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On a Fractional in Time Nonlinear Schrödinger Equation with Dispersion Parameter and Absorption Coefficient

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Abstract: This paper is concerned with the nonexistence of global solutions to fractional in time nonlinear Schrödinger equations of the form $i^\alpha\partial_t^\alpha\omega(t,z) + a_1(t)\Delta\omega(t,z) + i^\alpha a_2(t)\omega(t,z) = \xi|\omega(t,z)|^p, \quad (t,z) \in (0,\infty)\times\mathbb{R}^N$, where $N \geq 1$, $\xi \in \mathbb{C}\backslash\{0\}$ and $p > 1$, under suitable initial data. To establish our nonexistence theorem, we adopt the Pohozaev nonlinear capacity method, and consider the combined effects of absorption and dispersion terms. Further, we discuss in details some special cases of coefficient functions $a_1, a_2 \in L^1_{loc}([0,\infty),\mathbb{R})$, and provide two illustrative examples.

Keywords: fractional in time nonlinear Schrödinger equation; absorption coefficient; dispersion parameter; global solution

MSC: 26A33; 35B44; 35R11

1. Introduction and Preliminaries

In this paper, we study the following initial value problem for the fractional in time nonlinear Schrödinger equation

$$i^\alpha\partial_t^\alpha\omega(t,z) + a_1(t)\Delta\omega(t,z) + i^\alpha a_2(t)\omega(t,z) = \xi|\omega(t,z)|^p,\ (t,z) \in (0,\infty)\times\mathbb{R}^N, \tag{1}$$

where $N \geq 1$, $\xi \in \mathbb{C}\backslash\{0\}$ and $p > 1$, under the assumption that

$$\omega(0,z) = \varpi(z), \quad z \in \mathbb{R}^N,\ N \geq 1. \tag{2}$$

In the left-hand side of (1), $i^\alpha = e^{i\frac{\alpha\pi}{2}}$, and ∂_t^α means the Caputo fractional derivative in time of order $\alpha \in (0,1)$. Later on, we need some regularities of the coefficient functions: $a_1, a_2 \in L^1_{loc}([0,\infty),\mathbb{R})$, $a_1 \not\equiv 0$ (that is, not identically zero), and $\varpi \in L^1_{loc}(\mathbb{R}^N,\mathbb{C})$.

Schrödinger type equations arise naturally in the analysis of dispersive equations on large domains (for example, we refer to oceanic water waves). In addition, they are useful in the study of wave turbolence (as an application of statistical physics), see the comprehensive paper of Buckmaster–Germain–Hani–Shatah [1]. A significant topic for nonlinear partial Schrödinger equations is to establish sufficient conditions for the existence of solutions providing a localized behavior. Following, this feature, Rego–Monteiro [2] proved the existence of a traveling-wave solution, with solitary-wave behavior. Furthermore, also relevant to this study is the focus of qualitative research in symmetric domains to provide the symmetries of solutions (mainly, working with kinematical and dynamical algebras of Schrödinger type equations). A combination of suitable iteration methods, maximum principle and method of moving planes, is useful to detect symmetries of positive solutions and nonexistence results (see, for example, [3]). We also mention the recent works of Peng–Zhao [4] (global existence and blow-up of solutions) and Hoshino [5] (asymptotic behavior of solutions).

In [6], the authors considered the nonlinear Schrödinger equation in the form

$$i\partial_t\omega(t,z) + a_1(t)\Delta\omega(t,z) + ia_2(t)\omega(t,z) = \xi|\omega(t,z)|^p, \quad (t,z) \in (0,\infty)\times\mathbb{R}^N,$$

(that is, set $\alpha = 1$ in (1)), in the context of optical soliton systems, where $a_1(t)$ plays the role of dispersion parameter, and $a_2(t)$ means the absorption coefficient.

In absence of absorption (that is, $a_2 \equiv 0$), and setting the dispersion term equal to one (that is, $a_1 \equiv 1$), Ikeda-Wakasugi [7] studied the mathematical model

$$i\partial_t\omega(t,z) + \Delta\omega(t,z) = \xi|\omega(t,z)|^p, \quad (t,z) \in (0,\infty)\times\mathbb{R}^N. \tag{3}$$

In detail, the authors considered the global behavior of solutions to problem (3), then they established a finite-time blow-up result of an L^2-solution whenever $p \in (1, 1+\frac{2}{N})$. The same problem (3) was discussed by Ikeda–Inui [8]. This time, they established a small data blow-up result of H^1-solution, whenever $p \in (1, 1+\frac{4}{N})$.

Furthermore, a revival of interest to the study of Schrödinger equation is linked to the theory of fractional calculus (see, for example, the books of Kilbas–Srivastava–Trujillo [9] and Samko–Kilbas–Marichev [10]).

Let $h \in C([0,T])$ be a real-valued function and, as usual, denote by $\Gamma(\cdot)$ the Gamma function. Here, we recall that the Riemann–Liouville fractional integrals of order $\sigma > 0$, are given as

$$(I_0^\sigma h)(t) = \frac{1}{\Gamma(\sigma)}\int_0^t (t-u)^{\sigma-1}h(u)\,du, \quad 0 < t \le T$$

and

$$(I_T^\sigma h)(t) = \frac{1}{\Gamma(\sigma)}\int_t^T (u-t)^{\sigma-1}h(u)\,du, \quad 0 \le t < T.$$

We note that the limit of $(I_0^\sigma h)(t)$, as t approaches zero from the right, is zero. So, we can put $(I_0^\sigma h)(0) = 0$ to extend by continuity $I_0^\sigma h$ to $[0,T]$. The similar extended continuity holds for $I_T^\sigma h$, by taking $(I_T^\sigma h)(T) = 0$.

