SINGULAR QUASILINEAR ELLIPTIC SYSTEMS INVOLVING GRADIENT TERMS

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Abstract. In this paper we establish the existence of at least one smooth positive solution for a singular quasilinear elliptic system involving gradient terms. The approach combines the sub-supersolutions method and Schauder's fixed point theorem.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a bounded domain with smooth boundary $\partial\Omega$. We deal with the following quasilinear elliptic system

$$
\text{(P)} \qquad \begin{cases} -\Delta_p u = f(x,u,v,\nabla u,\nabla v) & \text{in } \Omega, \\ -\Delta_q v = g(x,u,v,\nabla u,\nabla v) & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u, v = 0 & \text{on } \partial\Omega, \end{cases}
$$

where Δ_p (resp. Δ_q) stands for the p-Laplacian (resp. q-Laplacian) differential operator on $W_0^{1,p}(\Omega)$ (resp. $W_0^{1,q}(\Omega)$) with $1 < p, q < N$.

The nonlinearity terms $f(x,u,v,\nabla u,\nabla v)$ and $g(x,u,v,\nabla u,\nabla v)$, which are often expressed as dealing with convection terms, can exhibit singularities when the variables u and v approach zero.

Specifically, we assume that $f, g : \Omega \times (0,+\infty) \times (0,+\infty) \times \mathbb{R}^{2N} \to (0,+\infty)$ are Carathéodory functions, that is, $f(\cdot, s_1, s_2, \xi_1, \xi_2)$ and $g(\cdot, s_1, s_2, \xi_1, \xi_2)$ are measurable for every $(s_1, s_2, \xi_1, \xi_2) \in (0,+\infty) \times (0,+\infty) \times \mathbb{R}^{2N}$ while $f(x,\cdot,\cdot,\cdot,\cdot)$ and $g(x,\cdot,\cdot,\cdot,\cdot)$ are continuous functions for a.e. $x \in \Omega$, and are subjected to the following hypotheses:

H(f) : *There exist constants $M_1, m_1 > 0$ and $-1 < \alpha_1 < 0 < \beta_1, \gamma_1, \theta_1$ with*

$$
0 \leq \alpha_1 + \beta_1 < -\alpha_1 + \beta_1 < p - 1 \quad \textit{and} \quad \max\{\gamma_1, \theta_1\} < p - 1, \textit{such that}
$$

$$
m_1 s_1^{\alpha_1} s_2^{\beta_1} \leq f(x, s_1, s_2, \xi_1, \xi_2) \leq M_1 s_1^{\alpha_1} s_2^{\beta_1} + |\xi_1|^{\gamma_1} + |\xi_2|^{\theta_1},
$$

for a.e. $x \in \Omega$, for all $s_1, s_2 > 0$ and for all $\xi_1, \xi_2 \in \mathbb{R}^N$.

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H(g) : *There exist constants* $M_2, m_2 > 0$ *and* $-1 < \beta_2 < 0 < \alpha_2, \gamma_2, \theta_2$ *with*

$$0 \leq \alpha_2 + \beta_2 < \alpha_2 - \beta_2 < q - 1 \quad \text{and} \quad \max\{\gamma_2, \theta_2\} < q - 1, \text{such that}$$

$$m_2 s_1^{\alpha_2} s_2^{\beta_2} \leq g(x, s_1, s_2, \xi_1, \xi_2) \leq M_2 s_1^{\alpha_2} s_2^{\beta_2} + |\xi_1|^{\gamma_2} + |\xi_2|^{\theta_2},$$

for a.e. $x \in \Omega$, *for all* $s_1, s_2 > 0$ *and for all* $\xi_1, \xi_2 \in \mathbb{R}^N$.

The main interest of this work lies in the dependence of the right hand side terms on the solution and its gradient. In particular, the presence of the latter makes nonstandard the applications of some classical tools of nonlinear analysis to study problem (P), as for instance, sub-super solutions and fixed point methods. Furthermore, the imposed hypotheses do not guarantee that the structure of the system is variational. Thus, also variational methods cannot be applied, at least directly.

Another important aspect of problem (P) is that the convection terms can exhibit singularities when the variables u and v approach zero. This occurs under hypotheses H(f) and H(g) where exponents α_1 and β_2 are allowed to be negative.

Actually, according to our knowledge, singular system (P) was examined only in [28] where the system is supposed to have a competitive structure. This means that the nonlinearities f and g are not increasing with respect to v and u, respectively. Beside that, the singularities appear in both the solution and its gradient through some specific growth conditions. It is worth pointing out that the assumptions imposed therein, precisely (1.2)-(1.5), are not satisfied for system (P) in our setting. Indeed, our hypotheses are compatible with the complementary situation called cooperative structure that is f and g are increasing with respect to v and u, respectively.

The semilinear case (i.e., $p = q = 2$) for a class of singular systems with convection terms was examined by Alves, Carriao and Faria [3], and by Alves and Moussaoui [4], by essentially using the linearity of the principal part. For singular elliptic systems without gradient terms, we refer to Alves and Corrêa [1], Alves, Corrêa and Gonçalves [2], Chu, Hai and Shivaji [9], El Manouni, Perera and Shivaji [13], Ghergu [15, 16], Hernández, Mancebo and Vega [19], Montenegro and Suarez [23], Motreanu and Moussaoui [25, 26, 27].

For completeness, on these topics, we mention the pioneering work [5] for elliptic equations with convection terms and the attractive introduction [11] on the elliptic systems for $p = 2$.

Here, $d(x)$ denotes the distance from a point $x \in \overline{\Omega}$ to the boundary $\partial\Omega$, where $\overline{\Omega} = \Omega \cup \partial\Omega$ is the closure of $\Omega \subset \mathbb{R}^N$, see (2.1).

Our main result is the following theorem which ensures the existence and localization of at least one smooth positive solution for the system (P).

Theorem 1. *Assume* H(f) *and* H(g) *hold. Then problem* (P) *admits at least one positive solution* (u, v) *in* $C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$ *satisfying*

$$\tilde{c}_0 d(x) \leq u(x) \leq \tilde{c}_1 d(x) \quad \text{and} \quad \tilde{c}_0' d(x) \leq v(x) \leq \tilde{c}_1' d(x) \ \text{in } \Omega, \tag{1.1}$$

for some positive constants $\tilde{c}_0, \tilde{c}_0', \tilde{c}_1$ *and* $\tilde{c}_1'$.

The proof of Theorem 1 is chiefly based on a suitable combination of the sub-supersolution method together with Schauder's fixed point Theorem. However, the sub-supersolution method cannot be directly implemented.

Specifically, the definition of sub-supersolutions pairs for system (P) (see [8]) seems to be hardly applicable because, a priori, no conclusions can be drawn on the comparison of the gradient of two comparable functions.

To handle problem (P), first we consider an auxiliary singular system where the nonlinearities do not involve the gradient of the solutions and for which, the sub-supersolution method involving singular terms is applicable, see [20, Theorem 2.1].

Here, we construct the sub and supersolution pair by choosing suitable functions with an adjustment of adequate constants. Then, focusing on the rectangle formed by these functions, we prove the existence of a smallest and a biggest positive solutions of the auxiliary problem (Theorem 2). The arguments are based on the Hardy-Sobolev inequality, Zorn's Lemma and the S_+-property of the negative p-Laplacian operator on $W_0^{1,p}(\Omega)$.

Next, we apply Theorem 2 to a second auxiliary system $P_{(z_1,z_2)}$, see Section 3, where, roughly speaking, the convention terms are "frozen".

Thereby, owing to the global gradient estimates (3.6), see also [10, Theorem 3.1], [6, Lemma 1], these allow to define a suitable operator whose fixed points, obtained via Schauder's fixed point theorem, are solutions of (P).

For sake of clarity, we mention that a similar approach is adopted for a single $p-$Laplacian equation with convention term and without singularity with respect to the solution in [14], where the Schaefer's fixed point Theorem is applied.

The rest of the paper is organized as follows. Section 2 presents auxiliary results related to sub-supersolutions and extremal solutions. Section 3 deals with the existence of a smallest positive solution for an auxialiary system. Section 4 contains the proof of the main result.

2. Preliminary results

Given $1 < p < +\infty$, the spaces $L^p(\Omega)$ and $W_0^{1,p}(\Omega)$ are endowed with the usual norms $\|u\|_p = (\int_\Omega |u|^p \; dx)^{1/p}$ and $\|u\|_{1,p} = (\int_\Omega |\nabla u|^p \; dx)^{1/p}$, respectively. Denote by $p' = \frac{p}{p-1}$ and $q' = \frac{q}{q-1}$. We will also utilize the spaces $C(\overline{\Omega})$ and $C_0^{1,\beta}(\overline{\Omega}) = \{u \in C^{1,\beta}(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}$ with $\beta \in (0,1)$.
For later use, we denote by $\lambda_{1,p}$ and $\lambda_{1,q}$ the first eigenvalue of $-\Delta_p$ on $W_0^{1,p}(\Omega)$ and of $-\Delta_q$ on $W_0^{1,q}(\Omega)$, respectively.
Let $\phi_{1,p}$ be the positive normalized eigenfunction of $-\Delta_p$ corresponding to $\lambda_{1,p}$, that is $-\Delta_p \phi_{1,p} = \lambda_{1,p}\phi_{1,p}^{p-1}$ in Ω, $\phi_{1,p} = 0$ on $\partial\Omega$. Similarly, let $\phi_{1,q}$ be the positive normalized eigenfunction of $-\Delta_q$ corresponding to $\lambda_{1,q}$, that is $-\Delta_q \phi_{1,q} = \lambda_{1,q}\phi_{1,q}^{q-1}$ in Ω, $\phi_{1,q} = 0$ on $\partial\Omega$. Denoted with

$$d(x) = d(x, \partial\Omega) = \inf_{y\in\partial\Omega} |x - y|, \text{for all } x \in \Omega, \tag{2.1}$$

the strong maximum principle ensures the existence of positive constants l_1, l_2, $\hat{l}$ and l such that (see also [17])

$$l_1\phi_{1,p}(x) \leq \phi_{1,q}(x) \leq l_2\phi_{1,p}(x) \text{ for all } x \in \Omega, \tag{2.2}$$

and

$$\hat{l}d(x) \geq \phi_{1,p}(x), \phi_{1,q}(x) \geq ld(x) \text{ for all } x \in \Omega. \tag{2.3}$$

