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Stability and bifurcations in ring's dynamics

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E-mail: sara.diruzza@unipa.it, celletti@mat.uniroma2.it and irene.deblasi@unito.it**Keywords:** ring's dynamics, perturbation theory, KAM tori, stability, bifurcation, corotation resonances, Lindblad resonances**Mathematics Subject Classification numbers:** 70F15, 37N05, 70H08, 34C23

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Abstract

We use perturbation theory and bifurcation theory to analyse the dynamical behaviour, associated to a model describing a particle moving within a ring around a celestial object. The central body is modelled as a homogeneous triaxial ellipsoid, rotating about its shortest physical axis at a constant angular velocity. It is assumed that the massless ring particle moves within the equatorial plane of the ellipsoid. The dynamics of the particle is studied using epicyclic variables, that lead to a straightforward definition of corotation and Lindblad resonances. These resonances are associated to a Hamiltonian function with two degrees of freedom, for which we compute appropriate expansions for the normal form and the resonant Hamiltonian. Initially, the normal form is verified to be multi-scale non-degenerate, thereby guaranteeing the existence of invariant KAM tori, providing the stability of the resonances, through their confinement in phase space. Subsequently, two test cases are examined: a nearly spherical ellipsoid and a highly aspherical ellipsoid. Furthermore, this study concentrates on three principal resonances: corotation, $1 : 2$, and $1 : 3$, for which we present results concerning their dynamical behaviour obtained analysing the Hamiltonian formulation of the model and the resonant normal form. Specifically, we examine the phase space structure, the amplitude of libration around the resonances, and the occurrence of bifurcations. Remarkably, in none of the two studied test cases, the resonance $1 : 3$ presents evidence of bifurcations for relevant values of the eccentricity. Our dynamical study in the current model problem thus supports a higher probability of selecting the resonance $1 : 3$ compared to the other resonances.

1. Introduction

The analytical study of planetary rings is a traditional subject, which has been investigated in great detail in several works; among the others, we quote [3, 12, 21, 32, 35] and we refer the reader to the references therein.

The recent discoveries of rings surrounding small bodies within the Solar System, for example Chariklo, Haumea and Quaoar (see [5, 20, 22, 23, 36]), significantly expanded the field of research in rings dynamics (see, e.g. [11, 19, 30, 31, 33]) and raised interesting questions from the dynamical systems point of view.

There is a fundamental difference between the dynamics of rings around planets and that around small celestial bodies. While planets have an almost spherical shape, small bodies of the Solar System, like dwarf planets or Centaurs, might have a marked irregular shape, whose non-axisymmetric gravity field generates a variety of dynamical behaviours. Furthermore, although space missions have yielded comprehensive insights into the structure of planets, the shapes of certain small celestial bodies continue

to warrant exploration, and fundamental quantities such as the approximate dimensions of these bodies remain largely undisclosed. Consequently, the dynamics of rings around small bodies needs the exploration of various models to accurately approximate their shapes. As frequently found in the literature, a small body can be approximated as a triaxial ellipsoid or alternatively as a sphere with relatively minor topographical features; the latter case is often called the *mass anomaly* model. A prolate ellipsoid and a mass anomaly model have been studied in [18, 27], using Poincaré surfaces of section with the aim to locate the resonances or to estimate the size of regular and chaotic regions. It is noteworthy to highlight that within ring's systems collisional interactions among particles play a crucial role and help to explain the flattening phenomena observed in the rings. This line of inquiry constitutes a substantial domain of research, see e.g. [2, 6, 12, 14, 16, 17, 29, 34].

The model considered in this work is that of a homogeneous body with the shape of a triaxial ellipsoid; we assume that the ellipsoid rotates with constant angular velocity around its shortest physical axis. To account for different shapes, from planets to small bodies, we consider two sample cases: an almost spherical body (with small values of oblateness and elongation) and a highly aspherical body (with high values of oblateness and elongation). We analyse the dynamics of a single particle, that we assume to belong to a ring system around the triaxial body. We constrain the particle to move on the equatorial plane. With these assumptions, we derive the potential in terms of the polar coordinates, namely, the radius and the true longitude. Afterwards, we split the potential into the symmetric and non-symmetric parts, and we introduce the two (classical) frequencies, called synodic frequency (related to the mean motion) and epicyclic frequency. Corotation and Lindblad resonances are defined in terms of these two frequencies. In the corotation case the orbital mean motion of the particle is equal to the frequency of rotation of the central body. A Lindblad resonance is a commensurability relation between the mean motion, the epicyclic frequency, and the frequency of rotation of the central body. Next, we introduce a system of coordinates which is well suited to study the dynamics, namely the epicyclic coordinates. We provide a 2 DOF Hamiltonian function, which is conveniently written in terms of action-angle variables associated to the epicyclic coordinates; in particular, the angle variables have frequency equal to the epicyclic and synodic frequencies. We stress here that the problem we are considering shares some similarities with that of a satellite orbiting the Earth (see, e.g. [8]). However, there are also remarkable differences and precisely: (i) we consider a tiny particle instead of a large object like a spacecraft, (ii) we do not include atmospheric drag as well as the attractions of Sun and Moon (which are essential in satellite theory), (iii) we consider an expansion of the gravitational potential with several coefficients and we do not limit to the zonal terms, (iv) we constrain the particle to move on the equatorial plane, (v) we do not limit our analysis to tesseral resonances (namely, commensurabilities between the mean motion of the satellite and the rotation of the Earth), but we rather consider Lindblad resonances (involving synodic and epicyclic frequencies), which motivates the choice of the epicyclic variables.

Our first task consists in checking a multi-scale non-degeneracy condition to ensure the existence of KAM invariant tori (for more details, see in [13, 25, 26]); in a 2 DOF system, like the one under study, such tori provide stability results on fixed energy levels, since they yield a confinement of the resonant motion, ensuring its stability for infinite times. The need for the multi-scale condition is motivated by the fact that the axisymmetric part of the Hamiltonian is composed by terms which depend on different smallness parameters. The verification of the multi-scale non-degeneracy condition requires a sequence of canonical transformations to eliminate the dependence on the angle up to a suitable order in such small parameters.

Using the averaging method, we proceed to introduce the resonant Hamiltonian, defined as the sum of the normal form (i.e. the Hamiltonian part that does not depend on the angles) and the resonant part (which depends on the resonant—either Lindblad or corotation—combination of the angles). We consider the two sample cases of almost spherical and highly aspherical body, and we compute the resonant Hamiltonian for three main case studies: corotation or 1 : 1 resonance (which turns out to be located very close to the central body), 1 : 3 resonance (which is far from the central body) and 1 : 2 resonance (whose location is between corotation and 1 : 3 resonance). For each resonance and for a constant value of the particle's eccentricity, orbits are shown in the phase space, and the amplitude of libration of the resonance is quantified, with consideration for the oblateness and elongation of the central body. The findings indicate that these shape factors significantly impact the amplitude of libration, thereby establishing a distinct disparity between celestial bodies such as planets, which are nearly spherical, and smaller bodies, which are highly aspherical. Notably, this behaviour is predominantly influenced by the elongation factor.