In addition, the Caputo derivative of order $\sigma \in (0,1)$ of $h \in C^1([0,T])$ is obtained as

$$(\partial_t^\sigma h)(t) = (I_0^{1-\sigma}h')(t), \quad 0 < t < T.$$

In such a framework setting, we mention that the fractional version of (3), that is,

$$i^\alpha\partial_t^\alpha\omega + \Delta\omega = \xi|\omega|^p, \quad t > 0,\, z \in \mathbb{R}^N,$$

received the attention of Zhang–Sun–Li [11], whose studies lead to nonexistence (blow-up) results of global solutions with suitable initial values and $p \in (1, 1+\frac{2}{N})$.

For further interesting contributions to the analysis of the blow-up behavior of solutions to fractional nonlinear Schrödinger problems, we mention the papers of Fino–Dannawi–Kirane [12] (semilinear equation with fractional Laplacian), Ionescu–Pusateri [13] (equation in dimension one with cubic nonlinearities) and Kirane–Nabti [14] (nonlocal in time equation). Finally, we recall the paper of Li–Ding–Xu [15] where a cubic non-polynomial spline method is implemented to solve the time-fractional nonlinear Schrödinger equation. Furthermore, the stability of the method is analyzed by Fourier analysis. Moreover, Shi–Ma–Ding [16] studied a fourth-order quasi-compact conservative difference scheme and provided precise informations on its stability. Resuming, now-a-day nonlinear Schödinger equations play a crucial role in modelling and controlling the behavior of optical soliton systems. The physical significance of considering the fractional in time version of such an equation, is mainly related to the description of the evolution of the above system in terms of Lévy motion, instead of the Brownian motion (see, for example, [17]).

Mathematically, we are concerned with the solvability of problem (1) and (2), depending on the behaviors of a_1 and a_2 at infinity. The approach is based on the nonlinear capacity method of Pohozaev [18], whose main skill is the ability to use specific test and cut-off functions related to the particular form of the nonlinear operator in the differential equation driving the problem. Following [18], and particularizing the method for Equation (1), we construct a nonexistence theorem and discuss some consequences, over the following definition.

Definition 1. *Let $Q_T = (0,T) \times \mathbb{R}^N$. Then, $\omega \in L^p_{loc}((0,\infty) \times \mathbb{R}^N, \mathbb{C})$ satisfying the integral equation*

$$\begin{aligned}
&- i^\alpha \int_{\mathbb{R}^N} \varpi(z) I_T^{1-\alpha}\varphi(0,z)\,dz - i^\alpha \int_{Q_T} \omega (I_T^{1-\alpha}\varphi)_t\,dz\,dt \\
&+ \int_{Q_T} a_1(t)\omega\Delta\varphi\,dz\,dt + i^\alpha \int_{Q_T} a_2(t)\omega\varphi\,dz\,dt \\
&= \xi \int_{Q_T} |\omega|^p \varphi\,dz\,dt,
\end{aligned} \tag{4}$$

for all $T > 0$ and $\varphi \in C^2(Q_T, \mathbb{R})$ with $supp_z\varphi \subset\subset \mathbb{R}^N$, is a global weak solution to problems (1)–(2).

In Definition 1, we made use of the following integration by parts rule:

$$\int_0^T (I_0^\sigma h_1)(t)h_2(t)\,dt = \int_0^T h_1(t)(I_T^\sigma h_2)(t)\,dt, \quad h_1, h_2 \in C([0,T]) \text{ and } \sigma > 0.$$

Remark 1. *Given two complex numbers ϖ, ξ, we denote by ϖ_1, ξ_1 (respectively, ϖ_2, ξ_2) the real part (respectively, the imaginary part) of ϖ, ξ. If $\omega \in L^p_{loc}((0,\infty) \times \mathbb{R}^N, \mathbb{C})$ is a global weak solution to (1)–(2), then we have the following facts:*
(i) (4) *implies that*

$$\begin{aligned}
&\int_{\mathbb{R}^N} (-r_\alpha\varpi_1 + s_\alpha\varpi_2)\, I_T^{1-\alpha}\varphi(0,z)\,dz + \int_{Q_T} (-r_\alpha\omega_1 + s_\alpha\omega_2)\,(I_T^{1-\alpha}\varphi)_t\,dz\,dt \\
&+ \int_{Q_T} a_1(t)\omega_1\Delta\varphi\,dz\,dt + \int_{Q_T} (r_\alpha\omega_1 - s_\alpha\omega_2)\,a_2(t)\varphi\,dz\,dt \\
&= \xi_1 \int_{Q_T} |\omega|^p\varphi\,dz\,dt,
\end{aligned} \tag{5}$$

for all $T > 0$ and $\varphi \in C^2(Q_T, \mathbb{R})$ with $supp_z\varphi \subset\subset \mathbb{R}^N$, where

$$r_\alpha = \cos\left(\frac{\alpha\pi}{2}\right) \quad \text{and} \quad s_\alpha = \sin\left(\frac{\alpha\pi}{2}\right).$$

(ii) $\nu = -i\omega \in L^p_{loc}((0,\infty) \times \mathbb{R}^N, \mathbb{C})$ *is a global weak solution to*

$$i^\alpha\partial_t^\alpha\nu + a_1(t)\Delta\nu + i^\alpha a_2(t)\nu = \tilde{\xi}|\nu|^p, \quad t > 0,\, z \in \mathbb{R}^N, \tag{6}$$

under the assumption that

$$\nu(0,z) = \tilde{\varpi}(z), \quad z \in \mathbb{R}^N, \tag{7}$$

where $\tilde{\xi} = -i\xi$ and $\tilde{\varpi}(z) = -i\varpi(z)$.