Throughout the paper, if (u_1, v_1), $(u_2, v_2) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ are such that $u_1 \leq u_2$ and $v_1 \leq v_2$ a.e. in Ω we will write $(u_1, v_1) \leq (u_2, v_2)$ and we will use the notation

$$[u_1, u_2] \times [v_1, v_2] =$$
$$= \{(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega) : \ u_1 \leq u \leq u_2, \ v_1 \leq v \leq v_2, \text{ a.e. in } \Omega\}.$$

A (weak) solution of (P) is any pair $(u, v) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\varphi\, dx = \int_\Omega f(x, u, v, \nabla u, \nabla v)\varphi\, dx,$$
$$\int_\Omega |\nabla v|^{q-2}\nabla v\nabla\psi\, dx = \int_\Omega g(x, u, v, \nabla u, \nabla v)\psi\, dx,$$

for all $(\varphi, \psi) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$.

We will study auxiliary problems with not convection terms, for this reason let us consider the following quasilinear elliptic problem

$$(P_{(f_1,f_2)}) \qquad \begin{cases} -\Delta_p u = f_1(x, u, v) & \text{in } \Omega, \\ -\Delta_q v = f_2(x, u, v) & \text{in } \Omega, \\ u, v > 0 & \text{in } \Omega, \\ u, v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f_i : \Omega \times (0, +\infty) \times (0, +\infty) \to \mathbb{R}$, $i = 1, 2$, are Carathéodory functions which can exhibit singularities near zero.

Recall that $(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v}) \in (W^{1,p}(\Omega) \cap L^\infty(\Omega)) \times (W^{1,q}(\Omega) \cap L^\infty(\Omega))$ form a pair of a sub-supersolution for $(P_{(f_1,f_2)})$ if $(\underline{u}, \underline{v}) \leq (\overline{u}, \overline{v})$ and

$$\begin{cases} \int_\Omega |\nabla \underline{u}|^{p-2} \nabla \underline{u}\nabla\varphi \ dx - \int_\Omega f_1(x, \underline{u}, w_2)\varphi \ dx \leq 0 \\ \int_\Omega |\nabla \underline{v}|^{q-2} \nabla \underline{v}\nabla\psi \ dx - \int_\Omega f_2(x, w_1, \underline{v})\psi \ dx \leq 0, \end{cases}$$

$$\begin{cases} \int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u}\nabla\varphi \ dx - \int_\Omega f_1(x, \overline{u}, w_2)\varphi \ dx \geq 0 \\ \int_\Omega |\nabla \overline{v}|^{q-2} \nabla \overline{v}\nabla\psi \ dx - \int_\Omega f_2(x, w_1, \overline{v})\psi \ dx \geq 0, \end{cases}$$

for all $(\varphi, \psi) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ with $\varphi, \psi \geq 0$ a.e. in Ω and for all $(w_1, w_2) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$.

Lemma 1. *Let* $(\underline{u}_i, \underline{v}_i), (\overline{u}_i, \overline{v}_i) \in (W^{1,p}(\Omega) \cap L^\infty(\Omega)) \times (W^{1,q}(\Omega) \cap L^\infty(\Omega))$, *for* $i = 1, 2$. *Put*

$$\underline{u} = \max\{\underline{u}_1, \underline{u}_2\}, \quad \underline{v} = \max\{\underline{v}_1, \underline{v}_2\},$$
$$\tilde{u} = \min\{\overline{u}_1, \overline{u}_2\}, \quad \tilde{v} = \min\{\overline{v}_1, \overline{v}_2\}$$

and assume that

$$\underline{u} \leq \tilde{u}, \quad \underline{v} \leq \tilde{v}.$$

Moreover, suppose that

$$f_1(x, w_1, w_2) \in W^{-1,p'}(\Omega), \quad f_2(x, w_1, w_2) \in W^{-1,q'}(\Omega)$$

for every $(w_1, w_2) \in [\underline{u}, \tilde{u}] \times [\underline{v}, \tilde{v}]$ *and* $(\underline{u}_i, \underline{v}_i)$, $(\overline{u}_i, \overline{v}_i)$ $(i = 1, 2)$ *form two pairs of sub-supersolutions for problem* $(P_{(f_1,f_2)})$.
Then $(\underline{u}, \underline{v})$, $(\tilde{u}, \tilde{v}) \in (W^{1,p}(\Omega) \cap L^\infty(\Omega)) \times (W^{1,q}(\Omega) \cap L^\infty(\Omega))$ *form also a pair of sub-supersolution for the problem* $(P_{(f_1,f_2)})$.

Proof. Inspired by the proof of [24, Lemma 3], for a fixed $\varepsilon > 0$, let us define the truncation function $\xi_\varepsilon(s) = \max\{-\varepsilon, \min\{s, \varepsilon\}\}$ for $s \in \mathbb{R}$. It is shown in [22] that $\xi_\varepsilon((\overline{u}_1 - \overline{u}_2)^-), \xi_\varepsilon((\underline{u}_1 - \underline{u}_2)^+) \in W^{1,p}(\Omega)$,

$$\nabla \xi_\varepsilon((\overline{u}_1 - \overline{u}_2)^-) = \xi'_\varepsilon((\overline{u}_1 - \overline{u}_2)^-) \nabla (\overline{u}_1 - \overline{u}_2)^-$$

and

$$\nabla \xi_\varepsilon((\underline{u}_1 - \underline{u}_2)^+) = \xi'_\varepsilon((\underline{u}_1 - \underline{u}_2)^+) \nabla (\underline{u}_1 - \underline{u}_2)^+.$$

For any test function $\varphi \in C^1_c(\Omega)$ with $\varphi \geq 0$, it holds

$$\left\langle -\Delta_p \overline{u}_1, \xi_\varepsilon((\overline{u}_1 - \overline{u}_2)^-)\varphi \right\rangle \geq \int_\Omega f_1(x, \overline{u}_1, \hat{w}_2) \xi_\varepsilon((\overline{u}_1 - \overline{u}_2)^-)\varphi \, dx, \tag{2.4}$$

$$\left\langle -\Delta_p \underline{u}_1, \xi_\varepsilon((\underline{u}_1 - \underline{u}_2)^+)\varphi \right\rangle \leq \int_\Omega f_1(x, \underline{u}_1, \hat{w}_2) \xi_\varepsilon((\underline{u}_1 - \underline{u}_2)^+)\varphi \, dx, \tag{2.5}$$

for all $\hat{w}_2 \in W^{1,q}(\Omega)$ with $\underline{v}_1 \leq \hat{w}_2 \leq \overline{v}_1$, and

$$\left\langle -\Delta_p \overline{u}_2, (\varepsilon - \xi_\varepsilon((\overline{u}_1 - \overline{u}_2)^-))\varphi \right\rangle \geq \int_\Omega f_1(x, \overline{u}_2, \check{w}_2) \left(\varepsilon - \xi_\varepsilon((\overline{u}_1 - \overline{u}_2)^-)\right) \varphi \, dx, \tag{2.6}$$

$$\left\langle -\Delta_p \underline{u}_2, (\varepsilon - \xi_\varepsilon((\underline{u}_1 - \underline{u}_2)^+))\varphi \right\rangle \leq \int_\Omega f_1(x, \underline{u}_2, \check{w}_2) \left(\varepsilon - \xi_\varepsilon((\underline{u}_1 - \underline{u}_2)^+)\right) \varphi \, dx, \tag{2.7}$$

for all $\check{w}_2 \in W^{1,q}(\Omega)$ with $\underline{v}_2 \leq \check{w}_2 \leq \overline{v}_2$. On the other hand, using the monotonicity of the p-Laplacian operator, we get

$$\begin{aligned} &\langle -\Delta_p \overline{u}_1, \xi_\varepsilon((\overline{u}_1 - \overline{u}_2)^-)\varphi \rangle + \langle -\Delta_p \overline{u}_2, (\varepsilon - \xi_\varepsilon((\overline{u}_1 - \overline{u}_2)^-))\varphi \rangle \\ &\leq \textstyle\int_\Omega |\nabla \overline{u}_1|^{p-2} (\nabla \overline{u}_1, \nabla \varphi)_{\mathbb{R}^N} \xi_\varepsilon((\overline{u}_1 - \overline{u}_2)^-) \, dx \\ &+ \textstyle\int_\Omega |\nabla \overline{u}_2|^{p-2} (\nabla \overline{u}_2, \nabla \varphi)_{\mathbb{R}^N} \left(\varepsilon - \xi_\varepsilon((\overline{u}_1 - \overline{u}_2)^-)\right) \, dx \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} &\langle -\Delta_p \underline{u}_1, (\xi_\varepsilon((\underline{u}_1 - \underline{u}_2)^+))\varphi \rangle + \langle -\Delta_p \underline{u}_2, \varepsilon - \xi_\varepsilon((\underline{u}_1 - \underline{u}_2)^+)\varphi \rangle \\ &\geq \textstyle\int_\Omega |\nabla \underline{u}_1|^{p-2} (\nabla \underline{u}_1, \nabla \varphi)_{\mathbb{R}^N} \xi_\varepsilon((\underline{u}_1 - \underline{u}_2)^+) \, dx \\ &+ \textstyle\int_\Omega |\nabla \underline{u}_2|^{p-2} (\nabla \underline{u}_2, \nabla \varphi)_{\mathbb{R}^N} \left(\varepsilon - \xi_\varepsilon((\underline{u}_1 - \underline{u}_2)^+)\right) \, dx. \end{aligned} \tag{2.9}$$