The dynamical study proceeds with the computation of the equilibrium points, the linear stability analysis, and the occurrence of bifurcations. We found that the 1 : 2 resonance undergoes pitchfork bifurcations as the eccentricity varies. Moreover, the amplitude of libration of the corotation is definitely

larger than those associated with the 1:2 and 1:3 resonances. On the other hand, the 1 : 3 resonance has a small amplitude of libration and does not present bifurcations within the considered range of eccentricity, thus making this resonance more stable than the other resonances.

It is important to note that the model currently under review serves as merely an approximation of the actual physical scenario. In a more precise approach, alternative models might have been employed, for example to more accurately depict the shape of the central body, or to include the gravitational effects of potential satellites of the central body. Regardless, our findings substantiate the conclusion that the central body's potential governs a distinct behaviour of the resonances, independently of any additional factor.

This work is organized as follows. In section 2, we describe the derivation of the potential function. In section 3 we introduce epicyclic variables, define the resonances, and provide the Hamiltonian function and its series expansion. In section 4, we perform a suitable normalization algorithm and discuss the multi-scale non-degeneracy conditions of the resulting Hamiltonian. In section 5, we present the results for the two test cases of an almost spherical and a highly aspherical body, and the three resonances given by the 1 : 1, 1 : 2 and 1 : 3 resonances.

2. Potential of a homogeneous triaxial ellipsoid

Let $\mathcal{E}$ be a body of mass M_p , whose shape is assumed to be given by an ellipsoid with semi-axes $1 \geq a > b > c > 0$. Let O be the centre of mass of the ellipsoid $\mathcal{E}$. Assume that $\mathcal{E}$ rotates around the shortest physical axis with constant angular velocity $\Omega_p = 2\pi/T_{\text{rot}}$, where T_{rot} is the rotation period. Following [33], assume that the mass distribution is symmetric with respect to the equatorial plane and that it has a symmetry plane containing the rotation axis.

Consider a massless particle $\mathcal{P}$ moving in the equatorial plane of the ellipsoid and let (r, L) be its polar coordinates in an inertial frame with centre in O , say (O, x, y, z) (see figure 1); r denotes its distance from O and L is the true longitude. In a quasi-inertial frame rotating with angular velocity Ω_p , we introduce the angle θ defined as $\theta = L - \Omega_p t$. Following [1, 30], it is possible to derive the three-dimensional gravitational potential generated by the above rotating body.

Proposition 1. *The gravitational potential generated by $\mathcal{E}$, moving around its minor axis with angular velocity Ω_p , is given by*

$$U(r, \theta, \phi) = -\frac{\mathcal{G}M_p}{r} \sum_{p=0}^{\infty} \cos(2p\theta) \sum_{\ell=p}^{\infty} \left(\frac{R}{r}\right)^{2\ell} C_{2\ell, 2p} P_{2\ell, 2p}(\sin \phi), \quad (2.1)$$

where ϕ denotes the latitude of the particle, $\mathcal{G}$ is the gravitational constant, R is the reference length, introduced to make the coefficients dimensionless, which we can define as⁴

$$R = \left(\frac{3}{a^{-2} + b^{-2} + c^{-2}} \right)^{\frac{1}{2}}, \quad (2.2)$$

$$P_{\ell, p}(x) = (-1)^p \frac{(1-x^2)^{\frac{p}{2}}}{2^{\ell} \ell!} \frac{d^{\ell+p}}{dx^{\ell+p}} ((x^2-1)^{\ell}),$$

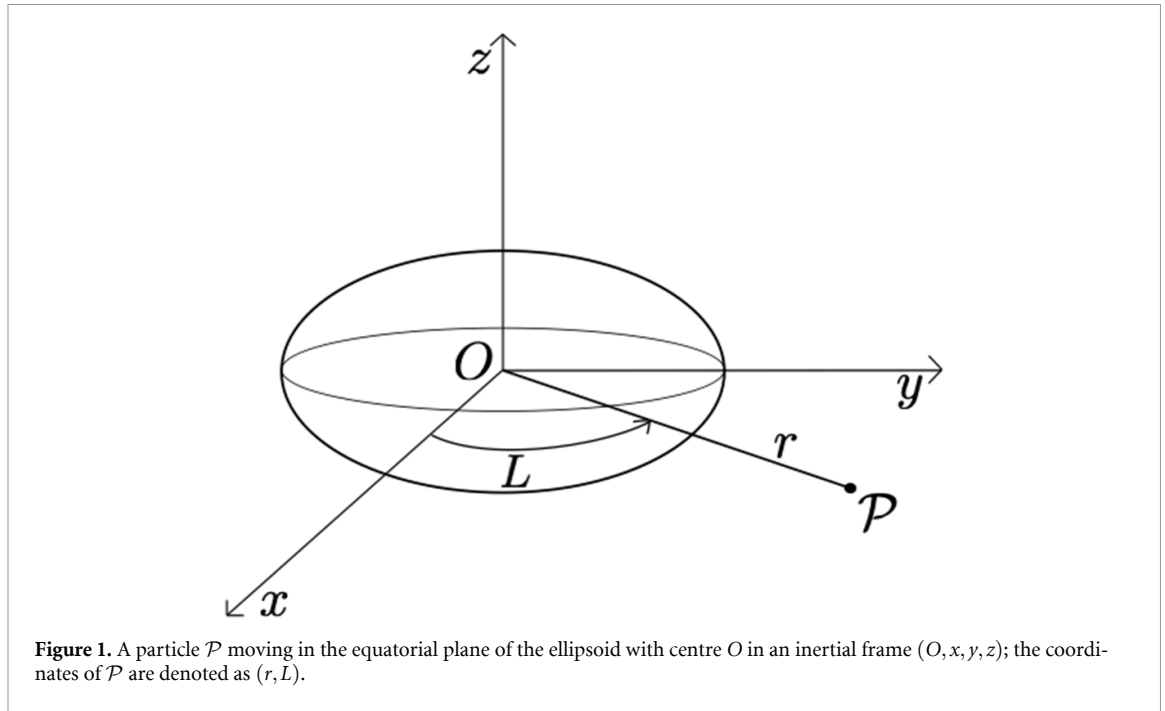
while the coefficients $C_{2\ell, 2p}$ are given by the following expression (compare with [4]):

$$C_{2\ell, 2p} = \frac{3}{R^{2\ell}} \frac{\ell!(2\ell-2p)!}{2^{2p}(2\ell+3)(2\ell+1)!} (2 - \delta_{0,p}) \sum_{k=0}^{\lfloor \frac{\ell-p}{2} \rfloor} \frac{(a^2 - b^2)^{p+2k} (c^2 - \frac{1}{2}(a^2 + b^2))^{\ell-p-2k}}{16^k (\ell-p-2k)! (p+k)! k!}. \quad (2.3)$$

The expansion (2.1) of the gravitational potential is a classical topic of Celestial Mechanics; the proof of proposition 1 can be found, e.g. following the results contained in [1, 30].

