Parametric nonlinear singular Dirichlet problems

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Abstract

We consider a nonlinear parametric Dirichlet problem driven by the p-Laplacian and a reaction which exhibits the competing effects of a singular term and of a resonant perturbation. Using variational methods together with suitable truncation and comparison techniques, we prove a bifurcation-type theorem describing the dependence on the parameter of the set of positive solutions.

Keywords: Singular term, resonant perturbation, p-Laplacian, variational eigenvalue, nonlinear regularity, nonlinear maximum principle, bifurcation-type theorem

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1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we study the following nonlinear singular Dirichlet problem

$$-\Delta_p u(z) = \lambda u(z)^{-\gamma} + f(z, u(z)) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0, \quad u \geq 0. \qquad (P_\lambda)$$

In the above equation, Δ_p denotes the p-Laplace differential operator defined by

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u) \quad \text{for all } u \in W_0^{1,p}(\Omega), \quad 1 < p < +\infty.$$

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In the right hand side (forcing term) $\lambda > 0$ is a parameter, $\lambda u^{-\gamma}$ is the singular term with $0 < \gamma < 1$ and $f(z, x)$ is a Carathéodory perturbation (that is, for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z, x)$ is continuous). We assume that $f(z, \cdot)$ exhibits $(p-1)$-linear growth near $+\infty$ and asymptotically at $+\infty$ is resonant with respect to any nonprincipal variational eigenvalue of the Dirichlet p-Laplacian. Here "resonant" means that asymptotically as $x \to +\infty$ the quotient $\dfrac{f(z, x)}{x^{p-1}}$ interacts with the spectrum of the Dirichlet p-Laplacian. This makes the analysis of (P_λ) more difficult, since among other things the verification of the compactness condition for the energy functional is highly nontrivial. So, in problem (P_λ) we have the competing effects of singular and resonant terms. We are looking for positive solutions and our aim is to obtain the precise dependence of the set of positive solutions as the parameter $\lambda > 0$ varies. We prove a bifurcation-type result which says that there exists a critical parameter value $\lambda^* > 0$ such that

- for all $\lambda \in (0, \lambda^*)$ problem (P_λ) admits at least two positive solutions;
- for $\lambda = \lambda^*$ problem (P_λ) has at least one positive solution;
- for all $\lambda > \lambda^*$ problem (P_λ) has no positive solutions.

In the past such studies for singular equations were conducted by Sun-Wu-Long [19] (semilinear equations) and by Papageorgiou-Smyrlis [16] (nonlinear equations). In both papers the competition is between a singular term and a superlinear perturbation. Moreover, the parameter multiplies the superlinear term, while in problem (P_λ) the parameter $\lambda > 0$ multiplies the singular term. In principle, it is easier to control the perturbation than the singular term. Sun-Wu-Long [19] prove the existence of a parameter $\lambda^* > 0$ such that for all $\lambda \in (0, \lambda^*)$ the problem has at least two positive solutions. A more precise description of the dependence on the parameter $\lambda > 0$ of the set of positive solutions, can be found in Papageorgiou-Smyrlis [16], who prove a bifurcation-type result as described above. Other multiplicity results for singular Dirichlet problems can be found in the works of Hirano-Saccon-Shioji [11], Papageorgiou-Rădulescu

[14] (semilinear equations) and Giacomoni-Schindler-Takáč [9], Papageorgiou-Rădulescu-Repovš [15], Papageorgiou-Smyrlis [17], Perera-Zhang [18]. We should mention the works of Cirmi-Leonardi [6], Cianci-Cirmi-D'Asero-Leonardi [4], Cirmi-D'Asero-Leonardi [5] which also deal with operators exhibiting some kind of degeneracy. It is worth examining whether our results here can be extended also to such operators. Finally there is also the recent work of Bonanno-Candito-Livrea-Papageorgiou [3] treating a similar parametric problem with no singular term.

2. Mathematical Background - Hypotheses

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X, \mathbb{R})$, we say that φ satisfies the "Cerami condition" (the "C-condition" for short), if the following property holds:

"Every sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ such that $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|)\varphi'(u_n) \to 0$ in X^* as $n \to +\infty$, admits a strongly convergent subsequence".

This is a compactness-type condition on the functional φ. It leads to a deformation theorem from which one can derive the minimax theory of the critical values of φ. One of the main results in that theory is the so-called "mountain pass theorem" of Ambrosetti-Rabinowitz [2]. Below we present a slightly more general version of that theorem (see, for example, Gasiński-Papageorgiou [7]).

Theorem 1. *If $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition, $u_0, u_1 \in X$, $\|u_1 - u_0\| > r$, $\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \|u - u_0\| = r\} = m_r$ and $c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} \varphi(\gamma(t))$ where $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$, then $c \geq m_r$ and c is a critical value of φ (that is, there exists $\widehat{u} \in X$ such that $\varphi(\widehat{u}) = c$, $\varphi'(\widehat{u}) = 0$).*

The study of (P_λ) involves the use of two spaces. The Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u\big|_{\partial\Omega} = 0\}$. By $\|\cdot\|$ we denote the norm of $W_0^{1,p}(\Omega)$. On account of the Poincaré inequality, we can have

$$\|u\| = \|\nabla u\|_p \text{ for all } u \in W_0^{1,p}(\Omega).$$

The Banach space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone given by

$$C_+ = \{u \in C_0^1(\overline{\Omega}) \,:\, u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\operatorname{int} C_+ = \left\{ u \in C_+ \,:\, u(z) > 0 \text{ for all } z \in \Omega, \quad \frac{\partial u}{\partial n}\bigg|_{\partial\Omega} < 0 \right\},$$

with $\dfrac{\partial u}{\partial n}$ being the normal derivative of u and $n(\cdot)$ is the outward unit normal on $\partial\Omega$.

Recall that $W_0^{1,p}(\Omega)^* = W^{-1,p'}(\Omega)$ with $\frac{1}{p} + \frac{1}{p'} = 1$. Consider the nonlinear map $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined by

$$\langle A(u), h\rangle = \int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla h)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W_0^{1,p}(\Omega).$$

Proposition 1. *The map $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$ (that is, if $u_n \xrightarrow{w} u$ in $W_0^{1,p}(\Omega)$ and $\limsup\limits_{n\to+\infty}\langle A(u_n), u_n - u\rangle \leq 0$, then $u_n \to u$ in $W_0^{1,p}(\Omega)$).*

The next result will be a useful tool in our arguments in Section 3. It can be found in Marano-Papageorgiou [13] (Proposition 2.1).

Proposition 2. *If X is an ordered Banach space with order cone K and $u_0 \in \operatorname{int} K$, then for every $v \in K$, we can find $t_v > 0$ such that $(t_v u_0 - v) \in K$.*

Next we fix our notation. Given $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then for $u \in W_0^{1,p}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. We have

$$u^\pm \in W_0^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

If $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function (for example, a Carathéodory function), we define

$$N_g(u)(\cdot) \equiv g(\cdot, u(\cdot)) \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

the Nemytskii (superposition) operator corresponding to g. Evidently, $z \to N_g(u)(z)$ is measurable.

