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Umbral Calculus and New Trigonometries

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ABSTRACT

This article presents an overview of Umbral calculus and of its wide range of applicability. The aim is to provide new tools, embedding umbral, symbolic and operatorial methods to be exploited in pure and applied mathematics, in the research of analytical or numerical solutions in different fields.

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1. INTRODUCTION

The term *Umbra*[1] has been diffused by S. Roman and G.C. Rota [1] in the second half of the 20th century to underline, in the emerging field of operational calculus, the practice of replacing the representation in series of certain function $f(x)$ with the formal exponential series[2]. Earlier, other mathematicians like J. Blissard in *Theory of Generic Equations* or Edouard Lucas in *Théorie nouvelle des nombres de Bernoulli et d'Euler*, established the rules which allowed for viewing different functions through the same abstract entity and showed its usefulness for a wide range of applications. In the current research, the same starting point has been used but with methods closer to the *Heaviside* point of view [2], by proposing strategies combining different technicalities and operational methods which yield the formulation of a powerful "different mathematical language" [3]. The umbral image, the key element to establish the rules to replace higher transcendental functions in terms of elementary functions, and rewrite complex problems into simplified exercises, will be the starting point to fix the criteria to take advantage from such a replacement, based on the Laplace and Borel transform Theory and the Principle of Permanence of the Formal Properties.

To introduce the umbral operator, we have to provide some preliminary definitions. The operator $\hat{c}$, called *umbral*, is a shift operator $\hat{c} = e^{\partial_z}$ acting on an appropriate chosen vacuum[3], which is a function that will change according to the problem to solve [3]. If the initial vacuum is, for example, the function:

[1] It is assonant to the term “Ombra” in Latin which means “shadow” in English.

[2] The promotion of the index n in c_n to a power exponent of the operator $\hat{c}$, the umbral operator, is the essence of umbra, since it is a kind of projection of one into the other: $f(x) = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!} \to \sum_{n=0}^{\infty} \hat{c}\, \frac{x^n}{n!} = e^{\hat{c}x}$

[3] This term is used to stress that the action of the operators, raised to some power, is that of acting on an appropriate set of functions by “filling” the initial state $\varphi_0 = \frac{1}{\Gamma(1)}$ (see [3] for a rigorous treatment of the topic).

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$$\varphi(\mu) := \varphi_\mu = \frac{1}{\Gamma(\mu+1)} \qquad \forall \mu \in \mathbb{R} \tag{1}$$

the umbral operator $\hat{c}^\mu$, $\forall \mu \in \mathbb{R}$, will be the action of the operator $\hat{c}$ on the vacuum φ_0 such that:[4]

$$\hat{c}^\mu \varphi_0 := \varphi_\mu = \frac{1}{\Gamma(\mu+1)} \tag{2}$$

By that definition, the operator $\hat{c}$ has the properties of the standard exponential $\hat{c}^{\pm\mu}\hat{c}^\nu = \hat{c}^{\pm\mu+\nu}$ and $(\hat{c}^{\pm\mu})^\nu = \hat{c}^{\pm\mu\,\nu}, \forall \mu, \nu \in \mathbb{R}$. In the following we list a plethora of examples of applications of the umbral operator. We start with the creation of the umbral image of some special function.

2. SPECIAL FUNCTIONS

By using the operator in Eq. (2) and the property of Γ-functions [4], we find the umbral image of the Bessel function [5]. Indeed, by exploiting the series representation of the Bessel function, we get, $\forall x \in \mathbb{R}$:

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r}}{(r!)^2} = \sum_{r=0}^{\infty} d \frac{(-1)^r \left(\frac{x}{2}\right)^{2r} \hat{c}^r}{r!} \varphi_0 = e^{-\hat{c}\left(\frac{x}{2}\right)^2} \varphi_0 \tag{3}$$

obtaining in this way a new formulation of Bessel function in terms of the Gaussian function. We can furthermore write the umbral n-order Bessel function:

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r \left(\frac{x}{2}\right)^{2r+n}}{r!\,(r+n)!} = \left(\hat{c}\frac{x}{2}\right) e^{-\hat{c}\left(\frac{x}{2}\right)^2} \varphi_0 \qquad \forall x \in \mathbb{R} \tag{4}$$

So we have formally reduced a transcendental function to an exponential form, downgrading the "complexity" of the function.

The umbral image is not unique indeed, if we chose another vacuum, the action of the umbral operator will yield a different function. For example, if we introduce the pair umbral operator/vacuum such that:

$$\hat{b}^r \Phi_0 := \Phi_r = \frac{1}{\left(\Gamma(r+1)\right)^2} \qquad \forall r \in \mathbb{R} \tag{5}$$

we obtain:

$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \left(\frac{x}{2}\right)^{2r} = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{2r} \left(\hat{b}^r \Phi_0\right) = \frac{1}{1+\hat{b}\left(\frac{x}{2}\right)^2} \Phi_0 \qquad \forall x \in \mathbb{R} \tag{6}$$

thus expressing the Bessel function in a rational form, according to operator $\hat{b}$ and the vacuum Φ_0. Obviously, the umbral images are "equivalent" in the sense that the mathematical problems including the initial function (in the present example, the Bessel function) can be treated indifferently through the first of the second proposed umbral image, as shown below.

