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Classes of operators satisfying a-Weyl's theorem

by

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Abstract. In this article Weyl's theorem and a-Weyl's theorem on Banach spaces are related to an important property which has a leading role in local spectral theory: the single-valued extension theory.

We show that if T has SVEP then Weyl's theorem and a-Weyl's theorem for T^* are equivalent, and analogously, if T^* has SVEP then Weyl's theorem and a-Weyl's theorem for T are equivalent. From this result we deduce that a-Weyl's theorem holds for classes of operators for which the quasi-nilpotent part $H_0(\lambda I - T)$ is equal to $\ker (\lambda I - T)^p$ for some $p \in \mathbb{N}$ and every $\lambda \in \mathbb{C}$, and for algebraically paranormal operators on Hilbert spaces. We also improve recent results established by Curto and Han, Han and Lee, and Oudghiri.

1. Notation and terminology. We begin with some standard notations in Fredholm theory. Throughout this note by $L(X)$ we denote the algebra of all bounded linear operators acting on an infinite-dimensional complex Banach space X. For every $T \in L(X)$ we denote by $\alpha(T)$ and $\beta(T)$ the dimension of the kernel $\ker T$ and the codimension of the range $T(X)$, respectively. The class of *upper semi-Fredholm* operators is defined by

$$\Phi_+(X) := \{T \in L(X) : \alpha(T) < \infty \text{ and } T(X) \text{ is closed}\},$$

whilst the class of *lower semi-Fredholm* operators is defined by

$$\Phi_-(X) := \{T \in L(X) : \beta(T) < \infty\}.$$

An operator $T \in L(X)$ is said to be *semi-Fredholm* if $T \in \Phi_+(X) \cup \Phi_-(X)$, whilst the class of *Fredholm operators* is $\Phi(X) := \Phi_+(X) \cap \Phi_-(X)$. The *index* of a semi-Fredholm operator is defined by $\operatorname{ind} T := \alpha(T) - \beta(T)$.

For a linear operator T the *ascent* $p := p(T)$ is defined as the smallest nonnegative integer p such that $\ker T^p = \ker T^{p+1}$. If such an integer does not exist we put $p(T) = \infty$. Analogously, the *descent* $q := q(T)$ is defined as the smallest nonnegative integer q such that $T^q(X) = T^{q+1}(X)$, and if such an integer does not exist we put $q(T) = \infty$. A classical result states that if

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$p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$ (see [24, Proposition 38.3]). Moreover, $\lambda \in \sigma(T)$ (the spectrum of T) is a pole of the resolvent precisely when $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ (see Proposition 50.2 of [24]), and in this case $X = \ker(\lambda I - T)^p \oplus (\lambda I - T)^p(X)$, with $p := p(\lambda I - T) = q(\lambda I - T)$. Two important classes of operators in Fredholm theory are the class of *upper semi-Browder operators* defined by

$$\mathcal{B}_+(X) := \{T \in \Phi_+(X) : p(T) < \infty\},$$

and the class of *lower semi-Browder operators* defined by

$$\mathcal{B}_-(X) := \{T \in \Phi_-(X) : q(T) < \infty\}.$$

The class of *Browder operators* (known in the literature also as the *Riesz–Schauder operators*) is defined by $\mathcal{B}(X) := \mathcal{B}_+(X) \cap \mathcal{B}_-(X)$. A bounded operator $T \in L(X)$ is called a *Weyl operator* if $T \in \Phi(X)$ and $\operatorname{ind} T = 0$. A Browder operator T is Weyl since the finiteness of $p(T)$ and $q(T)$ entails for a Fredholm operator T that T has index 0 (cf. [24, Proposition 38.5]).

The classes of operators defined above motivate the definition of several spectra. The *upper semi-Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\mathrm{ub}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}_+(X)\},$$

the *lower semi-Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\mathrm{lb}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}_-(X)\},$$

whilst the *Browder spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\mathrm{b}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{B}(X)\}.$$

Finally, the *Weyl spectrum* of $T \in L(X)$ is defined by

$$\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Weyl}\}.$$

It should be noted that $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{w}}(T^*)$, whilst

$$\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{lb}}(T^*), \qquad \sigma_{\mathrm{lb}}(T) = \sigma_{\mathrm{ub}}(T^*).$$

Moreover,

$$\sigma_{\mathrm{w}}(T) \subseteq \sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T) \cup \operatorname{acc} \sigma(T),$$

where we write $\operatorname{acc} K$ for the accumulation points of $K \subseteq \mathbb{C}$.

Recall that $T \in L(X)$ is said to be *bounded below* if T is injective and has closed range. Let $\sigma_{\mathrm{a}}(T)$ denote the classical *approximate point spectrum* of T defined as

$$\sigma_{\mathrm{a}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\},$$

and let

$$\sigma_{\mathrm{s}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\}$$

denote the *surjectivity spectrum* of T.

For a bounded operator $T \in L(X)$ set

$$p_{00}(T) := \sigma(T) \setminus \sigma_{\mathrm{b}}(T) = \{\lambda \in \sigma(T) : \lambda I - T \text{ is Browder}\}.$$

and, if we write $\operatorname{iso} K$ for the set of all isolated points of $K \subseteq \mathbb{C}$, then we define

$$\pi_{00}(T) := \{\lambda \in \operatorname{iso} \sigma(T) : 0 < \alpha(\lambda I - T) < \infty\}.$$

Obviously,

$$p_{00}(T) \subseteq \pi_{00}(T) \quad \text{for every } T \in L(X). \tag{1}$$

Following Coburn [10], we say that *Weyl's theorem holds* for $T \in L(X)$ if

$$\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T), \tag{2}$$

whilst T *satisfies Browder's theorem* if

$$\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = p_{00}(T),$$

or equivalently, $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$.

The *Weyl* (or *essential*) *approximate point spectrum* $\sigma_{\mathrm{wa}}(T)$ of a bounded operator $T \in L(X)$ is the complement of those $\lambda \in \mathbb{C}$ for which $\lambda I - T \in \Phi_+(X)$ and $\operatorname{ind}(\lambda I - T) \leq 0$. Note that $\sigma_{\mathrm{wa}}(T)$ is the intersection of all approximate point spectra $\sigma_{\mathrm{a}}(T + K)$ of compact perturbations K of T (see [32]). The *Weyl surjectivity spectrum* $\sigma_{\mathrm{ws}}(T)$ is the complement of those $\lambda \in \mathbb{C}$ for which $\lambda I - T \in \Phi_-(X)$ and $\operatorname{ind}(\lambda I - T) \geq 0$. The spectrum $\sigma_{\mathrm{wa}}(T)$ coincides with the intersection of all surjectivity spectra $\sigma_{\mathrm{s}}(T + K)$ of compact perturbations K of T (see [32] or [1, p. 151]). Clearly, the two spectra are dual to each other, i.e.,

$$\sigma_{\mathrm{wa}}(T) = \sigma_{\mathrm{ws}}(T^*) \quad \text{and} \quad \sigma_{\mathrm{ws}}(T) = \sigma_{\mathrm{wa}}(T^*).$$

