

# Non-Local Thermoelasticity: The fractional Heat Conduction

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**SUMMARY.**The paper is devoted to the analysis of a coupled thermoelastic 1D problem in presence of non-local thermal energy fluxes. The governing equation of temperature distribution involves fractional operators as far as a power-law decaying function is included in the transport equation of long-range fluxes that have been assumed proportional to the relative temperature among the interacting locations. Some numerical applications reporting the distribution of temperature in 1D elastic domain have been discussed in the paper.

## 1 INTRODUCTION

In recent years fractional differential calculus applications have been developed in physics, chemistry as well as in engineering fields. Fractional order integrals and derivatives extend the well-known definitions of integer-order primitives and derivatives of the ordinary differential calculus to real-order operators.

Engineering applications of these concepts dealt with viscoelastic models, stochastic dynamics as well as with the, recently developed, fractional-order thermoelasticity [1]. In these fields the main use of fractional operators has been concerned with the interpolation between the heat flux and its time-rate of change, that is related to the well-known second-sound effects. In other recent studies [2] a fractional, non-local thermoelastic model has been proposed as a particular case of the non-local, integral, thermoelasticity introduced at the mid of the seventies [3].

Very recently the authors provided a physical description of fractional, non-local effects for heat transfer in a rigid body introducing the long-range heat flux and, on this basis, a modified version of the Fourier heat flux equation is obtained. Such an equation involves spatial Marchaud fractional derivatives of the temperature field as well as Riemann-Liouville fractional derivatives of the heat flux with respect to time variable to account for second sound effects [4].

In this study the authors aim to extend the non-local model of fractional heat conduction to the case of a purely elastic material accounting for the thermoelastic coupling. Some numerical examples will be also discussed to show the effects of the long-range thermal energy exchange in an 1D unbounded domain.

## 2 FRACTIONAL-ORDER THERMODYNAMICS: THE LONG-RANGE HEAT FLUXES IN RIGID BODIES

In this section the fundamentals of heat transfer in rigid bodies will be shortly summarized in sec.(2.1) whereas sec.(2.2) will be devoted to the introduction of a physically consistent fractional-order model of thermal energy transfer in rigid bodies that will be used in the context of thermoelastic coupling as reported in sec.(3). The study will be confined to a one-dimensional problem involving a rigid body with uniform cross-section  $A$ , homogeneous thermal conductivity  $\kappa$  and uniform mass density of the material  $\rho$ . The body will be referred to a proper coordinate system, positive rightward as in fig.(1).

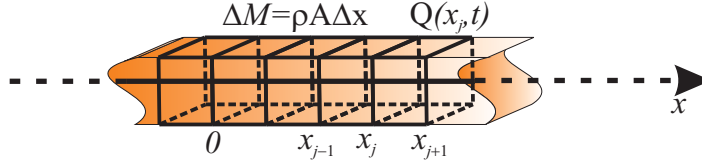


Figure 1: 1D domain with thermal energy transfer

### 2.1 Thermal energy transfer in rigid bodies: The Irreversible Thermodynamics

Let us refer the body internal energy to the extensive field of the absolute temperature denoted as  $\theta(x, t)$  with  $t$  denotes the time variable, whereas the specific thermal energy of the body at location  $x$  and time  $t$  will be denoted as  $\varphi(x, t)$ . The thermal energy associated with the mass  $\Delta M$  contained in the elementary volume element  $\Delta V = A\Delta x$  reads  $Q(x, t) = \varphi(x, t)\rho(x)\Delta V$ . The heat flux along the  $x$ -direction across the area  $A$  will be denoted  $q_l(x, t)$  (see fig.2) so that the energy balance between the ingoing and the outgoing fluxes reads:

$$A(-q_l(x + \Delta x) + q_l(x)) = \Delta \dot{Q}(x, t) = \rho\Delta V \left( \frac{\partial \varphi(x, t)}{\partial t} \right) \quad (1)$$

that holds true under the assumption of no energy source located at abscissa  $x$ . As we set  $\Delta x \rightarrow 0$ , then the balance equation between the heat energy stored at location  $x$  at time  $t$  and the difference between the ingoing and outgoing heat flux that reads:

$$-A \frac{\partial q_l(x, t)}{\partial x} = \frac{\partial \varphi(x, t)}{\partial t} = \dot{\varphi}(x, t) \quad (2)$$