Example 1. $\forall x \in \mathbb{R}$

$$\int_{-\infty}^{\infty} J_0(x)dx = \int_{-\infty}^{\infty} e^{-\hat{c}\left(\frac{x}{2}\right)^2} dx\varphi_0 = 2\sqrt{\frac{\pi}{\hat{c}}}\varphi_0 = 2\sqrt{\pi}\hat{c}^{-\frac{1}{2}}\varphi_0 = 2\sqrt{\pi}\frac{1}{\Gamma\left(-\frac{1}{2}+1\right)} = 2 \tag{7}$$

[4] $\hat{c}^\mu \varphi_0 = e^{\mu \partial_z} \varphi_z \,|_{z=0} = \varphi_{z+\mu}\big|_{z=0} = \frac{1}{\Gamma(z+\mu+1)}\Big|_{z=0} = \frac{1}{\Gamma(\mu+1)}$

$$\int_{-\infty}^{\infty} J_0(x)dx = \int_{-\infty}^{\infty} \frac{1}{1+\hat{b}\left(\frac{x}{2}\right)^2} dx\Phi_0 = 2\frac{\pi}{\sqrt{\hat{b}}}\Phi_0 = 2\pi\hat{b}^{-\frac{1}{2}}\Phi_0 = 2\frac{\pi}{\left(\Gamma\left(\frac{1}{2}\right)\right)^2} = 2 \tag{8}$$

This kind of new "interpretation" of a certain function opens a wide range of possibilities. To this aim, we provide some examples including the solution of non-trivial integrals.

Example 2. *We consider integrals involving the Bessel function* $J_0(\cdot)$ *and the umbral operator* $\hat{c}$ [3,6]. *Then* $\forall a, b \in \mathbb{R}: ax^2 + bx > 0$:

$$I_J(a,b) = \int_{-\infty}^{\infty} J_0\left(2\sqrt{ax^2+bx}\right) dx = \int_{-\infty}^{\infty} e^{-\hat{c}(ax^2+bx)} dx\varphi_0 = \sqrt{\frac{\pi}{\hat{c}a}} e^{\frac{\hat{c}b^2}{4a}} \varphi_0 = \sqrt{\frac{\pi}{a}}\sqrt{\frac{b}{2\sqrt{a}}} I_{-\frac{1}{2}}\left(\frac{b}{\sqrt{a}}\right) \tag{9}$$

where $I_\nu(x)$ *is a modified Bessel function of the first kind, or* $\forall a, b \in \mathbb{R}^+$:

$$I_{L,J}(a,b) = \int_{-\infty}^{\infty} \frac{J_0\left(\frac{2\sqrt{a}x}{\sqrt{1+bx^2}}\right)}{1+bx^2} dx = \sqrt{\frac{\pi}{b}} \sum_{r=0}^{\infty} \frac{\Gamma\left(r+\frac{1}{2}\right)}{r!^3}\left(-\frac{a}{b}\right)^r \tag{10}$$

Furthermore, we want to stress the possibility to evaluate the product of special functions by using the umbral images. However, we recall that in general the operator is not commutative. So if we want to calculate, for example, $(J_0(x))^2$ we cannot use the same operator raised to power 2, but we have to define two different operators acting on two different vacua in the following way. We get $\forall a, b, x \in \mathbb{R}$ [7]:

$$J_0(ax) = e^{-a^2\hat{c}_1\left(\frac{x}{2}\right)^2}\varphi_{0,1} \qquad J_0(bx) = e^{-b^2\hat{c}_2\left(\frac{x}{2}\right)^2}\varphi_{0,2} \tag{11}$$

and introduce the function $f(x; a, b) := J_0(ax)J_0(bx)$. Then [7]:

$$f(x; a, b) = e^{-(a^2\hat{c}_1 + b^2\hat{c}_2)\left(\frac{x}{2}\right)^2}\varphi_{0,1}\varphi_{0,2} \tag{12}$$

By expanding the exponential in MacLaurin series we can finally write:

$$\begin{aligned} f(x; a, b) &= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} l_r(a^2, b^2)\left(\frac{x}{2}\right)^{2r} \\ l_r(a,b) &= r! \sum_{s=0}^{r} \frac{a^{r-s}b^s}{(s!)^2[(r-s)!]^2} \end{aligned} \tag{13}$$

hence finding a closed formula for a non-trivial product of special functions.