⁴ Notice that in [30], the reference length is defined as

$$\frac{3}{R^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$



Definition 2. We introduce the quantities

$$\mathcal{O}_b = \frac{a^2 + b^2 - 2c^2}{4R^2}, \quad (2.4)$$

and

$$\mathcal{E}_\ell = \frac{a^2 - b^2}{2R^2} \quad (2.5)$$

which measure, respectively, the oblateness and the elongation of the body.

We notice that the coefficients $C_{2\ell, 2p}$ depend on oblateness and elongation.

When the particle moves in the equatorial plane, then $\phi = 0$; hence, the potential acting on $\mathcal{P}$ can be written as

$$U(r, \theta) = \sum_{p=0}^{\infty} U_{2p}(r) \cos(2p\theta), \quad (2.6)$$

which contains only even terms, since the ellipsoid is symmetric and the potential is invariant under a rotation of the angle π .

From equation (2.1) with $\phi = 0$, we obtain the following expression for the potential and, hence, for the coefficient $U_{2p}(r)$ in equation (2.6):

$$U(r, \theta) = -\frac{GM_p}{r} \sum_{p=0}^{\infty} \cos(2p\theta) \sum_{\ell=p}^{\infty} \left(\frac{R}{r}\right)^{2\ell} C_{2\ell, 2p} P_{2\ell, 2p}(0), \quad (2.7)$$

where, using the binomial formula for $(x^2 - 1)^\ell$ (see [30]), the associated Legendre polynomials take the form

$$P_{2\ell, 2p}(0) = (-1)^{\ell-p} \frac{(2\ell + 2p)!}{2^{2\ell} (\ell + p)! (\ell - p)!}, \quad \ell \geq p. \quad (2.8)$$

The convergence of the series defining the potential in (2.7) has been discussed in [24] (see also [1]), where it is proven that the series with harmonic coefficients as in (2.3) is uniformly convergent, provided the smaller semi-axis satisfies $c > a/\sqrt{2}$.

3. Epicyclic formulation of the equations of motion

In this section, we introduce the epicyclic formulation of the equations of motion for a ring's particle lying in the equatorial plane, specifying how the resonances are defined in such framework.

Proposition 3. *Let the momenta p_r and p_L , conjugated respectively to r and L , be*

$$p_r = \dot{r}, \quad p_L = r^2 \dot{L}.$$

Then, the Hamiltonian describing the dynamics of the particle is given by

$$\mathcal{H}(p_r, p_L, r, L - \Omega_p t) = \frac{p_r^2}{2} + \frac{p_L^2}{2r^2} + U(r, L - \Omega_p t). \quad (3.1)$$

The proof of $\mathcal{H}(p_r, p_L, r, L - \Omega_p t)$ in equation (3.1) follows from classical arguments of Hamiltonian and Keplerian dynamics, recalling that L is an angular coordinate and that the kinetic energy associated to it is given by $p_L^2/(2r^2)$.

Recalling that $\theta = L - \Omega_p t$ and defining the momentum $p_\theta \equiv p_L$, we obtain the Hamiltonian⁵

$$\mathcal{H}(p_r, p_\theta, r, \theta) = \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} - \Omega_p p_\theta + U(r, \theta). \quad (3.2)$$

The gravitational potential (2.7) can be written as the sum of an axisymmetric part $U_s(r)$ which is obtained taking the term with $p = 0$ in equation (2.7), plus a non-axisymmetric part $U_{ns}(r, \theta)$. The axisymmetric part is given by

$$U_s(r) = -\frac{\mathcal{G}M_p}{r} \sum_{\ell=0}^{\infty} \left(\frac{R}{r}\right)^{2\ell} C_{2\ell,0} P_{2\ell,0}(0). \quad (3.3)$$

We remark that $U_s(r)$ can be written as the sum of the Keplerian potential $U_{Kep}(r) = -\mathcal{G}M_p/r$ (corresponding to $\ell = 0$) and a potential $\tilde{U}_s = \tilde{U}_s(r)$ depending only on the radius. From equation (2.7), the non-axisymmetric part $U_{ns}(r, \theta)$ is given by

$$U_{ns}(r, \theta) = -\frac{\mathcal{G}M_p}{r} \sum_{\ell=1}^{\infty} \left(\frac{R}{r}\right)^{2\ell} \sum_{p=1}^{\ell} \cos(2p\theta) C_{2\ell,2p} P_{2\ell,2p}(0). \quad (3.4)$$

We now introduce two main concepts to describe the motion of the particle around the rotating body $\mathcal{E}$: first of all, we define the principal *frequencies* of the particle's motion, and then we use the frequencies to characterize the *resonances*, representing a key subject of this work. We introduce the *mean motion* n , representing the average angular speed of the particle around the central body, and the *epicyclic frequency* κ , representing the frequency of horizontal oscillation of the particle about its average position. In the spherical case, n and κ reduce to the Keplerian mean motion, namely $n^2(r) = \kappa^2(r) = \mathcal{G}M_p/r^3$ and the epicyclic frequency represents the frequency of oscillation around an unperturbed circular orbit. For an ellipsoidal central body, the two frequencies are introduced as follows.

Definition 4. Given the axisymmetric potential U_s as in equation (3.3), we define the mean motion n and the epicyclic frequency κ as (see [10])

$$n(r) = \left(\frac{1}{r} \frac{dU_s(r)}{dr}\right)^{\frac{1}{2}}, \quad (3.5)$$

and

$$\kappa(r) = \left(\frac{3}{r} \frac{dU_s(r)}{dr} + \frac{d^2 U_s(r)}{dr^2}\right)^{\frac{1}{2}}. \quad (3.6)$$

We define the synodic frequency of the particle as $n - \Omega_p$.

⁵ In fact, we have

$$\dot{\theta} = \frac{\partial \mathcal{H}}{\partial p_\theta} = \frac{p_\theta}{r^2} - \Omega_p = \frac{p_L}{r^2} - \Omega_p = \dot{L} - \Omega_p.$$

Notice that the synodic frequency denotes the frequency of the particle's passages through a fixed point relative to the central body and that the square of the quantity in equation (3.6) can be written also as

$$\kappa^2(r) = \frac{1}{r^3} \frac{d(r^4 n^2(r))}{dr}.$$

Expanding U_s as in equation (3.3), we can write the squares of the frequencies as

$$\begin{aligned} n^2(r) &= \frac{\mathcal{G}M_P}{r^3} \left(1 + \sum_{\ell=1}^{\infty} (2\ell+1) \left(\frac{R}{r} \right)^{2\ell} C_{2\ell,0} P_{2\ell,0}(0) \right) \\ &= \frac{\mathcal{G}M_P}{r^3} \left(1 - \frac{3}{2} C_{2,0} \left(\frac{R}{r} \right)^2 + \frac{15}{8} C_{4,0} \left(\frac{R}{r} \right)^4 + \dots \right) \\ \kappa^2(r) &= \frac{\mathcal{G}M_P}{r^3} \left(1 - \sum_{\ell=1}^{\infty} (2\ell+1) \left(\frac{R}{r} \right)^{2\ell} C_{2\ell,0} P_{2\ell,0}(0) \right) \\ &= \frac{\mathcal{G}M_P}{r^3} \left(1 + \frac{3}{2} C_{2,0} \left(\frac{R}{r} \right)^2 - \frac{45}{8} C_{4,0} \left(\frac{R}{r} \right)^4 + \dots \right). \end{aligned} \quad (3.7)$$