The rate of thermal energy at location  $x$  is related, in rigid bodies, to the extensive absolute temperature field  $\theta(x, t)$  by means of specific heat, dubbed  $c_V$  as:

$$d\varphi(x, t) = c_V d\theta(x, t) \quad (3)$$

we will assume, hereinafter that  $c_V$  is constant for the considered range of temperature, the differential relation reported in eq.(3) may be reverted into a proper relation involving the time variations as  $\dot{\varphi}(x, t) = c_V \dot{T}$ .

The temperature equation, that is a differential relation that rules the evolution of the space-time temperature field  $\theta(x, t)$  of the body is readily obtained as we introduce the transport equation in eq.(2). In this study we will confine the thermal energy exchange to cases dealt with the classical irreversible thermodynamics (CIT), corresponding to the use of the Fourier transport equation as:

$$q_l(x, t) = -\kappa \frac{\partial \theta(x, t)}{\partial x} \quad (4)$$

where  $\kappa > 0$  according to the second principle of thermodynamics and it is known as thermal conductivity of the material. The use of eq.(4) in combination with eq.(3) yields the well-known temperature equation of CIT as:

$$\frac{\partial \theta}{\partial t} = \lambda \frac{\partial^2 \theta(x, t)}{\partial x^2} \quad (5)$$

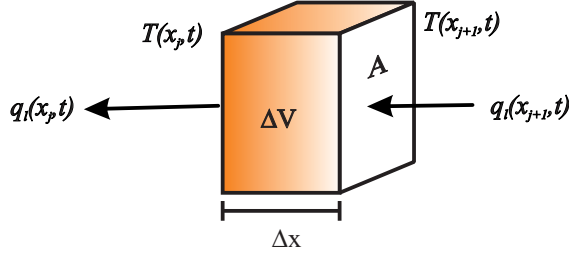


Figure 2: Thermal exchange among interacting volumes

where  $\lambda = \kappa/c_V\rho$  is the diffusion coefficient of the material. Eq.(5) must be supplemented by the relevant initial condition  $\theta(x, 0) = \bar{\theta}(x)$  and boundary conditions to be satisfied for the temperature field as:

$$\theta(0, t) = \theta_0(t); \theta(L, t) = \theta_L(t) \quad (6a)$$

The boundary conditions involving the temperature field may be replaced, alternatively, by boundary conditions involving the heat fluxes ingoing or outgoing from the considered solid as:

$$q_l(0, t) = -\kappa \left. \frac{\partial \theta(x, t)}{\partial x} \right|_{x=0}; q_l(L, t) = -\kappa \left. \frac{\partial \theta(x, t)}{\partial x} \right|_{x=L} \quad (7a)$$

We must consider, additionally, that the parabolic differential equation of the temperature distribution reported in eq.(5) does not account for the finite speed of temperature waves generated by disturbances in temperature fields. Such paradoxes are well-known in the context of thermal energy transfer in superfluids (He II) and they are usually accounted for as we introduce first-order time derivatives of the heat flux in transport equations ([7]) and/or fractional-order derivatives of the heat flux resorting to fractional-order thermodynamics ([1]).

