Designing and pricing guarantee options in defined contribution pension plans

Andrea Consiglio a,*, Michele Tumminello a, Stavros A. Zenios b

a University of Palermo, Palermo, Italy
b University of Cyprus, Nicosia, Cyprus

A B S T R A C T

The shift from defined benefit (DB) to defined contribution (DC) is pervasive among pension funds, due to demographic changes and macroeconomic pressures. In DB all risks are borne by the provider, while in plain vanilla DC all risks are borne by the beneficiary. However, for DC to provide income security some kind of guarantee is required. A minimum guarantee clause can be modeled as a put option written on some underlying reference portfolio and we develop a discrete model that selects the reference portfolio to minimize the cost of a guarantee. While the relation DB–DC is typically viewed as a binary one, the model shows how to price a wide range of guarantees creating a continuum between DB and DC. Integrating guarantee pricing with asset allocation decision is useful to both pension fund managers and regulators. The former are given a yardstick to assess if a given asset portfolio is fit-for-purpose; the latter can assess differences of specific reference funds with respect to the optimal one, signaling possible cases of moral hazard. We develop the model and report numerical results to illustrate its uses.

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1. Introduction

Developed and developing countries are currently debating comprehensive approaches for delivering adequate, sustainable and safe retirement incomes to their aging populations. Defined benefit (DB) pension plans, desirable as they may be for retirees, are not sustainable and shift all the risks to the provider, be it a corporate employer or future taxpayers. A consensus has emerged that retirees will “rely more on supplementary retirement savings”, European Commission (2012), and we are witnessing a trend favoring defined contribution (DC) that shifts risks to retirees. To make DC politically acceptable, encourage participation and increase savings, the retirement income must be safe. Hence, some type of guarantee is needed and success of DC hinges upon the design of appropriate guarantees. However, the difficulty does not stop in designing the guarantee. We then need asset allocation decisions that deliver on the guarantee or appropriate insurance in case the guarantee cannot be met. These interrelated decisions need to be “optimized for their safety and performance” in the words of the European Commission report cited above. Given the interactions of financial, economic and demographic risks, a guarantee may fail after all, as much as a “defined benefit” may be modified by government legislation, World Bank (2000). Complementary retirement plans make failures less likely.

In core-DB the provider commits to a set of rigid promises and assumes all risks. In DC there is no promise made to the beneficiary, who assumes all the risks. This is a binary division. Complementary plans range from DB-lite, i.e., plans with a floor on minimum benefit, to DC-plus, i.e., defined contribution plans with some guarantee on the contribution made during the working life. However, “defined ambition” plans – a term coined during the Dutch pension reform debate of the 2000’s and currently providing the basis for policy debates in the UK – argue that a tweak to the binary system cannot solve the problem and requires better risk sharing to ensure that DC is going to work and be sustainable. A comprehensive approach views pension plans as a hybrid of guarantees and ambitions: nominal annuities are guaranteed, but the degree to which pensions rise in line with prices and wages depends on the performance of investments of the pension funds. The Dutch reforms are discussed in Bovenberg and Nijman (2009), NAPF (2012) provides an overview of risk-sharing issues for the UK industry, and Smetters (2002) discusses the conversion of public pensions to DC in the United States.
Our contribution is in modeling DC plans with alternative guarantee options to compute ex ante mark-to-market risk premia and facilitate risk sharing in the design of guarantees.

1.1. The pensions challenge

The US Census Bureau reports that before baby boomers started turning 65 in 2010, 11% of the total population was between the ages of 65 and 84. Thereafter, this age group is projected to reach 18% of the population by 2030, Colby and Ortmann (2014). The US will experience a 45% increase of aging population by 2050. Data from the EC, European Commission (2012), and the IMF, Carone and Costello (2006), reveal even bigger challenges in Europe. Older Europeans are a significant and growing part of the EU population (24%) and by 2050 it is projected to grow by 77%. The fastest growing group in the EU are the very elderly (over 80) projected to grow by 174%, and the old-age dependency ratio is projected to double to 54%.

At the same period per capita growth rate slides to a projected 1.4%, Carone and Costello (2006) (these are pre-crisis estimates). Pensions represent a large share of public expenditure: more than 10% of GDP on average today, expected to rise to 12.5% by 2060 in the EU as a whole. Spending on public pensions ranges from 6% of GDP in Ireland to 15% in Italy, so countries are in different situations although they face similar demographic challenges. Projected changes between 2004 and 2050 ranges from −5.9% GDP in Poland to +12.9% GDP in Cyprus. Only three EU members experience a decrease and nine members expect increases over 5%.

According to the EC white paper for “adequate, safe and sustainable” pensions, European Commission (2012), a majority of member states have been reforming pension systems to put them on a more sustainable footing. Shifting from DB to DC is a significant component of the reforms. Velculescu (2010) reports that “on existing [pension] policies, the intertemporal net worth of the EU27 is deeply negative […] Europe’s current policies need to be significantly strengthened to bring future liabilities in line with the EU governments’ capacity to generate assets”. The challenges are not restricted to the US and EU. Latin American countries were pioneers in pension reforms in the 1990’s; the pricing literature reviewed below was motivated by DC plans introduced in Uruguay, Chile and Colombia. In India, DB plans were closed by the Government in 2004 and were replaced by a two-tier DC system. The introduction of DC plans in China appears to be modest but it represents a very recent and ongoing trend.

The challenges are addressed with a variety of policy tools: balancing time spent in work and retirement, enhancing productivity, indexing replacement rates, supporting the development of complementary retirement savings to enhance retirement incomes. Shifting away from DB is a way to develop complementary retirement savings and we focus now on DC plans.