Furthermore, $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{wa}}(T) \cup \sigma_{\mathrm{ws}}(T)$. Note that $\sigma_{\mathrm{wa}}(T) \subseteq \sigma_{\mathrm{ub}}(T)$ and $\sigma_{\mathrm{ws}}(T) \subseteq \sigma_{\mathrm{lb}}(T)$; precisely:

$$\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{wa}}(T) \cup \operatorname{acc} \sigma_{\mathrm{a}}(T), \tag{3}$$

$$\sigma_{\mathrm{lb}}(T) = \sigma_{\mathrm{ws}}(T) \cup \operatorname{acc} \sigma_{\mathrm{s}}(T) \tag{4}$$

(see [33]). Define

$$\pi^{\mathrm{a}}_{00}(T) := \{\lambda \in \operatorname{iso} \sigma_{\mathrm{a}}(T) : 0 < \alpha(\lambda I - T) < \infty\}.$$

Following Rakočević [32], we shall say that *a-Weyl's theorem holds* for $T \in L(X)$ if

$$\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{wa}}(T) = \pi^{\mathrm{a}}_{00}(T),$$

whilst we shall say that T satisfies *a-Browder's theorem* if

$$\sigma_{\mathrm{wa}}(T) = \sigma_{\mathrm{ub}}(T).$$

We have

$$a\text{-Browder's theorem} \;\Rightarrow\; \text{Browder's theorem},$$

and

$$a\text{-Weyl's theorem} \;\Rightarrow\; \text{Weyl's theorem} \;\Rightarrow\; \text{Browder's theorem}$$

(see for instance [1, Chapter 3]).

2. Single-valued extension property. The single-valued extension property dates back to the early days of local spectral theory and was introduced by Dunford [19], in his theory of spectral operators. This property plays a crucial role in local spectral theory (see the recent monograph of Laursen and Neumann [26]). We shall consider a local version of this property, which has been studied in recent papers [3], [4], [6], and previously by Finch [20] and Mbekhta [29].

DEFINITION 2.1. The operator $T \in L(X)$ is said to have *the single-valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0) if for every open disc U centered at λ_0 the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$.

An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$.

Trivially, an operator $T \in L(X)$ has SVEP at every point of the resolvent set $\varrho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity theorem for analytic functions it easily follows that $T \in L(X)$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum $\sigma(T)$. In particular, every operator has SVEP at the isolated point of its spectrum.

An important subspace in local spectral theory is the *glocal spectral subspace* $\mathcal{X}_T(F)$ associated with a closed subset $F \subseteq \mathbb{C}$. It is defined, for an arbitrary operator $T \in L(X)$ and a closed subset F of $\mathbb{C}$, as the set of all $x \in X$ for which there exists an analytic function $f : \mathbb{C} \setminus F \to X$ which satisfies the identity $(\lambda I - T)f(\lambda) = x$ for all $\lambda \in \mathbb{C} \setminus F$. The basic role of SVEP arises in local spectral theory since all decomposable operators enjoy this property. Recall $T \in L(X)$ has the *decomposition property* (δ) if $X = \mathcal{X}_T(\overline{U}) + \mathcal{X}_T(\overline{V})$ for every open cover $\{U, V\}$ of $\mathbb{C}$. Decomposable operators may be defined in several ways, for instance as the union of those with property (β) and property (δ) (see [26, Theorem 2.5.19] for relevant definitions). Note that property (β) implies that T has SVEP, whilst property (δ) implies SVEP for T^* (see [26, Theorem 2.5.19]).

Note that

$$p(\lambda I - T) < \infty \;\Rightarrow\; T \text{ has SVEP at } \lambda,$$

and dually

$$q(\lambda I - T) < \infty \;\Rightarrow\; T^* \text{ has SVEP at } \lambda$$

(see [5]). Furthermore,

$$\sigma_{\mathrm{a}}(T) \text{ does not cluster at } \lambda \;\Rightarrow\; T \text{ has SVEP at } \lambda,$$

and

$$\sigma_{\mathrm{s}}(T) \text{ does not cluster at } \lambda \ \Rightarrow\ T^* \text{ has SVEP at } \lambda$$

(see [6]).

Let us consider the *quasi-nilpotent part* of T, i.e. the set

$$H_0(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0\}.$$

It is easily seen that $\ker(T^m) \subseteq H_0(T)$ for every $m \in \mathbb{N}$ and T is quasi-nilpotent if and only if $H_0(T) = X$ (see [29, Remarque 1.1]). Moreover, if T is invertible then $H_0(T) = \{0\}$.

The *analytic core* of T is the set $K(T)$ of all $x \in X$ such that there exists a sequence $(u_n) \subset X$ and $\delta > 0$ for which $x = u_0$, and $Tu_{n+1} = u_n$ and $\|u_n\| \leq \delta^n \|x\|$ for every $n \in \mathbb{N}$. It easily follows, from the definition, that $K(T)$ is a linear subspace of X and $T(K(T)) = K(T)$.

Definition 2.2. An operator $T \in L(X)$, X a Banach space, is said to be *semi-regular* if $T(X)$ is closed and $\ker T \subseteq T^\infty(X)$. An operator $T \in L(X)$ is said to admit a *generalized Kato decomposition*, abbreviated GKD, if there exists a pair (M, N) of T-invariant closed subspaces such that $X = M \oplus N$, the restriction $T|M$ is semi-regular and $T|N$ is quasi-nilpotent.

A relevant case is obtained if we assume in the definition above that $T|N$ is nilpotent. In this case T is said to be of *Kato type* (see for details [1]). Recall that every semi-Fredholm operator is of Kato type, by the classical result of Kato [25] (see also Chapter 1 of [1]). The following characterizations of SVEP for operators of Kato type have been proved in [3] and [6] (see also Chapter 3 in [1]).