If $u_1, u_2 \in W_0^{1,p}(\Omega)$ and $u_1 \leq u_2$, then by $[u_1, u_2]$ we denote the order interval in $W_0^{1,p}(\Omega)$ defined by

$$[u_1, u_2] = \left\{ u \in W_0^{1,p}(\Omega) : u_1(z) \leq u(z) \leq u_2(z) \text{ for a.a. } z \in \Omega \right\}.$$

By $\mathrm{int}_{C_0^1(\overline{\Omega})}[u_1, u_2]$ we denote the interior in the $C_0^1(\overline{\Omega})$-norm topology of $[u_1, u_2] \cap C_0^1(\overline{\Omega})$. Finally if $\overline{u} \in W_0^{1,p}(\Omega)$, then we set

$$[\overline{u}) = \{u \in W_0^{1,p}(\Omega) : \overline{u}(z) \leq u(z) \text{ for a.a. } z \in \Omega\}.$$

Given $\varphi \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ by K_φ we denote the critical set of φ, that is,

$$K_\varphi = \{u \in W_0^{1,p}(\Omega) \,:\, \varphi'(u) = 0\}.$$

Now let us recall some basic facts about the spectrum of the Dirichlet p-Laplacian. So, we consider the following nonlinear eigenvalue problem

$$-\Delta_p u(z) = \widehat{\lambda}|u(z)|^{p-2}u(z) \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0. \tag{1}$$

We say that $\widehat{\lambda} \in \mathbb{R}$ is an "eigenvalue" of $(-\Delta_p, W_0^{1,p}(\Omega))$, if problem (1) admits a nontrivial solution $\widehat{u} \in W_0^{1,p}(\Omega)$. The nontrivial solution $\widehat{u}$ is an "eigenfunction" corresponding to the eigenvalue $\widehat{\lambda}$. The nonlinear regularity theory (see, for example, Gasiński-Papageorgiou [7], pp. 737-738), implies that $\widehat{u} \in C_0^1(\overline{\Omega})$. There is a smallest eigenvalue $\widehat{\lambda}_1$ with the following properties:

- $\widehat{\lambda}_1 > 0$ and it is isolated in the spectrum $\widehat{\sigma}(p)$ of $(-\Delta_p, W_0^{1,p}(\Omega))$ (that is, there exists $\varepsilon > 0$ such that $(\widehat{\lambda}_1, \widehat{\lambda}_1 + \varepsilon) \cap \widehat{\sigma}(p) = \emptyset$).

- $\widehat{\lambda}_1$ is simple (that is, if $\widehat{u}$, $\widehat{v}$ are two eigenfunctions corresponding to the eigenvalue $\widehat{\lambda}_1$, then $\widehat{u} = \xi\widehat{v}$ with $\xi \in \mathbb{R} \setminus \{0\}$).

- $$\lambda_1 = \inf \left[\frac{\|\nabla u\|_p^p}{\|u\|_p^p} : u \in W_0^{1,p}(\Omega), \quad u \neq 0 \right]. \tag{2}$$

In (2) the infimum is realized on the one-dimensional eigenspace corresponding to $\widehat{\lambda}_1 > 0$. From the above properties it follows that the elements of this eigenspace have fixed sign. Let $\widehat{u}_1$ be the L^p-normalized (that is, $\|\widehat{u}_1\|_p = 1$) positive eigenfunction corresponding to $\widehat{\lambda}_1 > 0$. We know that $\widehat{u}_1 \in C_+ \setminus \{0\}$. Moreover, the nonlinear strong maximum principle (see Gasiński-Papageorgiou [7], p. 738) implies that $\widehat{u}_1 \in \operatorname{int} C_+$.

Since $\widehat{\sigma}(p)$ is closed and $\widehat{\lambda}_1 > 0$ is isolated, the second eigenvalue $\widehat{\lambda}_2$ of $(-\Delta_p, W_0^{1,p}(\Omega))$ is well-defined by

$$\widehat{\lambda}_2 = \inf[\widehat{\lambda} \in \widehat{\sigma}(p) : \widehat{\lambda} > \widehat{\lambda}_1].$$

Employing the Ljusternik-Schnirelmann minimax scheme (see Gasiński-Papageorgiou [7]), we can produce a whole sequence $\{\widehat{\lambda}_k\}_{k\in\mathbb{N}}$ of distinct eigenvalues of $(-\Delta_p, W_0^{1,p}(\Omega))$ such that $\widehat{\lambda}_k \to +\infty$ as $k \to +\infty$. These eigenvalues are known as "variational eigenvalues". Depending on the index used in the execution of the Ljusternik-Schnirelmann scheme, we can have a corresponding sequence of variational eigenvalues. All these sequences are the same in the first two elements which are defined as above. For the rest we do not know if this is the case. Also, we do not know if there are other eigenvalues distinct from the variational eigenvalues. The variational eigenvalues exhaust the spectrum $\widehat{\sigma}(p)$, if $p = 2$ (linear eigenvalue problem) or if $N = 1$ (ordinary differential equation). We mention that, if $\widehat{u}$ is an eigenfunction corresponding to an eigenvalue $\widehat{\lambda} \neq \widehat{\lambda}_1$, then $\widehat{u} \in C_0^1(\overline{\Omega})$ is nodal (that is, sign changing). For details on these and related issues we refer to Gasiński-Papageorgiou [7].

Now we are ready to introduce our hypotheses on the perturbation term $f(z,x)$:

H(f): $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$ $f(z,0) = 0$, $f(z,x) \geq 0$ for all $x \geq 0$ and

(i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)$ such that

$$|f(z,x)| \leq a_\rho(z) \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \rho;$$

(ii) there exists $m \in \mathbb{N}$, $m \geq 2$ such that

$$\lim_{x \to +\infty} \frac{f(z,x)}{x^{p-1}} = \widehat{\lambda}_m \quad \text{uniformly for a.a. } z \in \Omega$$

and if $F(z,x) = \int_0^x f(z,s)ds$, then

$$pF(z,x) - f(z,x)x \to +\infty \text{ as } x \to +\infty, \text{ uniformly for a.a. } z \in \Omega;$$

(iii) for some $r > p$, we have

$$0 \leq \liminf_{x \to 0^+} \frac{f(z,x)}{x^{r-1}} \leq \limsup_{x \to 0^+} \frac{f(z,x)}{x^{r-1}} \leq c_0 \text{ uniformly for a.a. } z \in \Omega;$$

(iv) for every $\rho > 0$, there exists $\widehat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$

$$x \to f(z,x) + \widehat{\xi}_\rho x^{p-1}$$

is nondecreasing on $[0, \rho]$.

Remark 1. Since we are interested to find positive solutions and all the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0, +\infty)$, without any loss of generality we may assume that

$$f(z,x) = 0 \quad \text{for a.a. } z \in \Omega, \text{ all } x \leq 0. \tag{3}$$

Hypotheses $H(f)\,(ii)$ implies that the equation is resonant at $+\infty$ with respect to a nonprincipal variational eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$.

Example 1. The following function satisfies hypotheses $H(f)$. For the sake of simplicity we drop the z-dependence:

$$f(x) = \begin{cases} x^{r-1} & \text{if } 0 \leq x \leq 1, \\ \widehat{\lambda}_m x^{p-1} + x^{q-1} - \widehat{\lambda}_m & \text{if } 1 < x, \end{cases}$$

with $1 < q < p < r < +\infty$, $m \in \mathbb{N}$, $m \geq 2$.

3. Positive Solutions

Let $\mathcal{L} = \{\lambda > 0 : \text{problem } (P_\lambda) \text{ has a positive solution}\}$ (the set of admissible parameters) and S_λ is the set of positive solutions. The nonlinear regularity theory and the nonlinear strong maximum principle (see Gasiński-Papageorgiou [7], pp. 737-738), imply that

$$S_\lambda \subseteq \operatorname{int} C_+$$

(see Proposition 6 of Papageorgiou-Smyrlis [16]).