Beside such remarkable contributions, in the mid-seventies a complementary approach based upon spatial integro-differential mathematical description of the transport equations has been introduced in the relevant scientific literature dubbed as non-local thermodynamics ([3]). In such a theory, the transport equation, relating the heat flux to the gradient of temperature field, involves a convolution integral in the form:

$$q_l(x, t) = -\kappa_1 \frac{\partial \theta(x, t)}{\partial x} - \kappa_2 \int_L g_E(|x - \xi|) \left. \frac{\partial \theta(x, t)}{\partial x} \right|_{\xi} d\xi \quad (8)$$

where  $\kappa_1$  and  $\kappa_2$  are the material thermal conductivities and function  $g_E(|x - \xi|)$  is real a material-dependent attenuation function that is introduced to fade out the effects of the temperature gradients at large distances.

Eq.(8) is the basis to introduce the fractional-order non-local thermodynamics selecting the non-local kernel in the power-law decay functional class  $g_E(|x - \xi|) = |x - \xi|^{\alpha-1}$  with  $0 \leq \alpha \leq 2$  for physical reasons ([2]) and the dimensional coefficient  $\kappa_2 = \kappa_\alpha/\Gamma(1 - \alpha)$  with  $\kappa_\alpha$  the material thermal conductivity with anomalous dimensions. As we assume vanishing values of the absolute

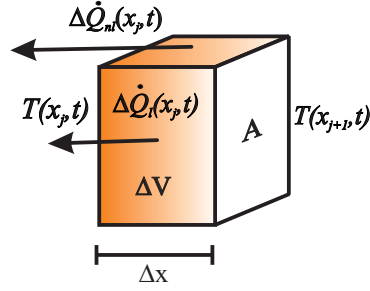


Figure 3: Balance of thermal energy including local and long-range contributions

temperature field at the borders of the body domain we get a non-local fractional transport equation that involves the left and right RL fractional derivatives as:

$$q_l(x, t) = -\kappa_1 \frac{\partial \theta(x, t)}{\partial x} - \kappa_\alpha [(D_{0+}^\alpha \theta)_x(x, t) + (D_{L-}^\alpha \theta)_x(x, t)] \quad (9)$$

where  $(D_{+}^\alpha \theta)_x(x, t)$  and  $(D_{-}^\alpha \theta)_x(x, t)$  are, respectively the left and right fractional-order Riemann-Liouville (RL) derivatives in unbounded domains that reads, respectively (see e.g. [6] for details):

$$(D_{0+}^\alpha \theta)_x(x, t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial x} \int_0^x \frac{\theta(\xi, t) d\xi}{(\xi - x)^{1-\alpha}}; (D_{L-}^\alpha \theta)_x(x, t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial x} \int_x^L \frac{\theta(\xi, t) d\xi}{(x - \xi)^{\alpha-1}} \quad (10)$$

where  $\Gamma[\bullet]$  is the Euler-Gamma function. It may be observed that eq.(9) coincides with the well-known Fourier transport equation as  $\kappa_1 = 0$ ,  $k_\alpha = \kappa/2$  and  $\alpha \rightarrow 1$  from below. Some straightforward manipulations involving the transport equation in eqs.(9,1) yield a fractional-order temperature equation that, however may be solved only in unbounded cases due to the presence of divergent contributions at the borders ([4]).

This consideration open the way to introduce a non-local fractional-order thermodynamics with underlying physical scheme that is not pathologically affected by the problem of non-homogeneous boundary conditions. This will be done in the next section.

## 2.2 The fractional model of thermal energy transfer: The long-range fluxes

In this section we introduce a different approach to fractional-order thermodynamics accounting for the existence, at a mesoscale, of long-range transport of the thermal energy. This concept is analogous to the idea of long-range interactions, recently introduced by the authors in the context of non-local elasticity [5]. The physical description of the model has been reported in fig.(1) where we introduced a spatial discretization grid of the body domain with  $N + 1$  nodal points located at abscissas  $x_j = (j - 1) \Delta x$  with  $j = 1, 2, \dots, N + 1$ . Let us consider a volume element extracted by the solid as in fig.(3), we assume that the thermal exchange of the considered element with the surrounding domain is ruled by a two-scale phenomenon:

- A thermal exchange between the considered and the adjacent elementary volumes represented by the local heat flux  $\Delta q_l(x_j, t) = q_l(x_j + \Delta x) - q_l(x_j, t)$  resulting in the time rate of change of the local thermal energy  $\Delta \dot{Q}_l(x_j, t)$ ;

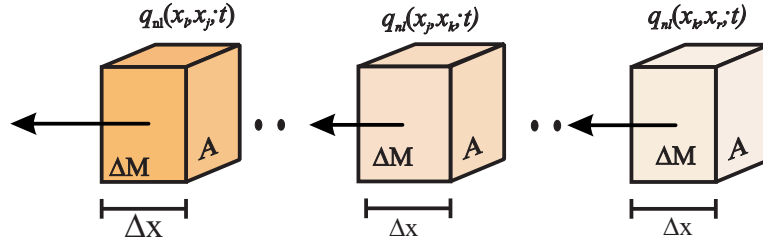


Figure 4: Long-range heat fluxes

- An overall long-range thermal energy transfer between volume  $\Delta V_j$  and the surrounding volume elements of the body, named  $\Delta \dot{Q}_{nl}(x_j, t)$ .

This latter contribution accounts for small scales heat transfer so that, considering both contributions, the balance equation of thermal energy of a volume element  $\Delta V$  reads:

$$\rho A \Delta x_j \frac{d\varphi}{dt} = -\Delta \dot{Q}_l(x_j, t) - \Delta \dot{Q}_{nl}(x_j, t) \quad (11)$$

In the following we will assume that the long-range overall flux  $\Delta \dot{Q}_{nl}(x_j, t)$  is the resultant of the elementary long-range fluxes occurring at smaller scales. Such contributions are provided by elementary fluxes of higher-order and henceforth are assumed proportional to the product of the interacting volumes as (fig.4):

$$\Delta \dot{Q}_{nl}(x_j, t) = A^2 \sum_{k=1}^{N+1} q_{nl}(x_j, x_k, t) \Delta x_j \Delta x_k = A^2 \sum_{k=1}^{N+1} q_{nl}(x_j, x_k, t) \Delta x^2 \quad (12)$$

where  $q_{nl}(x_j, x_k, t)$  ( $[q_{nl}] = F / (TL^7)$ ) is the elementary long-range flux exchanged by volumes located at abscissa  $x_j$  and  $x_k$ . Introducing eq.(12) and eq.(1) into eq.(11) the balance equation involving local and long-range contributions reads:

$$\rho \frac{d\varphi}{dt} = -\frac{\Delta q_l(x_j, t)}{\Delta x} - A \sum_{k=1}^{N+1} q_{nl}(x_j, x_k, t) \Delta x \quad (13)$$

that reverts as  $\Delta x \rightarrow 0$  to an integro-differential equation that reads:

$$\rho \frac{\partial \varphi(x, t)}{\partial t} = -\frac{\partial q_l(x, t)}{\partial x} - A \int_L q_{nl}(x, \xi, t) d\xi \quad (14)$$

where we replaced the running discrete abscissa  $x_k$  with its continuous counterpart  $\xi$  that is the dummy integration variable in the latter integral term of eq.(14). Eq.(14) is the non-local version of the first principle of thermodynamics in presence of long-range heat fluxes.