Theorem 2.3. *If $\lambda_0 I - T \in L(X)$ is of Kato type then the following statements are equivalent*:

(i) *T has SVEP at λ_0*;
(ii) $p(\lambda_0 I - T) < \infty$;
(iii) *$\sigma_{\mathrm{a}}(T)$ does not cluster at λ_0*;
(iv) *$H_0(\lambda_0 I - T)$ is closed.*

If $\lambda_0 I - T$ is semi-Fredholm then assertions (i)–(iv) *are equivalent to the following statement*:

(v) *$H_0(\lambda_0 I - T)$ is finite-dimensional.*

Dually, if $\lambda_0 I - T$ is of Kato type then the following statements are equivalent:

(vi) *T^* has SVEP at λ_0*;
(vii) $q(\lambda_0 I - T) < \infty$;
(viii) *$\sigma_{\mathrm{s}}(T)$ does not cluster at λ_0*;

If $\lambda_0 I - T$ *is semi-Fredholm then assertions* (vi)–(viii) *are equivalent to the following statement*:

(ix) $K(\lambda_0 I - T)$ *is finite-codimensional.*

Let λ_0 be an isolated point of $\sigma(T)$ and let P_0 denote the spectral projection $P_0 := (2\pi i)^{-1} \int_\Gamma (\lambda I - T)^{-1}\, d\lambda$ associated with $\{\lambda_0\}$, via the classical Riesz functional calculus. A classical result shows that the range $P_0(X)$ is $N := H_0(\lambda_0 I - T)$ (see [24, Proposition 49.1]), whilst $\ker P_0$ is the analytic core $M := K(\lambda_0 I - T)$ of $\lambda_0 I - T$ (see [34] and [29]). In this case, $X = M \oplus N$ and

$$\sigma(\lambda_0 I - T|N) = \{\lambda_0\}, \qquad \sigma(\lambda_0 I - T|M) = \sigma(T) \setminus \{\lambda_0\},$$

so $\lambda_0 I - T|M$ is invertible and hence $H_0(\lambda_0 I - T|M) = \{0\}$. Therefore from the decomposition $H_0(\lambda_0 I - T) = H_0(\lambda_0 I - T|M) \oplus H_0(\lambda_0 I - T|N)$ we deduce that $N = H_0(\lambda_0 I - T|N)$, so $\lambda_0 I - T|N$ is quasi-nilpotent. Hence the pair (M, N) is a GKD for $\lambda_0 I - T$.

COROLLARY 2.4. *Let* λ_0 *be an isolated point of* $\sigma(T)$. *Then*

$$X = H_0(\lambda_0 I - T) \oplus K(\lambda_0 I - T)$$

and the following assertions are equivalent:

(i) $\lambda_0 I - T$ *is semi-Fredholm*;
(ii) $H_0(\lambda_0 I - T)$ *is finite-dimensional*;
(iii) $K(\lambda_0 I - T)$ *is finite-codimensional.*

Proof. Since for every operator $T \in L(X)$, both T and T^* have SVEP at any isolated point, the equivalence of the assertions easily follows from the decomposition $X = H_0(\lambda_0 I - T) \oplus K(\lambda_0 I - T)$, and from Theorem 2.3. ∎

THEOREM 2.5. *Let* $T \in L(X)$ *and suppose that* T *or* T^* *has SVEP. Then*

$$\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{wa}}(T), \qquad \sigma_{\mathrm{lb}}(T) = \sigma_{\mathrm{ws}}(T) \tag{5}$$

and

$$\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{w}}(T). \tag{6}$$

Proof. Suppose first that T has SVEP. To show the first equality of (5) we only need to show the inclusion $\sigma_{\mathrm{ub}}(T) \subseteq \sigma_{\mathrm{wa}}(T)$. If $\lambda \notin \sigma_{\mathrm{wa}}(T)$ then $\lambda I - T \in \Phi_+(X)$ and the SVEP implies by Theorem 2.3 that $p(\lambda I - T) < \infty$. Hence $\lambda \notin \sigma_{\mathrm{ub}}(T)$.

Analogously, to prove the equality $\sigma_{\mathrm{lb}}(T) = \sigma_{\mathrm{ws}}(T)$ we only need to show that $\sigma_{\mathrm{lb}}(T) \subseteq \sigma_{\mathrm{ws}}(T)$. If $\lambda \notin \sigma_{\mathrm{ws}}(T)$ then $\lambda I - T \in \Phi_-(X)$ with $\beta(\lambda I - T) \le \alpha(\lambda I - T)$. Again, the SVEP at λ entails that $p(\lambda I - T) < \infty$, and hence from Proposition 38.5 of [24] we deduce that $\alpha(\lambda I - T) = \beta(\lambda I - T)$. At this point, the finiteness of $p(\lambda I - T)$ implies by Proposition 38.6 of [24] that also $q(\lambda I - T)$ is finite, so $\lambda \notin \sigma_{\mathrm{lb}}(T)$. Therefore $\sigma_{\mathrm{lb}}(T) \subseteq \sigma_{\mathrm{ws}}(T)$ and the proof of the second equality is complete in the case that T has SVEP.

Suppose now that T^* has SVEP. Then, by the first part, $\sigma_{\mathrm{ub}}(T^*) = \sigma_{\mathrm{wa}}(T^*)$ and $\sigma_{\mathrm{lb}}(T^*) = \sigma_{\mathrm{ws}}(T^*)$. By duality it follows that $\sigma_{\mathrm{lb}}(T) = \sigma_{\mathrm{ws}}(T)$ and $\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{wa}}(T)$. The last equality is clear from the equality $\sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{ub}}(T) \cup \sigma_{\mathrm{lb}}(T)$ and $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{wa}}(T) \cup \sigma_{\mathrm{ws}}(T)$. ∎

We shall denote by $\mathcal{H}(\sigma(T))$ the set of all analytic functions defined on a neighborhood of $\sigma(T)$. The next result shows that for operators having SVEP the spectral theorem holds for $\sigma_{\mathrm{w}}(T)$. This is not, in general, true for all operators, whilst the spectral theorem holds for $\sigma_{\mathrm{b}}(T)$, $\sigma_{\mathrm{ub}}(T)$ and $\sigma_{\mathrm{lb}}(T)$ for every $T \in L(X)$ (see [33] or also [1, Chapter 3]).

Corollary 2.6. *Suppose that T or T^* has SVEP and $f \in \mathcal{H}(\sigma(T))$. Then*

$$\sigma_{\mathrm{wa}}(f(T)) = f(\sigma_{\mathrm{wa}}(T)), \qquad \sigma_{\mathrm{ws}}(f(T)) = f(\sigma_{\mathrm{ws}}(T)), \tag{7}$$

and

$$\sigma_{\mathrm{w}}(f(T)) = f(\sigma_{\mathrm{w}}(T)). \tag{8}$$

Moreover, a-Browder's theorem holds for both $f(T)$ and $f(T^*)$.

Proof. By Theorem 2.5 if T has SVEP (respectively, if T^* has SVEP) then $f(\sigma_{\mathrm{ub}}(T)) = f(\sigma_{\mathrm{wa}}(T))$. From the spectral mapping theorem for $\sigma_{\mathrm{ub}}(T)$ we then infer that $f(\sigma_{\mathrm{wa}}(T)) = \sigma_{\mathrm{ub}}(f(T))$, and again by Theorem 2.5 the last set coincides with $\sigma_{\mathrm{wa}}(f(T))$, since $f(T)$ (respectively, $f(T^*) = f(T)^*$) has SVEP by Theorem 3.3.6 of [26]. Hence the first equality of (7) is proved. The second equality of (7) and the equality (8) follow in a similar way.