Then, gathering (2.4) together with (2.6) and (2.5) together with (2.7), by means of (2.8) and (2.9), one gets

$$\begin{aligned}&\textstyle\int_\Omega |\nabla \overline{u}_1|^{p-2}(\nabla \overline{u}_1, \nabla\varphi)_{\mathbb{R}^N} \frac{1}{\varepsilon}\xi_\varepsilon((\overline{u}_1-\overline{u}_2)^-)\ dx\\ &\textstyle+\int_\Omega |\nabla \overline{u}_2|^{p-2}(\nabla \overline{u}_2, \nabla\varphi)_{\mathbb{R}^N} \left(1-\frac{1}{\varepsilon}\xi_\varepsilon((\overline{u}_1-\overline{u}_2)^-)\right)\ dx\\ &\textstyle\geq \int_\Omega f_1(x,\overline{u}_1,w_2)\frac{1}{\varepsilon}\xi_\varepsilon((\overline{u}_1-\overline{u}_2)^-)\varphi\ dx\\ &\textstyle+\int_\Omega f_1(x,\overline{u}_2,w_2)\left(1-\frac{1}{\varepsilon}\xi_\varepsilon((\overline{u}_1-\overline{u}_2)^-)\right)\varphi\ dx,\end{aligned}$$

and

$$\begin{aligned}&\textstyle\int_\Omega |\nabla \underline{u}_1|^{p-2}(\nabla \underline{u}_1, \nabla\varphi)_{\mathbb{R}^N} \frac{1}{\varepsilon}\xi_\varepsilon((\underline{u}_1-\underline{u}_2)^+)\ dx\\ &\textstyle+\int_\Omega |\nabla \underline{u}_2|^{p-2}(\nabla \underline{u}_2, \nabla\varphi)_{\mathbb{R}^N} \left(1-\frac{1}{\varepsilon}\xi_\varepsilon((\underline{u}_1-\underline{u}_2)^+)\right)\ dx\\ &\textstyle\leq \int_\Omega f_1(x,\underline{u}_1,w_2)\frac{1}{\varepsilon}\xi_\varepsilon((\underline{u}_1-\underline{u}_2)^+)\varphi\ dx\\ &\textstyle+\int_\Omega f_1(x,\underline{u}_2,w_2)\left(1-\frac{1}{\varepsilon}\xi_\varepsilon((\underline{u}_1-\underline{u}_2)^+)\right)\varphi\ dx,\end{aligned}$$

for all $w_2 \in W^{1,q}(\Omega)$ such that $\underline{v} \leq w_2 \leq \tilde{v}$ a.e. in Ω. Passing to the limit as $\varepsilon \to 0$ and noticing that

$$\left\{\begin{array}{l}\frac{1}{\varepsilon}\xi_\varepsilon((\overline{u}_1-\overline{u}_2)^- \to \chi_{\{\overline{u}_1<\overline{u}_2\}}(x)\\ \frac{1}{\varepsilon}\xi_\varepsilon((\underline{u}_1-\underline{u}_2)^+ \to \chi_{\{\underline{u}_1>\underline{u}_2\}}(x)\end{array}\right., \text{ a.e. in } \Omega \text{ as } \varepsilon \to 0,$$

where $\chi_{\mathcal{A}}$ is the characteristic function of the set $\mathcal{A}$, we obtain

$$\int_\Omega |\nabla\tilde{u}|^{p-2}\nabla\tilde{u}\nabla\varphi\, dx \geq \int_\Omega f_1(x,\tilde{u},w_2)\varphi\ dx$$

and

$$\int_\Omega |\nabla\underline{u}|^{p-2}\nabla\underline{u}\nabla\varphi\, dx \leq \int_\Omega f_1(x,\underline{u},w_2)\varphi\ dx$$

for all $\varphi \in C_c^1(\Omega)$, $\varphi \geq 0$ a.e. in Ω and for all $w_2 \in W^{1,q}(\Omega)$ within $[\underline{v},\tilde{v}]$ a.e. in Ω.

In the same manner we get

$$\int_\Omega |\nabla\tilde{v}|^{q-2}\nabla\tilde{v}\nabla\psi\, dx \geq \int_\Omega f_2(x,w_1,\tilde{v})\psi\ dx$$

and

$$\int_\Omega |\nabla\underline{v}|^{q-2}\nabla\underline{v}\nabla\psi\, dx \leq \int_\Omega f_2(x,w_1,\underline{v})\psi\, dx$$

for all $\psi \in C_c^1(\Omega)$, $\psi \geq 0$ a.e. in Ω and for all $w_1 \in W^{1,p}(\Omega)$ within $[\underline{u},\tilde{u}]$ a.e. in Ω. Finally, since $C_c^1(\Omega)$ is dense in both $W^{1,p}(\Omega)$ and $W^{1,q}(\Omega)$, we achieve the desired conclusion. □

Theorem 2. *Let* $(\underline{u},\underline{v})$, $(\overline{u},\overline{v}) \in C^1(\overline{\Omega})\times C^1(\overline{\Omega})$ *be a pair of sub-supersolution* $(P_{(f_1,f_2)})$ *with* $\underline{u},\underline{v} \geq c_0 d(x)$ *in* Ω *for some constant* $c_0>0$ *and suppose there exist constants* $k_1,k_2>0$ *and* $-1<\alpha,\beta<0$ *such that*

$$|f_1(x,u,v)| \leq k_1 d(x)^\alpha \text{ and } |f_2(x,u,v)| \leq k_2 d(x)^\beta \tag{2.10}$$

a.e. in Ω *and for every* $(u,v)\in[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$.
Then problem $(P_{(f_1,f_2)})$ *has a smallest solution* (u^*,v^*) *and a biggest solution* (u^+,v^+) *in* $C_0^{1,\gamma}(\overline{\Omega})\times C_0^{1,\gamma}(\overline{\Omega})$ *for certain* $\gamma\in(0,1)$, *within* $[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$.

Proof. We only prove the existence of a smallest positive solution $(u^*,v^*)\in C_0^{1,\gamma}(\overline{\Omega})\times C_0^{1,\gamma}(\overline{\Omega})$ within $[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$. That of a biggest positive solution within $[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$ can be carried out in the similar way.

Denote by S the set of all $(w_1,w_2)\in[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$ that are solutions of $(P_{(f_1,f_2)})$. It is well know from [20, Theorem 2.1] that under assumption (2.10), system $(P_{(f_1,f_2)})$ has a (positive) solution $(u,v)\in C_0^{1,\gamma}(\overline{\Omega})\times C_0^{1,\gamma}(\overline{\Omega})$ for certain $\gamma\in(0,1)$, located in $[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$. Thus, S is not empty. Moreover, let $(u_1,v_1),(u_2,v_2)\in S$. Since $(\underline{u},\underline{v})$, (u_1,v_1) and $(\underline{u},\underline{v})$, (u_2,v_2) form two pairs of sub-supersolution, if we put $(\tilde{u},\tilde{v})=(\min\{u_1,u_2\},\min\{v_1,v_2\})\in(W_0^{1,p}(\Omega)\cap L^\infty(\Omega))\times(W_0^{1,q}(\Omega)\cap L^\infty(\Omega))$, by virtue of Lemma 1, $(\underline{u},\underline{v})$, $(\tilde{u},\tilde{v})$ form a pair of sub-supersolution for $(P_{(f_1,f_2)})$. Then, owing to [20, Theorem 2.1], there exists a solution of $(P_{(f_1,f_2)})$ in $([\underline{u},\tilde{u}]\cap C_0^1(\overline{\Omega}))\times([\underline{v},\tilde{v}]\cap C_0^1(\overline{\Omega}))$, which proves that S is downward directed.

Now, let us consider a chain C in S. Then there is a sequence $\{(u_k,v_k)\}_{k\geq1}\subset C$ such that $\inf C=\inf_{k\geq1}(u_k,v_k)$ (see [12, pag. 336]) and it is not restrictive assume $\{(u_k,v_k)\}_{k\geq1}$ to be decreasing. Hence, if we put $(\hat{u},\hat{v})=\inf C$, one has that $u_k\to\hat{u}$ and $v_k\to\hat{v}$ a.e. in Ω, that is

$$(\hat{u},\hat{v})\in[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]. \tag{2.11}$$

Moreover, because (u_k,v_k) for $k\geq1$ are solutions of $(P_{(f_1,f_2)})$ we have

$$\|\nabla u_k\|_p^p=\int_\Omega f_1(x,u_k,v_k)u_k\,dx\leq k_1\int_\Omega d(x)^\alpha\overline{u}\,dx \tag{2.12}$$

and

$$\|\nabla v_k\|_q^q=\int_\Omega f_2(x,u_k,v_k)v_k\,dx\leq k_2\int_\Omega d(x)^\beta\overline{v}\,dx. \tag{2.13}$$

Since $-1<\alpha,\beta<0$, by virtue of the Hardy-Sobolev inequality (see, e.g., [1] or [29]), the last integrals in (2.12) and (2.13) are finite which in turn imply that $\{u_k\}$ and $\{v_k\}$ are bounded in $W_0^{1,p}(\Omega)$ and $W_0^{1,q}(\Omega)$, respectively. So, passing to relabelled subsequences and recalling the Rellich embedding theorem, we have

$$(u_k,v_k)\rightharpoonup(\hat{u},\hat{v})\ \text{ in } W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)\,. \tag{2.14}$$