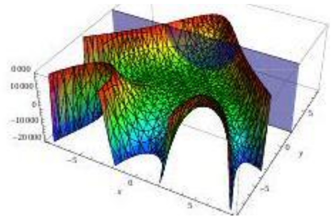
2.1. Hermite Polynomials

We consider now the two-variable polynomials of Hermite Kampé dè Fériét [4,8,9], also called heat polynomials:

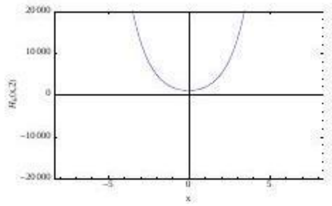
$$H_n(x,y) = n! \sum_{s=0}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{x^{n-2s}y^s}{(n-2s)!\,s!} \qquad \forall x, y \in \mathbb{R}, \forall n \in \mathbb{N} \tag{14}$$

with generating function [3]:

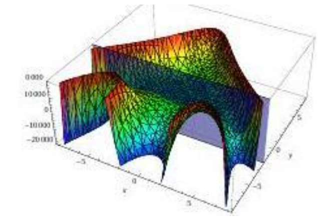
$$\sum_{r=0}^{\infty} \frac{t^n}{n!} H_n(x,y) = e^{xt+yt^2} \tag{15}$$



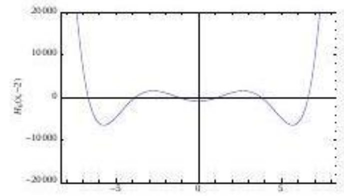
(a) $H_6(x, y)$ cut by the plane $y = 2$.



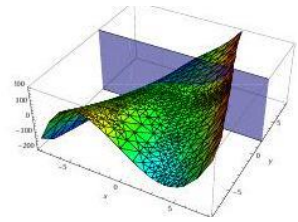
(b) $H_6(x, 2)$.



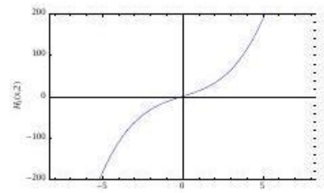
(c) $H_6(x, y)$ cut by the plane $y = -2$.



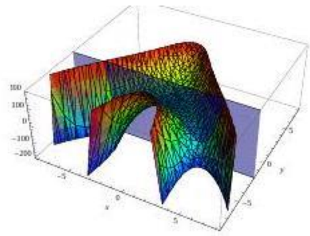
(d) $H_6(x, -2)$.



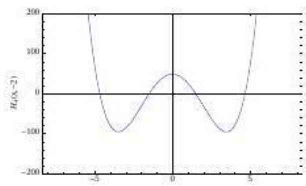
(e) $H_3(x, y)$ cut by the plane $y = 2$.



(f) $H_3(x, 2)$.



(g) $H_4(x, y)$ cut by the plane $y = -2$.



(h) $H_4(x, -2)$.

Figure 1. Geometrical representation of two-variable Hermite polynomials in 3D and 2D for different n and y values.

From an operational point of view, they can be expressed as the result of the action of the operator ∂_x^2 on the monomial x^n: $H_n(x, y) = e^{y\,\partial_x^2} x^n$[5,10]. In a geometrical sense we understand the operational rule as the transformation of the exponential operator on an ordinary monomial into a Hermite type special polynomial. The "evolution" from an ordinary monomial to the corresponding Hermite is shown by moving the cutting plane orthogonal to the y axis (Fig. 1). For a specific value of the polynomial degree n, the polynomials lie on the cutting plane. For negative values of y they realize an orthogonal set [3,11]. By using the umbral technique [1], we can extend the Hermite polynomial representation to Hermite function representation [10,11], considering the order $\nu \in \mathbb{R}$ and exploiting the Newton binomial and the Laplace transform [4,12,13,14], as shown below.

Proposition 1.

$$H_n(x, y) = \left(x + {}_y\hat{h}\right)^n \theta_0 \qquad \forall x, y \in \mathbb{R}, \forall n \in \mathbb{N} \tag{16}$$

with ${}_y\hat{h}$ *umbral operator and* θ_0 *vacuum such that:*

$$ {}_y\hat{h}^r\theta_0 := \theta_r = \frac{y^{\frac{r}{2}} r!}{\Gamma\left(\frac{r}{2}+1\right)}\left|\cos\left(r\frac{\pi}{2}\right)\right| = \begin{cases} 0 & r = 2s+1 \\ y^s\dfrac{(2s)!}{s!} & r = 2s \end{cases} \quad \forall s \in \mathbb{Z} \tag{17} $$

Proof. By considering the definition of the ${}_y\hat{h}$ operator we can write:

$$ \left(x + {}_y\hat{h}\right)^n\theta_0 = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}\,{}_y\hat{h}^r\theta_0 = \sum_{r=0}^{n}\binom{n}{r}x^{n-r}y^s\frac{(2s)!}{s!} = \sum_{s=0}^{\left\lfloor\frac{n}{2}\right\rfloor} x^{n-2s}y^s\frac{n!}{(n-2s)!\,s!} = H_n(x,y) \quad \square \tag{18} $$

Proposition 2. *The relevant Hermite function integral representation, with order real and negative too, can be written as:*

$$ H_{-\nu}(x,y) = \frac{1}{\Gamma(\nu)}\int_0^\infty s^{\nu-1}e^{-sx}e^{-ys^2}ds \qquad \forall x \in \mathbb{R}, \forall y,\nu \in \mathbb{R}^+ \tag{19} $$

Proof. By considering the definition of the ${}_y\hat{h}$ operator we can write:

$$ H_{-\nu}(x,y) = \frac{1}{\left(x + {}_y\hat{h}\right)^\nu}\theta_0 = \int_0^\infty e^{-xs}\frac{s^{\nu-1}e^{-y\hat{h}s}}{\Gamma(\nu)}ds\theta_0 = \frac{1}{\Gamma(\nu)}\int_0^\infty s^{\nu-1}e^{-sx}e^{-ys^2}ds \quad \square $$