The argument above shows that if T or T^* has SVEP then a-Browder's theorem holds for $f(T)$. Moreover, the SVEP for $f(T)$ (respectively, for $f(T^*)$) implies by Theorem 2.5 that $\sigma_{\mathrm{lb}}(f(T)) = \sigma_{\mathrm{ws}}(f(T))$, and hence by duality $\sigma_{\mathrm{ub}}(f(T^*)) = \sigma_{\mathrm{wa}}(f(T^*))$, so a-Browder's theorem also holds for $f(T^*)$. ∎

Note that Corollary 2.6 extends to a more general situation the result established in Theorem 3.2 of [12]. The spectral theorem for $\sigma_{\mathrm{w}}(T)$ in the case T or T^* has SVEP has been proved by using different methods by Curto and Han [11].

An operator $U \in L(X, Y)$ between the Banach spaces X and Y is said to be a *quasi-affinity* if U is injective and has dense range. The operator $S \in L(Y)$ is said to be a *quasi-affine transform* of $T \in L(X)$, notation $S \prec T$, if there is a quasi-affinity $U \in L(Y, X)$ such that $TU = US$. If both $S \prec T$ and $T \prec S$ hold then S, T are called *quasi-similar*.

Theorem 2.7. *If $T \in L(X)$ has SVEP at $\lambda_0 \in \mathbb{C}$ and $S \in L(Y)$ is a quasi-affine transform of T then S has SVEP at λ_0. In particular, if*

$T \in L(X)$ has SVEP and $S \prec T$ then $f(S)$ satisfies a-Browder's theorem for all $f \in \mathcal{H}(\sigma(T))$.

Proof. Let $f : \mathcal{U} \to Y$ be an analytic function defined on an open disc $\mathcal{U}$ of λ_0 such that $(\mu I - S)f(\mu) = 0$ for all $\mu \in \mathcal{U}$. Then $U(\lambda I - S)f(\mu) = (\mu I - T)Uf(\mu) = 0$ and the SVEP of T at λ_0 entails that $Uf(\mu) = 0$ for all $\mu \in \mathcal{U}$. Since U is injective it follows that $f(\mu) = 0$ for all $\mu \in \mathcal{U}$, hence S has SVEP at λ_0.

Thus if T has SVEP then S has SVEP. The last assertion is clear by Corollary 2.6. ∎

3. Weyl's theorems for Banach space operators. In this section we give a useful description of operators which satisfy Weyl's theorem, or a-Weyl's theorem, in terms of the SVEP. From these characterizations we shall deduce that a-Weyl's theorem holds for many classes of Banach space operators.

THEOREM 3.1. *If $T \in L(X)$ then the following assertions are equivalent*:

(i) *Weyl's theorem holds for T.*

(ii) *T has SVEP at every point $\lambda \notin \sigma_{\mathrm{w}}(T)$ and $\pi_{00}(T) = p_{00}(T)$.*

In particular, *if T or T^* has SVEP then Weyl's theorem holds for T if and only if $\pi_{00}(T) = p_{00}(T)$.*

Proof. (i)⇒(ii). Suppose that T satisfies Weyl's theorem. Let $\lambda \notin \sigma_{\mathrm{w}}(T)$. Since T has SVEP at every $\lambda \notin \sigma(T)$ we may assume that $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T)$. By definition of $\pi_{00}(T)$ we know that λ is isolated in $\sigma(T)$, so T has SVEP at λ.

To show that $\pi_{00}(T) = p_{00}(T)$ it suffices to prove the inclusion $\pi_{00}(T) \subseteq p_{00}(T)$. Suppose that $\lambda \in \pi_{00}(T) = \sigma(T) \setminus \sigma_{\mathrm{w}}(T)$. Since $\lambda I - T$ is Weyl, by Theorem 2.3 it follows that both $p(\lambda I - T)$ and $q(\lambda I - T)$ are finite. Consequently, $\lambda I - T$ is Browder and hence $\lambda \in p_{00}(T)$.

(ii)⇒(i). Let $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{w}}(T)$. By assumption T has SVEP at λ and $\lambda I - T$ is Weyl, so by Theorem 2.3 both $p(\lambda I - T)$ and $q(\lambda I - T)$ are finite. This shows that

$$\sigma(T) \setminus \sigma_{\mathrm{w}}(T) \subseteq p_{00}(T) = \pi_{00}(T).$$

On the other hand, if $\lambda \in \pi_{00}(T) = p_{00}(T)$ then $p(\lambda I - T) = q(\lambda I - T) < \infty$ and $\alpha(\lambda I - T) < \infty$, so by Proposition 38.6 of [24] we have $\beta(\lambda I - T) = \alpha(\lambda I - T) < \infty$. Hence $\lambda \in \sigma(T) \setminus \sigma_{\mathrm{w}}(T)$. Therefore $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T)$. The last assertion is clear in the case T has SVEP. Suppose that T^* has SVEP and $\pi_{00}(T) = p_{00}(T)$. By Theorem 2.5, $\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T)$, and hence

$$\pi_{00}(T) = p_{00}(T) = \sigma(T) \setminus \sigma_{\mathrm{b}}(T) = \sigma(T) \setminus \sigma_{\mathrm{w}}(T),$$

so the proof is complete ∎

In general, we cannot expect that Weyl's theorem holds for operators T for which T or T^* has SVEP. For instance, if $T \in L(\ell^2(\mathbb{N}))$ is defined by

$$T(x_0, x_1, \ldots) := \left(\frac{1}{2}x_1, \frac{1}{3}x_2, \ldots\right) \quad \text{for all } (x_n) \in \ell^2(\mathbb{N}),$$

then T is quasi-nilpotent and hence both T and T^* have SVEP. But T does not satisfy Weyl's theorem, since $p_{00}(T) = \emptyset$, whilst $\pi_{00}(T) = \{0\}$.

DEFINITION 3.2. A bounded operator $T \in L(X)$ on a Banach space X is said to have *property* (H_p) if for every $\lambda \in \mathbb{C}$ there exists an integer $p := p(\lambda) \geq 1$ such that

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p.$$

A bounded operator $T \in L(X)$ is said to be *isoloid* if every isolated point of $\sigma(T)$ is an eigenvalue of T.

LEMMA 3.3. *If $T \in L(X)$ has property (H_p) then T has SVEP and every isolated point of the spectrum is a pole of the resolvent. In particular, T is isoloid.*

Proof. T has SVEP by Theorem 1.6 of [3]. Furthermore, if $\lambda \in \operatorname{iso}\sigma(T)$ then by Theorem 2.4 we have

$$X = H_0(\lambda I - T) \oplus K(\lambda I - T) = \ker(\lambda I - T)^p \oplus K(\lambda I - T),$$

and consequently

$$(\lambda I - T)^p(X) = (\lambda I - T)^p(K(\lambda I - T)) = K(\lambda I - T).$$

Therefore $X = \ker(\lambda I - T)^p \oplus (\lambda I - T)^p(X)$, from which it follows by Proposition 38.4 of [24] that $p(\lambda I - T) = q(\lambda I - T) \leq p$, i.e. λ is a pole of the resolvent of T. ∎

We owe the following result to a recent work of M. Oudghiri [30].