Using $\varphi=u_k-\hat{u}$ and $\psi=v_k-\hat{v}$ as test functions we find that

$$\langle-\Delta_p u_k,u_k-\hat{u}\rangle=\textstyle\int_\Omega f_1(x,u_k,v_k)(u_k-\hat{u})\,dx$$

and

$$\langle-\Delta_q v_k,v_k-\hat{v}\rangle=\textstyle\int_\Omega f_2(x,u_k,v_k)(v_k-\hat{v})\,dx.$$

From (2.10) and since $(u_k,v_k)\in S$ for all $k\in\mathbb{N}$, in view of (2.11) we have

$$f_1(x,u_k,v_k)(u_k-\hat{u})\leq k_1d(x)^\alpha(u_k-\hat{u})\leq 2k_1d(x)^\alpha\|\overline{u}\|_\infty$$

and

$$f_2(x,u_k,v_k)(v_k-\hat{v})\leq k_2d(x)^\beta(v_k-\hat{v})\leq k_2d(x)^\beta\|\overline{v}\|_\infty\,.$$

Thank's to [21, Lemma], $f_1(x,u_k,v_k)(u_k-\hat{u})$ and $f_2(x,u_k,v_k)(v_k-\hat{v})$ are dominated by $L^1(\Omega)$ functions and using the dominated convergence theorem, we obtain

$$\lim_{k\to\infty}\langle -\Delta_p u_k, u_k-\hat{u}\rangle = \lim_{k\to\infty}\langle -\Delta_q v_k, v_k-\hat{v}\rangle = 0.$$

Then the S_+-property of $-\Delta_p$ and $-\Delta_q$ on $W_0^{1,p}(\Omega)$ and $W_0^{1,q}(\Omega)$, respectively, guarantees that

$$(u_k,v_k)\longrightarrow(\hat{u},\hat{v}) \text{ in } W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$$

and therefore $(\hat{u},\hat{v})$ is a positive solution of problem $(P_{(f_1,f_2)})$. Consequently, $(\hat{u},\hat{v})=\inf C$ belongs to S. Then Zorn's Lemma can be applied which provides a minimal element (u^*,v^*) of S. Furthermore, since $S\subset[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$, (2.10) enables us to apply the regularity theory (see [18]) to infer that $(u^*,v^*)\in C_0^{1,\gamma}(\overline{\Omega})\times C_0^{1,\gamma}(\overline{\Omega})$ for some $\gamma\in(0,1)$.

The proof is completed by showing that (u^*,v^*) is the smallest solution of $(P_{(f_1,f_2)})$ in S. To this end, let $(u,v)\in S$. Bearing in mind that S is downward directed, there is $(\mathring{u},\mathring{v})\in S$ with $\mathring{u}\le u^*$, $\mathring{v}\le v^*$ and $\mathring{u}\le u$, $\mathring{v}\le v$. Since (u^*,v^*) is a minimal element of S, it turns out that $(u^*,v^*)=(\mathring{u},\mathring{v})\le(u,v)$. The same reasoning can be used to prove the existence of a biggest solution (u^+,v^+) in $C_0^{1,\gamma}(\overline{\Omega})\times C_0^{1,\gamma}(\overline{\Omega})$, for certain $\gamma\in(0,1)$, within $[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$. This completes the proof. □

3. Auxiliary system

For every $z_1,z_2\in C_0^1(\Omega)$, let us state the auxiliary problem

$$(P_{(z_1,z_2)})\qquad \begin{cases} -\Delta_p u = f(x,u,v,\nabla z_1,\nabla z_2) & \text{in } \Omega,\\ -\Delta_q v = g(x,u,v,\nabla z_1,\nabla z_2) & \text{in } \Omega,\\ u,v>0 & \text{in } \Omega,\\ u,v=0 & \text{on } \partial\Omega.\end{cases}$$

With the aim of finding pairs of sub-supersolutions of problem $(P_{(z_1,z_2)})$, let us define ξ_1 and ξ_2 in $C_0^{1,\beta}(\overline{\Omega})$, $\beta\in(0,1)$, as the unique solutions of the problems

$$(3.1)\qquad \begin{cases} -\Delta_p\xi_1 = 1 & \text{in } \Omega,\\ \xi_1=0 & \text{on } \partial\Omega\end{cases} \quad\text{and}\quad \begin{cases} -\Delta_q\xi_2 = 1 & \text{in } \Omega,\\ \xi_2=0 & \text{on } \partial\Omega,\end{cases}$$

respectively, which are known to satisfy

$$(3.2)\qquad c_0 d(x)\le\xi_1(x)\le c_1 d(x) \quad\text{and}\quad c_0' d(x)\le\xi_2(x)\le c_1' d(x) \quad\text{in } \Omega,$$

with constants $c_i,c_i'>0$ (see [7]).

Set

$$(3.3)\qquad (\overline{u},\overline{v})=C(\xi_1,\xi_2) \quad\text{and}\quad (\underline{u},\underline{v})=C^{-1}\left(\phi_{1,p},\phi_{1,q}\right),$$

where $C>1$ is a constant that will be fixed large enough and denote by

$$(3.4)\qquad M=\max\{\max_\Omega\phi_{1,p},\max_\Omega\phi_{1,q}\}.$$

Obviously, as a consequence of the maximum principle, we have

$$(\overline{u}, \overline{v}) \geq (\underline{u}, \underline{v}) \text{ in } \overline{\Omega} \text{ for } C > 1 \text{ large.} \tag{3.5}$$

Recall from [10, Theorem 3.1], see also [6, Lemma 1], that if $h_1, h_2 \in L^\infty(\Omega)$ and $u \in W_0^{1,p}(\Omega)$, $v \in W_0^{1,q}(\Omega)$ are the weak solutions of problems

$$\begin{cases} -\Delta_p u = h_1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}, \quad \begin{cases} -\Delta_q v = h_2 & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

there exist positive constants $K_p = K_p(p, N, \Omega)$ and $K_q = K_q(q, N, \Omega)$ such that

$$\|\nabla u\|_\infty \leq K_p \|h_1\|_\infty^{\frac{1}{p-1}} \quad \text{and} \quad \|\nabla v\|_\infty \leq K_q \|h_2\|_\infty^{\frac{1}{q-1}}. \tag{3.6}$$

Denote by

$$R_1 = \max\{\|\xi_1\|_{C_0^{1,\beta}(\overline{\Omega})}, K_p\} \text{ and } R_2 = \max\{\|\xi_2\|_{C_0^{1,\beta}(\overline{\Omega})}, K_q\}.$$

Using the functions in (3.3), we introduce the sets

$$\mathcal{K}_1(C) = \left\{ y \in C_0^1(\overline{\Omega}) : \underline{u} \leq y \leq \overline{u} \text{ in } \Omega, \ \|\nabla y\|_\infty \leq CR_1 \right\}$$

and

$$\mathcal{K}_2(C) = \left\{ y \in C_0^1(\overline{\Omega}) : \underline{v} \leq y \leq \overline{v} \text{ in } \Omega, \ \|\nabla y\|_\infty \leq CR_2 \right\},$$

which are closed, bounded and convex in $C_0^1(\overline{\Omega})$.

Proposition 1. *Assume* H(f) *and* H(g). *Then, for $C > 1$ sufficiently large and for every $(z_1, z_2) \in \mathcal{K}_1(C) \times \mathcal{K}_2(C)$, problem $(P_{(z_1,z_2)})$ has a smallest solution $(u^*, v^*)_{(z_1,z_2)}$ in $C_0^{1,\gamma}(\overline{\Omega}) \times C_0^{1,\gamma}(\overline{\Omega})$, for certain $\gamma \in (0,1)$, within $[\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$.*

Proof. The proof is related to Theorem 2. First, let us prove that

<u>Claim</u>: *For every $(z_1, z_2) \in \mathcal{K}_1(C) \times \mathcal{K}_2(C)$, $(\underline{u}, \underline{v})$, $(\bar{u}, \bar{v})$ form a pair of sub-supersolution for $(P_{(z_1,z_2)})$ provided that C is large enough.*

Using H(f), H(g), (3.3), (3.4) and (2.2) we get

$$\begin{aligned} &\underline{u}^{-\alpha_1}\underline{v}^{-\beta_1}(-\Delta_p \underline{u}) = C^{-(p-1-\alpha_1-\beta_1)}\lambda_{1,p}\phi_{1,p}^{p-1-\alpha_1}\phi_{1,q}^{-\beta_1} \\ &\leq C^{-(p-1-\alpha_1-\beta_1)}\lambda_{1,p} l_1^{-\beta_1}\phi_{1,p}^{p-1-\alpha_1-\beta_1} \\ &\leq C^{-(p-1-\alpha_1-\beta_1)}\lambda_{1,p} l_1^{-\beta_1} M^{p-1-\alpha_1-\beta_1} \leq m_1 \text{ in } \Omega \end{aligned}$$

and

$$\begin{aligned} &\underline{u}^{-\alpha_2}\underline{v}^{-\beta_2}(-\Delta_q \underline{v}) = C^{-(q-1-\alpha_2-\beta_2)}\lambda_{1,q}\phi_{1,q}^{q-1-\beta_2}\phi_{1,p}^{-\alpha_2} \\ &\leq C^{-(q-1-\alpha_2-\beta_2)}\lambda_{1,q} l_2^{\alpha_2}\phi_{1,q}^{q-1-\alpha_2-\beta_2} \\ &C^{-(q-1-\alpha_2-\beta_2)}\lambda_{1,q} l_2^{\alpha_2} M^{q-1-\alpha_2-\beta_2} \leq m_2 \text{ in } \Omega, \end{aligned}$$

provided that $C > 1$ is large enough (such that (3.5) holds too). Then, it is readily seen from H(f) and H(g) that

$$\begin{aligned} \int_\Omega |\nabla \underline{u}|^{p-2}\, \nabla \underline{u} \nabla \varphi &\leq m_1 \int_\Omega \underline{u}^{\alpha_1} \underline{v}^{\beta_1} \varphi \leq m_1 \int_\Omega \underline{u}^{\alpha_1} w_2^{\beta_1} \varphi \\ &\leq \int_\Omega f(x, \underline{u}, w_2, \nabla z_1, \nabla z_2) \varphi \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \int_\Omega |\nabla \underline{v}|^{q-2}\, \nabla \underline{v} \nabla \psi &\leq m_2 \int_\Omega \underline{u}^{\alpha_2} \underline{v}^{\beta_2} \psi \leq m_2 \int_\Omega w_1^{\alpha_2} \underline{v}^{\beta_2} \psi \\ &\leq \int_\Omega g(x, w_1, \underline{v}, \nabla z_1, \nabla z_2) \psi, \end{aligned} \tag{3.8}$$

for all $(\varphi, \psi) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ with $\varphi, \psi \geq 0$ in Ω, for all $(w_1, w_2) \in W^{1,p}(\Omega) \times W^{1,q}(\Omega)$ satisfying $\underline{u} \leq w_1 \leq \overline{u}$ and $\underline{v} \leq w_2 \leq \overline{v}$ a.e. in Ω, and for $(z_1, z_2) \in \mathcal{K}_1(C) \times \mathcal{K}_2(C)$.