The equation that rules the evolution of the temperature field in a rigid body in presence of long-range fluxes may be obtained as we specify the transport equations for the local flux  $q_l(x, t)$  and for the non-local contributions  $q_{nl}(x, \xi, t)$ . The transport equation of the local heat flux is assumed as the fundamental equation of the EIT (eq.4). The transport equation of the long-range heat flux  $q_{nl}(x, \xi, t)$  is assumed analogous to the ballistic motion of photonic gases and henceforth it is assumed proportional to the relative absolute temperature between locations  $\xi$  and  $x$  as:

$$q_{nl}(x, \xi; t) = g(|x - \xi|) (\theta(\xi, t) - \theta(x, t)) \quad (15)$$

where function  $g(|x - \xi|)$  is a material-dependent distance-decaying function that accounts for smaller thermal exchange as the distance increases. The second principle of thermodynamics rules the direction of the thermal energy transfer and henceforth it may be satisfied only if  $g(|x - \xi|)$  is strictly positive. Despite several class of decaying function are allowed (exponential, gaussian, stretched exponential) in the following we will assume a fractional power-law as:

$$g(|x - \xi|) = \frac{\alpha c_\alpha}{\Gamma(1 - \alpha)} |x - \xi|^{-(1+\alpha)} \quad (16)$$

where  $\alpha \in \mathfrak{R}^+$  and  $c_\alpha$  is a material dependent proportionality coefficient ( $[c_\alpha] = F / (TL^{4-\alpha})$ ). Introducing the transport equations in eq.(4) and in eq.(15) into eq.(14) we get the equation:

$$\rho c_V \frac{\partial \theta}{\partial t} + \rho c_\alpha \left( \hat{\mathbf{D}}^\alpha \theta \right)_x (x, t) = -\kappa \frac{\partial^2 \theta}{\partial x^2} \quad (17)$$

where we denoted  $\left( \hat{\mathbf{D}}^\alpha \theta \right)_x (x, t) = \left( \hat{\mathbf{D}}_{0+}^\alpha \theta \right)_x (x, t) + \left( \hat{\mathbf{D}}_{L-}^\alpha \theta \right)_x (x, t)$  the sum of left and right integral parts of the Marchaud-type fractional operators that reads, respectively:

$$\left( \mathbf{D}_{0+}^\alpha \theta \right)_x (x, t) = \frac{\alpha}{\Gamma(1 - \alpha)} \left( \frac{\theta(x, t)}{x^\alpha} + \int_0^x \frac{\theta(x, t) - \theta(\xi, t) d\xi}{(\xi - x)^{1+\alpha}} \right) = \quad (18a)$$

$$= \frac{\alpha}{\Gamma(1 - \alpha)} \frac{\theta(x, t)}{x^\alpha} + \left( \hat{\mathbf{D}}_{0+}^\alpha \theta \right)_x (x, t) \quad (18b)$$

$$\left( \mathbf{D}_{L-}^\alpha \theta \right)_x (x, t) = \frac{\alpha}{\Gamma(1 - \alpha)} \left( \frac{\theta(x, t)}{(L - x)^\alpha} + \int_x^L \frac{\theta(x, t) - \theta(\xi, t) d\xi}{(x - \xi)^{1+\alpha}} \right) = \quad (18c)$$

$$= \frac{\alpha}{\Gamma(1 - \alpha)} \frac{\theta(x, t)}{(L - x)^\alpha} + \left( \hat{\mathbf{D}}_{L-}^\alpha \theta \right)_x (x, t) \quad (18d)$$

The field equation of the temperature distribution in the solid in eq.(17) must be supplemented with the Dirichlet boundary conditions about the prescribed temperature field at the borders and/or with the Neumann boundary conditions about the spatial gradients of the temperature field. This latter conditions do not involve the presence of long-range fluxes. In fact the overall incoming flux is of the same order of a volume heat source and therefore it does not appear in the position of the Neumann boundary conditions that are expressed in terms of the local heat flux  $q_l(x, t)$  only (see e.g. [4] for details).