THEOREM 3.4. *Let $T \in L(X)$, X a Banach space, and suppose that there exists an analytic function $h \in \mathcal{H}(\sigma(T))$ with domain $\mathcal{U}$, not identically constant in any component of $\mathcal{U}$, such that $h(T)$ has property (H_p). Then Weyl's theorem holds for both $f(T)$ and $f(T^*)$ for every $f \in \mathcal{H}(\sigma(T))$. In particular, if T has property (H_p) then Weyl's theorem holds for both T and T^*.*

The class of operators having property (H_p) is rather large. In fact, as observed in [30], every *generalized scalar* operator and every *subscalar* operator on a Banach space has property (H_p) (see [30] and [26] for relevant definitions). In particular, from Theorem 3.4 one may deduce that Weyl's theorem holds for the following classes of operators:

(a) An operator $T \in L(H)$, H a Hilbert space, is called *log-hyponormal* if T is invertible and satisfies $\log(T^*T) \geq \log(TT^*)$. Every log-hyponormal operator has property (H_1) (see [7]), so Weyl's theorem holds for this class of operators (see also [9]).

(b) An operator $T \in L(H)$ on a Hilbert space H is called *p-hyponormal*, with $0 < p \leq 1$, if $(T^*T)^p \geq (TT^*)^p$. Every p-hyponormal has property (H_p), since it is subscalar [28]. Weyl's theorem holds for operators on Hilbert spaces for which either T or T^* is a p-hyponormal operator [8].

(c) An operator $T \in L(H)$ is said to be *M-hyponormal* if there is $M > 0$ for which $TT^* \leq MT^*T$. Also every M-hyponormal operator T obeys Weyl's theorem since it is subscalar [28].

(d) An operator $T \in L(H)$ is said to be *$*$-paranormal* if $\|T^*x\|^2 \leq \|T^2x\|$ for every unit vector $x \in H$. If $\lambda I - T$ is $*$-paranormal for every $\lambda \in \mathbb{C}$ then T is said to be *totally $*$-paranormal.* If T is totally $*$-paranormal then T has property (H_1) (see [22, Lemma 2.2]).

(e) An important class of operators having property (H_1) is given by the class of all multipliers of commutative semi-simple Banach algebras [3]. In particular, every convolution operator on the group algebra $L^1(G)$, G a locally compact abelian group, has property (H_1).

Also transaloid operators on Banach spaces have property (H_1) [11, Theorem 2.3].

For a bounded operator $T \in L(X)$ on a Banach space X define

$$p_{00}^{\mathrm{a}}(T) := \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T) = \{\lambda \in \sigma_{\mathrm{a}}(T) : \lambda I - T \in \mathcal{B}_+(X)\}.$$

We have

$$p_{00}^{\mathrm{a}}(T) \subseteq \pi_{00}^{\mathrm{a}}(T) \quad \text{ for every } T \in L(X).$$

In fact, if $\lambda \in p_{00}^{\mathrm{a}}(T)$ then $\lambda I - T \in \Phi_+(X)$ and $p(\lambda I - T) < \infty$. By Theorem 2.3, λ is isolated in $\sigma_{\mathrm{a}}(T)$. Furthermore, $0 < \alpha(\lambda I - T) < \infty$ since $(\lambda I - T)(X)$ is closed and $\lambda \in \sigma_{\mathrm{a}}(T)$.

THEOREM 3.5. *If $T \in L(X)$ the following statements are equivalent*:

(i) *T satisfies a-Weyl's theorem*;
(ii) *T has SVEP at every point $\lambda \notin \sigma_{\mathrm{wa}}(T)$ and $p_{00}^{\mathrm{a}}(T) = \pi_{00}^{\mathrm{a}}(T)$.*

In particular, if T or T^* has SVEP, then a-Weyl's theorem holds for T if and only if $p_{00}^{\mathrm{a}}(T) = \pi_{00}^{\mathrm{a}}(T)$.

Proof. (i)$\Rightarrow$(ii). Suppose that T satisfies a-Weyl's theorem. Let $\lambda \notin \sigma_{\mathrm{wa}}(T)$. To show that T has SVEP at λ we may assume, since T has SVEP at every point $\lambda \notin \sigma_{\mathrm{a}}(T)$, that $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{wa}}(T) = \pi_{00}^{\mathrm{a}}(T)$. Since λ is isolated in $\sigma_{\mathrm{a}}(T)$ it follows that T has SVEP at λ. To prove that $p_{00}^{\mathrm{a}}(T) = \pi_{00}^{\mathrm{a}}(T)$ it suffices to prove $\pi_{00}^{\mathrm{a}}(T) \subseteq p_{00}^{\mathrm{a}}(T)$. Let $\lambda \in \pi_{00}^{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{wa}}(T)$. Then $\lambda I - T \in \Phi_+(X)$ and since λ is isolated in $\sigma_{\mathrm{a}}(T)$ it follows by Theorem 2.3 that $p(\lambda I - T) < \infty$. Hence $\lambda \in p_{00}^{\mathrm{a}}(T)$, and consequently $\pi_{00}^{\mathrm{a}}(T) \subseteq p_{00}^{\mathrm{a}}(T)$.

(ii)⇒(i). Let $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{wa}}(T)$. Then T has SVEP at λ and $\lambda I - T \in \Phi_+(X)$, so by Theorem 2.3 the ascent $p(\lambda I - T)$ is finite. This shows that

$$\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{wa}}(T) \subseteq p_{00}^{\mathrm{a}}(T) = \pi_{00}^{\mathrm{a}}(T).$$

On the other hand, if $\lambda \in \pi_{00}^{\mathrm{a}}(T) = p_{00}^{\mathrm{a}}(T)$ then $\lambda I - T \in \Phi_+(X)$ with $p(\lambda I - T) < \infty$. From Proposition 38.5 of [24] we deduce that $\alpha(\lambda I - T) \leq \beta(\lambda I - T)$, so $\operatorname{ind}(\lambda I - T) \leq 0$. Therefore, $\lambda \in \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{wa}}(T)$ and consequently $\sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{wa}}(T) = \pi_{00}^{\mathrm{a}}(T)$.

The last assertion is clear in the case where T has SVEP. Suppose that T^* has SVEP. If a-Weyl's theorem holds for T then $p_{00}^{\mathrm{a}}(T) = \pi_{00}^{\mathrm{a}}(T)$ by the first part of the proof. Conversely, suppose that $p_{00}^{\mathrm{a}}(T) = \pi_{00}^{\mathrm{a}}(T)$. The SVEP for T^* ensures by Theorem 2.5 that $\sigma_{\mathrm{wa}}(T) = \sigma_{\mathrm{ub}}(T)$, so

$$\pi_{00}^{\mathrm{a}}(T) = p_{00}^{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{wa}}(T),$$

and hence a-Weyl's theorem holds for T also in the case where T^* has SVEP. ∎

The next result has a crucial role in proving that many classes of operators satisfy a-Weyl's theorem.