Now, taking into account (3.1), (3.2), (3.4), H(f) and H(g), we derive the estimates

$$\begin{aligned} &\overline{u}^{-\alpha_1} \overline{v}^{-\beta_1} (-\Delta_p \overline{u} - (CR_1)^{\gamma_1} - (CR_2)^{\theta_1}) \\ &= C^{-\alpha_1-\beta_1} \xi_1^{-\alpha_1} \xi_2^{-\beta_1} (C^{p-1} - (CR_1)^{\gamma_1} - (CR_2)^{\theta_1}) \\ &= C^{p-1-\alpha_1-\beta_1} \xi_1^{-\alpha_1} \xi_2^{-\beta_1} \left[1 - C^{-(p-1)} (C^{\gamma_1} R_1^{\gamma_1} + C^{\theta_1} R_2^{\theta_1}\right] \\ &\geq C^{p-1-\alpha_1-\beta_1} (c_0 d(x))^{-\alpha_1} (c_1' d(x))^{-\beta_1} \left[1 - (C^{\gamma_1-(p-1)} R_1^{\gamma_1} + C^{\theta_1-(p-1)} R_2^{\theta_1})\right] \\ &\geq C^{p-1-\alpha_1-\beta_1} c_0^{-\alpha_1} (c_1')^{-\beta_1} d(x)^{-(\alpha_1+\beta_1)} \left[1 - (C^{\gamma_1-(p-1)} R_1^{\gamma_1} + C^{\theta_1-(p-1)} R_2^{\theta_1})\right] \\ &\geq M_1 \text{ in } \Omega \end{aligned}$$

and

$$\begin{aligned} &\overline{u}^{-\alpha_2} \overline{v}^{-\beta_2} (-\Delta_q \overline{v} - (CR_1)^{\gamma_2} - (CR_2)^{\theta_2}) \\ &\geq C^{q-1-\alpha_2-\beta_2} c_1^{-\alpha_2} (c_0')^{-\beta_2} d(x)^{-(\alpha_2+\beta_2)} \left[1 - (C^{\gamma_2-(q-1)} R_1^{\gamma_2} + C^{\theta_2-(q-1)} R_2^{\theta_2})\right] \\ &\geq M_2 \text{ in } \Omega, \end{aligned}$$

provided that $C > 1$ is sufficiently large. Consequently, it turns out that

$$\begin{aligned} \int_\Omega |\nabla \overline{u}|^{p-2}\, \nabla \overline{u} \nabla \varphi \; dx &\geq \int_\Omega (M_1 \overline{u}^{\alpha_1} \overline{v}^{\beta_1} + |\nabla z_1|^{\gamma_1} + |\nabla z_2|^{\theta_1}) \varphi \; dx \\ &\geq \int_\Omega (M_1 \overline{u}^{\alpha_1} w_2^{\beta_1} + |\nabla z_1|^{\gamma_1} + |\nabla z_2|^{\theta_1}) \varphi \; dx \\ &\geq \int_\Omega f(x, \overline{u}, w_2, \nabla z_1, \nabla z_2) \varphi \; dx \end{aligned} \tag{3.9}$$

and

$$\begin{aligned}(3.10)\int_\Omega |\nabla \overline{v}|^{q-2}\,\nabla \overline{v}\nabla \psi\; dx &\geq \int_\Omega M_2(\overline{u}^{\alpha_2}\overline{v}^{\beta_2}+|\nabla z_1|^{\gamma_2}+|\nabla z_2|^{\theta_2})\psi\; dx\\ &\geq \int_\Omega M_2(w_1^{\alpha_2}\overline{v}^{\beta_2}+|\nabla z_1|^{\gamma_2}+|\nabla z_2|^{\theta_2})\psi\; dx\\ &\geq \int_\Omega g(x,w_1,\overline{v},\nabla z_1,\nabla z_2)\psi\; dx\end{aligned}$$

for all $(\varphi,\psi)\in W_0^{1,p}(\Omega)\times W_0^{1,q}(\Omega)$ with $\varphi,\psi\geq 0$ in Ω, for all $(w_1,w_2)\in W^{1,p}(\Omega)\times W^{1,q}(\Omega)$ satisfying $\underline{u}\leq w_1\leq \overline{u}$ and $\underline{v}\leq w_2\leq \overline{v}$ a.e. in Ω, and for $(z_1,z_2)\in\mathcal{K}_1(C)\times\mathcal{K}_2(C)$.
Putting together (3.7), (3.8), (3.9) and (3.10) we get the Claim.

Furthermore, for every $(z_1,z_2)\in\mathcal{K}_1(C)\times\mathcal{K}_2(C)$ and for every $(u,v)\in[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$, from H$(f)$, H$(g)$, (3.3), (3.4) and (2.3) we have the estimates

$$\begin{aligned}&f(x,u,v,\nabla z_1,\nabla z_2)\leq M_1u^{\alpha_1}v^{\beta_1}+|\nabla z_1|^{\gamma_1}+|\nabla z_2|^{\theta_1}\\ &\leq M_1\underline{u}^{\alpha_1}\overline{v}^{\beta_1}+(CR_1)^{\gamma_1}+(CR_2)^{\theta_1}\\ (3.11)\quad&\leq M_1C^{-\alpha_1+\beta_1}\phi_{1,p}^{\alpha_1}\,\|\xi_2\|_\infty^{\beta_1}+(CR_1)^{\gamma_1}+(CR_2)^{\theta_1}\\ &\leq \phi_{1,p}^{\alpha_1}\left(M_1C^{-\alpha_1+\beta_1}\,\|\xi_2\|_\infty^{\beta_1}+M^{-\alpha_1}(CR_1)^{\gamma_1}+M^{-\alpha_1}(CR_2)^{\theta_1}\right)\\ &\leq C_1d(x)^{\alpha_1}\quad\text{in }\Omega\end{aligned}$$

and

$$\begin{aligned}&g(x,u,v,\nabla z_1,\nabla z_2)\leq M_2u^{\alpha_2}v^{\beta_2}+|\nabla z_1|^{\gamma_2}+|\nabla z_2|^{\theta_2}\\ &\leq M_2\overline{u}^{\alpha_2}\underline{v}^{\beta_2}+(CR_1)^{\gamma_2}+(CR_2)^{\theta_2}\\ (3.12)\quad&\leq M_2C^{\alpha_2-\beta_2}\,\|\xi_1\|_\infty^{\alpha_2}\,\phi_{1,q}^{\beta_2}+(CR_1)^{\gamma_2}+(CR_2)^{\theta_2}\\ &\leq \phi_{1,q}^{\beta_2}\left(M_2C^{\alpha_2+\beta_2}\,\|\xi_1\|_\infty^{\alpha_2}+M^{-\beta_2}(CR_1)^{\gamma_2}+M^{-\beta_2}(CR_2)^{\theta_2}\right)\\ &\leq C_2d(x)^{\beta_2}\quad\text{in }\Omega,\end{aligned}$$

for some positive constants C_1 and C_2 independent from u, v, z_1 and z_2. Then, owing to Theorem 2, it follows that if $C>1$ is large enough (according to the Claim), for every $(z_1,z_2)\in\mathcal{K}_1(C)\times\mathcal{K}_2(C)$ problem $(P_{(z_1,z_2)})$ has a smallest solution $(u^*,v^*)_{(z_1,z_2)}\in C_0^{1,\gamma}(\overline{\Omega})\times C_0^{1,\gamma}(\overline{\Omega})$ for certain $\gamma\in(0,1)$, within $[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$. This complete the proof. ☐

Remark 1. *We wish explicitly to point out that from the proof of Proposition 1 one can derive an estimate of the largeness of $C>1$. In particular, the choice of C, that first of all is related to (3.5), is crucial for verifying the Claim and, as a consequence, that for every $(z_1,z_2)\in\mathcal{K}_1(C)\times\mathcal{K}_2(C)$ the set of the solutions of problem $(P_{(z_1,z_2)})$ is nonempty.*

In what follows, $C>1$ will be assumed large enough such that for any $(z_1,z_2)\in\mathcal{K}_1(C)\times\mathcal{K}_2(C)$, put

$$S_{(z_1,z_2)}=\{(u,v)\in[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]:\ (u,v)\in C_0^1(\overline{\Omega})\times C_0^1(\overline{\Omega})\text{ solves }(P_{(z_1,z_2)})\},$$

then $S_{(z_1,z_2)}\neq\emptyset$.