Furthermore, if we introduce another application of the umbral methods, called Hermite calculus [15], we can reach the target to combine the previous results and calculate, for example, non-trivial integrals including combination of special functions. To give an idea of the methods we will use, we consider the following integral:

$$ I(\alpha,\beta,\gamma) = \int_{-\infty}^{\infty} e^{-(\alpha+\beta)x^2-\gamma x}dx \tag{20} $$

which can be evaluated through the Gauss-Weierstrass integral [3]:

$$ I(\alpha,\beta,\gamma) = \sqrt{\frac{\pi}{\alpha+\beta}}e^{\frac{\gamma^2}{4(\alpha+\beta)}} \tag{21} $$

but we recast the integral in Eq. (20) in umbral form:

$$ I(\alpha,\beta,\gamma) = \int_{-\infty}^{\infty} e^{-\alpha x^2-\hat{h}_{(\gamma,-\beta)}x}dx \tag{22} $$

where we introduced the umbral operator $\hat{h}_{(.,.)}$ and the vacuum η_0 such that:

$$ \hat{h}^r_{(x,y)}\eta_0 := H_r(x,y) \tag{23} $$

By that position, we can recast:

$$ e^{-\hat{h}_{(\gamma,-\beta)}x} = \sum_{r=0}^{\infty}\frac{(-x)^r}{r!}\hat{h}^r_{(\gamma,-\beta)} = \sum_{r=0}^{\infty}\frac{(-x)^r}{r!}H_r(\gamma,-\beta) \tag{24} $$

and by treating the umbral operator $\hat{h}_{(.,.)}$ as an ordinary algebraic quantity, as explained previously, we can solve the integral in Eq. (22) in the form:

$$ I(\alpha,\beta,\gamma) = \sqrt{\frac{\pi}{\alpha}}e^{\frac{\hat{h}^2_{(\gamma,-\beta)}}{4\alpha}}\eta_0 = \sqrt{\frac{\pi}{\alpha}}\sum_{r=0}^{\infty}\frac{1}{r!}\left(\frac{\hat{h}^2_{(\gamma,-\beta)}}{4\alpha}\right)^r\eta_0 = \sqrt{\frac{\pi}{\alpha}}\sum_{r=0}^{\infty}\frac{1}{r!}H_{2r}\left(\frac{\gamma}{2\sqrt{\alpha}},-\frac{\beta}{4\alpha}\right) = \sqrt{\frac{\pi}{\alpha+\beta}}e^{\frac{\gamma^2}{4(\alpha+\beta)}} \tag{25} $$

In this way, we have shown the flexibility of the technique. Indeed, we can raise the complexity of the integrals and solve them by methods as, for example, in the case of the anharmonic oscillator in the example below [15,16].

Example 3. *Let:*

$$J(a,b,c) = \int_{-\infty}^{\infty} e^{-(ax^4+bx^2+cx)} dx \qquad \text{Re}\,(a) > 0 \tag{26}$$

be the integral of the anharmonic oscillator. We can solve it by the Hermite calculus position:

$$J(a,b,c) = \int_{-\infty}^{\infty} e^{-\hat{h}_{(b,-a)}x^2-cx} dx = \sqrt{\frac{\pi}{\hat{h}}} e^{\frac{c^2}{4\hat{h}}} = \sqrt{\pi} \sum_{s=0}^{\infty} \frac{1}{s!} \left(\frac{c}{2}\right)^{2s} \hat{h}^{-\left(s+\frac{1}{2}\right)} \tag{27}$$

in which we used the fractional order of the umbral operator and of the Hermite function:

$$\hat{h}^{-\left(s+\frac{1}{2}\right)} = H_{-\left(s+\frac{1}{2}\right)}(b,-a) \tag{28}$$

The same consideration can be done for each special function through appropriate umbral images. We list in the following just some examples for the Laguerre, Jacobi, Legendre, Tricomi-Bessel and Chebyshev functions, and the Voigt transform [12,17,18].

Laguerre	$L_n(x,y) = (y - \hat{c}x)^n \varphi_0$
Jacobi	$P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!^2} R_n^{(\alpha,\beta)}\left(\frac{x-1}{2},\frac{x+1}{2}\right)$
	$R_n^{(\alpha,\beta)}(\xi,\eta) = n!\, \hat{c}_1^{\alpha} \hat{c}_2^{\beta} [\hat{c}_1\xi + \hat{c}_2\eta]^n \varphi_{1,0}\varphi_{2,0}$
Legendre	$P_n(x) = R_n^{(0,0)}(\xi,\eta)$
Tricomi-Bessel	$C_\nu(x) = \left(\frac{1}{x}\right)^{\frac{\nu}{2}} J_\nu(2\sqrt{x})$
Chebyshev	$U_n(x,y) = \frac{1}{n!}\int_0^{\infty} e^{-s}(-xs + {}_{(-ys)}\hat{h})^n \theta_0 ds$
Voigt	${}_V\hat{f}(x,y;z) = \int_0^{\infty} e^{-xt-yt^2} f(zt) dt$

and so on.