THEOREM 3.6. *If $T \in L(X)$ has SVEP then the following statements are equivalent*:

(i) *Weyl's theorem holds for T^**;
(ii) *a-Weyl's theorem holds for T^*.*

Analogously, if the dual T^* of T has SVEP then the following statements are equivalent:

(iii) *Weyl's theorem holds for T*;
(iv) *a-Weyl's theorem holds for T.*

Proof. (i)⇔(ii). We only have to show the implication (i)⇒(ii). Suppose that T^* satisfies Weyl's theorem, i.e., $\sigma(T^*) \setminus \sigma_{\mathrm{w}}(T^*) = \pi_{00}(T^*)$. Since T has SVEP we have $\sigma_{\mathrm{a}}(T^*) = \sigma(T^*)$ (see [26, Proposition 1.3.2]), hence $\pi_{00}^{\mathrm{a}}(T^*) = \pi_{00}(T^*)$. The SVEP for T also implies by Theorem 2.5 and [2, Corollary 2.8] that

$$\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{lb}}(T) = \sigma_{\mathrm{ws}}(T).$$

By duality we then obtain $\sigma_{\mathrm{w}}(T^*) = \sigma_{\mathrm{wa}}(T^*)$, so

$$\pi_{00}^{\mathrm{a}}(T^*) = \pi_{00}(T^*) = \sigma(T^*) \setminus \sigma_{\mathrm{w}}(T^*) = \sigma_{\mathrm{a}}(T^*) \setminus \sigma_{\mathrm{wa}}(T^*),$$

and hence a-Weyl's theorem holds for T^*.

To prove the equivalence (iii)⇔(iv) we proceed in a similar way. Suppose that the dual T^* has SVEP and that Weyl's theorem holds for T. Then $\sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \pi_{00}(T)$ and by [26, Proposition 1.3.2], $\sigma_{\mathrm{a}}(T) = \sigma(T)$, so that

$\pi_{00}^{\mathrm{a}}(T) = \pi_{00}(T)$. By Theorem 2.5 and [2, Corollary 2.8] we have

$$\sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{b}}(T) = \sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{wa}}(T).$$

From this it follows that

$$\pi_{00}^{\mathrm{a}}(T) = \pi_{00}(T) = \sigma(T) \setminus \sigma_{\mathrm{w}}(T) = \sigma_{\mathrm{a}}(T) \setminus \sigma_{\mathrm{wa}}(T),$$

so a-Weyl's theorem holds for T. ∎

In what follows, we shall denote by $M^{\perp}$, for every $M \subset X$, the *annihilator* of $M \subseteq X$, and by ${}^{\perp}N$ the pre-annihilator of $N \subseteq X^*$. The next result improves Theorem 3.4.

THEOREM 3.7. *If $T \in L(X)$ has property (H_p) then a-Weyl's holds for $f(T^*)$ for every $f \in \mathcal{H}(\sigma(T))$. Analogously, if T^* has property (H_p) then a-Weyl's holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$.*

Proof. If $T \in L(H)$ has property (H_p) then T has SVEP by Lemma 3.3, and hence by Theorem 3.3.6 of [26], $f(T)$ has SVEP for every $f \in \mathcal{H}(\sigma(T))$. Moreover, by Theorem 3.4 Weyl's theorem holds for $f(T)^* = f(T^*)$, and this by Theorem 3.6 is equivalent to saying that a-Weyl's theorem holds for $f(T^*)$.

Suppose now that T^* has property (H_p). We show first that Weyl's theorem holds for T. We know that T^* has SVEP, again by Lemma 3.3, so, in order to show that T satisfies Weyl's theorem it suffices by Theorem 3.1 to prove that $\pi_{00}(T) = p_{00}(T)$. Let $\lambda \in \pi_{00}(T)$. Then λ is an isolated point in $\sigma(T) = \sigma(T^*)$, and hence by Lemma 3.3, λ is a pole of the resolvent of T^*, i.e. $p := p(\lambda I^* - T^*) = q(\lambda I^* - T^*) < \infty$. Therefore, $X^* = \ker(\lambda I^* - T^*)^p \oplus (\lambda I^* - T^*)^p(X^*)$ and since $(\lambda I^* - T^*)^p(X^*)$ is closed it follows that also $(\lambda I - T)^p(X)$ is closed. By the classical closed range theorem we then have

$$X = {}^{\perp}\ker(\lambda I^* - T^*)^p \oplus {}^{\perp}(\lambda I^* - T^*)^p(X^*) = \ker(\lambda I - T)^p \oplus (\lambda I - T)^p(X),$$

so by Proposition 38.4 of [24] we conclude that $p(\lambda I - T) = q(\lambda I - T) < \infty$. Finally, $\alpha(\lambda I - T) < \infty$ by assumption and consequently $\beta(\lambda I - T) < \infty$, from which we conclude that $\lambda \in p_{00}(T)$. Hence Weyl's theorem holds for T.

The argument above shows that if T^* has property (H_p) then T is isoloid. We prove now that Weyl's theorem holds for $f(T)$. In fact, since T is isoloid and T satisfies Weyl's theorem, by [27, Lemma] we have

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(T)) = f(\sigma_{\mathrm{w}}(T)).$$

The SVEP for T^* entails that $f(\sigma_{\mathrm{w}}(T)) = \sigma_{\mathrm{w}}(f(T))$, by Theorem 2.5, and hence Weyl's theorem holds for $f(T)$. ∎

REMARK 3.8. Theorem 3.7 implies that if T is a multiplier of a commutative semi-simple Banach algebra, or if T is transaloid, then a-Weyl's theorem holds for $f(T^*)$ for all $f \in \mathcal{H}(\sigma(T))$.

REMARK 3.9. The following example shows that property (H_p) for an operator $T \in L(X)$ does not imply, in general, that T satisfies a-Weyl's theorem. Let T be the hyponormal operator T given by the direct sum of the 1-dimensional zero operator and the unilateral right shift R on $\ell^2(\mathbb{N})$. Then 0 is an isolated point of $\sigma_{\mathrm{a}}(T)$ and $0 \in \pi_{00}^{\mathrm{a}}(T)$, whilst $0 \notin p_{00}^{\mathrm{a}}(T)$, since $p(T) = p(R) = \infty$. Hence, by Theorem 3.5, T does not satisfy a-Weyl's theorem.