1.2. Type of guarantees

A survey of 1700 organizations in the nine largest EU economies, found 22% of the respondent’s undergoing pension reforms and 42% mentioned sustainability as a reason, Hewitt (2007). In the UK, 88% of DB schemes were open to new members in 2000 but by 2011 this had dropped to 19%, NAPF (2012). Shifting from DB to DC addresses sustainability issues, but shifts all risks to the beneficiaries. To mitigate risks DC plans typically offer some type of guarantee. In the core DB, the provider commits to a set of rigid promises and takes all the risk. A plain vanilla DC guarantees the nominal value of the contribution.

However, we are not restricted to a binary classification and hybrid forms met with success in Sweden, Denmark or Holland. Hybrid schemes come with a variety of guarantees, such as a return guarantee on the pension pot but no guarantee on what will buy in terms of income. From the Hewitt survey 50% of the plans were DB, 32% DC and 18% hybrid. From the NAPFA data 8% define themselves as hybrid. In a paper pricing the cost of public pension liabilities in the US, Biggs (2011) uses the database from the National Association of State Retirement Administrators covering 125 mostly state-level programs and finds that around 80% of the employees have a DB pension, 14% DC and 6% have both. Fig. 1 illustrates the prevalence of DC and hybrid plans in OECD countries.

Guarantees come in various forms, see, e.g., Antolín et al. (2011). There are significant legislative differences among countries on who backs the guarantee, such as the Government, the provider, a public pension protection fund, a collective DC trust and so on. We define a risk sharing ladder based on the level of protection to the beneficiary and the risks to the provider:

Rung 1. Money-safe accounts, that guarantee the contribution, either in nominal or real value upon retirement.
Rung 2. Guaranteed return plans, that guarantee a fixed rate of return on contribution, upon retirement.
Rung 3. Guaranteed return to match some industry average upon retirement.
Rung 4. Guaranteed return for each time period until retirement.
Rung 5. Guaranteed income past retirement.

Note an important distinction between the first four and the fifth level of protection. The first four provide the beneficiary with guarantees on level of wealth attained upon retirement while the fifth guarantees retirement income. Of course, wealth accumulation provides the means to buy an income upon retirement, but the connection between the two is not trivial. Plans with the first four levels of protection are focusing on the volatility of assets and returns rather than the risk of not realizing inflation-protected incomes. The “Defined Ambitions” debate climbs this risk ladder, offering some protection in the form of guarantees and some in the form of soft guarantees (ambitions).

Deciding how far to climb the risk ladder offers possibilities for risk-sharing, but this requires fair valuation of the risks. For instance, if the employer – or a public protection fund or a collective trust – provides asset volatility insurance for the retirees, the insurance premium should be determined ex-ante and priced using the markets. Risk transfers should be valued on a mark-to-market basis and whoever underwrites the guarantee – employer, future taxpayers or members of a collective trust – must be compensated (NAPF, 2012, pp. 25–30). We turn therefore to the pricing literature.

1.3. Pricing and asset management literature

A minimum guarantee clause can be modeled as a put option written on an underlying reference portfolio of assets whose returns determine the return on the contribution. Valuation of the guarantee option has attracted significant interest from academics and practitioners. The seminal papers, developed independently and simultaneously, are Pennacchi (1999) and Fischer (1999). Pennacchi used continuous martingale theory to price the guarantees offered in Uruguay and Chile, respectively; these lie on the second and third rung of our risk ladder. Fischer values Colombia’s guarantees using a discrete martingale model and obtained qualitatively similar results to Pennacchi. An interesting feature of Fischer’s model is the existence of a ceiling on the guarantee. Pennacchi recognized the similarity between pension guarantees and insurance participating products with embedded options, as priced by Brennan and Schwartz (1979); Boyle and Hardy (1997), see also Embrechts (2000). Advances in this field generated numerous studies extending the framework to
more general stochastic processes, see, e.g., Coleman et al. (2007); Consiglio and De Giovanni (2010), but they apply to insurance products where the obligation has significant differences from the pension obligation.

There is also literature on pricing the cost of benefit guarantees in DB plans. This is the cost to the public entity (e.g., Public Benefits Guarantee Corporation) that may have to step in to resolve a failing DB plan. This problem is different than the one facing the management of DC plans, as pointed out by Fischer. A discussion of the cost of different types of guarantees, focusing on policy instead of pricing models is Smetters (2002). Options pricing for public pension liabilities is given in Biggs (2011), who used data from State governments in the US to find that public pension shortfalls equal an average of 27% of State GDP; this finding provides a compelling argument for shifting towards DC.

A limitation of current literature is in assuming given the underlying reference portfolio. However, the construction of a reference portfolio is endogenous to the problem. Recognizing this limitation Bacinello (2003) reports sensitivity analyses to different model parameters, that would correspond to different asset portfolios.

An alternative strand of literature focuses on the problem of asset and liability management (ALM). The handbook Zenios and Ziemba (2007) contains papers for Dutch (ch. 18) and Swiss (ch. 20) pension funds, the Russell–Yasuda–Kasai model for insurance as adapted for pension funds (ch. 19) and a paper for simultaneously determining asset allocation and contribution rates (ch. 21). These papers follow a parallel stream of ALM models for the insurance industry, such as the seminal model developed for Japanese insurance, the Russell–Yasuda–Kasai model of Carinò and Ziemba (1998) and the Towers Perrin model of Mulvey et al. (2000).

In the context of ALM for insurance products with guarantees, the problem of structuring the reference portfolio was formulated as a stochastic programme in the PROMETEIA model, see ch. 15 in the handbook, and was applied to UK and Italian policies, Consiglio et al. (2006, 2008). This model addressed an important issue since the liabilities from the guarantee – with the associated bonus payments – depend on the asset portfolio, but while it extended ALM literature to account for guarantees it did not price the guarantee per se.

In this paper we endogenize the decision about the reference portfolio in a model for pricing the guarantee. We provide a model that extends the line of research started by Fischer–Pennacchi to select a reference portfolio that minimizes the cost of the guarantee. The cost is obtained using option pricing theory on a discrete tree. The model setup is innovative in two aspects. First, it integrates portfolio optimization and option pricing in a unified framework that is computationally tractable. Second, it provides a strategic tool to compare alternative guarantee designs to a yardstick reference portfolio with minimal guarantee costs, thereby facilitating risk sharing. The model is tractable for large-scale instances.