We start by considering the following purely singular Dirichlet problem

$$-\Delta_p u(z) = \lambda u(z)^{-\gamma} \quad \text{in } \Omega, \quad u\big|_{\partial\Omega} = 0, \quad u \geq 0. \tag{Q_λ}$$

From Proposition 5 of Papageorgiou-Smyrlis [16], we know that for every $\lambda > 0$, problem (Q_λ) has a unique solution $\widetilde{u}_\lambda \in \operatorname{int} C_+$. Using this unique solution of (Q_λ), we will show that $\mathcal{L} \neq \emptyset$.

Proposition 3. *If hypotheses $H(f)$ hold, then $\mathcal{L} \neq \emptyset$.*

Proof. Using $\widetilde{u}_\lambda \in \operatorname{int} C_+$ the unique solution of (Q_λ), we introduce the following truncation of the reaction in problem (P_λ)

$$\widehat{f}_\lambda(z,x) = \begin{cases} \lambda \widetilde{u}_\lambda(z)^{-\gamma} + f(z,x) & \text{if } x \leq \widetilde{u}_\lambda(z), \\ \lambda x^{-\gamma} + f(z,x) & \text{if } \widetilde{u}_\lambda(z) < x, \end{cases} \quad \text{(see (3))}. \tag{4}$$

This is a Carathéodory function. We set $\widehat{F}_\lambda(z,x) = \int_0^x \widehat{f}_\lambda(z,s)ds$ and consider the C^1-functional $\widehat{\varphi}_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega \widehat{F}_\lambda(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Given $u \in W_0^{1,p}(\Omega)$, $u \geq 0$, let

$$\Omega_\lambda^1 = \{z \in \Omega : u(z) \leq \widetilde{u}_\lambda(z)\} \quad \text{and} \quad \Omega_\lambda^2 = \{z \in \Omega : \widetilde{u}_\lambda(z) < u(z)\}.$$

We have

$$
\begin{aligned}
\int_\Omega \widehat{F}_\lambda(z,u)dz \leq & \int_{\Omega^1_\lambda} \lambda u^{1-\gamma}dz + \int_{\Omega^2_\lambda} \lambda u^{1-\gamma}dz + \frac{\lambda}{1-\gamma}\int_{\Omega^2_\lambda}[u^{1-\gamma} - \widetilde{u}_\lambda^{1-\gamma}]dz \\
& + \int_\Omega F(z,u)dz \quad \text{(see (4))} \\
& \leq \lambda c_1\|u\| + \frac{\lambda}{1-\gamma}\int_{\Omega^2_\lambda} u^{1-\gamma}dz + \int_\Omega F(z,u)dz \\
& \text{(see Theorem 13.17, p. 196, of Hewitt-Stromberg [10])} \\
& \leq \lambda c_1\|u\| + \frac{\lambda}{1-\gamma}\int_{\Omega^2_\lambda} \frac{u}{\widetilde{u}_\lambda^\gamma}dz + \int_\Omega F(z,u)dz. \qquad (5)
\end{aligned}
$$

Recall that $\widetilde{u}_\lambda \in \operatorname{int} C_+$. Then $\widetilde{u}_\lambda^{p'} \in \operatorname{int} C_+$ and so using Proposition 2, we can find $c_2 > 0$ such that

$$
\begin{aligned}
& \widehat{u}_1 \leq c_2 \widetilde{u}_\lambda^{p'}, \\
\Rightarrow \quad & \widehat{u}_1^{\frac{1}{p'}} \leq c_2^{\frac{1}{p'}} \widetilde{u}_\lambda, \\
\Rightarrow \quad & \widetilde{u}_\lambda^{-\gamma} \leq c_3 \widehat{u}_1^{-\frac{\gamma}{p'}} \quad \text{for some } c_3 > 0.
\end{aligned}
$$

Using the Lemma in Lazer-McKenna [12], we have

$$
\begin{aligned}
& \widehat{u}_1^{-\frac{\gamma}{p'}} \in L^{p'}(\Omega), \\
\Rightarrow \quad & \widetilde{u}_\lambda^{-\gamma} \in L^{p'}(\Omega).
\end{aligned}
$$

So, invoking Hölder's inequality, we have

$$
\frac{\lambda}{1-\gamma}\int_{\Omega^2_\lambda} \frac{u}{\widetilde{u}_\lambda^\gamma}dz \leq \frac{\lambda}{1-\gamma}\int_\Omega \frac{u}{\widetilde{u}_\lambda^\gamma}dz \leq \lambda c_4\|u\| \quad \text{for some } c_4 > 0 \text{ (recall } u \geq 0). \qquad (6)
$$

Hypotheses $H(f)\,(i), (ii), (iii)$ imply that

$$
\begin{aligned}
& F(z,x) \leq c_5|x|^r \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_5 > 0, \\
\Rightarrow \quad & \int_\Omega F(z,u)dz \leq c_6\|u\|^r \quad \text{for some } c_6 > 0. \qquad (7)
\end{aligned}
$$

Returning to (5) and using (6) and (7), we have

$$
\int_\Omega \widehat{F}_\lambda(z,u)dz \leq c_7[\lambda\|u\| + \|u\|^r] \quad \text{for some } c_7 > 0, \text{ all } u \in W_0^{1,p}(\Omega),\ u \geq 0.
$$

On the other hand, if $u \in W_0^{1,p}(\Omega)$, $u \leq 0$, then

$$\int_\Omega \widehat{F}_\lambda(z,u)dz \leq c_8[\lambda\|u\| + \|u\|^r] \quad \text{for some } c_8 > 0, \text{ (see (1))}.$$

Since for every $u \in W_0^{1,p}(\Omega)$, we have

$$u = u^+ - u^-,$$

we conclude that

$$\int_\Omega \widehat{F}_\lambda(z,u)dz \leq c_9[\lambda\|u\| + \|u\|^r] \quad \text{for some } c_9 > 0, \text{ all } u \in W_0^{1,p}(\Omega).$$

Therefore for all $u \in W_0^{1,p}(\Omega)$ we have

$$\begin{aligned}
\widehat{\varphi}_\lambda(u) &\geq \frac{1}{p}\|\nabla u\|_p^p - c_9[\lambda\|u\| + \|u\|^r] \\
&= \left[\frac{1}{p} - c_9(\lambda\|u\|^{1-p} + \|u\|^{r-p})\right]\|u\|^p. \qquad (8)
\end{aligned}$$

Let $\vartheta_\lambda(t) = \lambda t^{1-p} + t^{r-p}$, $t > 0$. Since $1 < p < r$, we see that

$$\vartheta_\lambda(t) \to +\infty \quad \text{as } t \to 0^+ \text{ and as } t \to +\infty.$$

So, there exists $t_0 \in (0, +\infty)$ such that

$$\begin{aligned}
& \vartheta_\lambda(t_0) = \inf[\vartheta_\lambda(t) : t \geq 0], \\
\Rightarrow \quad & \vartheta'_\lambda(t_0) = 0, \\
\Rightarrow \quad & \lambda(p-1)t_0^{-p} = (r-p)t_0^{r-p-1}, \\
\Rightarrow \quad & t_0 = \left[\frac{\lambda(p-1)}{r-p}\right]^{\frac{1}{r-1}}.
\end{aligned}$$