3. NUMBER THEORY

The umbral method can be applied in many different fields of pure and applied mathematics. A further example is indeed Number Theory. We remind that Harmonic Numbers are defined as [19,20,21,22]:

$$h_n := \sum_{r=1}^{n} \frac{1}{r} \qquad \forall n \in \mathbb{N}_0 \tag{29}$$

In terms of Laplace transform, we obtain:

$$h_n = \sum_{r=1}^{n} \int_0^{\infty} e^{-sr} ds \qquad \forall n \in \mathbb{N}_0 \tag{30}$$

thereby getting the n-th harmonic number through Euler's integral:

$$h_n = \int_0^1 \frac{1-x^n}{1-x}dx \tag{31}$$

valid more generally $\forall n \in \mathbb{R}^+$.

We can then introduce the function:

$$\varphi_h(z) := \varphi_{h_z} = \int_0^1 \frac{1-x^z}{1-x}dx \qquad \forall z \in \mathbb{R}^+ \tag{32}$$

as the *Harmonic Number Umbral Vacuum:*

$$\hat{h}^n \varphi_{h_0} = \hat{h}^n \varphi_{h_z}\Big|_{z=0} = e^{n\,\partial_z}\varphi_{h_z}\Big|_{z=0} = \varphi_{h_{z+n}}\Big|_{z=0} = \int_0^1 \frac{1-x^{z+n}}{1-x}dx\Bigg|_{z=0} = \int_0^1 \frac{1-x^n}{1-x}dx = h_n \tag{33}$$

and, without dwelling on the details[5], introduce the *Harmonic Based Exponential Function* as the series:

$$_h e(x) := e^{\hat{h}x} = 1 + \sum_{n=1}^{\infty} \frac{h_n}{n!}x^n \qquad \forall x \in \mathbb{R} \tag{34}$$

In an analogous way, we can treat other families of numbers like the Motzkin, telegraph, Stirling numbers, etc.

4. CLASSIC TRIGONOMETRY

Trigonometry can also be reinterpreted in terms of umbral images.

Definition 1. *We introduce,* $\forall \alpha, \beta \in \mathbb{R}^+$*, the umbral vacuum:*

$$\psi_\kappa := \frac{\Gamma(\kappa+1)}{\Gamma(\alpha\kappa+\beta)} \qquad \forall \kappa \in \mathbb{R} \tag{35}$$

and the shift operator $_{\alpha,\beta}\hat{d}$ *such that we can get:*

$$_{\alpha,\beta}\hat{d}^\kappa \psi_0 = \frac{\Gamma(\kappa+1)}{\Gamma(\alpha\kappa+\beta)} \tag{36}$$

Proposition 3. *(Cos-exponential umbral image) If* $\alpha = 2$ *and* $\beta = 1, \forall x \in \mathbb{R}$:

$$\cos(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{(2r)!} = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{r!}\ {}_{2,1}\hat{d}^r \psi_0 = e^{-_{2,1}\hat{d}x^2}\ \psi_0 \tag{37}$$

as we expect.

Corollary 1. $\forall x \in \mathbb{R}$:

$$\partial_x e^{-_{2,1}\hat{d}x^2}\psi_0 = -2x\ {}_{2,1}\hat{d}e^{-\,_{2,1}\hat{d}x^2}\psi_0 = -2x\sum_{r=0}^{\infty}\ (-1)^r \frac{(r+1)!}{(2r+2)!}\frac{x^{2r}}{r!} = -\sum_{r=0}^{\infty}\ (-1)^r \frac{x^{2r+1}}{(2r+1)!} = -\sin\,(x) \tag{38}$$

Fresnel Integral [17]:

$$C(x) = \int_x^{+\infty} \cos(\xi^2)d\xi \qquad \forall x \in \mathbb{R} \tag{39}$$

[5] See [20] for more details.

$$C(0)=\int_0^{+\infty}\left[e^{-{}_{2,1}\hat{d}x^4}\psi_0\right]dx=\left(\frac{1}{4}\int_0^{\infty}e^{-y}y^{\frac{1}{4}-1}dy\right)_{2,1}\hat{d}^{-\frac{1}{4}}\psi_0=\frac{1}{4}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{2}\right)}=\frac{1}{2}\sqrt{\frac{\pi}{2}} \tag{40}$$

By applying the methods discussed so far, we can put together integrals including trigonometric functions and Hermite polynomials [23] and solve them in a simple way.