2. The model setup

We assume that asset returns are stochastic processes in discrete space and time. The set of asset returns is labeled by index set \( J = \{1, 2, \ldots, J\} \) and are observed at finite time instances \( t \in \mathcal{T} \), where \( \mathcal{T} = \{1, 2, \ldots, T\} \):

\[ R = \left( R_t^1, \ldots, R_t^J \right)^T_{t=1} \]  

The return process is modeled on the probability space \((\Omega, \mathcal{F}, P)\), where the sample space \( \Omega \) is assumed to be finite. Such a formulation allows for a market representation through scenario trees, Pliska (1997). We denote by \( \mathcal{N}_t \) the set of nodes at \( t \), and by \( \mathcal{N} \equiv \bigcup_{t=0}^{T} \mathcal{N}_t \) the collection of all the nodes. Each node \( n \in \mathcal{N} \) corresponds one-to-one to an atom of the filtration \( \mathcal{F}_t \). (For simplicity, whenever we refer from now on to a node \( n \), it is understood to be a node from the set \( \mathcal{N}_t \) for all \( t \), unless specified otherwise.) These are possible future states of the economy at time \( t \). Not all nodes at \( t \) can be reached from every node at \( t - 1 \) and we define paths from the root node 0 to some final node in the set \( \mathcal{N}_T \) to denote the unique way of reaching a particular node. Each path

![Fig. 1. Relative shares of DB, DC and hybrid pension fund assets in selected OECD countries, 2011. Source: OECD Global Pension Statistics, http://dx.doi.org/10.1787/888932908079.](image)
is a scenario. A graphical representation of a tree is given in Fig. 2, for an example with 12 scenarios, two possible states at \( t = 1 \), three possible states at \( t = 2 \) and six at \( t \). We denote by \( P(n) \) the set of nodes on the unique path from the root node to \( n \in \mathcal{N} \), but excluding the root node itself, by \( p(n) \) the unique predecessor node for \( n \), with \( p(0) \) being empty, and by \( \mathcal{S}(n) \subset \mathcal{N} \) the non-empty set of successor nodes. Each successor node associated with a weight \( q_n \), interpreted as a probability. Given \( n \), all information contained in path \( P(n) \) is known. Without ambiguity we drop the time index when referring to data at a node, since each node \( n \) takes values from a time-indexed set \( \mathcal{N} \).

2.1. The basic minimum guarantee option

We assume that a DC fund guarantees a payment at maturity \( T \). The price of this guarantee is contingent on the value of a reference fund \( A_n \) for each \( n \in \mathcal{N} \). In the basic model we assume a closed fund with initial total contribution \( L_0 \). Typically, there will be some regulatory equity, such as \( E_0 = (1 - \alpha)A_0 \) and \( \alpha < 1 \), so that \( L_0 = \alpha A_0 \). The initial endowment \( A_0 = L_0 + E_0 \) is invested in a reference portfolio according to the asset allocation variables \( x_i \),

\[
\sum_{i \in \mathcal{I}} x_i = 1.
\]

Given the family of stochastic processes \( \{ R_t \}_{t \in T} \), defined as a \( J \)-dimensional vector of returns, \( R_s \equiv \{ R_n^1, \ldots, R_n^J \} \), the stochastic process of the asset value \( A_n \) is driven by the stochastic process of the portfolio return \( R_n \) for each \( n \in \mathcal{N} \setminus \{0\} \) as follows:

\[
A_n = A_{p(n)} e^{R_n^1},
\]

\[
R_n^1 = \sum_{i \in \mathcal{I}} x_i R_i^1,
\]

where \( A_{p(n)} \) is the asset value at the predecessor node. The main difference with the models reviewed earlier is that the reference portfolio performance depends on asset allocation decisions \( x_i \).

The liability process \( L_n \) also grows at a stochastic rate that, at each node \( n \in \mathcal{N} \setminus \{0\} \), is guaranteed not to be less than a minimum guarantee rate \( g \). That is,

\[
R_n^1 = \max \left( \delta R_n^1 - g, 0 \right) + g,
\]

where \( \delta \) is the participation rate and denotes the fraction of risky return which is passed to the beneficiary. Typically \( \delta \) would be 1 net management fees. It can be set to lower values to compensate the fund for the offered guarantee, by keeping a fraction of the portfolio upside to compensate for providing a guarantee in a downturn.

The stochastic process of the liability is given, for all \( n \in \mathcal{N} \setminus \{0\} \), by

\[
L_n = L_{p(n)} \exp \left[ g + \max \left( \delta R_n^1 - g, 0 \right) \right],
\]

where \( L_{p(n)} \) is the liability value at the predecessor node. For simplicity we ignore the effect of mortality on the liability process. This can be accommodated by introducing an intensity based model, where the survival probability at node \( n \) is a deterministic function of time, or by fitting a stochastic mortality process to the scenario tree.

3. The optimization model

We now formulate the mathematical program to determine the optimal composition of the reference portfolio, and show how to reduce its complexity.

3.1. The objective function

Denote by \( \Phi(A_n, L_n) \) the payoff function measuring the performance of the portfolio strategy at each final node \( n \). \( A_n \) and \( L_n \) are implicit functions of the asset allocation \( x_i \) and according to options theory the price of this contingent payoff is given by

\[
\Gamma = e^{-rT} \sum_{n \in \mathcal{N}} q_n \Phi(A_n, L_n),
\]

where the expectation is discounted at the risk free rate \( r \), and is taken under the risk-neutral measure. In our discrete probabilistic setting these are the weights \( q_n \), \( n \in \mathcal{N} \).