Definition 5. We introduce a Lindblad resonance for $j, m \in \mathbb{Z} \setminus \{0\}$, whenever the following condition is satisfied:

$$j\kappa(r) = m(n(r) - \Omega_P). \quad (3.8)$$

For given j, m, Ω_P , we define the Lindblad resonant radius as the quantity $r_{\text{lin}}(\Omega_P; m, j)$ which solves equation (3.8) (see also section 5.2 for the computation of the resonant radius).

Definition 6. We define a corotation resonance whenever, in equation (3.8), $j = 0$ (that is, $n(r) = \Omega_P$). In such case, the corresponding solution is called corotation radius and is denoted with $r_{\text{cor}}(\Omega_P)$.

Remark 7. Using $\kappa = n - \dot{\omega}$, for a Lindblad resonance (3.8) we have

$$\frac{n - \dot{\omega}}{\Omega_P - \dot{\omega}} = \frac{m}{m - j},$$

meaning that in a frame rotating at the particle precession rate $\dot{\omega}$, the particle completes m revolutions, while the central body completes $m - j$ rotations. For this reason, we will also refer to this kind of resonance as a $m : (m - j)$ Lindblad resonance.

Since our final aim is to study the local dynamics of the ring's particle in the resonant regime, it is convenient to introduce a new set of coordinates, centred at the resonant radius, which highlights the role of the synodic and epicyclic frequencies.

Proposition 8. Let r_* denote the value of the radius at the resonance, either r_{lin} or r_{cor} , and define $\kappa_* = \kappa(r_*)$ and $n_* = n(r_*)$. Then, there exists a set of canonical action–angle coordinates (J, I, φ, θ) , such that the particle's Hamiltonian, locally around r_* , can be expressed as

$$\begin{aligned} \mathcal{H}(J, I, \varphi, \theta) &= |\kappa_*| J + (n_* - \Omega_P) I \\ &\quad + \sum_{|i|=0}^{\infty} \sum_{j=0}^{\infty} (\alpha_{2i,j}(J, I) \cos(2i\theta + j\varphi) + \beta_{2i,j}(J, I) \sin(2i\theta + j\varphi)), \end{aligned} \quad (3.9)$$

where $\alpha_{2i,j}$ and $\beta_{2i,j}$ are suitable polynomials⁶ in $I, \sqrt{J}$.

Proof. In a neighborhood of the resonance, we define the distance ρ from the resonant radius r_* as

$$\rho = r - r_*. \quad (3.10)$$

Next, we introduce the angular momentum p_* at the resonant radius per unit mass in an orbit with axisymmetrical potential as

$$p_* = n_* r_*^2, \quad (3.11)$$

⁶ We remark that, although the Hamiltonian is not analytic at $J = 0$, for convenience, we keep the action-angle notation.

where $n_* = n(r_*)$. Let us introduce the momenta p_ρ (conjugated to ρ) and I (as the displacement from p_*) as

$$p_\rho = p_r, \quad I = p_\theta - p_* . \quad (3.12)$$

Expressing the Hamiltonian (3.2) in the translated variables (ρ, p_ρ) , (θ, I) , one gets

$$\mathcal{H}(p_\rho, I, \rho, \theta) = \frac{p_\rho^2}{2} + \frac{(I + p_*)^2}{2(r_* + \rho)^2} - (I + p_*)\Omega_p + U_s(\rho + r_*) + U_{ns}(\rho + r_*, \theta) . \quad (3.13)$$

We observe that, as stated in appendix B.1, proposition 21, the non-axisymmetric part of the potential is of the order of the quantity $\mathcal{O}_b(R^2/r_*^2)$, where the oblateness $\mathcal{O}_b$ is defined in (2.4). This motivates the introduction of a dimensionless parameter λ as

$$\lambda = \mathcal{O}_b \left(\frac{R^2}{r_*^2} \right)$$

and, assuming $\mathcal{E}_i \leq \mathcal{O}_b$ as it will be done in appendix B.1, the rescaling of U_{ns} as

$$U_{ns}(\rho + r_*, \theta) = \lambda \tilde{U}_{ns}(\rho + r_*, \theta) ,$$

where $\tilde{U}_{ns}(\rho + r_*, \theta)$ is a bounded function. Hence, we rewrite (3.13) as

$$\mathcal{H}(p_\rho, I, \rho, \theta) = \frac{p_\rho^2}{2} + \frac{(I + p_*)^2}{2(r_* + \rho)^2} - (I + p_*)\Omega_p + U_s(\rho + r_*) + \lambda \tilde{U}_{ns}(\rho + r_*, \theta) . \quad (3.14)$$

Let us now take the Taylor expansion of $(I + p_*)^2/(2(\rho + r_*)^2)$ and of the axisymmetric potential $U_s(\rho + r_*)$ around $I = 0$ and $\rho = 0$; ignoring constant terms and grouping together terms of the same degree, the Hamiltonian (3.14) takes the form

$$\begin{aligned} \mathcal{H}(p_\rho, I, \rho, \theta) = & \frac{p_\rho^2}{2} + \left[\frac{p_*}{r_*^2} - \Omega_p \right] I + \frac{I^2}{2r_*^2} + \left[\frac{dU_s}{dr}(r_*) - \frac{p_*^2}{r_*^3} \right] \rho - 2\frac{p_*}{r_*^3} I\rho - \frac{1}{r_*^3} I^2 \rho \\ & + \frac{1}{2} \left[\frac{d^2 U_s}{dr^2}(r_*) + 3\frac{p_*^2}{r_*^4} \right] \rho^2 + \frac{3p_*}{r_*^4} I\rho^2 + \frac{3}{2r_*^4} I^2 \rho^2 + G_s(I, \rho) + \lambda \tilde{U}_{ns}(\rho + r_*, \theta) , \end{aligned} \quad (3.15)$$

where $G_s(I, \rho)$ is a polynomial in ρ and I of order at least three in ρ and with terms of order 0, 1 and 2 in I .