COROLLARY 3.10. *Let $T \in L(X)$, X a Banach space, be a generalized scalar operator. Then a-Weyl's theorem holds for $f(T)$ and $f(T^*)$ for all $f \in \mathcal{H}(\sigma(T))$.*

Proof. Every generalized scalar operator has property (H_p), so by Theorem 3.7 Weyl's theorem holds for $f(T)$ and $f(T^*)$. Furthermore, since T is decomposable [26] both T and T^* have SVEP, and consequently both $f(T)$ and $f(T^*)$ have SVEP. By Theorem 3.6 we then conclude that a-Weyl's theorem holds for $f(T)$ and $f(T^*)$. ∎

THEOREM 3.11. *Suppose that $T \in L(X)$ has property (β) and $S \in L(Y)$ has property (δ). If T and S are quasi-similar then the following statements are equivalent*:

(i) *T satisfies Weyl's theorem*;
(ii) *S satisfies a-Weyl's theorem.*

Proof. Since T has property (β) we have $\sigma(T) = \sigma(S)$ by a result of Putinar [31], so $\operatorname{iso}\sigma(T) = \operatorname{iso}\sigma(S)$. Moreover, property (β) entails that T has SVEP and hence also S has SVEP, by Theorem 2.7. From Theorem 5 of [16], T satisfies Weyl's theorem precisely when S satisfies Weyl's theorem. Since property (δ) for S entails that S^* has SVEP, by Theorem 3.6 we conclude that (i) and (ii) are equivalent. ∎

COROLLARY 3.12. *Suppose that two quasi-similar operators $T \in L(X)$ and $S \in L(Y)$ are decomposable. Then T satisfies a-Weyl's theorem if and only if S does. In particular*, *every decomposable operator quasi-similar to a generalized scalar operator satisfies a-Weyl's theorem.*

Proof. If T is decomposable then T^* has SVEP, so a-Weyl's theorem and Weyl's theorem for T are equivalent. The statements are then clear from Theorem 3.11 and Corollary 3.10. ∎

4. Algebraically paranormal operators. In this section we shall denote by H a complex infinite-dimensional Hilbert space. In the case of operators defined on Hilbert spaces instead of the dual T^* it is more appropriate to consider the Hilbert adjoint T' of $T \in L(H)$. However, some of the basic results established in the previous section for T^* are also true for the adjoint T'. In fact, by means of the classical Fréchet–Riesz representation theorem we know that if U is the conjugate-linear isometry that associates to each $y \in H$ the linear form $x \mapsto \langle x, y\rangle$ then $UT' = T^*U$. From this equality and from Theorem 2.3 it easily follows that

$$q(\lambda I - T) < \infty \ \Rightarrow \ T' \text{ has SVEP at } \lambda.$$

Note that $\sigma_{\mathrm{w}}(T') = \overline{\sigma_{\mathrm{w}}(T)}$. Furthermore, using an argument similar to that in the proof of Theorem 2.7, from the equality $UT' = T^*U$ we easily deduce that

$$T' \text{ has SVEP at } \lambda_0 \ \Leftrightarrow \ T^* \text{ has SVEP at } \lambda_0.$$

Hence the SVEP of T' ensures by Corollary 2.6 that the equality $f(\sigma_{\mathrm{w}}(T)) = f(\sigma_{\mathrm{w}}(T))$ holds for all $f \in \mathcal{H}(\sigma(T))$.

THEOREM 4.1. *If T' has property (H_p) and $f \in \mathcal{H}(\sigma(T))$ then a-Weyl's theorem holds for $f(T)$.*

Proof. It is easily seen that if T' has property (H_p) then also T^* has property (H_p) (this property is preserved by quasi-affine transformations, and the same argument of [30, Lemma 3.2] works in our case, since $UT' = T^*U$ and U is an isometry). By Theorem 3.7 it then follows that a-Weyl's theorem holds for $f(T)$. ∎

REMARK 4.2. It should be noted that Theorem 4.1 provides a general framework for a-Weyl's theorem, from which all the results listed in the sequel follow as special cases. Note that in the literature a-Weyl's theorem has been proved separately for each class of operators.

(i) *If T' is log-hyponormal or p-hyponormal then a-Weyl's theorem holds for $f(T)$* [15, Theorem 3.3], [17, Theorem 4.2].

(ii) *If T' is M-hyponormal then a-Weyl's theorem holds for $f(T)$* [15, Theorem 3.6].

(iii) *If T' is totally $*$-paranormal then a-Weyl's theorem holds for $f(T)$* [22, Theorem 2.10].

A bounded operator $T \in L(X)$ on a Banach space X is said to be *paranormal* if

$$\|Tx\|^2 \le \|T^2x\|\,\|x\| \quad \text{for all } x \in X.$$

$T \in L(X)$ is called *totally paranormal* if $\lambda I - T$ is paranormal for all $\lambda \in \mathbb{C}$. Every totally paranormal T operator satisfies condition (H_1) (see [7]), and hence Weyl's theorem holds for T. By Theorem 4.1 we also have

(iv) *If $T' \in L(H)$ is totally paranormal then a-Weyl's theorem holds for $f(T)$.*

Theorem 3.4 and Theorem 4.1 do not work for paranormal operators. In fact, these operators do not have property (H_p) (see Remark following

Lemma 3 in [18]). However, we shall see that Weyl's theorem for paranormal operators may be deduced from Theorem 3.1.

Every paranormal operator on a Hilbert space has SVEP. To see this note first that for these operators we have $\ker(\lambda I - T) \subseteq \ker(\lambda I - T')$ for all $\lambda \in \mathbb{C}$ and from this it easily follows that $p(\lambda I - T) \le 1$ for all $\lambda \in \mathbb{C}$, so T has SVEP (see also [18]). Observe that every paranormal operator T is normaloid (i.e. $\|T\| = r(T)$, the spectral radius of T, see [24, Proposition 54.6]), so if T is quasi-nilpotent then $T = 0$.

An operator $T \in L(X)$ for which there exists a complex nonconstant polynomial h such that $h(T)$ is paranormal is said to be *algebraically paranormal.* Note that algebraic paranormality is preserved under translation by scalars and under restriction to closed invariant subspaces.

LEMMA 4.3. *If $T \in L(H)$ is algebraically paranormal then T has SVEP and every isolated point of the spectrum is a pole of the resolvent. In particular, both T and T' are isoloid.*

Proof. Let h be a nonconstant complex polynomial such that $h(T)$ is paranormal. Then $h(T)$ has SVEP and hence by Theorem 3.3.9 of [26] also T has SVEP. To prove the second assertion note first that every quasi-nilpotent algebraically paranormal operator T is nilpotent. In fact, $\sigma(h(T)) = h(\sigma(T)) = \{h(0)\}$, so $h(0)I - h(T)$ is quasi-nilpotent. Since $h(0)I - T$ is paranormal, there is some $n \in \mathbb{N}$ such that

$$0 = h(0)I - h(T) = a\,T^m \prod_{i=1}^{n} (\lambda_i I - T) \quad \text{ with } \lambda_i \neq 0.$$

Since all $\lambda_i I - T$ are invertible it follows that $T^m = 0$.