### 3 THE NON-LOCAL THERMOELASTIC PROBLEM

In this section we will consider the more general problem involved in the analysis of 1D linearly elastic solid in presence of non-local thermoelastic contributions. The elastic properties of the considered solid, described geometrically, by means of the cross-section  $A$  and of the length

$L$  are described by the Young modulus  $E$ . Kinematics of the 1D model is described by the axial displacement function  $u(x, t)$  of the volume element  $dV(x) = A dx$  positive rightward. The elastic solid is undergoing a temperature distribution, measured by means of the absolute temperature field introduced in previous section and is subjected to external and internal force field  $F_0$  and  $F_L$  and  $n(x) A dx$ , respectively. The rate of change of the internal energy function in an elastic solid with long-range heat transfer defined in the previous section must account for the rate of change of the mechanical work, dubbed  $\dot{W} = -F_0 \dot{u}_0 + F_L \dot{u}_L + \int_0^L n(x) \dot{u}(x) dx$  that is:

$$\begin{aligned} A\rho \int_0^L d\dot{\varphi}(x, t) &= A\rho \int_0^L \dot{\varphi}(x, t) dx = A \int_0^L \dot{Q}(x, t) dx + \dot{W} = \\ &= -A \int_0^L \frac{\partial q_l(x, t)}{\partial x} dx + A^2 \int_0^L \int_0^L q_{nl}(x, \xi; t) d\xi dx + \dot{W} \end{aligned} \quad (19)$$

so that, recalling that at the borders,  $\sigma(0) A = F_0$  and  $\sigma(L) A = F_L$ , and using the 1D version of the Gauss identity, the contribution of the rate of the mechanical work may be written as:

$$\dot{W} = A \left[ \int_0^L \left( \frac{\partial \sigma(x, t)}{\partial x} - n(x) \right) \dot{u}(x) dx + \int_0^L \sigma(x, t) \dot{\varepsilon}(x, t) dx \right] = \int_0^L \sigma(x, t) \dot{\varepsilon}(x, t) dx \quad (20)$$

where the first integral is identically vanishing for equilibrium equations among the stress and the body force field  $\sigma(x, t)$  and  $n(x)$ , respectively. Substitution of the expression of the rate of change of mechanical work in eq.(20) into eq.(19) yields the local version of the energy balance, namely the local version of the first principle of thermodynamics in presence of long-range fluxes that reads:

$$\rho \dot{\varphi} = -\frac{\partial q_l(x, t)}{\partial x} + A \int_0^L q_{nl}(x, \xi; t) d\xi + \sigma(x, t) \dot{\varepsilon}(x, t) \quad (21)$$

that corresponds, beside the contribution of the non-local long-range fluxes, to the well-known balance equation of thermoelastic solids ([8]). In the considered thermoelastic problem we introduce the entropy state function  $\dot{s}(\varepsilon, \theta)$  that is defined in local version, for the long-range heat transfer here considered as:

$$\rho \theta \dot{s} = -\frac{\partial q_l(x, t)}{\partial x} + A \int_0^L q_{nl}(x, \xi; t) d\xi \quad (22)$$

so that, the complete version of the thermoelastic problem at generic location  $x$  may now be formulated in terms of the rate of change of the entropy density as:

$$\rho \dot{\varphi} = \rho \theta \dot{s} + \sigma \dot{\varepsilon} \quad (23)$$

and then, introducing the Helmholtz free energy:

$$H(\varepsilon, \theta) = \varphi(\varepsilon, \theta) - \rho \theta s(\varepsilon, \theta) \quad (24)$$

we get, upon substitution in the energy balance equation in eq.(22):

$$\rho \left( \dot{H}(\varepsilon, \theta) + \dot{\theta} s(\varepsilon, \theta) \right) = \rho \left( \frac{\partial H}{\partial \varepsilon} \dot{\varepsilon} + \frac{\partial H}{\partial \theta} \dot{\theta} \right) + \rho \dot{\theta} s = \sigma \dot{\varepsilon} \quad (25)$$

and henceforth the constitutive equations for the state functions  $\sigma(\varepsilon, \theta)$  and  $s(\varepsilon, \theta)$  as:

$$\sigma = \frac{1}{\rho} \frac{\partial H}{\partial \varepsilon}; s = -\frac{\partial H}{\partial \theta} \quad (26)$$