To write the objective function, we first specify the payoff function \( \Phi(A_n, L_n) \). A natural choice is

\[
\Phi(A_n, L_n) = \max \left( L_n - A_n, 0 \right),
\]

that implicitly assumes that shareholders cover possible shortfalls at each state. A rationale strategy for DC fund management is to minimizing the expected value of these losses. Substituting in (6) we obtain

\[
\Gamma = e^{-rT} \sum_{n \in \mathcal{N}} q_n \max \left( L_n - A_n, 0 \right).
\]

This is the cost of a put option written on the value of the assets \( A_n \) with a stochastic strike price \( L_n \), i.e., it is the cost of the guarantee.

This cost can be considered as the fair risk premium charged by a pensions guarantee agency to the fund, to set minimum equity requirements by a regulator, to determine charges to the internal sub-division operating the specific DC fund, or to determine prices for risk-sharing between the beneficiary and the fund. The guarantee option contributes to the total risk of the company and minimizing its value is consistent with enterprise-wide risk management.

3.2. The bilinear constraints

The variables affecting the cost of the guarantee (8) are the values of the asset and liability accounts at \( T \). We denote by \( w_n \) and \( z_n \), respectively, the final cumulative returns of the asset and liability accounts \( A_n \) and \( L_n \). For all \( n \in \mathcal{N} \), we have that:

\[
w_n = \sum_{i \in P(n)} R_i^1,
\]

\[
z_n = \sum_{i \in P(n)} g + \max \left( \delta R_i^1 - g, 0 \right).
\]
The max operator in Eq. (10) implies that the problem is a discontinuous nonlinear programming model (DNLP), thus making the optimization model intractable for large-scale applications. To solve the DNLP we reformulate it as an equivalent smooth non-linear program. We can write the first argument of the max operator as the difference of two positive variables with the condition that only one of them is non-zero. Thus, for all $n \in \mathcal{N} \setminus \{0\}$, we introduce the following set of equations:

$$\delta R_{n}^{A} - g = \varepsilon_{n}^{+} - \varepsilon_{n}^{-},$$

$$\varepsilon_{n}^{+} - \varepsilon_{n}^{-} = 0,$$

$$\varepsilon_{n}^{+}, \varepsilon_{n}^{-} \geq 0.$$  

The bilinear constraints (12) still add complexity that is difficult to deal with when we wish to solve models with a large number of nodes on the tree. However, we prove in the next section that it is not necessary to explicitly add constraints (12).

Given Eq. (11), we re-write the definitional Eq. (10) as follows:

$$z_{n} = g T + \sum_{i \in P(n)} \varepsilon_{i}^{+}. \quad (14)$$

With this notation, the final value of assets and liabilities is given by

$$A_{n} = A_{0} e^{w_{n}},$$

$$L_{n} = \alpha A_{0} e^{w_{n}},$$

for each node $n \in \mathcal{N}$, and recall that $L_{0} = \alpha A_{0}$.

We also need to reduce the complexity introduced by the max operator of the payoff function (7). To do so, we must handle the exponentiation terms of $w_{n}$ and $z_{n}$. Let us consider the generic nth term of the summation in Eq. (8). Using (15)–(16) for $A_{n}$ and $L_{n}$, we obtain:

$$\max\left(L_{n} - A_{n}, 0\right) = A_{n} \left[\max\left(\frac{L_{n}}{A_{n}}, 1\right) - 1\right]$$

$$= A_{0} e^{w_{n}} \left[\max\left(\alpha e^{w_{n}}, 1\right) - 1\right].$$  

(17)

If $a, b > 0$, then $\max(ab, b) = e^{\max(ln a, ln b)}$ and we write the last term in (17) as

$$A_{0} e^{w_{n}} \left[e^{\max(ln a + z_{n} - w_{n}, 0)} - 1\right].$$

We can now redefine the max operator as a set of constraints by writing its first argument as the difference of two positive variables

$$\ln a + z_{n} - w_{n} = H_{n}^{+} - H_{n}^{-},$$

$$H_{n}^{+}, H_{n}^{-} \geq 0,$$

and expression (18) becomes

$$A_{0} e^{w_{n}} \left(e^{H_{n}^{+}} - 1\right).$$

(22)

The bilinear equations (20) also lead to intractable optimization problems, but as we will prove later these constraints also need not be added explicitly.

Given (22), the cost of the guarantee (8) becomes:

$$\Gamma(x_{1}, x_{2}, \ldots, x_{j}) = e^{-r T} A_{0} \sum_{n \in \mathcal{N}} q_{n} e^{w_{n}} \left(e^{H_{n}^{+}} - 1\right).$$

We now show that (23) is a convex function of the portfolio choices $x_{j}$. 

**Proposition 1.** The function $\Gamma$ is a convex function of the portfolio choices $x_{j}, j \in \mathcal{J}$. 

**Proof.** By the definitions of $H_{j}^{+}$ and $w_{n}$, it is easy to show that these quantities are affine functions of $x_{j}, j \in \mathcal{J}$, then $e^{w_{n}}$ and $\left(e^{H_{j}^{+}} - 1\right)$ are convex functions of $x_{j}, j \in \mathcal{J}$. Since $e^{w_{n}}$ and $\left(e^{H_{j}^{+}} - 1\right)$ are increasing and non-negative functions of $w_{n}$ and $H_{n}$, then their product is convex. Finally, given that the coefficients $q_{n}, n \in \mathcal{N}$, are probabilities, then $\Gamma$ is a linear combination of convex functions and it is also convex.  