2. Non-Existence Theorem and Implications

Let $p \in (1, 1 + \frac{2}{N})$, $\xi \in \mathbb{C} \backslash \{0\}$ and $\varpi \in L^1(\mathbb{R}^N, \mathbb{C})$. For the reader's convenience, we collect the hypotheses on the data of problem (1):

$$(P_1) \quad \liminf_{T\to\infty} T^{\alpha\left(\frac{N+2}{2}\right)} \left(T^{\frac{-\alpha p}{p-1}} \int_0^1 |a_1(Tu)|^{\frac{p}{p-1}}\, du + \int_0^1 |a_2(Tu)|^{\frac{p}{p-1}}\, du \right) = 0;$$

$$(P_2)_a \quad \xi_1 \int_{\mathbb{R}^N} (r_\alpha \varpi_1 - s_\alpha \varpi_2)\, dz > 0;$$

$$(P_2)_b \quad \xi_2 \int_{\mathbb{R}^N} (r_\alpha \varpi_2 + s_\alpha \varpi_1)\, dz > 0.$$

Using the above hypotheses we establish the following nonexistence result for problems (1)–(2).

Theorem 1. *If (P_1) and $(P_2)_a$ (or $(P_2)_b$) hold, then problems (1) and (2) admits no global weak solution.*

This theorem is convenient in many cases of practical interest. Consequently, we discuss in details some special classes of coefficient functions in (1), and provide two illustrative examples.

We assume $a_1 \in L^\infty([0,\infty), \mathbb{R})$, to deduce that

$$\begin{aligned} & T^{\alpha\left(\frac{N+2}{2}\right)} T^{\frac{-\alpha p}{p-1}} \int_0^1 |a_1(Tu)|^{\frac{p}{p-1}}\, du \le \|a_1\|_\infty^{\frac{p}{p-1}} T^{\alpha\left(\frac{N+2}{2} - \frac{p}{p-1}\right)}, \\ \Rightarrow \quad & \lim_{T\to\infty} T^{\alpha\left(\frac{N+2}{2}\right)} T^{\frac{-\alpha p}{p-1}} \int_0^1 |a_1(Tu)|^{\frac{p}{p-1}}\, du = 0 \quad (\text{recall } p \in (1, 1+\tfrac{2}{N})). \end{aligned}$$

Consequently, we may modify hypothesis (P_1) to the form

$$(P_1)' \quad \liminf_{T\to\infty} T^{\alpha\left(\frac{N+2}{2}\right)} \int_0^1 |a_2(Tu)|^{\frac{p}{p-1}}\, du = 0,$$

and so we have the result:

Corollary 1. *If $(P_1)'$ and $(P_2)_a$ (or $(P_2)_b$) hold, then problems (1) and (2), with $a_1 \in L^\infty([0,\infty), \mathbb{R})$, admits no global weak solution.*

Remark 2. *We point out that the constant choices $a_1 \equiv 1$ and $a_2 \equiv 0$, lead to interpret Corollary 1 as the nonexistence result of Zhang–Sun–Li ([11], Theorem 2.2).*

Fixing $a_1 \in L^\infty([0,\infty), \mathbb{R})$, we set

$$a_2(t) = (1+t)^{-q},\ t > 0,\ q > 0, \quad 0 < \alpha < \frac{2}{N+2}. \tag{8}$$

So, we focus particular attention on the following cases:
Case 1: q satisfies the inequality

$$q \ge \frac{p-1}{p}.$$

The choices in (8) lead us to obtain that

$$\begin{aligned} \int_0^1 |a_2(Tu)|^{\frac{p}{p-1}}\, du &= \int_0^1 (1+Tu)^{\frac{-qp}{p-1}}\, du \\ &= \begin{cases} T^{-1} \ln(1+T) & \text{if } q = \frac{p-1}{p}, \\ C_{q,p} T^{-1} \left[1 - (1+T)^{1-\frac{qp}{p-1}} \right] & \text{if } q > \frac{p-1}{p}, \end{cases} \end{aligned}$$

for some $C_{q,p} > 0$ (that is, a constant depending on q and p). This means that

$$T^{\alpha\left(\frac{N+2}{2}\right)} \int_0^1 |a_2(Tu)|^{\frac{p}{p-1}}\, du = \begin{cases} O\left(T^{\alpha\left(\frac{N+2}{2}\right)-1}\ln(1+T)\right) & \text{if } q = \frac{p-1}{p}, \\ O\left(T^{\alpha\left(\frac{N+2}{2}\right)-1}\right) & \text{if } q > \frac{p-1}{p}, \end{cases}$$

as T goes to infinity, which gives us hypothesis $(P_1)'$ (by the choice of α in (8)).

Case 2: q is restricted to positive values satisfying the inequalities

$$\frac{\alpha(N+2)(p-1)}{2p} < q < \frac{p-1}{p}. \tag{9}$$

Since by (8) we work with $\alpha \in (0, \frac{2}{N+2})$, then we have

$$\begin{aligned} & \{q : (9) \text{ holds true}\} \neq \emptyset, \\ \Rightarrow\ & \int_0^1 |a_2(Tu)|^{\frac{p}{p-1}}\, du = C_{q,p} T^{\frac{-qp}{p-1}} \left[(1+T^{-1})^{1-\frac{qp}{p-1}} - T^{\frac{qp}{p-1}-1}\right], \\ \Rightarrow\ & T^{\alpha\left(\frac{N+2}{2}\right)} \int_0^1 |a_2(Tu)|^{\frac{p}{p-1}}\, du = O\left(T^{\frac{-qp}{p-1}+\alpha\left(\frac{N+2}{2}\right)}\right), \text{ as } T \text{ goes to infinity,} \end{aligned}$$

which gives us again hypothesis $(P_1)'$.