Then we have

$$\begin{aligned}
& \vartheta_\lambda(t_0) = \lambda\frac{(r-p)^{\frac{p-1}{r-1}}}{\lambda^{\frac{p-1}{r-1}}(p-1)^{\frac{p-1}{r-1}}} + \left(\frac{\lambda(p-1)}{r-p}\right)^{\frac{r-p}{r-1}}, \\
\Rightarrow \quad & \vartheta_\lambda(t_0) = \lambda^{\frac{r-p}{r-1}}\left[\left(\frac{r-p}{p-1}\right)^{\frac{p-1}{r-1}} + \left(\frac{p-1}{r-p}\right)^{\frac{r-p}{r-1}}\right] \\
& = \lambda^{\frac{r-p}{r-1}}\left[a^{-\frac{p-1}{r-1}} + a^{\frac{r-p}{r-1}}\right] \quad \text{with } a = \frac{p-1}{r-p} > 0 \\
& = \lambda^{\frac{r-p}{r-1}}\frac{1+a}{a^{\frac{p-1}{r-1}}} \geq \lambda^{\frac{r-p}{r-1}} \\
\Rightarrow \quad & \vartheta_\lambda(t_0) \to 0 \quad \text{as } \lambda \to 0^+ \text{ (recall } 1 < p < r).
\end{aligned}$$

(We did this estimation in detail because it provides a lower estimate for the critical parameter λ^*).

So, we can find $\lambda_0 > 0$ such that

$$\vartheta_\lambda(t_0) < \frac{1}{p} \quad \text{for all } \lambda \in (0, \lambda_0).$$

Returning to (8), we see that

$$\widehat{\varphi}_\lambda(u) > 0 = \widehat{\varphi}_\lambda(0) \quad \text{for all } \|u\| = \rho_\lambda = t_0, \text{ all } \lambda \in (0, \lambda_0). \tag{9}$$

Hypothesis $H(f)\,(ii)$ implies that given any $\eta > 0$, we can find $M = M(\eta) > 0$ such that

$$pF(z,x) - f(z,x)x \geq \eta \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq M. \tag{10}$$

We have

$$\begin{aligned}
\frac{d}{dx}\left(\frac{F(z,x)}{x^p}\right) &= \frac{f(z,x)x^p - pF(z,x)x^{p-1}}{x^{2p}} \\
&= \frac{f(z,x)x - pF(z,x)}{x^{p+1}} \\
&\leq \frac{-\eta}{x^{p+1}} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq M \text{ (see (10))},
\end{aligned}$$

$$\Rightarrow \quad \frac{F(z,v)}{v^p} - \frac{F(z,x)}{x^p} \leq \frac{\eta}{p}\left[\frac{1}{v^p} - \frac{1}{x^p}\right] \quad \text{for a.a. } z \in \Omega, \text{ all } M \leq x \leq v. \tag{11}$$

Hypothesis $H(f)\,(ii)$ implies that

$$\lim_{v \to +\infty} \frac{F(z,v)}{v^p} = \frac{1}{p}\widehat{\lambda}_m \quad \text{uniformly for a.a. } z \in \Omega. \tag{12}$$

So, if in (11) we pass to the limit as $v \to +\infty$ and use (12), then

$$\widehat{\lambda}_m x^p - pF(z,x) \leq -\eta \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq M.$$

Since $\eta > 0$ is arbitrary, we conclude that

$$\widehat{\lambda}_m x^p - pF(z,x) \to -\infty \quad \text{as } x \to +\infty \text{ uniformly for a.a. } z \in \Omega. \tag{13}$$

We know that $\widehat{u}_1 \in \operatorname{int} C_+$. So, according to Proposition 2, we can find $t > 0$ big enough such that

$$\widetilde{u}_\lambda \leq t\widehat{u}_1.$$

Then using (4), we have

$$\widehat{\varphi}_\lambda(t\widehat{u}_1) \leq \frac{t^p}{p}\widehat{\lambda}_1 - \int_\Omega F(z, t\widehat{u}_1)dz = \frac{1}{p}\int_\Omega [\widehat{\lambda}_1(t\widehat{u}_1)^p - F(z, t\widehat{u}_1)]dz$$

$$\text{(recall that } \|\widehat{u}_1\|_p = 1\text{)},$$

$$\Rightarrow \quad \widehat{\varphi}_\lambda(t\widehat{u}_1) \to -\infty \quad \text{(see (13) and use Fatou's lemma).} \tag{14}$$

Claim: For every $\lambda > 0$, $\widehat{\varphi}_\lambda$ satisfies the C-condition.

Consider a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ such that

$$|\widehat{\varphi}_\lambda(u_n)| \leq M_1 \quad \text{for some } M_1 > 0, \text{ all } n \in \mathbb{N},$$

$$(1 + \|u_n\|)\widehat{\varphi}'_\lambda(u_n) \to 0 \quad \text{in } W^{-1,p'}(\Omega) \text{ as } n \to +\infty. \tag{15}$$

From (15) we have

$$\left|\langle A(u_n), h\rangle - \int_\Omega \widehat{f}_\lambda(z, u_n)hdz\right| \leq \frac{\varepsilon_n\|h\|}{1 + \|u_n\|} \quad \text{for all } h \in W_0^{1,p}(\Omega), \text{ with } \varepsilon_n \to 0^+. \tag{16}$$

In (16) we choose $h = -u_n^- \in W_0^{1,p}(\Omega)$. Then

$$\|\nabla u_n^-\|_p^p + \int_\Omega \widetilde{u}_\lambda^{-\gamma} u_n^- dz \leq \varepsilon_n \quad \text{(see (3), (4))},$$

$$\Rightarrow \quad u_n^- \to 0 \quad \text{in } W_0^{1,p}(\Omega) \text{ as } n \to +\infty. \tag{17}$$

Suppose that $\{u_n^+\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is not bounded. Then we may assume that

$$\|u_n^+\| \to +\infty.$$

We set $y_n = \dfrac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. We have $\|y_n\| = 1$, $y_n \geq 0$ for all $n \in \mathbb{N}$. We may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \text{ in } L^p(\Omega), \text{ with } y \geq 0.$$

From (16) and (17), we have

$$\left|\langle A(u_n^+), h\rangle - \int_\Omega \widehat{f}_\lambda(z, u_n^+)hdz\right| \leq \varepsilon_n'\|h\| \text{ for all } h \in W_0^{1,p}(\Omega), \text{ with } \varepsilon_n' \to 0^+,$$

$$\Rightarrow \quad \left|\langle A(y_n), h\rangle - \int_\Omega \frac{\widehat{f}_\lambda(z, u_n^+)}{\|u_n^+\|^{p-1}}hdz\right| \leq \frac{\varepsilon_n'\|h\|}{\|u_n^+\|^{p-1}} \quad \text{for all } n \in \mathbb{N}. \tag{18}$$

Recall that

$$\widetilde{u}_\lambda^{-\gamma} \in L^{p'}(\Omega), \tag{19}$$

while hypotheses $H(f)\,(i),(ii)$ imply that

$$0 \leq f(z,x) \leq c_{10}[1+|x|^{p-1}] \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ some } c_{10} > 0 \text{ (see (3))}. \tag{20}$$

Then from (19) and (20) it follows that

$$\left\{ \frac{N_{\widehat{f}_\lambda}(u_n^+)}{\|u_n^+\|^{p-1}} \right\}_{n \in \mathbb{N}} \subseteq L^{p'}(\Omega) \quad \text{is bounded.}$$

Therefore by passing to a subsequence if necessary and using hypothesis $H(f)\,(ii)$, we have

$$\frac{N_{\widehat{f}_\lambda}(u_n^+)}{\|u_n^+\|^{p-1}} \xrightarrow{w} \widehat{\lambda}_m y^{p-1} \quad \text{in } L^{p'}(\Omega) \tag{21}$$

(see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 16).