### 3.3. Convex model for minimizing the cost of guarantee option

If bilinear equations (12) and (20) are omitted, the remaining constraints are linear. Moreover, given the convexity of the objective function, the minimization of the cost of the guarantee is a convex programming problem. In convex optimization if a local minimum exists then it is a global minimum, and effective computational methods are available, even for large scale instances. Hence, the portfolio which minimizes the cost of the guarantee option is obtained as the solution to the following convex non-linear program:

**Problem 1 (Convex Optimization of the Guarantee Option).**

Minimize

$$e^{-r T} A_{0} \sum_{n \in \mathcal{N}} q_{n} e^{w_{n}} \left(e^{H_{n}^{+}} - 1\right)$$

subject to

$$\ln a + z_{n} - w_{n} = H_{n}^{+} - H_{n}^{-},$$

$$\delta R_{n}^{A} - g = \varepsilon_{n}^{+} - \varepsilon_{n}^{-},$$

$$\varepsilon_{n}^{+}, \varepsilon_{n}^{-} \geq 0,$$

$$z_{n} = g T + \sum_{i \in P(n)} \varepsilon_{i}^{+},$$

$$w_{n} = \sum_{i \in P(n)} R_{i},$$

$$R_{A}^{A} = \sum_{n \in \mathcal{N}} R_{n}^{A},$$

$$\sum_{j \in \mathcal{J}} x_{j} = 1,$$

$$H_{n}^{+}, H_{n}^{-} \geq 0,$$

$$x_{j} \geq 0,$$

$$n \in \mathcal{N}, j \in \mathcal{J}. \quad (24)-(33)$$

To establish the validity of this model we need to prove that the minimum value of the objective function of Problem 1, coincides with that of the non-convex problem obtained by adding the nonlinear equations (12) and (20) to the constraints set (25)–(33).

**Lemma 1.** Let us assume that $x_{j}^{*}, x_{j}^{*}, \ldots, x_{j}^{*}$ is an optimal portfolio choice for Problem 1. Then,

$$H_{j}^{+}, H_{j}^{-} = 0,$$

for all $n \in \mathcal{N}$. 

**Proof.** We will prove the lemma by negating the thesis and showing that this contradicts the hypothesis that $x_{j}^{*}, x_{j}^{*}, \ldots, x_{j}^{*}$ is a minimum of Problem 1.

Let us assume that there exists a $k \in \mathcal{N}$ such that $H_{j}^{+}, H_{j}^{-} > 0$. We define $\lambda = \min(H_{j}^{+}, H_{j}^{-})$. Since $\lambda > 0$, subtracting $\lambda$ from both $H_{j}^{+}$ and $H_{j}^{-}$ will keep the non-negativity of both variables, modify the objective function, and leave unaffected the rest of variables and constraints.
Let us denote by $\Gamma(\lambda)$ the optimal level of the objective function after we have subtracted $\lambda$ from $H^+_n$ and $H^-_n$. We have

$$\Gamma(\lambda) = e^{-rt} A_0 \left[ \sum_{n \in \mathcal{N}_T^e} q_n e^{w_n} \left( e^{H^+_n} - 1 \right) + q_n e^{w_n} \left( e^{H^-_n} - 1 \right) \right].$$

By simple algebra, it is possible to show that the change in the optimal objective function due to $\lambda$, $\Delta \Gamma = \Gamma(\lambda) - \Gamma(\lambda | \lambda = 0)$, is given by

$$\Delta \Gamma = e^{-rt} A_0 q_n e^{w_n} e^{H^+_n} \left( e^{\lambda} - 1 \right).$$

We observe that $\Delta \Gamma < 0$ since $(e^{\lambda} - 1) < 0$ and the other terms are all positive. Therefore, the objective function can be further reduced, but this contradicts the main hypothesis that $x_1^*, x_2^*, \ldots, x_N^*$ is a minimum for Problem 1. Hence, the assumption that there exists a $k$ such that $H^+_k = H^-_k > 0$ must be false. \hfill $\Box$

**Remark 1.** Lemma 1 only ensures that if an optimal solution of Problem 1 exists, then (20) holds for all $n \in \mathcal{N}_T$. To claim that such a minimum coincides with that of the non-convex problem, we have to prove that at the minimum of Problem 1 also conditions (12) hold. If the latter result is true, since the objective function is convex, and therefore the minimum level of the objective function is unique, we can conclude that the minimum of the convex and the non-convex problem are the same. In general, however, this is not true for the optimal portfolio $x_1^*, x_2^*, \ldots, x_N^*$. Only strict convexity of the objective function implies uniqueness of the optimal portfolio choices.

**Lemma 2.** Let us assume that $x_1^*, x_2^*, \ldots, x_N^*$ is an optimal portfolio choice for Problem 1. Then, it exists a non empty subset of nodes $B \subset \mathcal{N}$ such that for all $n \in B$ we have

$$e^{w_n} e^{u_n} = 0.$$

**Proof.** Let us assume that for all $n \in \mathcal{N}$ we have $e^{w_n} e^{u_n} > 0$. Denote by $\xi = \min \left\{ \left( e^{w_n} \right)_{n \in \mathcal{N}} \cup \left( e^{u_n} \right)_{n \in \mathcal{N}} \right\}$ and subtract $\xi$ from $e^{w_n}$ and $e^{u_n}$, for all $n \in \mathcal{N}$. Such a subtraction will affect the objective function value and the constraints (25). We define by $\lambda = \xi T$ the total change in the left-hand-side of the constraints set (25), where $T$ is the number of time steps. In particular, we have

$$\lambda + w_n - \lambda - u_n = H^+_n(\lambda) - H^-_n(\lambda), \quad \text{for all } n \in \mathcal{N}_T.$$

The effect of $\lambda$ is counterbalanced by a change in the variables $H^+_n$ and $H^-_n$. By Lemma 1, we have that $H^+_n = H^-_n = 0$ at the optimum. Let us assume that the change due to $\lambda$ preserves such a property, and therefore, $H^+_n(\lambda) = H^-_n(\lambda) = 0$, for all $n \in \mathcal{N}_T$. The final nodes are then partitioned as follows:

a. if $H^+_n > \lambda$, then $H^+_n(\lambda) = H^+_n - \lambda$ and $H^-_n(\lambda) = 0$, for all $n \in \mathcal{N}_T \subset \mathcal{N}_T$;

b. if $H^+_n > 0$ and $H^-_n \leq \lambda$, then $H^+_n(\lambda) = 0$ and $H^-_n(\lambda) \geq 0$, for all $n \in \mathcal{N}_T \subset \mathcal{N}_T$;

c. if $H^+_n > 0$, then $H^-_n(\lambda) = H^-_n + \lambda$ and $H^+_n(\lambda) = 0$, for all $n \in \mathcal{N}_T \subset \mathcal{N}_T$.