In both the cases, by Corollary 1 we can conclude that there is no global weak solution to problems (1) and (2).

Summarizing the above facts, we have the following result.

Corollary 2. *If $(P_2)_a$ (or $(P_2)_b$), (8), and $q > \frac{\alpha(N+2)(p-1)}{2p}$ hold, then problems (1) and (2), with $a_1 \in L^\infty([0,\infty),\mathbb{R})$, admits no global weak solution.*

Relaxing the hypothesis on a_1 (that is, considering again $a_1 \in L^1_{loc}([0,\infty),\mathbb{R})$, $a_1 \not\equiv 0$), we set $a_2 \equiv 0$ in (1). This means that we reduce hypothesis (P_1) to the following

$$(P_1)'' \qquad \liminf_{T\to\infty} T^{\alpha\left(\frac{N+2}{2}-\frac{p}{p-1}\right)} \int_0^1 |a_1(Tu)|^{\frac{p}{p-1}}\, du = 0.$$

So, Theorem 1 is restated in the following form.

Corollary 3. *If $(P_1)''$ and $(P_2)_a$ (or $(P_2)_b$) hold, then problems (1) and (2), with $a_2 \equiv 0$, admits no global weak solution.*

To illustrate the above corollary, we choice the coefficient function

$$a_1(t) = (1+t)^r, \quad t > 0, \text{ for some } r > 0. \tag{10}$$

We note that $a_1 \notin L^\infty([0,\infty),\mathbb{R})$ and

$$\begin{aligned} & \int_0^1 |a_1(Tu)|^{\frac{p}{p-1}}\, du = C_{r,p} T^{\frac{rp}{p-1}} \left[(1+T^{-1})^{\frac{rp}{p-1}+1} - T^{-\frac{rp}{p-1}-1}\right] \\ & \qquad \text{(for some } C_{r,p} > 0\text{)}, \\ \Rightarrow\ & T^{\alpha\left(\frac{N+2}{2}-\frac{p}{p-1}\right)} \int_0^1 |a_1(Tu)|^{\frac{p}{p-1}}\, du = O\left(T^{\alpha\left(\frac{N+2}{2}-\frac{p}{p-1}\right)+\frac{rp}{p-1}}\right), \\ & \qquad \text{as } T \text{ goes to infinity,} \end{aligned}$$

which gives us easily hypothesis $(P_1)''$ by the choice

$$0 < r < \alpha \quad \text{and} \quad 1 < p < 1 + \frac{2(\alpha - r)}{\alpha N + 2r}. \tag{11}$$

In such a situation, Corollary 3 is restated in the following way:

Corollary 4. *If $(P_1)''$, $(P_2)_a$ (or $(P_2)_b$), (10) and (11) hold, then problems (1) and (2), with $a_2 \equiv 0$, admits no global weak solution.*

The last situation to consider is that of

$$a_1(t) = (1+t)^r \quad \text{and} \quad a_2(t) = (1+t)^{-q}, \quad t > 0, \qquad \text{for some } r, q > 0. \tag{12}$$

From the above calculations, we know that

$$\begin{cases} r < \alpha \text{ and } 1 < p < 1 + \frac{2(\alpha - r)}{\alpha N + 2r} & \Rightarrow (P_1)'' \text{ holds true,} \\ 0 < \alpha < \frac{2}{N+2} \text{ and } q > \frac{\alpha(N+2)(p-1)}{2p} & \Rightarrow (P_1)' \text{ holds true.} \end{cases}$$

This set of parameters leads us to introduce the hypothesis:

$(P_1)'''$ $\quad 0 < r < \alpha < \frac{2}{N+2}, \quad q > \frac{\alpha(N+2)(p-1)}{2p}, \quad 1 < p < 1 + \frac{2(\alpha - r)}{\alpha N + 2r}.$

Indeed, by Theorem 1 we deduce the next result.

Corollary 5. *If (12), $(P_1)'''$, and $(P_2)_a$ (or $(P_2)_b$) hold, then problems (1) and (2) admits no global weak solution.*

Let us provide two examples to illustrate the above mentioned results.

Example 1. *Consider the fractional in time nonlinear Schrödinger equation*

$$i^{\frac{1}{3}} \partial_t^{\frac{1}{3}} \omega(t,z) + (1+t^2)^{-1} \omega_{zz}(t,z) + i^{\frac{1}{3}} (1+t)^{\frac{-1}{2}} \omega(t,z) = |\omega(t,z)|^2, \tag{13}$$

$(t,z) \in (0,\infty) \times \mathbb{R}$, under the initial condition

$$\omega(0,z) = e^{-z^2}, \quad z \in \mathbb{R}, \tag{14}$$

where $\omega_{zz} = \frac{\partial^2 \omega}{\partial z^2}$. Problems (13) and (14) is a special case of problems (1) and (2) with $N = 1$, $p = 2$, $\alpha = \frac{1}{3}$, $a_1(t) = (1+t^2)^{-1}$, $a_2(t) = (1+t)^{\frac{-1}{2}}$, $\xi = 1$, and $\varpi(z) = e^{-z^2}$. Notice that

$$\xi_1 \int_{\mathbb{R}} (r_\alpha \varpi_1 - s_\alpha \varpi_2)\, dz = \frac{\sqrt{3}}{2} \int_{\mathbb{R}} e^{-z^2}\, dz = \frac{\sqrt{3\pi}}{2} > 0,$$

which shows that condition $(P_2)_a$ is satisfied. Moreover, since

$$0 < \alpha = \frac{1}{3} < \frac{2}{N+2} = \frac{2}{3},$$

then (8) is satisfied with $q = \frac{1}{2}$. On the other hand, we have $a_1 \in L^\infty([0,\infty), \mathbb{R})$ and

$$\frac{\alpha(N+2)(p-1)}{2p} = \frac{1}{4} < \frac{1}{2} = q.$$

Therefore, by Corollary 2, we deduce that problems (13) and (14) admits no global weak solution.

