)\phi$ is a lower semi-frame of $\mathcal{H}_\mathsf{h}$ if and only if $\mathsf{h} \preceq \mathsf{g} \preceq \mathsf{i}^{1/2}\mathsf{h}$;*
- (iii) *$\widetilde{\mathsf{g}}(\mathsf{T}_\phi)\phi$ is a frame of $\mathcal{H}_\mathsf{h}$ if and only if $\widetilde{\mathsf{g}}(\mathsf{T}_\phi)\phi$ is a Parseval frame of $\mathcal{H}_\mathsf{h}$, if and only if $\mathsf{i}^{1/2}\mathsf{h} \preceq \mathsf{g}$.*

Proof. The proof is similar to that of Theorem 4.1. In fact, for $f \in \mathcal{H}_\mathsf{h}$, using the functional calculus for T_ϕ, we have:

$$\begin{aligned}\int_X |\langle f|\widetilde{\mathsf{g}}(\mathsf{T}_\phi)\phi_x\rangle|^2_\mathsf{h}\, \mathrm{d}\mu(x) &= \int_X |\langle \mathsf{h}(\mathsf{T}_\phi)f|(\mathsf{h}\widetilde{\mathsf{g}})(\mathsf{T}_\phi)\phi_x\rangle|^2\, \mathrm{d}\mu(x)\\ &= \int_X |\langle (\mathsf{h}^2\widetilde{\mathsf{g}})(\mathsf{T}_\phi)f|\phi_x\rangle|^2\, \mathrm{d}\mu(x)\\ &= \|\mathsf{T}_\phi^{1/2}(\mathsf{h}^2\widetilde{\mathsf{g}})(\mathsf{T}_\phi)f\|^2 = \|(\mathsf{h}^2\widetilde{\mathsf{g}}\mathsf{i}^{1/2})(\mathsf{T}_\phi)f\|^2. \end{aligned} \tag{4.2}$$

Thus, for instance, if $\mathsf{i}^{1/2}\mathsf{h} \preceq \mathsf{g}$, then $\mathsf{i}^{1/2}\mathsf{h}^2\widetilde{\mathsf{g}} \preceq \mathsf{h}$; hence, $\widetilde{\mathsf{g}}(\mathsf{T}_\phi)\phi$ is a Bessel mapping of $\mathcal{H}_\mathsf{h}$. The rest of the proof is analogous. □

5. Metric Operators

The generalized frame operator of a total weakly measurable function ϕ with $D(C_\phi)$ dense is an example of metric operator in the sense of the following definition [10].

Definition 5.1. By a metric operator in a Hilbert space $\mathcal{H}$, we mean a strictly positive self-adjoint operator G, that is, $G > 0$ or $\langle Gf|f\rangle \geq 0$ for every $f \in D(G)$ and $\langle Gf|f\rangle = 0$ if and only if $f = 0$.

Let S be an unbounded closed operator with dense domain $D(S)$. As usual, define the graph norm of S:

$$\langle f|g\rangle_{\rm gr} := \langle f|g\rangle + \langle Sf|Sg\rangle,\ f,g \in D(S),$$
$$\|f\|_{\rm gr}^2 = \|f\|^2 + \|Sf\|^2 .$$

Then, the norm $\|\cdot\|_{\rm gr}$ makes $D(S)$ into a Hilbert space continuously embedded into $\mathcal{H}$. For $f \in D(S^*S)$, we may write $\|f\|_{\rm gr}^2 = \langle f|(I + S^*S)f\rangle$. Note that the operator $S^*S = |S|^2$ is self-adjoint and non-negative: $|S|^2 \geq 0$. In addition, $D(|S|) = D(S)$ and $N(S^*S) = N(S)$ [28, Theor. 5.39 and 5.40].

Given S as above, the operator $R_S := I + S^*S$ is self-adjoint, with domain $D(S^*S)$, and $R_S \geq 1$. Hence, R_S is an unbounded metric operator, with bounded inverse $R_S^{-1} = (I + S^*S)^{-1}$. In our previous works [10,11,14], we have analyzed the lattice of Hilbert spaces generated by such a metric operator. In the sequel, we summarize this discussion, keeping the same notations.