Using equation (3.11), recalling equation (3.5), and omitting constant terms, the Hamiltonian (3.15) can be written as

$$\begin{aligned} \mathcal{H}(p_\rho, I, \rho, \theta) = & \frac{p_\rho^2}{2} + (n_* - \Omega_p)I + \frac{I^2}{2r_*^2} - 2\frac{n_*}{r_*} I\rho - \frac{1}{r_*^3} I^2 \rho \\ & + \frac{1}{2} \left(\frac{d^2 U_s}{dr^2}(r_*) + \frac{3}{r_*} \frac{dU_s}{dr}(r_*) \right) \rho^2 + \frac{3n_*}{r_*^2} I\rho^2 + \frac{3}{2r_*^4} I^2 \rho^2 \\ & + G_s(I, \rho) + \lambda \tilde{U}_{ns}(\rho + r_*, \theta) . \end{aligned}$$

Keeping the explicit form of just the terms of second degree and putting together all terms of total degree at least three, one arrives at the expression

$$\begin{aligned} \mathcal{H}(p_\rho, I, \rho, \theta) = & \frac{p_\rho^2}{2} + (n_* - \Omega_p)I + \frac{1}{2} \left(\frac{d^2 U_s}{dr^2}(r_*) + \frac{3}{r_*} \frac{dU_s}{dr}(r_*) \right) \rho^2 \\ & - \frac{2n_*}{r_*} I\rho + \frac{I^2}{2r_*^2} + F_s(I, \rho) + \lambda F_{ns}(\rho, \theta) , \end{aligned} \quad (3.16)$$

where F_s is given by

$$F_s(I, \rho) = \sum_{i=0}^2 \sum_{j=3-i}^{\infty} c_{i,j} I^i \rho^j , \quad (3.17)$$

which is a polynomial in I and ρ with total degree greater or equal than 3, where I varies from zero to second degree and $c_{i,j}$ are suitable coefficients depending on $C_{i,j}$ (in appendix A, we report the coefficients

$c_{i,j}$ for $i = 0, 1, 2$ and for j up to 4). The function F_{ns} is a polynomial in ρ of any degree and trigonometric in $\cos(2p\theta)$; it is the contribution of the non-axisymmetric part given by U_{ns} , namely, $F_{ns}(\rho, \theta) = \tilde{U}_{ns}(\rho + r_*, \theta)$, and it can be written as

$$\lambda F_{ns}(\rho, \theta) = \sum_{j=0}^{\infty} \sum_{i=1}^{\infty} V_{j,i} \rho^j \cos(2i\theta) \quad (3.18)$$

with $V_{j,i}$ suitable coefficients (in appendix A, we report the coefficients $V_{j,i}$ multiplied by λ , for j up to 4 and for i up to 3).

Let $\kappa_* = \kappa(r_*)$; we note that the coefficient of ρ^2 in equation (3.16) turns out to be (see equation (3.6))

$$\frac{d^2 U_s}{dr^2}(r_*) + \frac{3}{r_*} \frac{dU_s}{dr}(r_*) = \kappa_*^2.$$

Thus, the Hamiltonian (3.16) becomes

$$\mathcal{H}(p_\rho, I, \rho, \theta) = \frac{p_\rho^2}{2} + \frac{1}{2} \kappa_*^2 \rho^2 + (n_* - \Omega_P)I - \frac{2n_*}{r_*} I \rho + \frac{I^2}{2r_*^2} + F_s(I, \rho) + \lambda F_{ns}(\rho, \theta). \quad (3.19)$$

Let us introduce the epicyclic action-angle variables (J, φ) related to (ρ, p_ρ) by the canonical transformations:

$$\begin{cases} \rho = \sqrt{\frac{2J}{|\kappa_*|}} \sin \varphi \\ p_\rho = \sqrt{2|\kappa_*|J} \cos \varphi. \end{cases} \quad (3.20)$$

Then, the Hamiltonian (3.19) can be written in terms of the epicyclic variables as

$$\mathcal{H}(J, I, \varphi, \theta) = |\kappa_*|J + (n_* - \Omega_P)I + \tilde{F}_s(J, I, \varphi) + \lambda \tilde{F}_{ns}(J, \varphi, \theta), \quad (3.21)$$

where $\tilde{F}_s$ is due to the axisymmetrical part of the potential and can be written as

$$\begin{aligned} \tilde{F}_s(J, I, \varphi) &= -2 \frac{n_*}{r_*} I \sqrt{\frac{2J}{|\kappa_*|}} \sin \varphi + \frac{I^2}{2r_*^2} + F_s \left(I, \sqrt{\frac{2J}{|\kappa_*|}} \sin \varphi \right) \\ &= -2 \frac{n_*}{r_*} I \sqrt{\frac{2J}{|\kappa_*|}} \sin \varphi + \frac{I^2}{2r_*^2} + \sum_{i=0}^2 \sum_{j=3-i}^{\infty} d_{i,j} I^i J^{j/2} \sin^j \varphi, \end{aligned} \quad (3.22)$$

where

$$d_{i,j} = c_{i,j} \left(\frac{2}{|\kappa_*|} \right)^{j/2}, \quad (3.23)$$

while $\tilde{F}_{ns}$ is due to the non-axisymmetrical part of the potential F_{ns} and can be written as

$$\begin{aligned} \tilde{F}_{ns}(J, \varphi, \theta) &= \tilde{U}_{ns} \left(\sqrt{\frac{2J}{|\kappa_*|}} \sin \varphi + r_*, \theta \right) \\ &= \sum_{j=0}^{\infty} \sum_{i=1}^{\infty} \tilde{V}_{j,i} J^{j/2} \cos(2i\theta) \sin^i \varphi, \end{aligned} \quad (3.24)$$

where

$$\tilde{V}_{j,i} = V_{j,i} \left(\frac{2}{|\kappa_*|} \right)^{j/2}.$$

Observe that, in view of proposition 21 in appendix B.1, the function $\tilde{F}_{ns}$ is bounded. Let us rewrite the sum of equation (3.22) and of equation (3.24) as a linear combination of the angles θ, φ :

$$\tilde{F}_s(J, I, \varphi) + \lambda \tilde{F}_{ns}(J, \varphi, \theta) = \sum_{|i|=0}^{\infty} \sum_{j=0}^{\infty} (\alpha_{2i,j} \cos(2i\theta + j\varphi) + \beta_{2i,j} \sin(2i\theta + j\varphi)), \quad (3.25)$$

where $\alpha_{2i,j}, \beta_{2i,j}$ are polynomial functions in I, J depending on $\mathcal{G}, M_P$ and on the coefficients $P_{2\ell,2p}, C_{2\ell,2p}$ (in appendix A, we report the coefficients $\alpha_{2i,j}, \beta_{2i,j}$ for $|i|$ up to 3 and j from 0 to 4). Once again, we remark that the function (3.25) depends on multiples of 2θ , due to the symmetry of the central body $\mathcal{E}$, which is assumed to be an ellipsoid. Casting together (3.21), (3.24), (3.25), we obtain (3.9). $\square$

As a consequence of the definition of the epicyclic action–angle variables (J, φ) given in equations (3.20), it is possible to relate the action J to the eccentricity of the orbit of the particle revolving about the central body. Thus, we have the following result.