Example 2. *Consider the fractional in time nonlinear Schrödinger equation*

$$i^{\frac{1}{4}}\partial_t^{\frac{1}{4}}\omega(t,z) + (1+t)^{\frac{1}{8}}\omega_{zz}(t,z) + i^{\frac{1}{4}}(1+t)^{\frac{-1}{19}}\omega(t,z) = i|\omega(t,z)|^{\frac{19}{18}}, \tag{15}$$

$(t,z) \in (0,\infty) \times \mathbb{R}^2$, *under the initial condition*

$$\omega(0,z) = ie^{-|z|^2}, \quad z \in \mathbb{R}^2. \tag{16}$$

Problems (15) *and* (16) *is a special case of problems* (1) *and* (2) *with* $N = 2$, $p = \frac{19}{18}$, $\alpha = \frac{1}{4}$, $a_1(t) = (1+t)^{\frac{1}{8}}$, $a_2(t) = (1+t)^{\frac{-1}{19}}$, $\xi = i$, *and* $\varpi(z) = ie^{-|z|^2}$. *Notice that*

$$\xi_2 \int_{\mathbb{R}^2} (r_\alpha \varpi_2 + s_\alpha \varpi_1)\, dz = \cos\left(\frac{\pi}{8}\right) \int_{\mathbb{R}^2} e^{-|z|^2}\, dz = \pi \cos\left(\frac{\pi}{8}\right) > 0,$$

which shows that condition $(P_2)_b$ *is satisfied. On the other hand,* (12) *is satisfied with* $r = \frac{1}{8}$ *and* $q = \frac{1}{19}$. *Moreover, we have*

$$0 < r = \frac{1}{8} < \alpha = \frac{1}{4} < \frac{2}{N+2} = \frac{1}{2},$$
$$\frac{\alpha(N+2)(p-1)}{2p} = \frac{1}{38} < q = \frac{1}{19}$$

and

$$1 < p = \frac{19}{18} < 1 + \frac{2(\alpha - r)}{\alpha N + 2r} = \frac{4}{3},$$

which show that $(P_1)'''$ *is satisfied. Hence, by Corollary 5, we deduce that problems* (15) *and* (16) *admits no global weak solution.*

3. Proof of Theorem 1

In this section we give the complete proof of Theorem 1. To construct the nonexistence result by contradiction, we assume that $\omega \in L^p_{loc}((0,\infty) \times \mathbb{R}^N, \mathbb{C})$ is a global weak solution to (1) and (2). Then, we focus on some characteristic truncation and comparison functions required by the Pohozaev nonlinear capacity method in [18], and distinguish two cases.

Proof. (P_1) and $(P_2)_a$ hold.

Hypothesis $(P_2)_a$, gives us $\xi_1 \neq 0$. So, involving (5) for all $T > 0$ and $\varphi \in C^2(Q_T, \mathbb{R})$, $\varphi \geq 0$, $\mathrm{supp}_z \varphi \subset\subset \mathbb{R}^N$, we get the equation

$$\begin{aligned}
&\int_{Q_T} |\omega|^p \varphi\, dz\, dt + \frac{1}{\xi_1} \int_{\mathbb{R}^N} (r_\alpha \varpi_1 - s_\alpha \varpi_2)\, I_T^{1-\alpha} \varphi(0,z)\, dz \\
&= \frac{1}{\xi_1} \int_{Q_T} (-r_\alpha \omega_1 + s_\alpha \omega_2)\, (I_T^{1-\alpha}\varphi)_t\, dz\, dt + \frac{1}{\xi_1} \int_{Q_T} a_1(t) \omega_1 \Delta\varphi\, dz\, dt \\
&+ \frac{1}{\xi_1} \int_{Q_T} (r_\alpha \omega_1 - s_\alpha \omega_2)\, a_2(t) \varphi\, dz\, dt,
\end{aligned}$$

which yields the following inequality

$$\begin{aligned}
&\int_{Q_T} |\omega|^p \varphi\, dz\, dt + \frac{1}{\xi_1} \int_{\mathbb{R}^N} (r_\alpha \varpi_1 - s_\alpha \varpi_2)\, I_T^{1-\alpha} \varphi(0,z)\, dz \\
&\leq \frac{2}{|\xi_1|} \int_{Q_T} |\omega||(I_T^{1-\alpha}\varphi)_t|\, dz\, dt + \frac{1}{|\xi_1|} \int_{Q_T} |a_1(t)||\omega||\Delta\varphi|\, dz\, dt \\
&+ \frac{2}{|\xi_1|} \int_{Q_T} |\omega||a_2(t)|\varphi\, dz\, dt.
\end{aligned} \tag{17}$$