The constitutive equations reported in eq.(26) may be used to introduce the thermoelastic stress-strain relations as we introduce a proper expression for the Helmholtz free energy that, assuming that there exists a reference state, stress free, of the elastic solid at absolute temperature  $\theta_0$ , may be expressed as:

$$H(\varepsilon, \bar{\theta}) = \frac{1}{2} E \varepsilon^2 - E \gamma \bar{\theta} \varepsilon + \frac{1}{2} \gamma^2 \bar{\theta}^2 \quad (27)$$

where  $\bar{\theta} = (\theta - \theta_0) / \theta_0$ . Substitution of eq.(27) into eqs.(26) yields the stress-strain relation in the form  $\sigma = E(\varepsilon - \gamma(\theta - \theta_0))$ , whereas the state equation for the entropy balance in terms of thermal energy fluxes reads:

$$\rho \dot{\theta} \left( \frac{\partial s}{\partial \varepsilon} \dot{\varepsilon} + \frac{\partial s}{\partial \theta} \dot{\theta} \right) = \rho \dot{\theta} \left( \frac{\partial^2 H}{\partial \varepsilon \partial \theta} \dot{\varepsilon} + \frac{\partial^2 H}{\partial \theta^2} \dot{\theta} \right) = -\frac{\partial q_l(x, t)}{\partial x} + A \int_0^L q_{nl}(x, \xi; t) d\xi \quad (28)$$

The temperature equation involving thermoelastic effects as well as long-range contributions may then be written as:

$$\rho c_V \dot{\theta} + E \gamma \theta_0 \dot{\varepsilon} = \kappa \frac{\partial^2 \theta}{\partial x^2} - \kappa_\alpha \left( \hat{\mathbf{D}}^\alpha \theta \right)_x(x, t) \quad (29)$$

where we introduced the specific heat at constant volume  $c_V$  and we introduced the transport equations reported in sec.(2) assuming  $\kappa_\alpha = \rho c_\alpha$ . It may be observed that, assuming a rigid body analysis, the temperature equation with long-range thermal energy transfer is perfectly coalescing with the temperature equation in eq.(17)

The temperature equation yields the evolution of the temperatures in presence of long-range fluxes, accounting for thermoelastic coupling so that, the complete version of the thermoelastic 1D problem is ruled by the governing equations:

$$\frac{\partial^2 u}{\partial x^2} - \gamma \frac{\partial \theta}{\partial x} = -\frac{n(x)}{EA} \quad (30a)$$

$$\rho c_V \dot{\theta} + E \gamma \theta_0 \frac{\partial^2 u}{\partial x^2} = \kappa \frac{\partial^2 \theta}{\partial x^2} - \kappa_\alpha \left( \hat{\mathbf{D}}^\alpha \theta \right)_x(x, t) \quad (30b)$$

where we assumed a uniform temperature of the reference state along the bar domain  $\theta_0$ . Eq.(30) must be supplemented by Dirichlet or Neumann boundary conditions involving temperature and displacement fields and/or their spatial gradients. Some numerical applications showing the effect of the long-range fluxes in a 1D unbounded domain will be reported in the next section.



#### 4 NUMERICAL APPLICATION

Let us consider in this section a decoupled thermoelastic problem in which the transformation among thermal and mechanical energy is neglected. In this latter case, the decoupled, thermoelastic 1D problem is ruled by the following system of differential equations:

$$\frac{\partial^2 u}{\partial x^2} - \gamma \frac{\partial \theta}{\partial x} = -\frac{n(x)}{EA} \quad (31a)$$

$$\rho c_V \dot{\theta} + \kappa_\alpha \left( \hat{\mathbf{D}}^\alpha \theta \right)_x (x, t) = \kappa \frac{\partial^2 \theta}{\partial x^2} \quad (31b)$$