The objective function $\Gamma(\lambda)$ is then partitioned accordingly as

$$\Gamma(\lambda) = \sum_{n \in \mathcal{N}_T^e} q_n e^{w_n} \left( e^{H^+_n} - 1 \right) + \sum_{n \in \mathcal{N}_T^c} q_n e^{w_n} \left( e^{H^-_n} - 1 \right).$$

Observe that $\Delta \Gamma < 0$ since it is made up by two negative terms (the summation over the set of nodes $\mathcal{N}_T$ is not displayed as it has a null impact in the total change of the objective function). This contradicts the hypothesis that $x_1^*, x_2^*, \ldots, x_N^*$ is an optimal portfolio choice for Problem 1, and then, for each $k \in \mathcal{B}$ it must be that $e^{w_k} e^{u_k} = 0$, where $\mathcal{B}$ is the union of the nodes belonging to $\mathcal{P}(n)$, with $n \in \mathcal{N}_T \cup \mathcal{N}_T^c$.

$$\mathcal{B} = \bigcup_{n \in \mathcal{N}_T^c \cup \mathcal{N}_T^c} \mathcal{P}(n). \quad (34)$$

**Remark 2.** A further result following from Lemma 2 is that, at the optimum, the set of nodes $k \in \mathcal{B}$ correctly defines the expression $\max \left( \delta H_n^+ - g \right)$ since $e^{w_n} e^{u_n} = 0$. Moreover, they belong to paths leading to final nodes characterized by $H^+_n \geq 0$. This implies that the objective function value $\Gamma^*$ is correctly computed. The rest of the nodes $k \in \mathcal{N} \setminus \mathcal{B}$, where the max operator is not properly defined since it might occur that $e^{w_n} e^{u_n} > 0$, belong to paths leading to final nodes for which $H^-_n$ is surely greater than zero (or at most $H^-_n = H^+_n = 0$), thus making null their contribution in the computation of $\Gamma^*$.

**Corollary 1.** Let $x_1^*, x_2^*, \ldots, x_N^*$ be an optimal portfolio choice for Problem 1, if $e^{w_k} e^{u_k} > 0$, for any $k \in \mathcal{N}$, then it exists $n \in \mathcal{N}_T$ such that $k \in \mathcal{P}(n)$ and $H^-_n > 0$ or $H^-_n = H^+_n = 0$.

We now assemble the above results to prove that solving Problem 1 is equivalent to solving the same problem with the additional non-convex constraints (12) and (20).

**Theorem 1.** Let $x_1^*, x_2^*, \ldots, x_N^*$ be an optimal portfolio choice for Problem 1, with optimal objective value $\Gamma^*$. Let $x_1^{**}, x_2^{**}, \ldots, x_N^{**}$ be an optimal portfolio choice of Problem 1 with the non-convex equations (12) and (20) included, with optimal objective value $\Gamma^{**}$. Then $\Gamma^* = \Gamma^{**}$.

**Proof.** Lemma 1 ensures that the conditions $H^+_n = H^-_n = 0$ holds for all $n \in \mathcal{N}_T$. Lemma 2 implies that the conditions $e^{w_n} e^{u_n} = 0$ only hold partially (for all $k \in \mathcal{B}$). However, by Corollary 1 we can claim that if $e^{w_k} e^{u_k} > 0$ then $k \in \mathcal{P}_n$, i.e. the path leading to $H^-_n > 0$. So, even if $\max \left( \delta H_n^+ - g \right)$ is not properly defined, its contribution to the value of $\Gamma^*$ is nil. Finally, the convexity of the objective function implies that its optimal value is unique, hence, $\Gamma^* = \Gamma^{**}$. \hfill $\Box$

### 3.4. Extensions

The model was formulated for Rung 4 of the risk ladder described in the introduction, but it can be modified to represent the other rungs. Some extensions are straightforward while others are more elaborate and are only sketched here for further research. The model was formulated for Rung 4 of the risk ladder described in the introduction, but it can be modified to represent the other rungs. Some extensions are straightforward while others are more elaborate and are only sketched here for further research.

**Rung 1.** For money-safe accounts we eliminate Eq. (4) from the model. The stochastic process of the liability is simplified to

$$L_n = L_{(n)} \exp \left( \lambda R_n \right),$$

and constraints $L_n \geq L_0$ are added for each terminal node $n \in \mathcal{N}_T$. The resulting model is a simplified version of the model developed above.
Rung 2. To guarantee total return $g_t$ upon retirement, eliminate Eq. (4), use the simplified liability process (35), and add constraints for each terminal node $n \in N_f$:

$$L_n \geq g_t L_0.$$ 

(36)

The resulting model is also a simplified version of the model developed above.

Rung 3. To deliver the industry average upon retirement, replace $g$ by $g_{stn}$ in Eq. (36), where $g_{stn}$ is the state-dependent industry average. Implementation of this model requires estimates of the industry portfolio returns. Given scenarios of returns on the tree, this information is computed from the composition of industry portfolios. The model is structurally identical to the one developed in this paper but has additional data requirements.