Looking at the right-hand side of (17), together with the ε-Young inequality, choosing $\varepsilon \in (0, \frac{1}{3})$ and a suitable $C > 0$ changing value from line to line (but not depending on T), we have

$$\frac{2}{|\xi_1|} \int_{Q_T} |\omega||(I_T^{1-\alpha}\varphi)_t|\, dz\, dt \le \varepsilon \int_{Q_T} |\omega|^p \varphi\, dz\, dt + C \int_{Q_T} \varphi^{\frac{-1}{p-1}} |(I_T^{1-\alpha}\varphi)_t|^{\frac{p}{p-1}}\, dz\, dt, \tag{18}$$

$$\frac{1}{|\xi_1|} \int_{Q_T} |a_1(t)||\omega||\Delta\varphi|\, dz\, dt \le \varepsilon \int_{Q_T} |\omega|^p \varphi\, dz\, dt + C \int_{Q_T} |a_1(t)|^{\frac{p}{p-1}} \varphi^{\frac{-1}{p-1}} |\Delta\varphi|^{\frac{p}{p-1}}\, dz\, dt, \tag{19}$$

$$\frac{2}{|\xi_1|} \int_{Q_T} |\omega||a_2(t)|\varphi\, dz\, dt \le \varepsilon \int_{Q_T} |\omega|^p \varphi\, dz\, dt + C \int_{Q_T} |a_2(t)|^{\frac{p}{p-1}} \varphi\, dz\, dt. \tag{20}$$

Combining (18), (19) and (20) in the inequality (17), we get

$$\begin{aligned}
&(1-3\varepsilon) \int_{Q_T} |\omega|^p \varphi\, dz\, dt + \frac{1}{\xi_1} \int_{\mathbb{R}^N} (r_\alpha \varpi_1 - s_\alpha \varpi_2)\, I_T^{1-\alpha}\varphi(0,z)\, dz \\
&\le C \left(\int_{Q_T} \varphi^{\frac{-1}{p-1}} |(I_T^{1-\alpha}\varphi)_t|^{\frac{p}{p-1}}\, dz\, dt + \int_{Q_T} |a_1(t)|^{\frac{p}{p-1}} \varphi^{\frac{-1}{p-1}} |\Delta\varphi|^{\frac{p}{p-1}}\, dz\, dt \right. \\
&\qquad \left. + \int_{Q_T} |a_2(t)|^{\frac{p}{p-1}} \varphi\, dz\, dt \right),
\end{aligned}$$

which means

$$\frac{1}{\xi_1} \int_{\mathbb{R}^N} (r_\alpha \varpi_1 - s_\alpha \varpi_2)\, I_T^{1-\alpha}\varphi(0,z)\, dz \le C \left(A_\varphi(T) + B_\varphi(T) + C_\varphi(T) \right), \tag{21}$$

where we use the following notation, to compact the formula:

$$\begin{aligned}
A_\varphi(T) &:= \int_{Q_T} \varphi^{\frac{-1}{p-1}} |(I_T^{1-\alpha}\varphi)_t|^{\frac{p}{p-1}}\, dz\, dt, \\
B_\varphi(T) &:= \int_{Q_T} |a_1(t)|^{\frac{p}{p-1}} \varphi^{\frac{-1}{p-1}} |\Delta\varphi|^{\frac{p}{p-1}}\, dz\, dt, \\
C_\varphi(T) &:= \int_{Q_T} |a_2(t)|^{\frac{p}{p-1}} \varphi\, dz\, dt.
\end{aligned}$$

To continue the proof we use a suitable cut-off function $f \in C^\infty([0,\infty))$ assuming values in the interval $[0,1]$ with

$$f(\sigma) = \begin{cases} 1 & \text{if} \quad \sigma \in [0,1], \\ 0 & \text{if} \quad 2 \le \sigma. \end{cases}$$

Now, we work with the $C^2(Q_T, \mathbb{R})$-function $x(t,z)$ given as

$$x(t,z) = \vartheta(t) w(z), \quad (t,z) \in Q_T, \text{ for } T \gg 1, \tag{22}$$

where

$$\begin{aligned}
\vartheta(t) &= T^{-\ell}(T-t)^\ell, \quad t \in [0,T], \\
w(z) &= f\left(\frac{|z|^2}{T^{2\rho}}\right)^\ell, \quad z \in \mathbb{R}^N.
\end{aligned}$$

Here, we need $\ell \gg 1$ with $\rho > 0$ to be chosen opportunely.

Since $x \geq 0$ and $\mathrm{supp}_z x \subset\subset \mathbb{R}^N$, then we can set $\varphi = x$ in (21) to obtain

$$\frac{1}{\xi_1}\int_{\mathbb{R}^N}(r_\alpha\varpi_1 - s_\alpha\varpi_2)\, I_T^{1-\alpha}x(0,z)\,dz \leq C\left(A_x(T)+B_x(T)+C_x(T)\right),\ T \gg 1. \tag{23}$$

The definition of x in (22) (recall $t \in [0,T]$ and $z \in \mathbb{R}^N$) leads us to

$$\begin{aligned}
I_T^{1-\alpha}x(t,z) &= w(z)(I_T^{1-\alpha}\vartheta)(t)\\
&= w(z)\frac{1}{\Gamma(1-\alpha)}\int_t^T (u-t)^{-\alpha}\vartheta(u)\,du\\
&= w(z)\frac{1}{\Gamma(1-\alpha)}\int_t^T (u-t)^{-\alpha}T^{-\ell}(T-u)^{\ell}\,du\\
&= w(z)\frac{T^{-\ell}}{\Gamma(1-\alpha)}\int_t^T \left((T-t)-(T-u)\right)^{-\alpha}(T-u)^{\ell}\,du\\
&= w(z)\frac{T^{-\ell}(T-t)^{-\alpha}}{\Gamma(1-\alpha)}\int_t^T \left(1-\frac{T-u}{T-t}\right)^{-\alpha}(T-u)^{\ell}\,du.
\end{aligned}$$