In the general case where both the metric operator G and its inverse G^{-1} are unbounded, the lattice is given in Fig. 1. Given the metric operator G, equip the domain $D(G^{1/2})$ with the following norm:

$$\|f\|_{R_G}^2 = \left\|(I + G)^{1/2} f\right\|^2,\ f \in D(G^{1/2}). \tag{5.1}$$

Since this norm is equivalent to the graph norm of $G^{1/2}$, this makes $D(G^{1/2})$ into a Hilbert space, denoted $\mathcal{H}(R_G)$, dense in $\mathcal{H}$. Next, we equip $\mathcal{H}(R_G)$ with the norm $\|f\|_G^2 := \left\|G^{1/2} f\right\|^2$ and denote by $\mathcal{H}(G)$ the completion of $\mathcal{H}(R_G)$ in that norm and corresponding inner product $\langle\cdot|\cdot\rangle_G := \langle G^{1/2}\cdot|G^{1/2}\cdot\rangle$. Hence, we have $\mathcal{H}(R_G) = \mathcal{H} \cap \mathcal{H}(G)$, with the so-called projective norm [5, Sec. I.2.1], which here is simply the graph norm of $G^{1/2}$.

Next, we proceed in the same way with the inverse operator G^{-1}, and we obtain another Hilbert space, $\mathcal{H}(G^{-1})$. Then, we consider the lattice generated by $\mathcal{H}(G)$ and $\mathcal{H}(G^{-1})$ with the operations:

$$\mathcal{H}_1 \wedge \mathcal{H}_2 := \mathcal{H}_1 \cap \mathcal{H}_2 , \tag{5.2}$$
$$\mathcal{H}_1 \vee \mathcal{H}_2 := \mathcal{H}_1 + \mathcal{H}_2 , \tag{5.3}$$

as shown in Fig. 1. Here, every embedding, denoted by an arrow, is continuous and has dense range. Taking conjugate duals, it is easy to see that one has:

$$\mathcal{H}(R_G)^\times = \mathcal{H}(R_G^{-1}) = \mathcal{H} + \mathcal{H}(G^{-1}), \tag{5.4}$$
$$\mathcal{H}(R_{G^{-1}})^\times = \mathcal{H}(R_{G^{-1}}^{-1}) = \mathcal{H} + \mathcal{H}(G). \tag{5.5}$$

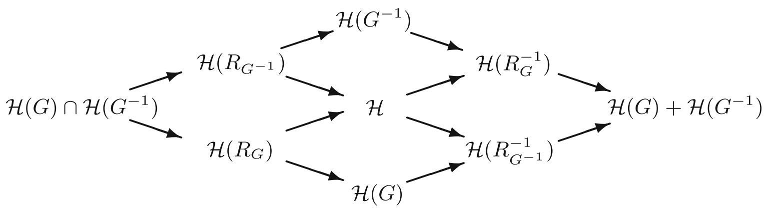

Figure 1. The lattice of Hilbert spaces generated by a metric operator

In these relations, the r.h.s. is meant to carry the inductive norm (and topology) [5, Sec. I.2.1], so that both sides are in fact unitary equivalent, hence identified.

At this stage, we return to the construction in terms of the closed unbounded operator S. We have to envisage two cases.

(i) An Unbounded Metric Operator

We take as metric operator $G_1 = I + S^*S$, which is unbounded, with $G_1 > 1$ and bounded inverse. Then, the norm $\|\cdot\|_{G_1}$ is equivalent to the norm $\|\cdot\|_{R_{G_1}}$ on $D(G_1^{1/2}) = D(S)$, so that $\mathcal{H}(G_1) = \mathcal{H}(R_{G_1})$ as vector spaces and thus also $\mathcal{H}(G_1^{-1}) = \mathcal{H}(R_{G_1}^{-1})$. On the other hand, G_1^{-1} is bounded. Hence, we get the triplet:

$$\mathcal{H}(G_1) \subset \mathcal{H} \subset \mathcal{H}(G_1^{-1}) = \mathcal{H}(G_1)^{\times}. \tag{5.6}$$

Next, following Example 3 of [21], if we take an ONB $\{e_n\}$ of $D(G_1^{1/2}) = D(S)$, contained in $D(S^*S) = \mathcal{H}(G_1)$, then $\{Ge_n\} = \{(1 = I + S^*S)e_n\}$ is a lower semi-frame of $\mathcal{H}$.