Now, if $\lambda \in \operatorname{iso}\sigma(T)$, $M := K(\lambda I - T)$ and $N := H_0(\lambda I - T)$ then (M, N) is a GKD for $\lambda I - T$. Since $\lambda I - T|N$ is quasi-nilpotent and algebraically paranormal it follows that $\lambda I - T|N$ is nilpotent and hence $\lambda I - T$ is of Kato type. The SVEP for T and T' at λ then implies, by Theorem 2.3, that both $p(\lambda I - T)$ and $q(\lambda I - T)$ are finite. Hence λ is a pole of the resolvent of T. This implies that T is isoloid.

Analogously, to prove that T' is isoloid we prove that every isolated point of $\sigma(T')$ is a pole of the resolvent of T'. Let λ be an isolated point in $\sigma(T') = \overline{\sigma(T)}$. Then $\overline{\lambda}$ is isolated in $\sigma(T)$, and hence by the first part of the proof the point $\overline{\lambda}$ is a pole of the resolvent of T, hence $p := p(\overline{\lambda} I - T) = q(\overline{\lambda} I - T) < \infty$. Consequently, $H = \ker(\overline{\lambda} I - T)^p \oplus (\overline{\lambda} I - T)^p(X)$ and the range $(\overline{\lambda} I - T)^p(X)$ is closed. From this it follows that

$$H = (\ker(\overline{\lambda} I - T)^p)^{\perp} \oplus ((\overline{\lambda} I - T)^p(H))^{\perp} = (\lambda I - T')^p(H) \oplus \ker(\lambda I - T')^p,$$

where now $N^{\perp}$ denotes the orthogonal of $N \subseteq H$. Therefore $p(\lambda I - T') = q(\lambda I - T') < \infty$, or equivalently λ is a pole of the resolvent of T'. ■

The next result improves Corollary 4 of [23] and Theorem 2.4 of [12].

THEOREM 4.4. *Let $T \in L(H)$. Then the following statements hold*:

(i) *If $T \in L(H)$ is algebraically paranormal then Weyl's theorem holds for $f(T)$ for all $f \in \mathcal{H}(\sigma(T))$.*

(ii) *If T' is algebraically paranormal then a-Weyl's theorem holds for $f(T)$ for all $f \in \mathcal{H}(\sigma(T))$.*

Proof. (i) Suppose that T is algebraically paranormal. We show first that Weyl's theorem holds for T. Since T has SVEP it suffices by Theorem 3.1 to show that $p_{00}(T) = \pi_{00}(T)$. Suppose that $\lambda \in \pi_{00}(T)$. By assumption $\alpha(\lambda I - T) < \infty$ and λ is isolated in $\sigma(T)$, so, by Lemma 4.3, λ is a pole and hence $p(\lambda I - T) = q(\lambda I - T) < \infty$. From [24, Proposition 38.6] it then follows that $\beta(\lambda I - T) < \infty$, i.e. $\lambda \in p_{00}(T)$. Therefore Weyl's theorem holds for T.

To show that Weyl's theorem holds for $f(T)$ note that, T being isoloid, by [27, Lemma] we have

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = f(\sigma(T) \setminus \pi_{00}(T)) = f(\sigma_{\mathrm{w}}(T)),$$

where the last equality holds since T satisfies Weyl's theorem. By Corollary 2.6 the SVEP for T implies that $f(\sigma_{\mathrm{w}}(T)) = \sigma_{\mathrm{w}}(f(T))$, and consequently

$$\sigma(f(T)) \setminus \pi_{00}(f(T)) = \sigma_{\mathrm{w}}(f(T)),$$

so Weyl's theorem holds for $f(T)$.

(ii) Suppose now that T' is algebraically paranormal. We show first that Weyl's theorem holds for T. Since T' is algebraically paranormal it follows that T', and hence also T^*, has SVEP. In order to show that T satisfies Weyl's theorem it then suffices, by Theorem 3.1, to prove that $\pi_{00}(T) = p_{00}(T)$.

Let $\lambda \in \pi_{00}(T)$. Then λ is an isolated point in $\sigma(T) = \overline{\sigma(T')}$, and hence by Lemma 4.3, $\overline{\lambda}$ is a pole of the resolvent of T', i.e. $p := p(\overline{\lambda} I - T') = q(\overline{\lambda} I - T') < \infty$. We have $H = \ker(\overline{\lambda} I - T')^p \oplus (\overline{\lambda} I - T')^p(H)$ and since $(\overline{\lambda} I - T')^p(H)$ is closed it follows that $(\lambda I - T)^p(H)$ is closed. We also have

$$H = (\ker(\overline{\lambda} I - T')^p)^{\perp} \oplus ((\overline{\lambda} I - T')^p(H))^{\perp} = (\lambda I - T)^p(H) \oplus \ker(\lambda I - T)^p,$$

and again by Proposition 38.4 of [24] we conclude that $p(\lambda I - T) = q(\lambda I - T) < \infty$, i.e. λ is a pole of the resolvent of T.

Finally, $\alpha(\lambda I - T) < \infty$ by assumption and consequently $\beta(\lambda I - T) < \infty$, from which we conclude that $\lambda \in p_{00}(T)$. Hence Weyl's theorem holds for T.

The argument above also proves that if T' is algebraically paranormal then T is isoloid. Since the SVEP for T' implies $f(\sigma_{\mathrm{w}}(T)) = \sigma_{\mathrm{w}}(f(T))$, arguing as in the proof of part (i) it readily follows that Weyl's theorem holds for $f(T)$ for all $f \in \mathcal{H}(\sigma(T))$.

Finally, since T' has SVEP, so does $f(T')$, and hence also $f(T^*)$. By Theorem 3.6 it follows that a-Weyl's theorem holds for $f(T)$. ■

Notice that Theorem 4.4 implies Weyl's theorem for paranormal operators. Weyl's theorem for an algebraically paranormal operator has been established by Curto and Han [12] by using different methods. Since every p-hyponormal operator is paranormal, Weyl's theorem for p-hyponormal operators ([8]) and algebraically hyponormal operators ([23]) may also be deduced from Theorem 4.4.

The operator defined in Remark 3.9 shows that, in general, we cannot expect that a-Weyl's theorem holds for any algebraically paranormal operator.

A bounded operator $T \in L(H)$ is said to be *quasi-hyponormal* if $\|T^*Tx\| \le \|T^2x\|$ for all $x \in H$. Every quasi-hyponormal operator is paranormal [21], so part (ii) of Theorem 4.4 subsumes the following result of S. V. Djordjević and D. S. Djordjević [14, Theorem 3.4] and improves Corollary 5.7 of D. S. Djordjević [13].

COROLLARY 4.5. *If $T' \in L(H)$ is quasi-hyponormal then a-Weyl's theorem holds for $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$.* ■

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