Lemma 2. *Assume* H(f) *and* H(g) *hold and let* $C > 1$ *be large enough. If* $\{(z_{1,n}, z_{2,n})\}$ *is a sequence in* $\mathcal{K}_1(C) \times \mathcal{K}_2(C)$ *such that* $(z_{1,n}, z_{2,n}) \to (z_1, z_2)$ *in* $\mathcal{K}_1(C) \times \mathcal{K}_2(C)$*, then for any* $(\breve{u}, \breve{v}) \in S_{(z_1,z_2)}$*, there exists* $(\breve{u}_n, \breve{v}_n) \in S_{(z_{1,n},z_{2,n})}$ *such that* $(\breve{u}_n, \breve{v}_n) \to (\breve{u}, \breve{v})$ *in* $C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$.

Proof. Fix $C > 1$ large enough, put

$$\hat{\rho} = \max\left\{-\frac{\alpha_1 M_1}{p-1}(C^{-1}l)^{\alpha_1-p+1}\hat{l}^{\beta_1}C^{-\beta_1}, -\frac{\beta_2 M_2}{q-1}(C^{-1}l)^{\beta_2-q+1}\hat{l}^{\alpha_2}C^{-\alpha_2}\right\},$$

and observe that

$$\left\{\begin{array}{l} \alpha_1\mu_1 t^{\alpha_1-1}\underline{v}(x)^{\beta_1} + \hat{\rho}(p-1)d(x)^{\alpha_1+\beta_1-(p-1)}t^{p-2} \geq 0 \\ \beta_2\mu_2\underline{u}(x)^{\alpha_2}t^{\beta_2-1} + \hat{\rho}(q-1)d(x)^{\alpha_2+\beta_2-(q-1)}t^{q-2} \geq 0, \end{array}\right. \tag{3.13}$$

uniformly in $x \in \Omega$, for all $t \geq \min\{\underline{u}(x), \underline{v}(x)\}$, where $\mu_1 \in \{m_1, M_1\}$ and $\mu_2 \in \{m_2, M_2\}$.
Indeed, from (2.3) and bearing in mind that $\alpha_1 - p + 1 < 0 < \beta_1$ one has that

$$\left(\frac{\underline{v}(x)}{d(x)}\right)^{\beta_1} \leq C^{-\beta_1}\hat{l}_1^{\beta}, \quad \max\left\{\left(\frac{\underline{u}(x)}{d(x)}\right)^{\alpha_1-p+1}, \left(\frac{\underline{v}(x)}{d(x)}\right)^{\alpha_1-p+1}\right\} \leq (C^{-1}l)^{\alpha_1-p+1}$$

for all $x \in \Omega$. Hence,

$$\begin{aligned} \hat{\rho} &\geq -\frac{\alpha_1 M_1}{p-1}\left(\frac{t}{d(x)}\right)^{\alpha_1-p+1}\left(\frac{\underline{v}(x)}{d(x)}\right)^{\beta_1} \\ &= \frac{-\alpha_1 M_1 t^{\alpha_1-1}\underline{v}(x)^{\beta_1}}{(p-1)d(x)^{\alpha_1+\beta_1-(p-1)}t^{p-2}} \\ &\geq \frac{-\alpha_1 m_1 t^{\alpha_1-1}\underline{v}(x)^{\beta_1}}{(p-1)d(x)^{\alpha_1+\beta_1-(p-1)}t^{p-2}}. \end{aligned}$$

for all $t \geq \min\{\underline{u}(x), \underline{v}(x)\}$ and uniformly in Ω, so that the first inequality in (3.13) holds. The second inequality can be verified arguing in analogy.
Here, condition (3.13) guaranties that for all $(u, v) \in \mathcal{K}_1(C) \times \mathcal{K}_2(C)$ the functions

$$\mu_1 t^{\alpha_1}v(x)^{\beta_1} + \hat{\rho}d(x)^{\alpha_1+\beta_1-(p-1)}t^{p-1}, \quad \mu_2 u(x)^{\alpha_2}t^{\beta_2} + \hat{\rho}d(x)^{\alpha_2+\beta_2-(q-1)}t^{q-1} \tag{3.14}$$

are monotone with respect to $t \geq \min\{\underline{u}(x), \underline{v}(x)\}$.
Let now $\hat{f}$, $\hat{g}$ be the functions defined by

$$\hat{f}(x, s_1, s_2, \xi_1, \xi_2) = f(x, s_1, s_2, \xi_1, \xi_2) + \hat{\rho}d(x)^{\alpha_1+\beta_1-(p-1)}s_1^{p-1}$$

$$\hat{g}(x, s_1, s_2, \xi_1, \xi_2) = g(x, s_1, s_2, \xi_1, \xi_2) + \hat{\rho}d(x)^{\alpha_2+\beta_2-(q-1)}s_2^{q-1}$$

for $(x, s_1, s_2, \xi_1, \xi_2) \in \Omega \times (0, +\infty) \times (0, +\infty) \times \mathbb{R}^{2N}$.
Arguing as in (3.11) and (3.12), bearing in mind (3.2) and (3.3), there exist two positive constants $\hat{C}_1$ and $\hat{C}_2$ such that

$$\hat{f}(x, u, v, \nabla y_1, \nabla y_2) \leq \hat{C}_1 d(x)^{\alpha_1}, \quad \hat{g}(x, u, v, \nabla y_1, \nabla y_2) \leq \hat{C}_2 d(x)^{\beta_2} \tag{3.15}$$

a.e. in Ω, for every $(u,v) \in [\underline{u},\overline{u}] \times [\underline{v},\overline{v}]$ and every $(y_1,y_2) \in \mathcal{K}_1(C) \times \mathcal{K}_2(C)$.

Let us consider now the following differential operators $L_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$, $L_q : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$ defined by

$$L_p(u) = -\Delta_p u + \hat{\rho} d(x)^{\alpha_1+\beta_1-(p-1)} |u|^{p-2} u$$

for all $u \in W_0^{1,p}(\Omega)$, and

$$L_q(u) = -\Delta_q u + \hat{\rho} d(x)^{\alpha_2+\beta_2-(q-1)} |u|^{q-2} u$$

for all $u \in W_0^{1,q}(\Omega)$. Observing that $p'(\alpha_1+\beta_1)-p > -p$ and $q'(\alpha_2+\beta_2)-q > -q$, one can apply [29, Theorem 19.8] ($\partial\Omega$ is assumed to be smooth enough) in order to obtain

$$d(x)^{\alpha_1+\beta_1-(p-1)} |u|^{p-2} u \in L^{p'}(\Omega), \quad d(x)^{\alpha_2+\beta_2-(q-1)} |u|^{q-2} u \in L^{q'}(\Omega),$$

namely L_p and L_q are well defined.
A direct computation shows that L_p and L_q are demicontinuous, coercive and strictly monotone. Hence, in view of (3.15), one can apply the Minty-Browder theorem and conclude that for every $(u,v) \in [\underline{u},\overline{u}] \times [\underline{v},\overline{v}]$, for every $(y_1,y_2) \in \mathcal{K}_1(C) \times \mathcal{K}_2(C)$ the problem

$$(P_{(u,v,y_1,y_2)}) \qquad \begin{cases} L_p(w_1) = \hat{f}(x,u,v,\nabla y_1,\nabla y_2) & \text{in } \Omega, \\ L_q(w_2) = \hat{g}(x,u,v,\nabla y_1,\nabla y_2) & \text{in } \Omega, \\ w_1, w_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

admits a unique solution.

At this point fix $(z_1,z_2) \in \mathcal{K}_1(C) \times \mathcal{K}_2(C)$, $(\breve{u},\breve{v}) \in S_{(z_1,z_2)}$ and let $\{(z_{1,n},z_{2,n})\}$ be a sequence in $\mathcal{K}_1(C) \times \mathcal{K}_2(C)$ such that $(z_{1,n},z_{2,n}) \to (z_1,z_2)$. Obviously, $(\breve{u},\breve{v}) \in S_{(z_1,z_2)}$ implies that

$$(3.16) \qquad (\breve{u},\breve{v}) \text{ is the unique solution of } (P_{(\breve{u},\breve{v},z_1,z_2)}).$$

Fix $n \in \mathbb{N}$ and let $(w_{1,n}^0, w_{2,n}^0)$ be the unique solution of the problem $(P_{(\breve{u},\breve{v},z_{1,n},z_{2,n})})$.