In (18) we choose $h = y_n - y \in W_0^{1,p}(\Omega)$ and pass to the limit as $n \to +\infty$. We obtain

$$\begin{aligned} &\lim_{n \to +\infty} \langle A(y_n), y_n - y \rangle = 0, \\ \Rightarrow \quad & y_n \to y \text{ in } W_0^{1,p}(\Omega), \text{ hence } \|y\| = 1,\ y \geq 0 \text{ (see Proposition 1).} \end{aligned} \tag{22}$$

If in (18) we pass to the limit as $n \to +\infty$ and use (22) and (21), then

$$\begin{aligned} &\langle A(y), h \rangle = \int_\Omega \widehat{\lambda}_m y^{p-1} h dz \quad \text{for all } h \in W_0^{1,p}(\Omega), \\ \Rightarrow \quad & -\Delta_p y(z) = \widehat{\lambda}_m y(z)^{p-1} \quad \text{for a.a. } z \in \Omega, \quad y\big|_{\partial\Omega} = 0. \end{aligned}$$

Since $y \neq 0$ (see (22)) and $m \geq 2$, it follows that y must be nodal, a contradiction (see (22)). Therefore

$$\begin{aligned} &\{u_n^+\}_{n \in \mathbb{N}} \subseteq W_0^{1,p}(\Omega) \quad \text{is bounded,} \\ \Rightarrow \quad & \{u_n\}_{n \in \mathbb{N}} \subseteq W_0^{1,p}(\Omega) \quad \text{is bounded (see (17)).} \end{aligned}$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u \text{ in } L^p(\Omega). \tag{23}$$

In (16) we choose $h = (u_n - u) \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (23). Then

$$
\begin{aligned}
& \lim_{n\to+\infty} \langle A(u_n), u_n - u\rangle = 0, \\
\Rightarrow \quad & u_n \to u \text{ in } W_0^{1,p}(\Omega) \text{ (see Proposition 1)}, \\
\Rightarrow \quad & \widehat{\varphi}_\lambda \text{ satisfies the } C\text{-condition.}
\end{aligned}
$$

This proves the Claim.

Then (9), (14) and the Claim permit the use of Theorem 1 (the mountain pass theorem). So, for $\lambda \in (0, \lambda_0)$, we can find $u_\lambda \in W_0^{1,p}(\Omega)$ such that

$$
u_\lambda \in K_{\widehat{\varphi}_\lambda} \quad \text{and} \quad u_\lambda \neq 0.
$$

We have

$$
\begin{aligned}
& \widehat{\varphi}'_\lambda(u_\lambda) = 0, \\
\Rightarrow \quad & \langle A(u_\lambda), h\rangle = \int_\Omega \widehat{f}_\lambda(z, u_\lambda) h dz \quad \text{for all } h \in W_0^{1,p}(\Omega).
\end{aligned}
\tag{24}
$$

In (24) we choose $h = (\widetilde{u}_\lambda - u_\lambda)^+ \in W_0^{1,p}(\Omega)$. Then

$$
\begin{aligned}
\langle A(u_\lambda), (\widetilde{u}_\lambda - u_\lambda)^+\rangle &= \int_\Omega [\lambda \widetilde{u}_\lambda^{-\gamma} + f(z, u_\lambda)](\widetilde{u}_\lambda - u_\lambda)^+ dz \quad \text{(see (4))} \\
&\geq \int_\Omega \lambda \widetilde{u}_\lambda^{-\gamma} (\widetilde{u}_\lambda - u_\lambda)^+ dz \quad (\text{since } f \geq 0) \\
&= \langle A(\widetilde{u}_\lambda), (\widetilde{u}_\lambda - u)^+\rangle,
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad & \langle A(\widetilde{u}_\lambda) - A(u_\lambda), (\widetilde{u}_\lambda - u)^+\rangle \leq 0, \\
\Rightarrow \quad & \widetilde{u}_\lambda \leq u_\lambda.
\end{aligned}
$$

Then from (24) and (4) it follows that

$$
\begin{aligned}
& u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+ \quad \text{for } \lambda \in (0, \lambda_0) \\
\Rightarrow \quad & (0, \lambda_0) \subseteq \mathcal{L} \neq \emptyset.
\end{aligned}
$$

□

Now we prove a structural property of $\mathcal{L}$, namely we show that $\mathcal{L}$ is an interval.

Proposition 4. *If hypotheses $H(f)$ hold, $\lambda \in \mathcal{L}$ and $\mu \in (0, \lambda)$, then $\mu \in \mathcal{L}$.*

Proof. Since $\lambda \in \mathcal{L}$ we can find $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$. We have

$$\begin{aligned} -\Delta_p u_\lambda(z) &= \lambda u_\lambda(z)^{-\gamma} + f(z, u_\lambda(z)) \\ &\geq \mu u_\lambda(z)^{-\gamma} + f(z, u_\lambda(z)) \quad \text{for a.a. } z \in \Omega \text{ (since } \mu < \lambda). \end{aligned}$$

We know that $u_\lambda \in \operatorname{int} C_+$. Therefore, on account of Proposition 2, we can find $\widehat{t} \in (0, 1)$ small such that

$$\widehat{t}\, \widetilde{u}_\mu \leq u_\lambda. \tag{25}$$

We have

$$-\Delta_p(\widehat{t}\, \widetilde{u}_\mu) = -\widehat{t}^p \Delta_p \widetilde{u}_\mu = \mu \widehat{t}^p \widetilde{u}_\mu^{-\gamma} \leq \mu(\widehat{t}\, \widetilde{u}_\mu)^{-\gamma}. \tag{26}$$

We set $\overline{u}_\mu = \widehat{t}\, \widetilde{u}_\mu$ and consider the following truncation of the reaction in problem (P_μ)

$$\widehat{f}_\mu(z, x) = \begin{cases} \mu \overline{u}_\mu(z)^{-\gamma} + f(z, \overline{u}_\mu(z)) & \text{if } x < \overline{u}_\mu(z), \\ \mu x^{-\gamma} + f(z, x) & \text{if } \overline{u}_\mu(z) \leq x \leq u_\lambda(z), \\ \mu u_\lambda(z)^{-\gamma} + f(z, u_\lambda(z)) & \text{if } u_\lambda(z) < x, \end{cases} \quad \text{(see (25))}. \tag{27}$$

This is a Carathéodory function. We set $\widehat{F}_\mu(z, x) = \int_0^x \widehat{f}_\mu(z, s) ds$ and consider the C^1-functional $\widehat{\varphi}_\mu : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\varphi}_\mu(u) = \frac{1}{p} \|\nabla u\|_p^p - \int_\Omega \widehat{F}_\mu(z, u) dz \quad \text{for all } u \in W_0^{1,p}(\Omega)$$