Example 4.

$$\begin{aligned}
&\int_a^x H_n(\xi,y)\cos(\xi)d\xi=-\sum_{s=0}^{n}\cos\left(x+(s+1)\frac{\pi}{2}\right)\frac{n!}{(n-s)!}H_{n-s}(x,y)+\\
&-n!\left|\cos\left((n+1)\frac{\pi}{2}\right)\right|(-1)^{\left\lfloor\frac{n-1}{2}\right\rfloor}e_{\left\lfloor\frac{n-1}{2}\right\rfloor}(-y)\\
&e_m(x)=\sum_{r=0}^{m}\frac{x^r}{r!}\qquad \textit{truncated exponential function}\\
&\int_a^y H_n(x,\eta)\cos(\eta)d\eta=-\sum_{s=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\cos\left(y+(s+1)\frac{\pi}{2}\right)\frac{n!}{(n-2s)!}H_{n-2s}(x,y)+\\
&-n!\,(-1)^{-\frac{n-2}{4}}\,{}_{[4]}e_{n-2}\left((-1)^{\frac{1}{4}}x\right)\\
&{}_{[k]}e_m(x)=\sum_{r=0}^{\left\lfloor\frac{m}{k}\right\rfloor}\frac{x^{m-kr}}{(m-kr)!}\qquad \textit{truncated exponential function of order } k
\end{aligned} \tag{41}$$

5. NEW TRIGONOMETRIES AND THE GIELIS SUPERFORMULA

We wish to close this contribution with a view on new trigonometries and the Gielis Superformula.

Definition 2. *We introduce the composition rule* $\forall x,y\in\mathbb{R},\forall n\in\mathbb{N}$:

$${}_le(y_l\,\partial_x)x^n=\sum_{r=0}^{n}\binom{n}{r}^2x^{n-r}y^r:=(x\oplus_l y)^n \tag{42}$$

called the Laguerre binomial sum.

Eq. (42) is based on the Laguerre exponential [24,25]:

$${}_le(\eta):=\sum_{r=0}^{\infty}\frac{\eta^r}{r!^2}\qquad\forall\eta\in\mathbb{R} \tag{43}$$

in which the argument of the function includes the Laguerre derivative ${}_l\,\partial_x$ and the binomial coefficient has exponent 2 instead of 1 in the ordinary Newton binomial. Then, in full analogy with the ordinary Euler formulae, we introduce the following:

Definition 3. *We introduce l-trigonometric (l-t) functions through the identity:*

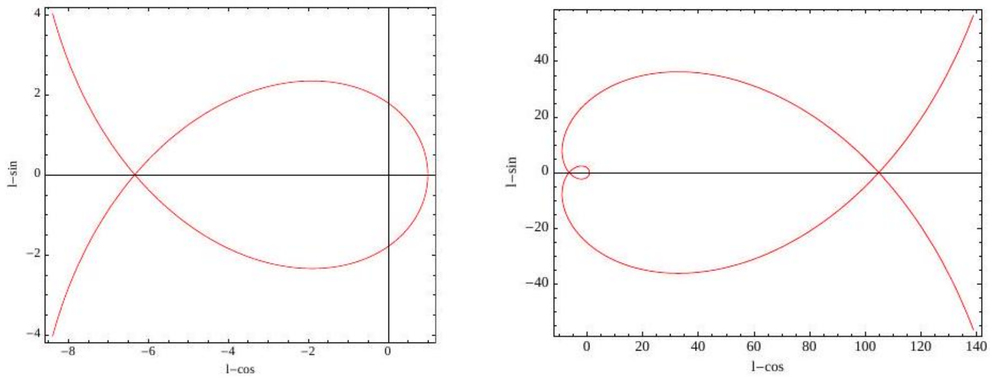
$${}_le(ix)={}_lc(x)+i_ls(x) \tag{44}$$

where l-t cosine and l-t sine functions are specified by the series[6] (Fig. 2):

[6] The l-t functions are defined by the corresponding series expansion ${}_lch\,(x)=\sum_{r=0}^{\infty}\frac{x^{2r}}{[(2r)!]^2}$, ${}_l\mathrm{sh}\,(x)=\sum_{r=0}^{\infty}\frac{x^{2r+1}}{[(2r+1)!]^2}$

$$ {}_{l}c(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{(2r)!^2}, \qquad {}_{l}s(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r+1}}{(2r+1)!^2} \tag{45} $$

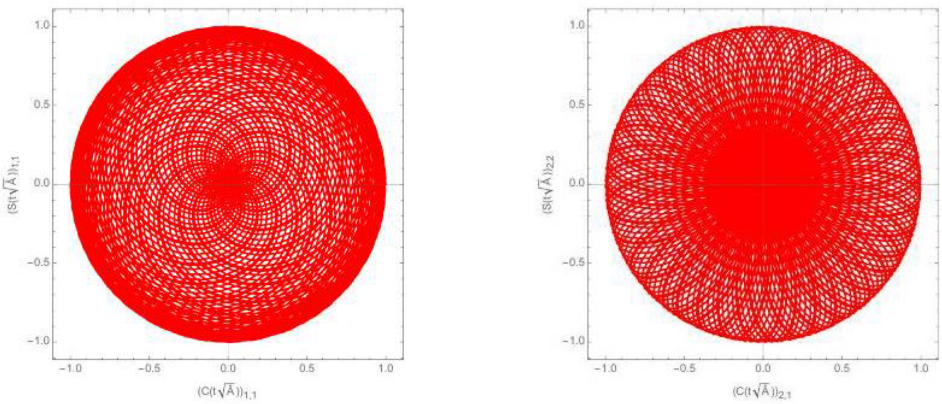
We can also use matrices as arguments of the *l-sin* or *l-cos* functions (like in PDE system problems): see Fig. 3.