Actually, the triplet (5.6) is the central part of the discrete scale of Hilbert spaces $V_{\mathcal{G}}$ built on the powers of $G_1^{1/2}$. This means that $V_{\mathcal{G}_1} := \{\mathcal{H}_n, n \in \mathbb{Z}\}$, where $\mathcal{H}_n = D(G_1^{n/2}), n \in \mathbb{N}$, with a norm equivalent to the graph norm, and $\mathcal{H}_{-n} = \mathcal{H}_n^{\times}$:

$$\cdots \subset \mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H} \subset \mathcal{H}_{-1} \subset \mathcal{H}_{-2} \subset \cdots \tag{5.7}$$

Thus, $\mathcal{H}_1 = \mathcal{H}({G_1}^{1/2}) = D(S)$, $\mathcal{H}_2 = \mathcal{H}(G_1) = D(S^*S)$, and $\mathcal{H}_{-1} = \mathcal{H}({G_1}^{-1})$. Then, $G_1^{1/2}$ is a unitary operator from $\mathcal{H}_1$ onto $\mathcal{H}$ and, more generally, from $\mathcal{H}_n$ onto $\mathcal{H}_{n-1}$. In the same way, G_1 is a unitary operator from $\mathcal{H}_n$ onto $\mathcal{H}_{n-2}$ and G_1^{-1} is a unitary operator from $\mathcal{H}_n$ onto $\mathcal{H}_{n+2}$.

Moreover, one may add the end spaces of the scale, namely:

$$\mathcal{H}_\infty(G_1) := \bigcap_{n\in\mathbb{Z}} \mathcal{H}_n, \qquad \mathcal{H}_{-\infty}(G_1) := \bigcup_{n\in\mathbb{Z}} \mathcal{H}_n. \tag{5.8}$$

In this way, we get a genuine Rigged Hilbert Space:

$$\mathcal{H}_\infty(G_1) \subset \mathcal{H} \subset \mathcal{H}_{-\infty}(G_1). \tag{5.9}$$

In fact, one can go one more step. Namely, following [5, Sec. 5.1.2], we can use quadratic interpolation theory [20] and build a continuous scale of Hilbert spaces $\mathcal{H}_\alpha, 0 \le \alpha \le 1$, between $\mathcal{H}_1$ and $\mathcal{H}$, where $\mathcal{H}_\alpha = D(G_1^{\alpha/2})$, with the graph norm $\|\xi\|_\alpha^2 = \|\xi\|^2 + \|G_1^{\alpha/2}\xi\|^2$ or, equivalently, the norm $\|(I+G_1)^{\alpha/2}\xi\|^2$. Indeed every $G_1^\alpha, \alpha \geq 0$, is an unbounded metric operator.

Next, we define $\mathcal{H}_{-\alpha} = \mathcal{H}_\alpha^\times$ and iterate the construction to the full continuous scale $V_{\widetilde{\mathcal{G}}_1} := \{\mathcal{H}_\alpha, \alpha \in \mathbb{R}\}$. Then, of course, one can replace $\mathbb{Z}$ by $\mathbb{R}$ in the definition (5.8) of the end spaces of the scale.

In the general case, $R_{G_1} = I + G_1 > 1$ is also an unbounded metric operator. Thus, we have:

$$\mathcal{H}(R_{G_1}) \subset \mathcal{H} \subset \mathcal{H}(R_{G_1}^{-1}) = \mathcal{H}(R_{G_1})^\times, \tag{5.10}$$

and we get another Hilbert–Gel'fand triplet. Then, one can repeat the construction and obtain the Hilbert scale built on the powers of $R_{G_1}^{1/2}$, as well as other lower semi-frames of $\mathcal{H}$.