(see Proposition 3 of Papageorgiou-Smyrlis [16]). From (27) it is clear that $\widehat{\varphi}_\mu(\cdot)$ is coercive. Also, using the Sobolev embedding theorem, we see that $\widehat{\varphi}_\mu(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_\mu \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} & \widehat{\varphi}_\mu(u_\mu) = \inf[\widehat{\varphi}_\mu(u) : u \in W_0^{1,p}(\Omega)], \\ \Rightarrow \quad & \widehat{\varphi}'_\mu(u_\mu) = 0, \\ \Rightarrow \quad & \langle A(u_\mu), h \rangle = \int_\Omega \widehat{f}_\mu(z, u_\mu) h dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{28}$$

In (28) we choose $h = (\overline{u}_\mu - u_\mu)^+ \in W_0^{1,p}(\Omega)$. We have

$$
\begin{aligned}
&\langle A(u_\mu), (\overline{u}_\mu - u_\mu)^+ \rangle \\
&= \int_\Omega [\mu \overline{u}_\mu^{-\gamma} + f(z, \overline{u}_\mu)](\overline{u}_\mu - u_\mu)^+ dz \quad \text{(see (27))} \\
&\geq \int_\Omega \mu \overline{u}_\mu^{-\gamma} (\overline{u}_\mu - u_\mu)^+ dz \quad \text{(since } f \geq 0) \\
&\geq \langle A(\overline{u}_\mu), (\overline{u}_\mu - u_\mu)^+ \rangle \quad \text{(see (26))}, \\
\Rightarrow \quad &\langle A(\overline{u}_\mu) - A(u_\mu), (\overline{u}_\mu - u_\mu)^+ \rangle \leq 0, \\
\Rightarrow \quad &\overline{u}_\mu \leq u_\mu \quad \text{(see Proposition 1).}
\end{aligned} \tag{29}
$$

Next in (28) we choose $h = (u_\mu - u_\lambda)^+ \in W_0^{1,p}(\Omega)$. Then

$$
\begin{aligned}
&\langle A(u_\mu), (u_\mu - u_\lambda)^+ \rangle \\
&= \int_\Omega [\mu u_\lambda^{-\gamma} + f(z, u_\lambda)](u_\mu - u_\lambda)^+ dz \quad \text{(see (27))} \\
&\leq \int_\Omega [\lambda u_\lambda^{-\gamma} + f(z, u_\lambda)](u_\mu - u_\lambda)^+ dz \quad \text{(since } \mu < \lambda) \\
&= \langle A(u_\lambda), (u_\mu - u_\lambda)^+ \rangle \quad \text{(since } u_\lambda \in S_\lambda), \\
\Rightarrow \quad &\langle A(u_\mu) - A(u_\lambda), (u_\mu - u_\lambda)^+ \rangle \leq 0, \\
\Rightarrow \quad &u_\mu \leq u_\lambda \quad \text{(see Proposition 1).}
\end{aligned} \tag{30}
$$

From (29) and (30) we infer that

$$
u_\mu \in [\overline{u}_\mu, u_\lambda].
$$

Then from (28) and (27), we have

$$
\begin{aligned}
&\langle A(u_\mu), h \rangle = \int_\Omega [\mu u_\mu^{-\gamma} + f(z, u_\mu)] h dz \quad \text{for all } h \in W_0^{1,p}(\Omega), \\
\Rightarrow \quad &u_\mu \in S_\mu \subseteq \operatorname{int} C_+ \text{ and so } \mu \in \mathcal{L}.
\end{aligned}
$$

□

Proposition 4 implies that $\mathcal{L}$ is an interval. Moreover, as a byproduct of the above proof, we have the following corollary which establishes a kind of monotonicity property for the solution multifunction $\lambda \to S_\lambda$.

Corollary 1. *If hypotheses $H(f)$ hold, $\lambda \in \mathcal{L}$, $\mu \in (0,\lambda)$ and $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$, then $\mu \in \mathcal{L}$ and there exists $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ such that $(u_\lambda - u_\mu) \in C_+ \setminus \{0\}$.*

With little additional effort, we can improve this corollary.

Proposition 5. *If hypotheses $H(f)$ hold, $\lambda \in \mathcal{L}$, $\mu \in (0,\lambda)$ and $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$, then $\mu \in \mathcal{L}$ and there exists $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ such that $(u_\lambda - u_\mu) \in \operatorname{int} C_+$.*

Proof. From Corollary 1, we already know that $\mu \in \mathcal{L}$ and there exists $u_\mu \in S_\mu \subseteq \operatorname{int} C_+$ such that

$$0 \leq u_\mu \leq u_\lambda. \tag{31}$$

Let $\rho = \|u_\lambda\|_\infty$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis $H(f)\,(iv)$. We have

$$\begin{aligned}
&-\Delta_p u_\lambda(z) - \lambda u_\lambda(z)^{-\gamma} + \widehat{\xi}_\rho u_\lambda(z)^{p-1} \\
&= f(z, u_\lambda(z)) + \widehat{\xi}_\rho u_\lambda(z)^{p-1} \\
&\geq f(z, u_\mu(z)) + \widehat{\xi}_\rho u_\mu(z)^{p-1} \quad \text{(see (31) and hypothesis } H(f)\,(iv)) \\
&> f(z, u_\mu(z)) + \widehat{\xi}_\rho u_\mu(z)^{p-1} - (\lambda-\mu) u_\mu(z)^{-\gamma} \quad (\text{since } \lambda > \mu,\ u_\mu \in \operatorname{int} C_+) \\
&= -\Delta_p u_\mu(z) - \lambda u_\mu(z)^{-\gamma} + \widehat{\xi}_\rho u_\mu(z)^{p-1} \quad \text{for a.a. } z \in \Omega \text{ (since } u_\mu \in S_\mu \subseteq \operatorname{int} C_+).
\end{aligned}$$

Invoking Proposition 4 of Papageorgiou-Smyrlis [16], we conclude that $(u_\lambda - u_\mu) \in \operatorname{int} C_+$. □

We set $\lambda^* = \sup \mathcal{L}$.

Proposition 6. *If hypotheses $H(f)$ hold, then $\lambda^* < +\infty$.*

Proof. We claim that there exists $\widehat{\lambda} > 0$ such that

$$\widehat{\lambda} x^{-\gamma} + f(z,x) \geq \widehat{\lambda}_1 x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq 0. \tag{32}$$

To see this, note that hypothesis $H(f)\,(ii)$ implies that we can find $\eta > \widehat{\lambda}_1$ and $M_2 > 0$ such that

$$f(z,x) \geq \eta x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq M_2. \tag{33}$$

Note that

$$\frac{\lambda}{x^\gamma} > \frac{\lambda}{M_2^\gamma} \quad \text{for all } x \in (0, M_2). \tag{34}$$

Moreover, for $\lambda > 0$ big we will have

$$\frac{\lambda}{M_2^\gamma} \geq \eta M_2^{p-1}. \tag{35}$$

Let $\widehat{\lambda} = \eta M_2^{p+\gamma-1}$. Then from (33), (34), (35) and since $f \geq 0$, we have that (32) holds for this $\widehat{\lambda} > 0$.