(a) Fish-like Lissajous diagram of l-t functions ${}_{l}s(x)$ vs. ${}_{l}c(x)$.

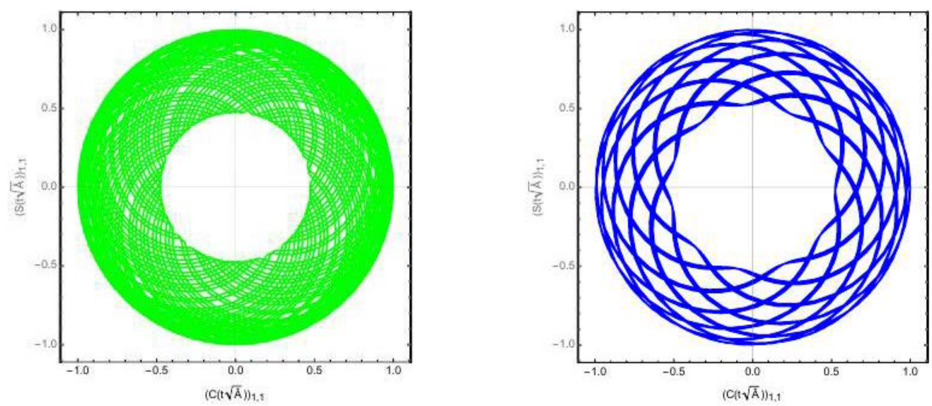
(b) Fish-like Lissajous diagram of l-t functions ${}_{l}s(x)$ vs. $l_{l}c(x)$.

Figure 2. Examples of l-trigonometric functions.



(a) $\hat{A}_1 = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$

(b) $\hat{A}_1 = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$



(c) $\hat{A}_2 = \begin{pmatrix} 0.62 & 0.71 \\ 0.41 & 2 \end{pmatrix}$

(d) $\hat{A}_3 = \begin{pmatrix} 0.71 & 1.62 \\ 0.2 & 1.41 \end{pmatrix}$

Figure 3. Examples of l-trigonometric functions with matrices as arguments: $S(t\sqrt{\hat{A}})$ vs. $C(t\sqrt{\hat{A}})$.

Finally, by using the Superformula of Johan Gielis [26], a wide range of novel shapes can be generated (Fig. 4). $\forall n_1, n_2, n_3, m, a, b \in \mathbb{R}^+, a, b \neq 0, \forall \theta \in \mathbb{R}, r = r(\theta)$ radius:

$$\frac{1}{r} = \sqrt[n_1]{\left|\frac{1}{a}\cos\left(\frac{m}{4}\theta\right)\right|^{n_2} \left|\frac{1}{b}\sin\left(\frac{m}{4}\theta\right)\right|^{n_3}} \tag{46}$$

We can identify $cos \rightarrow {}_{\iota}cos$ and $sin \rightarrow {}_{\iota}sin$ and repeat the procedure as in the previous sections, so yielding "strange" and lovely new figures in 2D and 3D (Fig. 5).

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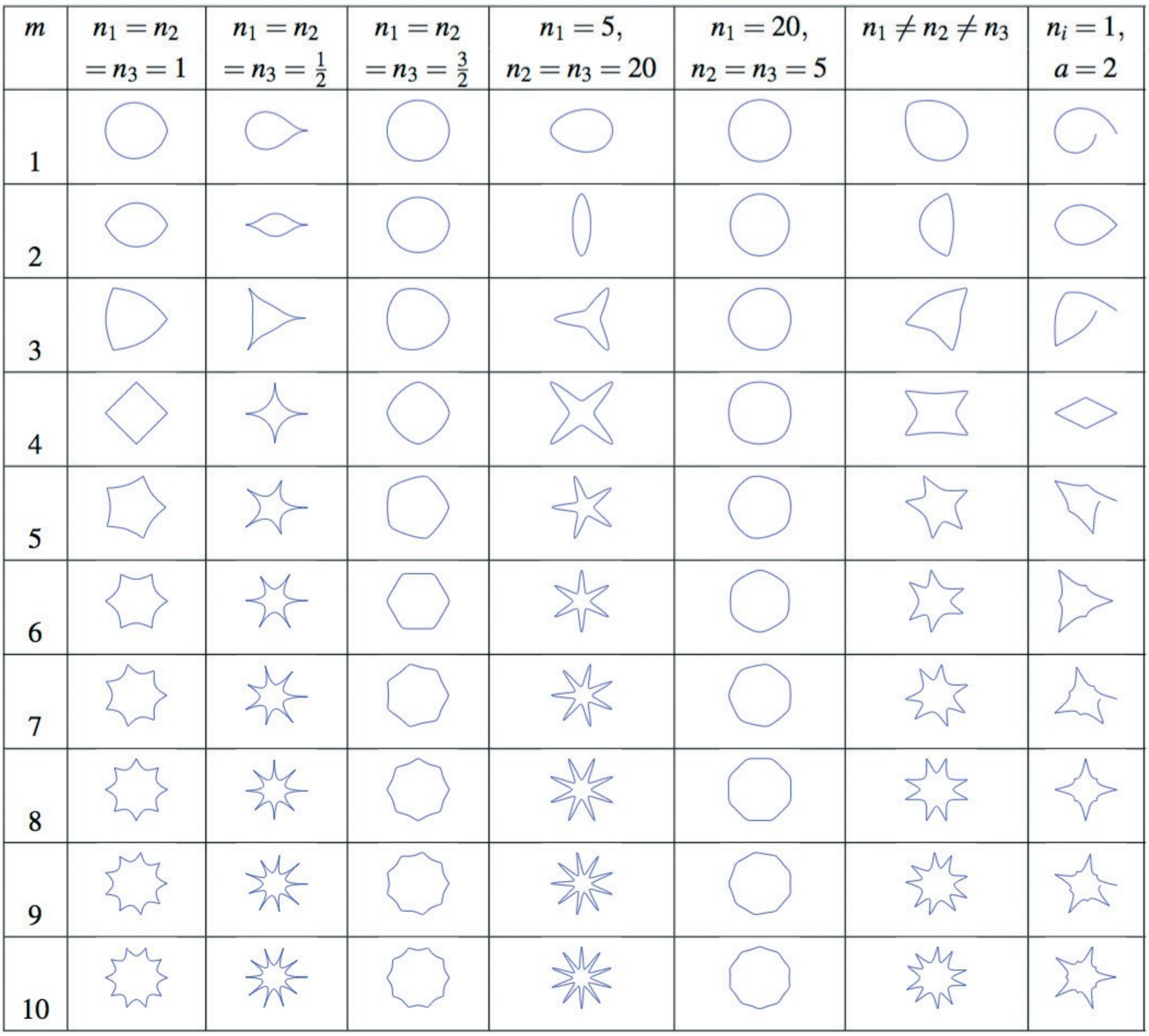