Now, if S is injective, i.e., $N(S) = \{0\}$, then $|S|^2 > 0$ is also an unbounded metric operator. Since $|S| > 0, R_S = I + |S|^2 > 1$ and it is another unbounded metric operator, with bounded inverse R_S^{-1}. In both cases, one may build the corresponding Hilbert scale corresponding to the powers of S or $R_S^{1/2}$.

At this stage, we have recovered the formalism based on metric operators that we have developed for the theory of quasi-Hermitian operators, in particular non-self-adjoint Hamiltonians, as encountered in the so-called $\mathcal{PT}$-symmetric quantum mechanics. We refer to [10,11,14] for a complete treatment. However, the case of an unbounded metric operator does not lead to many results, unless one considers a *quasi-Hermitian* operator [9, Def.3.1].

(ii) A Bounded Metric Operator

We take as metric operator $G_2 = (I + S^*S)^{-1}$, which is bounded, with unbounded inverse.

Since G_2 is *bounded*, things simplify, because now $D(G_2) = \mathcal{H}$. Similarly, one gets $\mathcal{H}(R_{G_2^{-1}}) = \mathcal{H}(G_2^{-1})$ and $\mathcal{H}(R_{G_2^{-1}}^{-1}) = \mathcal{H}(G_2)$. Therefore, we are left with the triplet

$$\mathcal{H}(G_2^{-1}) \subset \mathcal{H} \subset \mathcal{H}(G_2). \tag{5.11}$$

Then, $G_2^{1/2}$ is a unitary operator from $\mathcal{H}(G_2)$ onto $\mathcal{H}$ and from $\mathcal{H}$ onto $\mathcal{H}(G_2^{-1})$, whereas $G_2^{-1/2}$ is a unitary operator $\mathcal{H}(G_2^{-1})$ onto $\mathcal{H}$ and from $\mathcal{H}$ onto $\mathcal{H}(G_2)$.

(iii) A Bounded Metric Operator with Bounded Inverse

There is a third case, which is almost trivial. If the operator S is bounded, $G_1 = I + S^*S$ and $G_2 = (I + S^*S)^{-1}$ are both bounded metric operators, with bounded inverse. Then, all nine Hilbert spaces in the lattice of Fig. 1 coincide as vector spaces, with equivalent, but different, norms.

The advantage of this situation is that it leads to strong results on the similarity of two operators. As mentioned in [8, Sec. 3], up to unitary equivalence, one may always consider that the intertwining operator defining the similarity is in fact a metric operator. Let us briefly recall these notions.

Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces, and $D(A)$ and $D(B)$ be dense subspaces of $\mathcal{H}$ and $\mathcal{K}$, respectively, $A : D(A) \to \mathcal{H}$, $B : D(B) \to \mathcal{K}$ two linear operators.

A (metric) bounded operator $T : \mathcal{H} \to \mathcal{K}$ is called a *intertwining operator* for A and B if:

(i) $T : D(A) \to D(B)$;
(ii) $BT\xi = TA\xi$, $\forall\, \xi \in D(A)$.

We say that A and B are *similar*, and write $A \sim B$, if there exists an intertwining operator T for A and B with bounded inverse $T^{-1} : \mathcal{K} \to \mathcal{H}$, intertwining for B and A.

A parallel definition (quasi-similarity) may be given in case the inverse T^{-1} of the intertwining operator T is not bounded.

From the relation $A \sim B$, there follow many interesting results about the respective spectra of A and B, as described in detail in [8].

6. Transforming Functions into Frames by Metric Operators

This section concerns the second main aim of our paper. By Proposition 3.4 and Theorem 4.1, we get the following result: if ϕ is a lower semi-frame of $\mathcal{H}$ and $D(C_\phi)$ is dense, then $\mathsf{T}_\phi^{-1/2} : \mathcal{H} \to \mathcal{H}$ and $\psi = \mathsf{T}_\phi^{-1/2}\phi$ is a Parseval frame for $\mathcal{H}_\phi = \mathcal{H}$. As we already remarked, $\mathsf{T}_\phi^{-1/2} : \mathcal{H} \to \mathcal{H}$ is a metric operator. We now want to relax the condition for ϕ of being a lower semi-frame. Thus, we ask the following question: for which weakly measurable functions $\phi : X \to \mathcal{H}$ does there exist a metric operator G on $\mathcal{H}$, such that $G\phi$ is a frame?[3]

The motivation of this question is that, in general, one tries to pass from a less regular situation to a more regular, possibly in a smaller space.