Proposition 9. *Let r_* denote the value of the radius at the resonance and define $\kappa_* = \kappa(r_*)$. Then, locally around r_* , the action variable J can be expressed as*

$$J = \frac{1}{2} |\kappa_*| r_*^2 e^2,$$

where e is the eccentricity of the orbit of the particle $\mathcal{P}$ around $\mathcal{E}$.

Proof. From Kepler's relations, taking as semimajor axis $a = r_*$, we have $r = r_*(1 - e \cos u)$, where u is the eccentric anomaly. Recalling that $\rho = r - r_*$ and using the first of equations (3.20), we obtain

$$\sqrt{\frac{2J}{|\kappa_*|}} \sin \varphi = -r_* e \cos u.$$

From the last relation, we can set $u = \varphi + \pi/2$ and $\sqrt{\frac{2J}{|\kappa_*|}} = r_* e$, which implies

$$J = \frac{1}{2} |\kappa_*| r_*^2 e^2.$$

$\square$

4. Multi-scale KAM non-degeneracy

Motivated by the fact that the Hamiltonian (3.21) is composed by parts with different orders of magnitude (see (4.9) below), in this section we check that after suitable normalizations the Hamiltonian satisfies the multi-scale KAM non-degeneracy condition.

4.1. Definitions of multi-scale KAM non-degeneracy condition

In the following, we provide two definitions of the multi-scale condition. For the first one we refer to [13], while for the second to [25] (see also [26]).

Definition 10. (From [13]) Let us consider a real analytic nearly-integrable Hamiltonian with N DOF of the form

$$\mathcal{H}(y, x; \epsilon) = h_0(y^{n_0}) + \epsilon^{m_1} h_1(y^{n_1}) + \dots + \epsilon^{m_a} h_a(y^{n_a}) + \epsilon^{m_a+1} p(y, x; \epsilon),$$

where $(y, x) \in \mathbb{R}^N \times \mathbb{T}^N$, $\epsilon > 0$ sufficiently small, a positive integer, $m_1, \dots, m_a$ positive integers such that $m_1 \leq \dots \leq m_a$, $n_0, \dots, n_a$ positive integers such that $n_0 \leq \dots \leq n_a = N$, and $y^{n_i} = (y_1, \dots, y_{n_i})$ for $i = 1, \dots, a$. Let us define

$$\Omega(y) = (\nabla_{y^{n_0}} h_0(y^{n_0}), \dots, \nabla_{y^{n_a}} h_a(y^{n_a})).$$

The angle-independent part of $\mathcal{H}$ is said to satisfy the *multi-scale KAM non-degeneracy condition* in an open set of the actions $G \subset \mathbb{R}^N$ (Condition (A) in [13]), if there exists a positive integer M such that

$$\text{Rank} \left\{ \partial_y^\alpha \Omega(y) : 0 \leq |\alpha| \leq M \right\} = N, \quad \forall y \in G. \quad (4.1)$$

Definition 11. (From [25]) Let us consider a real analytic nearly-integrable Hamiltonian with N DOF of the form

$$\mathcal{H}(y, x) = \epsilon_0 h_0(y) + \epsilon_1 h_1(y) + \dots + \epsilon_m h_m(y) + \epsilon^2 P(y, x),$$

where $(y, x) \in G \times \mathbb{T}^N$ with $G \subset \mathbb{R}^N$ a bounded closed region, ϵ_i and ϵ are parameters such that $0 < \epsilon < \min\{\epsilon_0, \dots, \epsilon_m\}$, $|\epsilon_0| \leq 1$, $|\epsilon_i| \ll 1$ for $i = 1, \dots, m$. The angle-independent part of $\mathcal{H}$ is said to satisfy the

multi-scale KAM non-degeneracy condition (Condition (A'') of [25]), if there exists a positive integer M such that

$$\text{Rank} \left\{ \partial_y^\alpha \Omega(y) : 0 \leq |\alpha| \leq M, \forall y \in G \right\} = N \quad \text{for} \quad |\epsilon_i| > 0, 0 \leq i \leq m, \quad (4.2)$$

where

$$\Omega(y) = \epsilon_0 \partial_y h_0 + \cdots + \epsilon_m \partial_y h_m.$$

For ϵ sufficiently small, conditions (4.1) or (4.2) ensure the existence of a Cantor set of invariant tori for the complete Hamiltonian, which are small deformations of the invariant curves of the integrable dynamics induced by the full Hamiltonian (see [13, 25]).

4.2. Expansion of the axisymmetric part

In this section, we define the axisymmetric Hamiltonian, we expand it in powers of I and $\sqrt{J}$, and then we express the Hamiltonian in terms of rescaled variables.

Remark 12. For the following discussion, it is important to note that the Hamiltonian (3.21) depends on several small parameters, precisely:

- (1) it is typically assumed that the oblateness ($\mathcal{O}_b$) and the elongation ($\mathcal{E}_\ell$), respectively introduced in (2.4) and (2.5), are small;
- (2) the distance r of the particle from the centre of the ellipsoid is greater than the reference length R introduced in (2.2), thus the ratio R/r is small;
- (3) close to the resonance, the quantity ρ in (3.10) is small.

This remark leads us to the conclusion that the norm of the axisymmetric potential U_s in (3.3) is typically larger than the norm of the non-axisymmetric part U_{ns} in (3.4). Evidence of this statement, along with rigorous estimates, is given in appendix B, and in particular in proposition 21 and corollary 22. Later, we will specify under which conditions the non-axisymmetric part $\lambda \tilde{F}_{ns}$ in (3.24) can be considered as a perturbation of $\tilde{F}_s$ in (3.22).

Definition 13. With reference to the Hamiltonian (3.21), and to equation (3.22), we define the axisymmetric Hamiltonian as

$$\mathcal{H}^{\text{axi}}(J, I, \varphi) = |\kappa_*| J + (n_* - \Omega_p) I + \tilde{F}_s(J, I, \varphi),$$

which can be expanded in powers of I and $\sqrt{J}$ as

$$\mathcal{H}^{\text{axi}}(J, I, \varphi) = |\kappa_*| J + (n_* - \Omega_p) I - 2 \frac{n_*}{r_*} I \sqrt{\frac{2J}{|\kappa_*|}} \sin \varphi + \frac{I^2}{2r_*^2} + \sum_{j_1=0}^2 \sum_{j_2=3-j_1}^{\infty} d_{j_1, j_2} I^{j_1} J^{j_2/2} \sin^{j_2} \varphi \quad (4.3)$$

with d_{j_1, j_2} as in (3.23).