Rung 4. This is the type of guarantee already modeled. One significant extension is to open funds, whereby contributions $L_n$ are made to the fund at $t = 2, 3, \ldots, T$. This is trivially modeled on the liability side by rewriting the stochastic process as:

$$L_n = L_{p(n)} \exp \left[ g + \max \left( \delta r_n^A - g, 0 \right) \right] + L_n.$$ 

(37)

The contributions must also be accounted for on the asset side. A simple approach is to assume that incremental contributions are invested proportionately with the original portfolio. A more realistic approach is to rebalance the portfolio as new contributions arrive. The model setup on a multi-period tree permits the extension. Indeed, an advantage of the model is that it extends to multi-period optimization; this is an active research area, Zenios (2007), and has been used successfully in the ALM literature cited earlier. However, the linearizations developed in this paper need to be reworked.

Rung 5. To model guarantee income past retirement the model needs a past-retirement horizon $T' > T$. The asset at each state $n$ of the retirement horizon $T'$ can then be used to finance income for all nodes emanating from $n$ until $T'$, and lower bounds imposed on what this income would be. However, it is not clear what additional nonlinearities may be introduced with this extension and how to linearize them. Also, multi-factor trees are needed to calibrate risk-neutral probabilities of financial variables together with objective probabilities of economic variables to model inflation-adjusted income; such trees are available, see, e.g., Consiglio et al. (in press) and Mulvey (1996).

4. Implementation and results

We run experiments for $T = 30$ years and $f = 12$ financial asset indices. Without loss of generality – and appropriately in the current economic environment – we set the risk free rate $r = 0$. The indices represent the broad asset classes of sovereign bonds, corporate bonds and stocks; see Appendix. We simulate the risk-neutral process of asset returns using a standard Monte Carlo approach. The variance–covariance matrix is estimated from the monthly historical series of the indices. The yearly equivalent volatilities are obtained using the square root rule. All data are summarized in the Appendix.

The model was implemented on a simulated fan of 1000 risk-neutral paths. The size of the scenario set is chosen to limit computational times for the extensive experiments we run. Problems with more scenarios are solvable with modest computer resources, and variance reduction techniques can reduce sampling errors when using fewer scenarios. While we use a simple scenario generation method, we point out that alternative discrete representations, e.g., Geyer et al. (2010); Consiglio et al. (in press); Heyland and Wallace (2001), can be readily implemented.

The model is tested for parameters $\alpha$ and $\delta$ in the range 0.7–1.0, and $g$ in 0–5%.

4.1. The effect of policy parameters on the cost of the guarantee

The relationship between the cost of the guarantee $\Gamma^*$ and the guarantee rate $g$ is shown in Fig. 3. Within each panel, we display the effect of the parameter $\alpha$ on the cost, where higher values correspond to less equity and for $\alpha = 1$ there is no equity. Each panel corresponds to different participation rate $\delta$. The lower the $\delta$ the higher is the proportion of portfolio upside kept by fund management and this reduces the cost of the guarantee, especially for higher guarantees and lower equity. Fig. 4 shows the changes in the cost of the guarantee for varying participation rates, for $g = 3\%$ and equity $0$ and $0.3$.

We now focus on the effect of $\alpha$. This parameter controls risk sharing, as it determines the amount of equity, which can be viewed as regulatory requirement for solvency. For fixed $\delta = 0.9$ we show in Fig. 5 the cost of different guarantees as $\alpha$ varies from 0.7 to 1.0. The cost of the guarantee increases with the guarantee, but this increase is lower for lower $\alpha$, i.e., for more equity. This result is intuitive. By reducing $\alpha$ we increase the fraction of equity which composes the initial endowment $A_0$. Moreover, from Eq. (16), we observe that the final liability is due only on a fraction $\alpha$ of the initial value of the reference fund, $A_0$. Hence, as $\alpha$ decreases, an increasing part of the guarantee cost, given by $(L_n - A_0)^\alpha$, is borne by the equity-holders.

The curves in Fig. 5 trace the tradeoff between the option cost and the minimum guarantee. For a fixed value of $\alpha$, the points on each curve determine a Pareto frontier. For any guarantee rate $g$ the option cost is at its minimum and, given the convexity of the objective function (see Proposition 1), this value is unique. These curves can be used as a yardstick to assess how effective is a given reference portfolio vis-a-vis those on the efficient frontier and regulators can assess the risk of each fund. For instance, Solvency II regulations require embedded options to be marked-to-market, and, given the universe of the assets that characterizes the fund’s reference portfolio, a shift away from the optimal portfolio is a warning for potentially failed commitments that should be monitored by regulators. This point is elaborated next.

4.2. Portfolio composition and moral hazard

The portfolio compositions for $\alpha$ set to 0.7 and 0.9, and for changing levels of guaranteed return, are shown in Fig. 6. In all instances, the largest proportion is allocated in asset B00DS_1_3 of sovereign bonds with maturities less than 3 years; this is the asset with the lowest volatility, see Table 1.

This allocation is the consequence of the minimum guarantee mechanism. Whenever there is a downside deviation from the guarantee, the liability will increase by a function of $g$ and of the prior, compounded, performance. At the same time, the asset account will perform worse than the guarantee, and its value could turn out to be less than the value of the liability implying insolvency. This is illustrated in Fig. 7, where four panels show the dynamics of assets and liabilities for $g = 3\%$ and $\alpha = \delta = 0.9$. We consider three asset classes, respectively, the S&P500, the 3-month T-Bill and the 10-year T-Bond, and consider three reference portfolios corresponding to each asset class, plus a portfolio with weights 0, 0.9 and 0.1, respectively.
The yearly returns of the four portfolios are shown in Fig. 8.\(^4\) Although S&P500 outperforms the minimum guarantee rate, its high volatility leads to a situation in which the final asset value is well below the guaranteed liability. More suitable reference portfolios are obtained by allocating the initial capital over asset classes with low volatilities, or their combination, as shown in Fig. 7 and confirmed by the portfolio results in Fig. 6.