Let $\lambda > \widehat{\lambda}$ and suppose that $\lambda \in \mathcal{L}$. Then we can find $u_\lambda \in S_\lambda \subseteq \operatorname{int} C_+$. Using Proposition 2, let $t > 0$ be the biggest positive real such that

$$t\widehat{u}_1 \leq u_\lambda. \tag{36}$$

Let $\rho = \|u_\lambda\|_\infty$ and consider $\widehat{\xi}_\rho > 0$ as postulated by hypothesis $H(f)\,(iv)$. We have

$$\begin{aligned}
& -\Delta_p(t\widehat{u}_1) + \widehat{\xi}_p(t\widehat{u}_1)^{p-1} \\
& = [\widehat{\lambda}_1 + \widehat{\xi}_\rho](t\widehat{u}_1)^{p-1} \\
& \leq \widehat{\lambda}(t\widehat{u}_1)^{-\gamma} + f(z, t\widehat{u}_1) + \widehat{\xi}_\rho(t\widehat{u}_1)^{p-1} \quad \text{for a.a. } z \in \Omega \text{ (see (32))}, \\
\Rightarrow \quad & -\Delta_p(t\widehat{u}_1) - \lambda(t\widehat{u}_1)^{-\gamma} + \widehat{\xi}_\rho(t\widehat{u}_1)^{p-1} \\
& < f(z, t\widehat{u}_1) + \widehat{\xi}_\rho(t\widehat{u}_1)^{p-1} \quad (\text{since } \lambda > \widehat{\lambda}) \\
& \leq f(z, u_\lambda) + \widehat{\xi}_\rho u_\lambda^{p-1} \quad (\text{see (36) and hypothesis } H(f)\,(iv)) \\
& = -\Delta_p u_\lambda - \lambda u_\lambda^{-\gamma} + \widehat{\xi}_\rho u_\lambda^{p-1} \quad \text{for a.a } z \in \Omega, \\
\Rightarrow \quad & (u_\lambda - t\widehat{u}_1) \in \operatorname{int} C_+ \quad \text{(see Proposition 4 of Papageorgiou-Smyrlis [16])}.
\end{aligned}$$

This contradicts the maximality of $t > 0$. Therefore $\lambda \notin \mathcal{L}$ and so $\lambda^* \leq \widehat{\lambda} < +\infty$. □

Proposition 7. *If hypotheses $H(f)$ hold and $\lambda \in (0, \lambda^*)$, then problem (P_λ) admits at least two positive solutions u_0, $\widehat{u} \in \operatorname{int} C_+$.*

Proof. Let $0 < \lambda_1 < \lambda < \lambda_2 < \lambda^*$. According to Proposition 5 we can find $u_{\lambda_2} \in S_{\lambda_2} \subseteq \operatorname{int} C_+$ and $u_{\lambda_1} \in S_{\lambda_1} \subseteq \operatorname{int} C_+$ such that $(u_{\lambda_2} - u_{\lambda_1}) \in \operatorname{int} C_+$.

We introduce the following Carathéodory function

$$g_\lambda(z,x) = \begin{cases} \lambda u_{\lambda_1}(z)^{-\gamma} + f(z, u_{\lambda_1}(z)) & \text{if } x < u_{\lambda_1}(z), \\ \lambda x^{-\gamma} + f(z,x) & \text{if } u_{\lambda_1}(z) \leq x \leq u_{\lambda_2}(z), \\ \lambda u_{\lambda_2}(z)^{-\gamma} + f(z, u_{\lambda_2}(z)) & \text{if } u_{\lambda_2}(z) < x. \end{cases} \tag{37}$$

We set $G_\lambda(z,x) = \int_0^x g_\lambda(z,s)ds$ and consider the functional $\psi_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega G_\lambda(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Proposition 3 of Papageorgiou-Smyrlis [16] implies that $\psi_\lambda \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$. From (37) it is clear that $\psi_\lambda(\cdot)$ is coercive, while using the Sobolev embedding theorem, we see that $\psi_\lambda(\cdot)$ is sequentially weakly lower semicontinuous. Therefore, we can find $u_0 \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} & \psi_\lambda(u_0) = \inf[\psi_\lambda(u) : u \in W_0^{1,p}(\Omega)], \\ \Rightarrow \quad & \psi_\lambda'(u_0) = 0, \\ \Rightarrow \quad & \langle A(u_0), h\rangle = \int_\Omega g_\lambda(z,u_0)hdz \quad \text{for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{38}$$

In (38) we choose $h = (u_{\lambda_1} - u_0)^+ \in W_0^{1,p}(\Omega)$. Then

$$\begin{aligned} & \langle A(u_0), (u_{\lambda_1} - u_0)^+\rangle \\ & = \int_\Omega [\lambda u_{\lambda_1}^{-\gamma} + f(z, u_{\lambda_1})](u_{\lambda_1} - u_0)^+ dz \quad \text{(see (37))} \\ & = \langle A(u_{\lambda_1}), (u_{\lambda_1} - u_0)^+\rangle \quad (\text{since } u_{\lambda_1} \in S_{\lambda_1}), \\ \Rightarrow \quad & \langle A(u_{\lambda_1}) - A(u_0), (u_{\lambda_1} - u_0)^+\rangle = 0, \\ \Rightarrow \quad & u_{\lambda_1} \leq u_0 \quad \text{(see Proposition 1)}. \end{aligned}$$

Similarly choosing $h = (u_0 - u_{\lambda_2})^+ \in W_0^{1,p}(\Omega)$ in (38), we obtain

$$\begin{aligned} & u_0 \leq u_{\lambda_2}, \\ \Rightarrow \quad & u_0 \in [u_\lambda, u_{\lambda_2}]. \end{aligned} \tag{39}$$

From (37), (38) and (39) it follows that

$$u_0 \in S_\lambda \subseteq \operatorname{int} C_+.$$

Moreover, as before (see the proof of Proposition 5), using hypothesis $H(f)\,(iv)$ with $\rho = \|u_{\lambda_2}\|_\infty$ and Proposition 4 (the strong comparison principle) of Papageorgiou-Smyrlis [16], we show that

$$\begin{aligned} & u_0 - u_{\lambda_1} \in \operatorname{int} C_+ \quad \text{and} \quad u_{\lambda_2} - u_0 \in \operatorname{int} C_+, \\ \Rightarrow \quad & u_0 \in \operatorname{int}_{C_0^1(\overline{\Omega})}[u_{\lambda_1}, u_{\lambda_2}]. \end{aligned} \tag{40}$$

Next we consider the Carathéodory function $e_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by

$$e_\lambda(z,x) = \begin{cases} \lambda u_{\lambda_1}(z)^{-\gamma} + f(z, u_{\lambda_1}(z)) & \text{if } x \leq u_{\lambda_1}(z), \\ \lambda x^{-\gamma} + f(z,x) & \text{if } u_{\lambda_1}(z) < x. \end{cases} \tag{41}$$

We set $E_\lambda(z,x) = \int_0^x e_\lambda(z,s)ds$ and consider the functional $\sigma_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_\lambda(u) = \frac{1}{p}\|\nabla u\|_p^p - \int_\Omega E_\lambda(z,u)dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

We know that $\sigma_\lambda \in C^1(W_0^{1,p}(\Omega), \mathbb{R})$ (see Proposition 3 of [16]). From (37) and (41) it is clear that

$$\sigma_\lambda\big|_{[u_{\lambda_1}, u_{\lambda_2}]} = \psi_\lambda\big|_{[u_{\lambda_1}, u_{\lambda_2}]}. \tag{42}$$

From the first part of the proof, we know that $u_0 \in \operatorname{int} C_+$ is a minimizer of ψ_λ. This fact together with (40) and (42) imply that

$$\begin{aligned} & u_0 \text{ is a local } C_0^1(\overline{\Omega})\text{-minimizer of } \sigma_\lambda, \\ \Rightarrow \quad & u_0 \text{ is a local } W_0^{1,p}(\Omega)\text{-minimizer of } \sigma_\lambda \end{aligned} \tag{43}$$

(see Giacomoni-Saoudi [8], Theorem 1.1).