It is important to note that from (3.20), the variable $\sqrt{J}$ is of the order of the parameter ρ , which represents the distance of the reference point from the exact resonance. Moreover, expanding I in (3.12) around $\rho = 0$, one finds

$$I = n(r_* + \rho)(r_* + \rho)^2 - n_* r_*^2 = [2n_* r_* + n'(r_*) r_*^2] \rho + O(\rho^2).$$

We will order the terms of the expansion (4.3) on the basis of the total power in the action variables as well as their dependence on $1/r_*$. We start by writing the Hamiltonian (4.3) as

$$\mathcal{H}^{\text{axi}}(J, I, \varphi) = \sum_{j=2}^{+\infty} \mathcal{H}_j(J, I, \varphi),$$

where $\mathcal{H}_j$ are trigonometric polynomials in the product $I^{j_1} J^{j_2/2}$ with $j = 2j_1 + j_2$. In particular, we have:

$$\begin{aligned} \mathcal{H}_2(J) &= (n_* - \Omega_p) I + |\kappa_*| J, \\ \mathcal{H}_3(J, I, \varphi) &= -2 \sqrt{\frac{2}{|\kappa_*|}} \frac{n_*}{r_*} I J^{1/2} \sin \varphi + d_{0,3} J^{3/2} \sin^3 \varphi \end{aligned}$$

$$\begin{aligned}
&= \left(-2\frac{n_*}{r_*}\sqrt{\frac{2}{|\kappa_*|}}I^{1/2} + \frac{3}{4}d_{0,3}J^{3/2}\right)\sin\varphi - \frac{1}{4}d_{0,3}J^{3/2}\sin(3\varphi), \\
\mathcal{H}_4(J, I, \varphi) &= \frac{I^2}{2r_*^2} + d_{1,2}IJ\sin^2\varphi + d_{0,4}J^2\sin^4\varphi \\
&= \frac{1}{2r_*^2}I^2 + \frac{1}{2}d_{1,2}IJ + \frac{3}{8}d_{0,4}J^2 \\
&\quad - \left(\frac{1}{2}d_{1,2}IJ + \frac{1}{2}d_{0,4}J^2\right)\cos(2\varphi) + \frac{1}{8}d_{0,4}J^2\cos(4\varphi), \\
\mathcal{H}_5(J, I, \varphi) &= d_{2,1}I^2J^{1/2}\sin\varphi + d_{1,3}IJ^{3/2}\sin^3\varphi \\
&= (d_{2,1}I^2J^{1/2} + \frac{3}{4}d_{1,3}IJ^{3/2})\sin\varphi - \frac{1}{4}d_{1,3}IJ^{3/2}\sin(3\varphi),
\end{aligned}$$

where $d_{i,j}$ are defined as

$$d_{1,2} = \frac{6n_*}{|\kappa_*| r_*^2}, \quad d_{2,1} = -\frac{1}{r_*^3}\sqrt{\frac{2}{|\kappa_*|}}, \quad d_{1,3} = -\frac{8\sqrt{2}n_*}{|\kappa_*|^{3/2}r_*^3} \quad (4.4)$$

while, keeping only the terms depending on powers $1/r_*^\ell$ with $0 < \ell \leq 5$,

$$d_{0,3} = -\frac{2\sqrt{2}\mathcal{G}M_P}{|\kappa_*|^{3/2}r_*^4}, \quad d_{0,4} = \frac{6\mathcal{G}M_P}{|\kappa_*|^2r_*^5}. \quad (4.5)$$

From (3.7), (4.4) and (4.5), we observe that the quantities n_* , κ_* , $d_{i,j}$ depend on different powers of the inverse of r_* . Together with the dependence of I, J on ρ , this motivates the following rescalings:

$$\begin{aligned}
J &= \eta^2 \bar{J}, \quad I = \eta \bar{I}, \quad n_* = \frac{\bar{n}_*}{r_*^{3/2}}, \quad \kappa_* = \frac{\bar{\kappa}_*}{r_*^{3/2}}, \\
d_{0,3} &= \frac{\bar{d}_{0,3}}{r_*^{7/4}}, \quad d_{1,2} = \frac{\bar{d}_{1,2}}{r_*^2}, \quad d_{0,4} = \frac{\bar{d}_{0,4}}{r_*^2}, \quad d_{2,1} = \frac{\bar{d}_{2,1}}{r_*^{9/4}}, \quad d_{1,3} = \frac{\bar{d}_{1,3}}{r_*^{9/4}},
\end{aligned} \quad (4.6)$$

where $\eta > 0$ is a small parameter denoting the maximal distance to the resonance. In this way, all the barred quantities are of order one. In the rescaled variables, the Hamiltonian $\mathcal{H}^{\text{axi}}$ takes the form

$$\mathcal{H}^{\text{axi}}(\bar{J}, \bar{I}, \varphi) = \sum_{j=2}^{+\infty} \mathcal{H}_j(\bar{J}, \bar{I}, \varphi) \quad (4.7)$$

with

$$\begin{aligned}
\mathcal{H}_2(\bar{J}) &= \frac{\eta}{r_*^{3/2}} \left(\bar{n}_* - r_*^{3/2} \Omega_P \right) \bar{I} + \frac{\eta^2}{r_*^{3/2}} |\bar{\kappa}_*| \bar{J}, \\
\mathcal{H}_3(\bar{J}, \bar{I}, \varphi) &= -\frac{\eta^2}{r_*^{7/4}} \frac{2\sqrt{2}\bar{n}_*}{|\bar{\kappa}_*|^{1/2}} \bar{I} \bar{J}^{1/2} \sin\varphi \\
&\quad + \frac{\eta^3}{r_*^{7/4}} \left[\frac{3}{4} \bar{d}_{0,3} \sin\varphi - \frac{1}{4} \bar{d}_{0,3} \sin(3\varphi) \right] \bar{J}^{3/2}, \\
\mathcal{H}_4(\bar{J}, \bar{I}, \varphi) &= \frac{\eta^2}{r_*^2} \frac{\bar{I}^2}{2} + \frac{\eta^3}{r_*^2} \frac{3\bar{n}_*}{|\bar{\kappa}_*|} (1 - \cos(2\varphi)) \bar{I} \bar{J} \\
&\quad + \frac{\eta^4}{r_*^2} \frac{\bar{d}_{0,4}}{2} \left[\frac{3}{4} - \cos(2\varphi) + \frac{1}{4} \cos(4\varphi) \right] \bar{J}^2, \\
\mathcal{H}_5(\bar{J}, \bar{I}, \varphi) &= -\frac{\eta^3}{r_*^{9/4}} \sqrt{\frac{2}{|\bar{\kappa}_*|}} \bar{I}^2 \bar{J}^{1/2} \sin\varphi \\
&\quad + \frac{\eta^4}{r_*^{9/4}} \left[-\frac{6\sqrt{2}\bar{n}_*}{|\bar{\kappa}_*|^{3/2}} \sin\varphi + \frac{2\sqrt{2}\bar{n}_*}{|\bar{\kappa}_*|^{3/2}} \sin(3\varphi) \right] \bar{I} \bar{J}^{3/2},
\end{aligned} \quad (4.8)$$

while $\mathcal{H}_j(\bar{J}, \bar{I}, \varphi)$ is of order greater than or equal to $\eta^4/r_*^{5/2}$ for $j \geq 6$. In section 4.3, we will use a normalization algorithm to remove the dependence in $\mathcal{H}^{\text{axi}}$ on the angle φ up to terms of order $\eta^3/r_*^{5/2}$. To this