We observe that, neglecting mechanical-thermal energy coupling, the temperature equation in eq.(31b) reverts to the thermal energy transfer in rigid bodies yet discussed in sec.(2) with attendant initial and boundary conditions discussed in eqs.(6,7). The initial conditions for the axial displacement function read  $u(x, 0) = \bar{u}(x)$  whereas the Dirichlet boundary conditions read:

$$u(0, t) = u_0(t); u(L, t) = u_L(t) \quad (32)$$

that may be alternatively replaced by the or Neumann boundary conditions as:

$$EA \left( \frac{\partial u}{\partial x} \Big|_0 - \gamma \theta(0, t) \right) = -F_0 \quad (33a)$$

$$EA \left( \frac{\partial u}{\partial x} \Big|_L - \gamma \theta(L, t) \right) = F_L \quad (33b)$$

In the following analysis we will study the temperature distribution in an unbounded 1D rigid domain contrasting the temperature distribution for different values of the fractional differentiation index  $\alpha$ . The analysis is conducted under the assumption of vanishing local heat fluxes so that, the temperature equation reads:

$$\frac{\partial \theta}{\partial t} = -\frac{c_\alpha}{c_V} \left( \hat{\mathbf{D}}^\alpha \theta \right)_x (x, t) \quad (34)$$

The solution of eq.(34) will be obtained by means of Fourier Transform in the space of wavenumber:

$$\theta(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\theta}_0(k) e^{-C_\alpha |k|^\alpha t} e^{ikx} dk \quad (35)$$

where  $C_\alpha = \cos(\alpha\pi/2) c_\alpha/c_V$  and  $\theta(x, 0) = \theta_0(x)$ . The temperature distribution due to a Dirac delta value of temperature  $\theta_0(x) = \delta(x)$  is then obtained with  $\hat{\theta}_0(k) = 1$  as the inverse Fourier Transform of eq.(35) in figs.(5) for  $\alpha = 1$  and  $\alpha = 2$  have been represented.

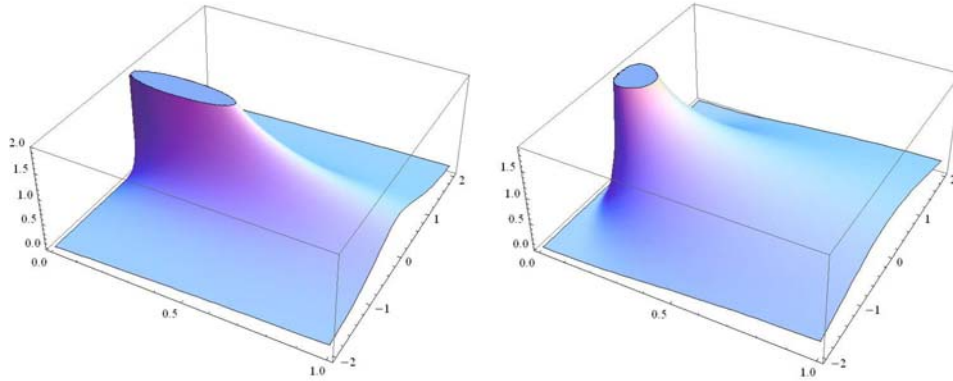


Figure 5: Evolution of a concentrated temperature impulse:Left  $\alpha = 1$ ; Right  $\alpha = 2$

## 5 CONCLUSIONS

In this paper, a first approach to fractional thermoelasticity has been presented for a 1D case. A non-local model involving long-range fluxes, recently proposed by the authors, has been used to represent the non-local thermal energy transfer in the body. The governing equations of thermoelastic model have been obtained resorting to the Helmholtz free energy and they involve Marchaud fractional derivatives beside ordinary second-order derivatives of the temperature field, that appear as the decaying function of the long-range fluxes is provided as power-law of the inter distance.

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