Setting $s = \dfrac{T-u}{T-t}$ and introducing the Beta function

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}, \quad u,v>0,$$

we deduce that

$$\begin{aligned}
I_T^{1-\alpha}x(t,z) &= w(z)\frac{T^{-\ell}(T-t)^{-\alpha+\ell+1}}{\Gamma(1-\alpha)}\int_0^1(1-s)^{-\alpha}s^{\ell}\,ds\\
&= w(z)T^{-\ell}(T-t)^{-\alpha+\ell+1}\frac{B(1-\alpha,\ell+1)}{\Gamma(1-\alpha)}\\
&= \frac{\Gamma(\ell+1)}{\Gamma(2+\ell-\alpha)}T^{-\ell}(T-t)^{-\alpha+\ell+1}w(z).
\end{aligned} \tag{24}$$

If we assume $t=0$ in (24), then we have

$$\frac{1}{\xi_1}\int_{\mathbb{R}^N}(r_\alpha\varpi_1 - s_\alpha\varpi_2)\, I_T^{1-\alpha}x(0,z)\,dz = \frac{\Gamma(\ell+1)}{\Gamma(2+\ell-\alpha)}T^{1-\alpha}\frac{1}{\xi_1}\int_{\mathbb{R}^N}(r_\alpha\varpi_1 - s_\alpha\varpi_2)\,w(z)\,dz,$$

and hence

$$\frac{1}{\xi_1}\int_{\mathbb{R}^N}(r_\alpha\varpi_1 - s_\alpha\varpi_2)\, I_T^{1-\alpha}x(0,z)\,dz = C_{\ell,\alpha}T^{1-\alpha}\left[\frac{1}{\xi_1}\int_{\mathbb{R}^N}(r_\alpha\varpi_1 - s_\alpha\varpi_2)\,f\left(\frac{|z|^2}{T^{2\rho}}\right)^{\ell}dz\right], \tag{25}$$

where $C_{\ell,\alpha} = \frac{\Gamma(\ell+1)}{\Gamma(2+\ell-\alpha)}$. Taking the absolute value, by (24) we have

$$|(I_T^{1-\alpha}x)_t| = \widetilde{C_{\ell,\alpha}}T^{-\ell}(T-t)^{\ell-\alpha}w(z), \quad (t,z)\in Q_T,$$

where $\widetilde{C_{\ell,\alpha}} = (1+\ell-\alpha)C_{\ell,\alpha}$. Using appropriate changes of variables and some manipulations, we get

$$\begin{aligned}
& A_x(T) = \int_{Q_T} x^{\frac{-1}{p-1}} |(I_T^{1-\alpha} x)_t|^{\frac{p}{p-1}}\, dz\, dt \\
& = C\left(T^{-\ell}\int_0^T (T-t)^{\ell-\frac{\alpha p}{p-1}}\, dt\right)\left(\int_{\mathbb{R}^N} w(z)\, dz\right) \\
& = C\left(T^{-\ell}\int_0^T (T-t)^{\ell-\frac{\alpha p}{p-1}}\, dt\right)\left(\int_{\mathbb{R}^N} f\left(\frac{|z|^2}{T^{2\rho}}\right)^{\ell} dz\right) \\
& = CT^{N\rho-\frac{\alpha p}{p-1}+1}\left(\int_0^1 (1-s)^{\ell-\frac{\alpha p}{p-1}}\, ds\right)\left(\int_{\mathbb{R}^N} f(|y|^2)^{\ell}\, dz\right), \\
\Rightarrow\quad & A_x(T) \le CT^{N\rho-\frac{\alpha p}{p-1}+1}.
\end{aligned} \tag{26}$$

The definition of x (see (22)) leads us to

$$\begin{aligned}
|\Delta x(t,z)| &= \vartheta(t)|\Delta w(z)| \\
&= \vartheta(t)\left|\Delta f\left(\frac{|z|^2}{T^{2\rho}}\right)^{\ell}\right| \\
&\le C\vartheta(t)T^{-2\rho} f\left(\frac{|z|^2}{T^{2\rho}}\right)^{\ell-2}.
\end{aligned}$$

Therefore, one obtains

$$\begin{aligned}
& B_x(T) = \int_{Q_T} |a_1(t)|^{\frac{p}{p-1}} x^{\frac{-1}{p-1}} |\Delta x|^{\frac{p}{p-1}}\, dz\, dt \\
& \le CT^{\frac{-2\rho p}{p-1}} \int_{Q_T} |a_1(t)|^{\frac{p}{p-1}} \vartheta(t) f\left(\frac{|z|^2}{T^{2\rho}}\right)^{\ell-\frac{2p}{p-1}} dz\, dt \\
& = CT^{\frac{-2\rho p}{p-1}}\left(\int_0^T T^{-\ell}(T-t)^{\ell}|f(t)|^{\frac{p}{p-1}}\, dt\right)\left(\int_{\mathbb{R}^N} f\left(\frac{|z|^2}{T^{2\rho}}\right)^{\ell-\frac{2p}{p-1}} dz\right) \\
& = CT^{\frac{-2\rho p}{p-1}+1+N\rho}\left(\int_0^1 (1-s)^{\ell}|a_1(Ts)|^{\frac{p}{p-1}}\, ds\right)\left(\int_{\mathbb{R}^N} f(|y|^2)^{\ell-\frac{2p}{p-1}}\, dy\right), \\
\Rightarrow\quad & B_x(T) \le CT^{\frac{-2\rho p}{p-1}+1+N\rho}\int_0^1 |a_1(Ts)|^{\frac{p}{p-1}}\, ds.
\end{aligned} \tag{27}$$