By H(f) and H(g), since $(\breve{u},\breve{v}) \in [\underline{u},\overline{u}] \times [\underline{v},\overline{v}]$, using the monotonicity of the functions introduced in (3.14) and the computations pointed out in (3.7) and (3.8), it follows that

$$\begin{aligned} L_p(w_{1,n}^0) &= \hat{f}(x,\breve{u},\breve{v},\nabla z_{1,n},\nabla z_{2,n}) = f(x,\breve{u},\breve{v},\nabla z_{1,n},\nabla z_{2,n}) + \hat{\rho} d(x)^{\alpha_1+\beta_1-(p-1)} \breve{u}^{p-1} \\ &\geq m_1 \breve{u}^{\alpha_1} \breve{v}^{\beta_1} + \hat{\rho} d(x)^{\alpha_1+\beta_1-(p-1)} \breve{u}^{p-1} \\ &\geq m_1 \underline{u}^{\alpha_1} \breve{v}^{\beta_1} + \hat{\rho} d(x)^{\alpha_1+\beta_1-(p-1)} \underline{u}^{p-1} \\ &\geq m_1 \underline{u}^{\alpha_1} \underline{v}^{\beta_1} + \hat{\rho} d(x)^{\alpha_1+\beta_1-(p-1)} \underline{u}^{p-1} \\ &\geq -\Delta_p \underline{u} + \hat{\rho} d(x)^{\alpha_1+\beta_1-(p-1)} \underline{u}^{p-1} = L_p(\underline{u}) \end{aligned}$$

and similarly, we obtain

$$\begin{aligned} L_q(w_{2,n}^0) &= \hat{g}(x,\breve{u},\breve{v},\nabla z_{1,n},\nabla z_{2,n}) \\ &\geq -\Delta_q \underline{v} + \hat{\rho} C_2 d(x)^{\beta_2-q+1} \underline{v}^{q-1} = L_q(\underline{v}). \end{aligned}$$

The same reasoning can be exploited for assuring that

$$L_p(w^0_{1,n}) \le L_p(\overline{u}) \text{ and } L_p(w^0_{1,n}) \le L_p(\overline{v}).$$

Accordingly, the weak comparison principle in [30] implies that $(w^0_{1,n}, w^0_{2,n}) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$. Furthermore, from (3.15), by the regularity theory (see [18, Lemma 3.1]), it follows $(w^0_{1,n}, w^0_{2,n}) \in C^{1,\gamma}_0(\overline{\Omega}) \times C^{1,\gamma}_0(\overline{\Omega})$ for some $\gamma \in (0,1)$, and, in particular, $\{(w_{1,n}, w_{2,n})\}$ is bounded in $C^{1,\gamma}_0(\overline{\Omega}) \times C^{1,\gamma}_0(\overline{\Omega})$. Then, since $C^{1,\gamma}_0(\overline{\Omega}) \subset C^1_0(\overline{\Omega})$ is compact, there exist a subsequence, denoted by the same symbol, $\{(w^0_{1,n}, w^0_{2,n})\}$ and $(\hat{u}, \hat{v})$ such that

$$(w^0_{1,n}, w^0_{2,n}) \to (\hat{u}, \hat{v}) \text{ in } C^1_0(\overline{\Omega}) \times C^1_0(\overline{\Omega}). \tag{3.17}$$

Hence, passing to the limit in $(P_{(\breve{u},\breve{v},z_{1,n},z_{2,n})}))$, one has that $(\hat{u}, \hat{v})$ is a solution of the problem $(P_{\breve{u},\breve{v},z_1,z_2})$. Namely, in view of (3.16), $(\hat{u}, \hat{v}) = (\breve{u}, \breve{v})$ and by the strong convergence (3.17) we infer that

$$\lim_{n\to\infty} (w^0_{1,n}, w^0_{2,n}) = (\breve{u}, \breve{v}) \text{ in } C^1_0(\overline{\Omega}) \times C^1_0(\overline{\Omega}).$$

Now, let $(w^1_{1,n}, w^1_{2,n})$ be the unique solution of the problem $(P_{w^0_{1,n},w^0_{2,n},z_{1,n},z_{2,n}})$. Following the same argument as before we obtain

$$(w^1_{1,n}, w^1_{2,n}) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$$

and

$$\lim_{n\to\infty} (w^1_{1,n}, w^1_{2,n}) = (\breve{u}, \breve{v}) \text{ in } C^1_0(\overline{\Omega}) \times C^1_0(\overline{\Omega}).$$

Inductively, for each $n \in \mathbb{N}$, we construct the sequences $\{(w^k_{1,n}, w^k_{2,n})\}_k$ in $C^1_0(\overline{\Omega}) \times C^1_0(\overline{\Omega})$ as a unique solution of

$$\begin{cases} -\Delta_p u = \hat{f}(x, w^{k-1}_{1,n}, w^{k-1}_{2,n}, \nabla z_{1,n}, \nabla z_{2,n}) & \text{in } \Omega, \\ -\Delta_q v = \hat{g}(x, w^{k-1}_{1,n}, w^{k-1}_{2,n}, \nabla z_{1,n}, \nabla z_{2,n}) & \text{in } \Omega, \\ u, v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.18}$$

such that for each $k \in \mathbb{N}$, we have

$$(w^k_{1,n}, w^k_{2,n}) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$$

and

$$\lim_{n\to\infty} (w^k_{1,n}, w^k_{2,n}) = (\breve{u}, \breve{v}) \text{ in } C^1_0(\overline{\Omega}) \times C^1_0(\overline{\Omega}).$$

The task is now to show that $\{(w^k_{1,n}, w^k_{2,n})\}_{n,k}$ is relatively compact in $C^1_0(\overline{\Omega}) \times C^1_0(\overline{\Omega})$. Indeed, bearing in mind that $(w^{k-1}_{1,n}, w^{k-1}_{2,n}) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$ and $(z_{1,n}, z_{2,n}) \in \mathcal{K}_1(C) \times \mathcal{K}_2(C)$, on account of (3.15), one gets

$$\hat{f}(x, w^k_{1,n}, w^k_{2,n}, z_{1,n}, z_{2,n}) \le \hat{C}_1 d(x)^{\alpha_1}, \quad \hat{g}(x, w^k_{1,n}, w^k_{2,n}, z_{1,n}, z_{2,n}) \le \hat{C}_2 d(x)^{\beta_2},$$

for every n, $k \in \mathbb{N}$, with $\hat{C}_1$, $\hat{C}_2 > 0$ independent from n and k. Applying [18, Lemma 3.1], $\{(w^k_{1,n}, w^k_{2,n})\}_{n,k}$ is bounded in $C^{1,\gamma}_0(\overline{\Omega}) \times C^{1,\gamma}_0(\overline{\Omega})$ and our task is achieved, being $C^{1,\gamma}_0(\overline{\Omega})$ compactly embedded in $C^1_0(\overline{\Omega})$. Finally, the conclusion follows by proceeding analogously to the proof of [14, Lemma 2.5, page 535]. □

4. Proof of the main result

According to Proposition 1, for $C > 1$ large enough, for all $(z_1, z_2) \in \mathcal{K}_1(C) \times \mathcal{K}_2(C)$, there exists $(u^*, v^*)_{(z_1,z_2)}$ in $C_0^{1,\gamma}(\overline{\Omega}) \times C_0^{1,\gamma}(\overline{\Omega})$ for certain $\gamma \in (0,1)$ that is the smallest solution within $[\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$ for system $(P_{(z_1,z_2)})$. Thus, the operator

$$\begin{array}{rll} \mathcal{T}: & \mathcal{K}_1(C) \times \mathcal{K}_2(C) \to & C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega}) \\ & (z_1, z_2) \ \mapsto & \mathcal{T}(z_1, z_2) = (u^*, v^*)_{(z_1,z_2)} \end{array}$$

is well defined and clearly the fixed points of the map $\mathcal{T}$ are solutions of problem (P).

Lemma 3. *The map $\mathcal{T}$ is continuous and compact.*

Proof. First, observe that $\mathcal{T}$ is compact, namely, taking in mind that $\mathcal{K}_1(C) \times \mathcal{K}_2(C)$ is bounded with respect to the $(C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega}))$-topology, for any sequence $\{(z_{1,n}, z_{2,n})\}_n$ in $\mathcal{K}_1(C) \times \mathcal{K}_2(C)$ one has that $(u_n^*, v_n^*) = \mathcal{T}(z_{1,n}, z_{2,n})$ is relatively compact in $C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$. This follows readily from (3.11) and (3.12), since $(\underline{u}, \underline{v}) \leq (u_n^*, v_n^*) \leq (\overline{u}, \overline{v})$ in Ω, Indeed, as in the proof of Lemma 2, applying [18, Lemma 3.1] one has that $\{(u_n, v_n)\}$ is bounded in $C_0^{1,\gamma}(\overline{\Omega}) \times C_0^{1,\gamma}(\overline{\Omega})$ and we can conclude again invoking the compactness of the embedding $C_0^{1,\gamma}(\overline{\Omega}) \hookrightarrow C_0^1(\overline{\Omega})$.

Let us show that $\mathcal{T}$ is continuous. Let $(z_{1,n}, z_{2,n}) \to (z_1, z_2)$ in $\mathcal{K}_1(C) \times \mathcal{K}_2(C)$ and put $(u_n^*, v_n^*) = \mathcal{T}(z_{1,n}, z_{2,n})$. Then, we already know that there exist a subsequence $\{(u_{n_k}^*, v_{n_k}^*)\}_k$ and an element $(u^*, v^*) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$ such that

$$(u_{n_k}^*, v_{n_k}^*) \to (u^*, v^*) \text{ in } C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega}). \tag{4.1}$$

Passing to the limit in the equations

$$-\Delta_p u_{n_k}^* = f(x, u_{n_k}^*, v_{n_k}^*, \nabla z_{1,n_k}, \nabla z_{2,n_k}),$$
$$-\Delta_q v_{n_k}^* = g(x, u_{n_k}^*, v_{n_k}^*, \nabla z_{1,n_k}, \nabla z_{2,n_k})$$

one gets that $(u^*, v^*) \in S_{(z_1,z_2)}$.