Using (41), we can easily show that

$$K_{\sigma_\lambda} \subseteq [u_{\lambda_1}) \cap \operatorname{int} C_+. \tag{44}$$

Hence, we may assume that K_{σ_λ} is finite or otherwise it is clear from (44) and (41) that we already have an infinity of positive smooth solutions of (P_λ) and so we are done. On account of (43), we can find $\rho \in (0,1)$ small such that

$$\sigma_\lambda(u_0) < \inf[\sigma_\lambda(u) : \|u - u_0\| = \rho] = m_\rho \tag{45}$$

(see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 29).

As in the proof of Proposition 3, we show that

$$\sigma_\lambda(\cdot) \text{ satisfies the } C\text{-condition (see the Claim in the proof of Proposition 3);} \tag{46}$$

$$\sigma_\lambda(t\widehat{u}_1) \to -\infty \text{ as } t \to +\infty \text{ (see (14) in the proof of Proposition 3).} \tag{47}$$

Then (45), (46), (47) permit the use of Theorem 1 (the mountain pass theorem). So, we can find $\widehat{u} \in W_0^{1,p}(\Omega)$ such that

$$\widehat{u} \in K_{\sigma_\lambda} \subseteq [u_{\lambda_1}) \cap \operatorname{int} C_+ \text{ (see (44))} \quad \text{and} \quad \sigma_\lambda(u_0) < m_\rho \le \sigma_\lambda(\widehat{u}).$$

Therefore $\widehat{u} \neq u_0$, $\widehat{u} \in \operatorname{int} C_+$ is the second positive solution of (P_λ). ☐

Finally we examine what can be said about the critical parameter value.

Proposition 8. *If hypotheses $H(f)$ hold, then $\lambda^* \in \mathcal{L}$.*

Proof. Consider a sequence $\{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathcal{L}$ such that $\lambda_n \to (\lambda^*)^-$. Let $u_n \in S_{\lambda_n} \subseteq \operatorname{int} C_+$ for all $n \in \mathbb{N}$. We have

$$\langle A(u_n), h\rangle = \int_\Omega \lambda_n u_n^{-\gamma} h dz + \int_\Omega f(z, u_n) h dz \quad \text{for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}. \tag{48}$$

We claim that $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded. Otherwise, we may assume that

$$\|u_n\| \to +\infty \quad \text{as } n \to +\infty. \tag{49}$$

Let $y_n = \dfrac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$, $y_n \ge 0$ for all $n \in \mathbb{N}$. So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad y_n \to y \text{ in } L^p(\Omega),\ y \ge 0. \tag{50}$$

From (48) we have

$$\langle A(y_n), h\rangle = \int_\Omega \lambda_n \frac{u_n^{-\gamma}}{\|u_n\|^{p-1}} h dz + \int_\Omega \frac{N_f(u_n)}{\|u_n\|^{p-1}} h dz \quad \text{for all } h \in W_0^{1,p}(\Omega), \text{ all } n \in \mathbb{N}. \tag{51}$$

From (20) and (50) it is clear that

$$\left\{ \frac{N_f(u_n)}{\|u_n\|^{p-1}} \right\}_{n\in\mathbb{N}} \subseteq L^{p'}(\Omega) \quad \text{is bounded.} \tag{52}$$

If in (51) we choose $h = (u_n - u) \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (49), (50), (52), then we obtain

$$\begin{aligned} &\lim_{n\to+\infty} \langle A(y_n), y_n - y\rangle = 0, \\ \Rightarrow \quad & y_n \to y \text{ in } W_0^{1,p}(\Omega), \text{ hence } \|y\| = 1,\ y \geq 0 \text{ (see Proposition 1).} \end{aligned} \tag{53}$$

On account of (52) and of hypothesis $H(f)\,(ii)$, we may assume that

$$\frac{N_f(u_n)}{\|u_n\|^{p-1}} \xrightarrow{w} \widehat{\lambda}_m y^{p-1} \quad \text{in } L^{p'}(\Omega) \tag{54}$$

(see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 16).

Therefore, if in (51) we let $n \to +\infty$ and use (49), (53), (54), we obtain

$$\begin{aligned} &\langle A(y), h\rangle = \int_\Omega \widehat{\lambda}_m y^{p-1} h dz \quad \text{for all } h \in W_0^{1,p}(\Omega), \\ \Rightarrow &- \Delta_p y(z) = \widehat{\lambda}_m y(z)^{p-1} \quad \text{for a.a. } z \in \Omega,\ y\big|_{\partial\Omega} = 0, \\ \Rightarrow \quad & y \text{ is nodal (recall } m \geq 2\text{), a contradiction, see (53).} \end{aligned}$$

Therefore, $\{u_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u^* \text{ in } W_0^{1,p}(\Omega) \quad \text{and} \quad u_n \to u^* \text{ in } L^p(\Omega). \tag{55}$$

From the proof of Proposition 7, we see that we can always have that $\{u_n\}_{n\in\mathbb{N}}$ is increasing (see (39)). Hence $u^* \neq 0$. Since $\{N_f(u_n)\}_{n\in\mathbb{N}} \subseteq L^{p'}(\Omega)$ is bounded, if in (48) we choose $h = (u_n - u^*) \in W_0^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (55), then

$$\begin{aligned} &\lim_{n\to+\infty} \langle A(u_n), u_n - u^*\rangle = 0, \\ \Rightarrow \quad & u_n \to u^* \quad \text{in } W_0^{1,p}(\Omega) \quad \text{(see Proposition 1).} \end{aligned}$$

So, from (48) in the limit as $n \to +\infty$, we obtain

$$\langle A(u^*), h\rangle = \int_\Omega \lambda^*(u^*)^{-\gamma} h dz + \int_\Omega f(z, u^\star) h dz \quad \text{for all } h \in W_0^{1,p}(\Omega),$$

$$\Rightarrow \quad u^* \in S_{\lambda^*} \subseteq \operatorname{int} C_+ \quad \text{ and so } \lambda^* \in \mathcal{L}.$$

□

Proposition 8 implies that $\mathcal{L} = (0, \lambda^*]$.

Summarizing the situation for problem (P_λ), we have the following bifurcation-type result.

Theorem 2. *If hypotheses $H(f)$ hold, then there exists $\lambda^* > 0$ such that*

(a) for all $\lambda \in (0, \lambda^*)$ problem (P_λ) has at least two positive solutions $u_0, \widehat{u} \in \operatorname{int} C_+$;

(b) for $\lambda = \lambda^\star$ problem (P_λ) has at least one positive solution $u^* \in \operatorname{int} C_+$;

(c) for all $\lambda > \lambda^*$ problem (P_λ) has no positive solutions.

Remark 2. A careful reading of the proof reveals that in fact in hypothesis $H(f)\,(ii)$, it is enough to assume that

$$\eta \geq \limsup_{x \to +\infty} \frac{f(z, x)}{x^{p-1}} \geq \liminf_{x \to +\infty} \frac{f(z, x)}{x^{p-1}} > \widehat{\lambda}_1$$

that is, the problem need not be resonant. If resonance occurs we use the second part of hypothesis $H(f)\,(ii)$. Otherwise that condition is redundant. To emphasize that the interesting case is the resonant one, we have decided to proceed with the particular formulation of $H(f)\,(ii)$.

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