In the next result, we find some necessary or sufficient conditions for an answer to our question. As we are going to see, the recourse of the generalized frame operator is again useful.

Theorem 6.1. *Let $\phi : X \to \mathcal{H}$ be a weakly measurable function with generalized frame operator T_ϕ. The following statements hold.*

(i) *If ϕ is not total, then there does not exist a metric operator G on $\mathcal{H}$, such that $G\phi$ is a frame and G^{-1} is bounded.*
(ii) *If ϕ is a Bessel mapping and not total, then there does not exist a metric operator G on $\mathcal{H}$, such that $G\phi$ is a frame.*
(iii) *If $D(C_\phi)$ is not dense, then there does not exist a metric operator G on $\mathcal{H}$, such that $G\phi$ is a frame.*
(iv) *If $D(C_\phi)$ is dense and ϕ is a lower semi-frame, then there exists a metric operator G on $\mathcal{H}$, such that $G\phi$ is a Parseval frame. In particular, a possible choice is $G = \mathsf{T}_\phi^{-1/2}$.*
(v) *If ϕ is total, $D(C_\phi)$ is dense and $\phi_x \in R(\mathsf{T}_\phi)$ for all $x \in X$, then there exists a metric operator G on $\mathcal{H}$, such that $G\phi$ is a Parseval frame. In particular, a possible choice is $G = \mathsf{T}_\phi^{-1/2}$.*

[3]Here, and in the rest of paper, writing $G\phi$ means implicitly that $\phi_x \in D(G)$ for all $x \in X$.

Proof. (i) Suppose that there exists a metric operator G on $\mathcal{H}$, such that $G\phi$ is a frame and G^{-1} is bounded. By hypothesis, there exists $f \in \mathcal{H}, f \neq 0$, such that $\langle \phi_x|f\rangle = 0$ for a.e. $x \in X$. Since G is self-adjoint and G^{-1} is bounded, then G^{-1} is defined on $\mathcal{H}$. Hence $\langle G\phi_x|G^{-1}f\rangle = \langle \phi_x|GG^{-1}f\rangle = 0$ for a.e. $x \in X$, which implies that $G^{-1}f = 0$, i.e., the contradiction $f = 0$.

(ii) Suppose that there exists a metric operator G, such that $G\phi$ is a frame, and let m be a lower bound of $G\phi$. Then, for all $f \in D(G)$:

$$\mathsf{m}\|f\|^2 \leq \int_X |\langle f|G\phi_x\rangle|^2 d\mu(x) = \int_X |\langle Gf|\phi_x\rangle|^2 d\mu(x) \leq \mathsf{M}\|Gf\|^2, \tag{6.1}$$

where M is an upper bound of ϕ. Thus, (6.1) implies that G^{-1} is bounded. By the previous point, we get then a contradiction.

(iii) Suppose that there exists a metric operator G on $\mathcal{H}$, such that $G\phi$ is a frame. Then, $\int_X |\langle f|G\phi_x\rangle|^2 d\mu(x) < \infty$ for all $f \in \mathcal{H}$. In particular, for $f \in D(G)$, we have:

$$\int_X |\langle Gf|\phi_x\rangle|^2 d\mu(x) < \infty,$$

so $R(G) \subseteq D(C_\phi)$. This means that $N(G) \supseteq D(C_\phi)^\perp \neq \{0\}$ which contradicts the property that G is a metric operator.

(iv) This follows by Proposition 3.4.