The proof is completed by showing that (u^*, v^*) is the smallest solution of $(P_{(z_1,z_2)})$ within $[\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$. Indeed, fix a solution (w_1, w_2) of $(P_{(z_1,z_2)})$ such that $(w_1, w_2) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$. We can conclude verifying that

$$u^* \leq w_1, \quad v^* \leq w_2. \tag{4.2}$$

According to Lemma 2, there exists $(w_{1,n}, w_{2,n}) \in S_{(z_{1,n},z_{2,n})}$ such that

$$(w_{1,n}, w_{2,n}) \to (w_1, w_2) \text{ in } C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega}) \text{ as } n \to \infty. \tag{4.3}$$

Then, since $(u_{n_k}^*, v_{n_k}^*)$ is the smallest solution in $[\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$ of $(P_{(z_{1,n_k}, z_{2,n_k})})$, it is clear that

$$u_{n_k}^* \leq w_{1,n_k}, \quad v_{n_k}^* \leq w_{2,n_k},$$

for all $k \in \mathbb{N}$. Passing to the limit in the previous inequalities and bearing in mind (4.1) and (4.3) one directly achieves (4.2). This ends the proof. □

Lemma 4. *$\mathcal{T}(\mathcal{K}_1(C)\times\mathcal{K}_2(C))\subset\mathcal{K}_1(C)\times\mathcal{K}_2(C)$ provided $C>1$ is large enough.*

Proof. For any $(z_1,z_2)\in\mathcal{K}_1(C)\times\mathcal{K}_2(C)$, let $(u^*,v^*)_{(z_1,z_2)}=\mathcal{T}(z_1,z_2)$ be the smallest solution of $(P_{(z_1,z_2)})$ in $[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$. Then, according to H$(f)$, H$(g)$, (3.3), (3.2) and (2.3), we get

$$\begin{aligned} f(x,u^*,v^*,\nabla z_1,\nabla z_2) &\leq M_1(u^*)^{\alpha_1}(v^*)^{\beta_1}+|\nabla z_1|^{\gamma_1}+|\nabla z_2|^{\theta_1}\\ &\leq M_1\underline{u}^{\alpha_1}\overline{v}^{\beta_1}+(CR_1)^{\gamma_1}+(CR_2)^{\theta_1}\\ &\leq M_1C^{-\alpha_1+\beta_1}(ld(x))^{\alpha_1}(c_1'd(x))^{\beta_1}+(CR_1)^{\gamma_1}+(CR_2)^{\theta_1}\\ &\leq M_1C^{-\alpha_1+\beta_1}l^{\alpha_1}(c_1')^{\beta_1}\,\|d(x)\|_\infty^{\alpha_1+\beta_1}+(CR_1)^{\gamma_1}+(CR_2)^{\theta_1}\\ &\leq (K_p^{-1}CR_1)^{p-1} \end{aligned}$$

a.e. in Ω and, analogously,

$$\begin{aligned} g(x,u^*,v^*,\nabla z_1,\nabla z_2) &\leq M_2(u^*)^{\alpha_2}(v^*)^{\beta_2}+|\nabla z_1|^{\gamma_2}+|\nabla z_2|^{\theta_2}\\ &\leq M_2\overline{u}^{\alpha_2}\underline{v}^{\beta_2}+(CR_1)^{\gamma_2}+(CR_2)^{\theta_2}\\ &\leq M_2C^{\alpha_2-\beta_2}\,\|\xi_1\|_\infty^{\alpha_2}\,\phi_{1,q}^{\beta_2}+(CR_1)^{\gamma_2}+(CR_2)^{\theta_2}\\ &\leq M_2C^{\alpha_2-\beta_2}c_1^{\alpha_2}l^{\beta_2}\,\|d(x)\|_\infty^{\alpha_2+\beta_2}+(CR_1)^{\gamma_2}+(CR_2)^{\theta_2}\\ &\leq (K_q^{-1}CR_2)^{q-1} \end{aligned}$$

a.e. in Ω, provided that $C>1$ is sufficiently large. Then, using the inequality in (3.6), it follows that

$$\|\nabla u^*\|_\infty\leq CR_1 \quad\text{and}\quad \|\nabla v^*\|_\infty\leq CR_2,$$

namely $(u^*,v^*)_{(z_1,z_2)}\in\mathcal{K}_1(C)\times\mathcal{K}_2(C)$. This ends the proof of lemma. □

Now we are in position to prove our main result.

Proof of Theorem 1. On the basis of Lemmas 3 and 4, Schauder's fixed point theorem (see, e.g., [31, p. 57]) garantees the existence of $(u,v)\in\mathcal{K}(C)_1\times\mathcal{K}_2(C)$ satisfying $(u,v)=\mathcal{T}(u,v)$. Taking into account the definition of $\mathcal{T}$, it turns out that $(u,v)\in C_0^1(\overline{\Omega})\times C_0^1(\overline{\Omega})$ is a (positive) solution of problem (P). Since $(u,v)\in\mathcal{K}_1(C)\times\mathcal{K}_2(C)$, in particular, $(u,v)\in[\underline{u},\overline{u}]\times[\underline{v},\overline{v}]$ and on account of (3.3), (3.2) and (2.3), the property (1.1) is fullfield. This completes the proof. □

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References

[1] C. O. Alves & F. J. S. A. Corrêa, *On the existence of positive solution for a class of singular systems involving quasilinear operators*, Appl. Math. Comput. **185** (2007), 727-736.

[2] C. O. Alves, F. J. S. A. Corrêa & J. V. A. Gonçalves, *Existence of solutions for some classes of singular Hamiltonian systems*, Adv. Nonlinear Stud. **5** (2005), 265-278.

[3] C.O. Alves, P.C. Carrião & L.F.O. Faria, *Existence of solutions to singular elliptic equations with convection terms via the Galerkin method*, Electron. J. Differential Equations **12** (2010), 12 pp.

[4] C. O. Alves & A. Moussaoui, *Existence of solutions for a class of singular elliptic systems with convection term*, Asymptot. Anal. **90** (2014), 237-248.

[5] H. Brézis, R. E. L. Turner, *On a class of superlinear elliptic problems*, Comm. Partial Differential Equations **2** (1977), no. 6, 601-614.

[6] H. Bueno & G. Ercole, *A quasilinear problem with fast growing gradient*, Appl. Math. Lett. **26** (2013), 520-523.

[7] P. Candito, S. A. Marano & A. Moussaoui, Nodal solutions to a Neumann problem for a class of (p_1, p_2)-Laplacian systems, Preprint.

[8] S. Carl & D. Motreanu, Extremal solutions for nonvariational quasilinear elliptic systems via expanding trapping regions, Monatsh. Math. (4) **182** (2017), 801- 821.

[9] K. D. Chu, D. D. Hai & R. Shivaji, *Positive solutions for a class of non-cooperative pq-Laplacian systems with singularities*, Appl. Math. Lett. **85** (2018), 103–109.

[10] A. Cianchi, V. Maz'ya, *Global gradient estimates in elliptic problems under minimal data and domain regularity*, Commun. Pure Appl. Anal. **14** (2015), 285–311.

[11] D.G. de Figueiredo, *Nonlinear elliptic systems,* An. Acad. Brasil. Ciênc. **72** (2000).

[12] N. Dunford & J. T. Schwartz, *Linear Operators. I. General Theory*, Interscience Publishers, Inc., New York, (1958).

[13] S. El Manouni, K. Perera & R. Shivaji, *On singular quasimonotone (p,q)-Laplacian systems*, Proc. Roy. Soc. Edinburgh Sect. A **142** (2012), 585–594.

[14] F. Faraci, D. Motreanu & D. Puglisi, *Positive solutions of quasi-linear elliptic equations with dependence on the gradient*, Calc. Var. Partial Differential Equations **54** (2015), 525-538.

[15] M. Ghergu, *Lane-Emden systems with negative exponents*, J. Funct. Anal. **258** (2010), 3295-3318.

[16] M. Ghergu, *Lane-Emden systems with singular data*, Proc. Roy. Soc. Edinburgh Sect. A **141** (2011), 1279-1294.

[17] J. Giacomoni, I. Schindler & P. Takac, *Sobolev versus Hölder local minimizers and existence of multiple solutions for a singular quasilinear equation*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **6** (2007), 117-158.

[18] D. D. Hai, *On a class of singular p-Laplacian boundary value problems*, J. Math. Anal. Appl. **383** (2011), 619-626.

[19] J. Hernández, F.J. Mancebo & J.M. Vega, *Positive solutions for singular semilinear elliptic systems*, Adv. Differential Equations **13** (2008), 857-880.

[20] B. Khodja & A. Moussaoui, *Positive solutions for infinite semipositone/positone quasilinear elliptic systems with singular and superlinear terms*, Differ. Equ. Appl. **8** (4) (2016), 535-546.

[21] A. C. Lazer & P. J. Mckenna, *On a singular nonlinear elliptic boundary-value problem*, Proc. Amer. Math. Soc. **3** (111), 1991.

[22] M. Marcus & V. J. Mizel, *Absolute continuity on tracks and mappings of Sobolev spaces*, Arch. Rational Mech. Anal. **45** (1972), 294-320.

[23] M. Montenegro & A. Suarez, *Existence of a positive solution for a singular system*, Proc. Roy. Soc. Edinburgh Sect. A **140** (2010), 435–447.

[24] D. Motreanu, V.V. Motreanu & N.S. Papageorgiou, *A unified approach for multiple constant sign and nodal solutions*, Adv. Differential Equations **12** (2007), 1363-1392.
[25] D. Motreanu & A. Moussaoui, *Existence and boundedness of solutions for a singular cooperative quasilinear elliptic system*, Complex Var. Elliptic Equ. **59** (2014), 285-296.
[26] D. Motreanu & A. Moussaoui, *A quasilinear singular elliptic system without cooperative structure*, Act. Math. Sci. **34 B** (2014), 905-916.
[27] D. Motreanu & A. Moussaoui, *An existence result for a class of quasilinear singular competitive elliptic systems*, Appl. Math. Lett. **38** (2014), 33–37.
[28] D. Motreanu, A. Moussaoui & Z. Zhang, *Positive solutions for singular elliptic systems with convection term*, J. Fixed Point Theory App. **19** (3) (2017), 2165-2175.
[29] B. Opic & A. Kufner, *Hardy-type inequalities*, Pitman Res. Notes Math., Longman, Harlow, 1990.
[30] P. Tolksdorf, *On the Dirichlet problem for quasilinear equations in domains with conical boundary points*, Comm. Partial Differential Equations **8** (1983), 773-817.
[31] E. Zeidler, *Nonlinear functional analysis and its applications. I. Fixed-point theorems*, Springer-Verlag, New York, 1986.

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