The average asset allocations from our model are 89% government bonds, 9% corporates and 2% stocks. By comparison, the average State pension portfolios reported in Biggs (2011) has 58% stocks, 26% bonds, 5% real estate, 2% cash and 9% other. There is significantly more exposure to risky assets in the funds studied by Biggs, compared to the optimal reference portfolio. We use the model to benchmark State pension fund practices. We assume a portfolio of our universe of assets of the same broad asset allocation as the State funds: 60% stocks, 35% bonds and 5% cash, and use the model to price the cost of the guarantee; see Fig. 9.

Comparing the frontiers of the State sponsored portfolios, with those of the optimized reference fund we observe significantly higher costs for the guarantee of the former. It is not surprising that Biggs finds funding ratios of 45% and concludes that the

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\(^4\) Data available from http://www.stern.nyu.edu/~adamo/pc/datasets/histretSPx.xls. The example is illustrative of the mechanism governing the DC with guaranteed returns. We do not draw any conclusion about the value of the guarantee option that must be addressed under a risk neutral measure.
Fig. 6. Portfolios are mostly composed of low volatility securities. Participation rate is set \( \delta = 0.9 \) and \( \alpha \) is set to 0.7 and 0.9 as shown in the strip above each panel.

Fig. 7. An illustrative example of the dynamics of assets and liabilities of a DC fund with 3% guaranteed return. Results based on historical data of three asset classes and a portfolio with weights \( w_1 = 0 \), \( w_2 = 0.9 \) and \( w_3 = 0.1 \). The less volatile assets and the portfolio better meet the guarantee, consistently with the findings of our model.
Fig. 8. Historical returns of the three assets and the portfolio assumed to be the reference fund of DC fund. The horizontal line denotes the 3% guarantee.

Fig. 9. The cost of the guarantee of the State pension funds using some benchmark portfolios from Biggs (2011) against optimized reference portfolio.
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typical State has significant unfunded public pension liabilities. The differences of cost between the actual and the optimized reference portfolios is a signal of moral hazard. The State guarantees the fund and will have to cover any shortfalls, while the funds take risky positions and pass the upside to the beneficiaries. This could also be a strategy of “gambling for redemption”: underfunded funds take a big gamble hoping for the big upsing.

4.3. Risk sharing

The cost of the guarantee can be used to set risk sharing premiums. From the example of Fig. 5 we obtain for $g = 3\%$ and $\alpha = 1$ (zero equity) a cost of 0.84. There are three possible ways to share this cost, depending on the risk sharing arrangements:

1. The pension fund bears the risk of the guarantee. Under this arrangement, the fund will charge the beneficiary – or the employer, or the government – the cost of the option. Hence, the beneficiary will pay 1.84 and will get a return only on 1 euro, since the 0.84 is the cost of the option. Probably this is not viable, because 3% guarantee is impossible, but the charge is fair. (Of course, since they bear the risk, they have to hedge the option, otherwise they will loose money, or default, even if they get the option premium.)

2. There is risk sharing between the beneficiary and a third party – such as the employer or the government. Let us assume that the sharing consist in paying 0.3 of equity ($\alpha = 0.7$), and investing it in the reference portfolio. We note from the figure that now the cost of the guarantee is 0.09, which can be charged to the employee. The pension fund will bear the option risk, but the cost is significantly lower than the previous risk sharing scheme. Again they have to hedge the option but the advantage is clear: instead of paying 0.89 for each euro invested in the asset portfolio, there will be a total cost of 0.39 (equity plus cost of the option).

We can also take the regulators’ view. Let us assume that a pension fund pursues a very aggressive marketing policy (a kind of dumping) promising $g = 3\%$. Then, the regulators should ask the pension fund to invest 0.3 from shareholders capital in the risky portfolio, and, at the same time, hedge the option. The latter means that in subsequent years the cost of the option must be covered by enough capital. Hence, the fund is “penalized” with a fair amount of capital requirement to account for the generous guarantee it promises.

5. Conclusions

This paper develops a general and computationally tractable model for pricing the cost of alternative embedded guarantee options in DC pension funds. The model determines the asset allocation choice that is optimal for a given guarantee, in that it minimizes the cost of the guarantee. The model is tested using real-world data to illustrate the effect of the design parameters of the guarantee on the cost of offering the option.

Results illustrate the effect on the option of (1) level of guarantee, (2) amount of equity, and (3) participation of the beneficiaries in any portfolio upsing above the guarantee. We also show how the model can be used to benchmark existing portfolios by applying it to test portfolios of State and local government pension funds from the literature. Our results are in agreement with empirical findings of existing literature, but also attribute the precise cost of the guarantee. We also show how the model can be used to calculate risk premia for risk sharing.

The model we implement and test can be extended to cover a broad range of guarantees that increasingly resemble DB, thus providing a continuum of funds in the hitherto dichotomous relation DC–DB. Some extensions are straightforward to build and calibrate, while others are provided as areas for further research.

Appendix. Indices, volatilities and correlations

To generate the risk neutral paths, we compute volatilities and correlations of 12 indices, representing three broad classes: sovereign bonds, corporate bonds and stocks. The data are obtained from the GAMS/FINLIB library.\footnote{Available at http://www.gams.de/finlib/libhtml/Estimate.htm, last accessed Sept.–Oct. 2015.} We denote by BONDS – 1–3, BONDS – 3–5, BONDS – 5–7 and BOND – 7–10 the J.P. Morgan aggregate indices of sovereign bonds issued by European countries with the indicated maturity ranges. The corporate bond classes are Salomon indices – CORP – FIN, CORP – ENE and CORP – INS – of bonds issued globally by financial, energy and insurance companies, respectively. Stock market indices are the Morgan Stanley Capital International Global, partitioned according to geo-political areas: EMU markets STOCKS_EMU, non-EMU STOCKS – EX_EMU, Pacific rim STOCKS – PAC, emerging economies STOCKS – EMER, and North-American STOCKS – NA. We estimate volatilities and correlations on 62 monthly observations, from Feb. 1995 to March 2000. Yearly volatilities are obtained by the square root rule (see Table 2).

References