Now, we are going to estimate $C_x(T)$. Again, using (22), we get

$$\begin{aligned}
& C_x(T) = \int_{Q_T} |a(t)|^{\frac{p}{p-1}} x\, dz\, dt \\
& = \left(\int_0^T |a_2(t)|^{\frac{p}{p-1}} T^{-\ell}(T-t)^{\ell}\, dt\right)\left(\int_{\mathbb{R}^N} f\left(\frac{|z|^2}{T^{2\rho}}\right)^{\ell} dz\right) \\
& = T^{N\rho+1}\left(\int_0^1 (1-s)^{\ell}|a_2(Ts)|^{\frac{p}{p-1}}\, ds\right)\left(\int_{\mathbb{R}^N} f(|y|^2)^{\ell}\, dy\right), \\
\Rightarrow\quad & C_x(T) \le CT^{N\rho+1}\int_0^1 |a_2(Ts)|^{\frac{p}{p-1}}\, ds.
\end{aligned} \tag{28}$$

The combined effects of (23), (25), (26), (27) and (28) give us

$$\begin{aligned}&\frac{1}{\xi_1}\int_{\mathbb{R}^N}(r_\alpha\varpi_1 - s_\alpha\varpi_2)\, f\left(\frac{|z|^2}{T^{2\rho}}\right)^\ell\, dz\\ &\leq C\left(T^{N\rho-\frac{\alpha p}{p-1}+\alpha} + T^{\frac{-2\rho p}{p-1}+\alpha+N\rho}\int_0^1 |a_1(Ts)|^{\frac{p}{p-1}}\, ds + T^{N\rho+\alpha}\int_0^1 |a_2(Ts)|^{\frac{p}{p-1}}\, ds\right).\end{aligned}$$

If we choose $\rho = \frac{\alpha}{2}$, then the above inequality reduces to

$$\begin{aligned}&\frac{1}{\xi_1}\int_{\mathbb{R}^N}(r_\alpha\varpi_1 - s_\alpha\varpi_2)\, f\left(\frac{|z|^2}{T^{2\rho}}\right)^\ell\, dz\\ &\leq C\left(T^{\alpha\left(\frac{N+2}{2}-\frac{p}{p-1}\right)} + T^{\alpha\left(\frac{N+2}{2}-\frac{p}{p-1}\right)}\int_0^1 |a_1(Ts)|^{\frac{p}{p-1}}\, ds + T^{\alpha\left(\frac{N+2}{2}\right)}\int_0^1 |a_2(Ts)|^{\frac{p}{p-1}}\, ds\right)\\ &= CT^{\alpha\left(\frac{N+2}{2}-\frac{p}{p-1}\right)} + T^{\alpha\left(\frac{N+2}{2}\right)}\left(T^{\frac{-\alpha p}{p-1}}\int_0^1 |a_1(Ts)|^{\frac{p}{p-1}}\, ds + \int_0^1 |a_2(Ts)|^{\frac{p}{p-1}}\, ds\right).\end{aligned} \tag{29}$$

Finally, we pass to the infimum limit as T goes to infinity in (29), use hypothesis (P_1), (recall $1 < p < 1 + \frac{2}{N}$, $\varpi \in L^1(\mathbb{R}^N, \mathbb{C})$), and the dominated convergence theorem, then we have

$$\frac{1}{\xi_1}\int_{\mathbb{R}^N}(r_\alpha\varpi_1 - s_\alpha\varpi_2)\, dz < 0,$$

which contradicts hypothesis $(P_2)_a$. □

Proof. (P_1) and $(P_2)_b$ hold.

We note that

$$\begin{aligned}&\omega \in L^p_{loc}((0,\infty)\times\mathbb{R}^N, \mathbb{C}) \text{ is a global weak solution to (1) and (2)},\\ \Rightarrow\quad &\nu = -i\omega \text{ is a global weak solution to (6) and (7) (by Remark 1-(ii))}.\end{aligned}$$

Now, the existence of no global weak solution to auxiliary problem (6) and (7) can be established by contradiction, on the same lines of the proof of previous Case 1, with hypothesis $(P_2)_a$ in the form

$(\widetilde{P}_2)_a$ $\quad \widetilde{\xi}_1\int_{\mathbb{R}^N}(r_\alpha\widetilde{\varpi}_1 - s_\alpha\widetilde{\varpi}_2)\, dz > 0.$

Finally, we point out that

$$\begin{cases}\widetilde{\xi} = -i\xi & \Rightarrow \quad \widetilde{\xi}_1 = \xi_2,\\ \widetilde{\varpi} = -i\varpi & \Rightarrow \quad \widetilde{\varpi}_2 = -\varpi_1.\end{cases}$$

We conclude that $(\widetilde{P}_2)_a$ and $(P_2)_b$ are equivalent.

In both the cases, we can exclude the existence of a global weak solution to problems (1) and (2). □

4. Conclusions

The interest for nonlinear Schrödinger equations is dictated by various applications in physics. Two important directions of research are aimed to prove:

- Existence and nonexistence (blow-up) results of global weak solutions.
- Improve the analysis of the regularity and asymptotic behavior of solutions.

The existing literature provides some basic approaches and other useful tools to develop a full analysis of the dynamical properties of specific classes of nonlinear Schrödinger equations. The fractional in time nonlinear Schrödinger equation provides us with a general point of view on the relationship between the effects of absorption and dispersion terms, in optical soliton systems evolving over Lévy trajectories. This subject may be relevant for the approximate and exact controllability of certain nonlinear equations and their solutions. Here, we obtained the nonexistence of global weak solutions to problems (1) and (2), adopting the Pohozaev nonlinear capacitary method. Consequently, we discussed in details some particular choices of absorption and dispersion terms, also with the help of examples.

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