(v) The statement can be proved with a similar argument to that of Proposition 3.4 (Proposition 3.2), and thus, we give only a sketch of the proof. First of all, we note that T_ϕ has domain dense in $\mathcal{H}$, $D(\mathsf{T}_\phi) \subseteq D(\mathsf{T}_\phi^{1/2})$ and also that T_ϕ is injective (ϕ is total). Let $f \in D(\mathsf{T}_\phi)$, and then:

$$\int_X |\langle f|\mathsf{T}_\phi^{-1/2}\phi_x\rangle|^2 \, \mathrm{d}\mu(x) = \|\mathsf{T}_\phi^{1/2}\mathsf{T}_\phi^{-1/2}f\|^2 = \|f\|^2.$$

Now, a standard argument of density concludes that ψ is a Parseval frame of $\mathcal{H}$. □

Example 6.2. Let $\{e_n\}_{n\in\mathbb{N}}$ be an orthonormal basis, $\phi = \{e_1 + e_n\}_{n\geq 2}$ and $\psi = \{e_n\}_{n\geq 2}$. Both ϕ and ψ cannot be transformed into frames of $\mathcal{H}$ by a metric operator. Indeed, $D(C_\phi) = \{e_1\}^\perp$ and ψ is a Bessel sequence but not total.

We will consider more examples (concerning, in particular, lower semi-frames) in the next section. As an immediate consequence of Theorem 6.1, we get the following result (compare with [25, Corollary II.1]). We recall that two sequences $\{\phi_n\}_{n\in\mathbb{N}}, \{\psi_n\}_{n\in\mathbb{N}}$ are said *bi-orthogonal* if $\langle \psi_n|\phi_m\rangle = \delta_{n,m}$, the Kronecker symbol.

Corollary 6.3. *Let $\phi := \{\phi_n\}_{n\in\mathbb{N}}, \psi := \{\psi_n\}_{n\in\mathbb{N}}$ be bi-orthogonal and total sequences of $\mathcal{H}$. Then, $D(C_\phi)$ is dense. Moreover, if T_ϕ is the generalized frame operator of ϕ, then $\phi_n \in R(\mathsf{T}_\phi) \subset R(\mathsf{T}_\phi^{1/2})$ for all $n \in \mathbb{N}$ and $\{\mathsf{T}_\phi^{-1/2}\phi_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $\mathcal{H}$.*

Proof. Then, $D(C_\phi)$ is dense, because it contains the total sequence ψ, and moreover, $\psi_n \in D(T_\phi)$ and $\mathsf{T}_\phi \psi_n = \phi_n$ for all $n \in \mathbb{N}$ (which also gives $\phi_n \in R(\mathsf{T}_\phi) \subset R(\mathsf{T}_\phi^{1/2})$). Now, by Theorem 6.1(iv), $\{\mathsf{T}_\phi^{-1/2}\phi_n\}_{n\in\mathbb{N}}$ is a Parseval frame of $\mathcal{H}$, but since $\langle \mathsf{T}_\phi^{-1/2}\phi_n | \mathsf{T}_\phi^{-1/2}\phi_m\rangle = \langle \psi_n | \phi_m\rangle = \delta_{n,m}$, we conclude that $\{\mathsf{T}_\phi^{-1/2}\phi_n\}_{n\in\mathbb{N}}$ is in particular an orthonormal basis of $\mathcal{H}$. □

The problem about transforming functions in frames is still open. However, in the light of Theorem 6.1, one may formulate a new version of the problem: given a weakly measurable function $\phi : X \to \mathcal{H}$, is it true that there exists a metric operator G on $\mathcal{H}$, such that $G\phi$ is a frame if and only if ϕ is total and $D(C_\phi)$ is dense?

7. Examples

In this final section, we exhibit several examples of lower semi-frames, mostly taken from our previous works.