Figure 4. Table of various shapes.

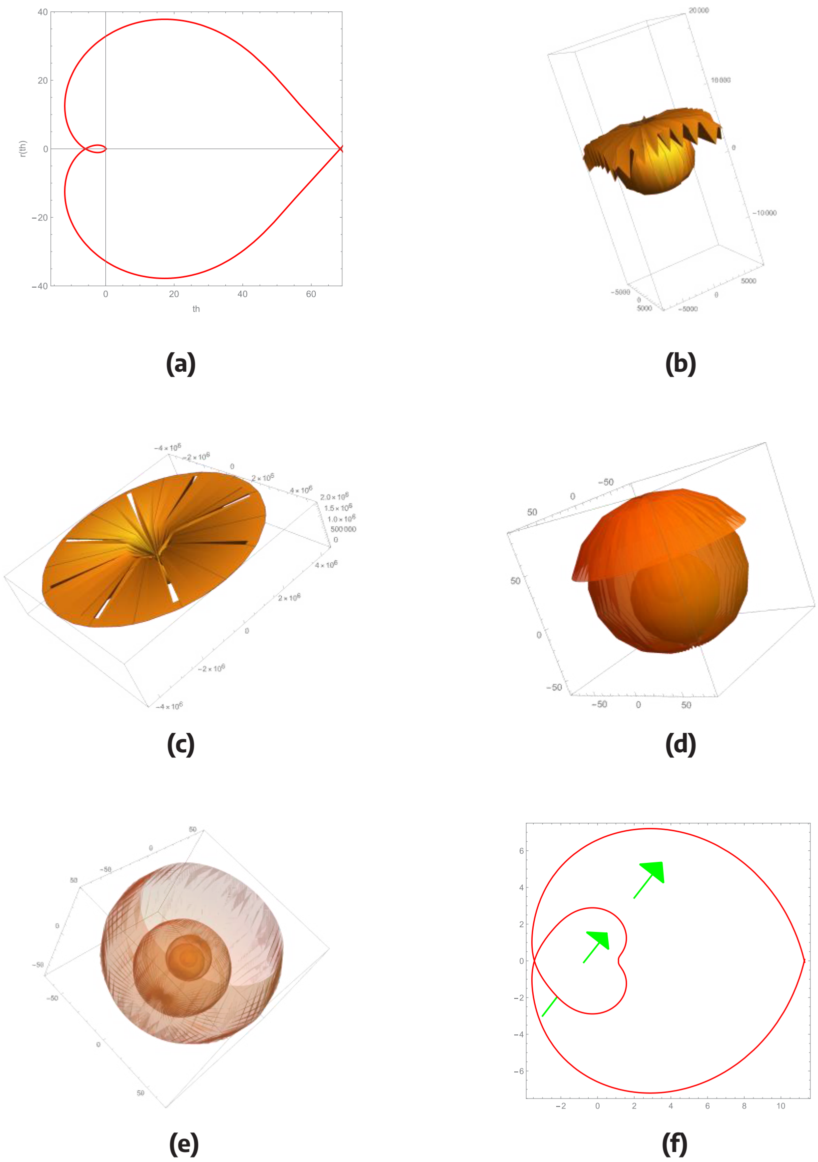

Figure 5. (a) Heart. (b) Strange fruit. (c) Strange leaf. (d) Nut. (e) Tulip. (f) Loving hearts.

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