(1) Sequences of Exponential Functions

Let $\mathcal{H} = L^2(0,1)$ and $g \in L^2(0,1)$. A weighted exponential sequence of g is $\mathcal{E}(g,b) := \{g_n\}_{n\in\mathbb{Z}} = \{g(x)e^{2\pi i n b x}\}_{n\in\mathbb{Z}}$ with $b > 0$. In [22, Remark 6.6], it was proved that if $0 < b \le 1$, then $\mathcal{E}(g,b)$ is a lower semi-frame if and only if g is bounded away from zero, i.e., $|g(x)| \ge A$ for some $A > 0$ and a.e. $x \in (0,1)$. Moreover, by [22, Corollary 6.5], the analysis operator of $\mathcal{E}(g,b)$ is densely defined and the generalized frame operator T_g of $\mathcal{E}(g,b)$ is the multiplication operator by $\frac{1}{b}|g|^2$, that is:

$$D(\mathsf{T}_g) = \{f \in L^2(0,1) : f|g|^2 \in L^2(0,1)\} \quad \text{and} \quad \mathsf{T}_g f = \frac{1}{b}|g|^2 f, \quad \forall f \in D(\mathsf{T}_g).$$

Thus, for $k \ge 0$ and $n \in \mathbb{Z}$, $(\mathsf{T}_g^{-k} g_n)(x) = g(x)/|g(x)|^{-2k} e^{2\pi i n b x}$, i.e., $\mathsf{T}_g^{-k}\mathcal{E}(g, b) = \mathcal{E}(g/|g|^{-2k}, b)$. It is not difficult to check that the conclusions of Theorem 4.1 hold.

(2) A Reproducing Kernel Hilbert space

Following [13], we consider a reproducing kernel Hilbert space of (nice) functions on a measure space (X,μ), with kernel function $k_x, x \in X$; that is, $f(x) = \langle f|k_x\rangle_K, \forall f \in \mathcal{H}_K$. Choose the weight function $m(x) > 1$ and define the Hilbert scale $\mathcal{H}_k$, $k \in \mathbb{Z}$, determined by the multiplication operator $Af(x) = m(x)f(x)$, $\forall x \in X$. Thus, we have, for $n \ge 1$ $(\overline{n} = -n)$:

$$\mathcal{H}_{2n} \subset \mathcal{H}_n \subset \mathcal{H}_K \subset \mathcal{H}_{\overline{n}} \subset \mathcal{H}_{2\overline{n}}. \tag{7.1}$$

Then, define the measurable functions $\phi_x = k_x m^n(x), \psi_x = k_x m^{-n}(x)$, so that $C_\psi : \mathcal{H}_K \to \mathcal{H}_n$, $C_\phi : \mathcal{H}_K \to \mathcal{H}_{\overline{n}}$ continuously, and they are dual of each other. One has indeed $\langle \phi_x|g\rangle_K = \overline{g(x)}\, m^n(x) \in \mathcal{H}_{\overline{n}}$ and $\langle \psi_x|g\rangle_K = \overline{g(x)}\, m^{-n}(x) \in \mathcal{H}_n$, which implies duality. Thus, (ψ, ϕ) is a reproducing pair with $S_{\psi,\phi} = I$, where ψ is an upper semi-frame and ϕ a lower semi-frame.

Now, we concentrate on the lower semi-frame ϕ. First, we have $D(C_\phi) = \mathcal{H}_n$, which is dense, so that $\mathcal{H}_\phi = \mathcal{H}$. Let us compute the sesquilinear form:

$$\begin{aligned}\Omega_\phi(f,g) = \langle C_\phi f | C_\phi g\rangle_K &= \int_X f(x) m(x)^n \,\overline{g(x)} m(x)^n \,\mathrm{d}\mu(x), \quad f,g \in D(C_\phi)\\ &= \langle \mathsf{T}_\phi^{1/2} f | \mathsf{T}_\phi^{1/2} g\rangle_K.\end{aligned}$$

Therefore, we have $(\mathsf{T}_\phi^{1/2} f)(x) = f(x)\, m(x)^n$, and therefore, $(\mathsf{T}_\phi f)(x) = f(x)\, m(x)^{2n}$. Hence, $D(\mathsf{T}_\phi) = \mathcal{H}_{2n} \subset D(C_\phi) = \mathcal{H}_n$ and $D(\mathsf{T}_\phi^{1/2}) = \mathcal{H}_n$, see (7.1) above.

Next, $\mathsf{T}_\phi^{-1/2}$ is the operator of multiplication by m^{-n}, and indeed, it is bounded in $\mathcal{H}$. Finally, since ϕ is a lower semi-frame, $\chi := \mathsf{T}_\phi^{-1/2}\phi$ is a frame in $\mathcal{H}$, by Proposition 3.2.

(3) Wavelets on the Sphere

The continuous wavelet transform on the 2-sphere $\mathbb{S}^2$ has been analyzed in [4]. For an axisymmetric (zonal) mother wavelet $\phi \in \mathcal{H} = L^2(\mathbb{S}^2, \mathrm{d}\mu)$, define the family of spherical wavelets:

$$\phi_{\varrho,a} := R_\varrho D_a \phi, \text{ where } (\varrho, a) \in X := SO(3) \times \mathbb{R}^+.$$

Here, D_a denotes the stereographic dilation operator and R_ρ the unitary rotation on $\mathbb{S}^2$.

Given the wavelet ϕ, it is known that the operator S_ϕ is diagonal in Fourier space (harmonic analysis on the 2-sphere reduces to expansions in spherical harmonics Y_l^m, $l \in \mathbb{N}_0, m = -l, \dots, l$), and thus, it is given by a Fourier multiplier $\widehat{S_\phi f}(l,n) = s_\phi(l)\widehat{f}(l,n)$ with the symbol s_ϕ given by:

$$s_\phi(l) := \frac{8\pi^2}{2l+1} \sum_{|m|\le l} \int_0^\infty \left|\widehat{D_a\phi}(l,m)\right|^2 \frac{\mathrm{d}a}{a^3}, \quad l \in \mathbb{N}_0,$$

where $\widehat{D_a\phi}(l,m) := \langle Y_l^m | D_a \phi\rangle$ is the Fourier coefficient of $D_a\phi$.

If one has $\mathsf{d} \le s_\phi(l) \le \mathsf{c}$, for every $l \in \mathbb{N}$, then the wavelet ϕ is admissible and a frame in $L^2(\mathbb{S}^2, \mathrm{d}\mu)$. However, it has been shown in [29] that the reconstruction formula converges under the weaker condition $\mathsf{d} \le s_\phi(l) < \infty$ for all $l \in \mathbb{N}_0$. In that case, ϕ is *not* admissible and is a lower semi-frame, with S_ϕ unbounded and densely defined. The domain of S_ϕ is the following:

$$\begin{aligned}D(S_\phi) = \{f \in L^2(\mathbb{S}^2, \mathrm{d}\mu) : |\widehat{f}(l,n)|\\ \le \frac{1}{s_\phi(l)} |\widehat{h}(l,n)|, \text{ for some } h \in L^2(\mathbb{S}^2, \mathrm{d}\mu)\}.\end{aligned}$$

This domain contains, in particular, the set of band-limited functions, i.e., functions f, such that $\widehat{f}(l,n) = 0$, $\forall\, l \ge N_1$, for some $N_1 < \infty$, which is dense in L^2. Since $S_\phi = D_\phi C_\phi$, it follows that $D(S_\phi) \subset D(C_\phi)$, and hence, $D(C_\phi)$ is dense as well and $\mathcal{H}_\phi = \mathcal{H}$.

We proceed as in Case (2) and consider the sesquilinear form:

$$\Omega_\phi(f,g) = \langle C_\phi f | C_\phi g\rangle = \langle \mathsf{T}_\phi^{1/2} f | \mathsf{T}_\phi^{1/2} g\rangle, \quad f,g \in D(C_\phi).$$

Since $D(C_\phi)$ is dense, we get $\mathsf{T}_\phi := |C_\phi|^2 = C_\phi^* C_\phi$, $\mathsf{T}_\phi^{1/2} = |C_\phi|$ and $D(\mathsf{T}_\phi) \subset D(\mathsf{T}_\phi^{1/2}) = D(C_\phi)$.

Finally, since ϕ is a lower semi-frame, $\chi := \mathsf{T}_\phi^{-1/2}\phi$ is a frame in $\mathcal{H}$, by Proposition 3.2, as before.

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