end, it is useful to split $\mathcal{H}_3, \mathcal{H}_4$ and $\mathcal{H}_5$ in different parts, according to the dependence on η and r_* :

$$\begin{aligned}\mathcal{H}_3 &= \mathcal{H}_3^{(1)} + \mathcal{H}_3^{(2)}, \\ \mathcal{H}_4 &= \mathcal{H}_4^{(1)} + \mathcal{H}_4^{(2)} + \mathcal{H}_4^{(3)}, \\ \mathcal{H}_5 &= \mathcal{H}_5^{(1)} + \mathcal{H}_5^{(2)},\end{aligned}$$

with

$$\begin{aligned}\mathcal{H}_3^{(1)} &= -\frac{\eta^2}{r_*^{7/4}} \frac{2\sqrt{2}\bar{n}_*}{|\bar{\kappa}_*|^{1/2}} \bar{I} \bar{J}^{1/2} \sin \varphi, \\ \mathcal{H}_3^{(2)} &= \frac{\eta^3}{r_*^{7/4}} \left[\frac{3}{4} \bar{d}_{0,3} \sin \varphi - \frac{1}{4} \bar{d}_{0,3} \sin(3\varphi) \right] \bar{J}^{3/2}, \\ \mathcal{H}_4^{(1)} &= \frac{\eta^2}{r_*^2} \frac{\bar{I}^2}{2}, \\ \mathcal{H}_4^{(2)} &= \frac{\eta^3}{r_*^2} \frac{3\bar{n}_*}{|\bar{\kappa}_*|} (1 - \cos(2\varphi)) \bar{I} \bar{J}, \\ \mathcal{H}_4^{(3)} &= \frac{\eta^4}{r_*^2} \frac{\bar{d}_{0,4}}{2} \left[\frac{3}{4} - \cos(2\varphi) + \frac{1}{4} \cos(4\varphi) \right] \bar{J}^2, \\ \mathcal{H}_5^{(1)} &= -\frac{\eta^3}{r_*^{9/4}} \frac{\sqrt{2}}{|\bar{\kappa}_*|^{1/2}} \bar{I}^2 \bar{J}^{1/2} \sin \varphi, \\ \mathcal{H}_5^{(2)} &= \frac{\eta^4}{r_*^{9/4}} \left[-\frac{6\sqrt{2}\bar{n}_*}{|\bar{\kappa}_*|^{3/2}} \sin \varphi + \frac{2\sqrt{2}\bar{n}_*}{|\bar{\kappa}_*|^{3/2}} \sin(3\varphi) \right] \bar{I} \bar{J}^{3/2}.\end{aligned}\quad (4.9)$$

4.3. Normal form transformations

We proceed to construct suitable coordinate changes to remove the dependence on the angle of the axisymmetric part of the Hamiltonian up to a suitable order.

Proposition 14. *Given the Hamiltonian (4.7) with the terms up to order 5 as in (4.8), equivalently (4.9), assuming a smallness condition on η by requiring that*

$$\eta < \frac{1}{r_*^{1/2}}, \quad (4.10)$$

then there exist canonical transformations, such that the normalized Hamiltonian up to terms of order $\eta^3/r_^{5/2}$ takes the form*

$$h^{\text{norm}}(\bar{J}, \bar{I}) = \eta \tilde{k}_{11} \bar{I} + \frac{\eta^2}{r_*^{3/2}} \left(\tilde{k}_{21} \bar{J} + \frac{1}{r_*^{1/2}} \tilde{k}_{22} \bar{I}^2 \right) + \frac{\eta^3}{r_*^2} \left(\tilde{k}_{31} \bar{I} \bar{J} + \frac{1}{r_*^{1/2}} \tilde{k}_{32} \bar{I}^3 \right), \quad (4.11)$$

where

$$\begin{aligned}\tilde{k}_{11} &= \frac{\bar{n}_*}{r_*^{3/2}} - \Omega_P, & \tilde{k}_{21} &= |\bar{\kappa}_*|, & \tilde{k}_{22} &= \frac{1}{2} - \frac{2\bar{n}_*^2}{|\bar{\kappa}_*|^2}, \\ \tilde{k}_{31} &= \frac{3\sqrt{2}\bar{d}_{0,3}\bar{n}_*}{2|\bar{\kappa}_*|^{3/2}} + \frac{3\bar{n}_*}{|\bar{\kappa}_*|}, & \tilde{k}_{32} &= \frac{2\sqrt{2}\bar{d}_{0,3}\bar{n}_*^3}{|\bar{\kappa}_*|^{9/2}} + \frac{12\bar{n}_*^3}{|\bar{\kappa}_*|^4} - \frac{2\bar{n}_*}{|\bar{\kappa}_*|^2}.\end{aligned}$$

Proof. Condition (4.10) implies that $\eta^4/r_*^2 < \eta^3/r_*^{5/2}$, which means that the term $\mathcal{H}_4^{(3)}$ is of higher order and will be considered as part of the remainder. We also note that the average over φ of $\mathcal{H}_3^{(1)}$ and $\mathcal{H}_3^{(2)}$ is zero. Using the Lie series operator, we transform $\mathcal{H}^{\text{axi}}$ into a new Hamiltonian $\mathcal{H}^{(1)}$ defined as

$$\mathcal{H}^{(1)} = \exp(L_{\chi_1}) \mathcal{H}^{\text{axi}} \quad (4.12)$$

where $L_{\chi_1}(f) = \{f, \chi_1\}$, being $\{\cdot, \cdot\}$ the usual Poisson bracket operator, and χ_1 a function to be determined. Using (4.7), we have that (4.12) can be written as

$$\begin{aligned}\mathcal{H}^{(1)} &= \sum_{i=0}^{+\infty} \sum_{j=2}^{+\infty} \frac{L_{\chi_1}^i \mathcal{H}_j}{i!} \\ &= \mathcal{H}^{\text{axi}} + L_{\chi_1} \mathcal{H}^{\text{axi}} + \frac{1}{2} L_{\chi_1}^2 \mathcal{H}^{\text{axi}} + \frac{1}{6} L_{\chi_1}^3 \mathcal{H}^{\text{axi}} + \dots\end{aligned}$$

As a first step, we want to determine $\mathcal{H}^{(1)}$ in order to remove the dependence on the angle from the term $\mathcal{H}_3^{(1)}$; for this reason, we look for χ_1 such that

$$\mathcal{H}_3^{(1)} + \{\mathcal{H}_2, \chi_1\} = 0,$$

yielding the following expression for χ_1 :

$$\chi_1(\bar{J}, \bar{I}, \varphi) = \frac{2\sqrt{2}}{r_*^{1/4}} \frac{\bar{n}_*}{|\bar{\kappa}_*|^{3/2}} \bar{I} \bar{J}^{1/2} \cos \varphi.$$

Let us now write the transformed Hamiltonian as

$$\begin{aligned}\mathcal{H}^{(1)}(\bar{J}, \bar{I}, \varphi) &= \mathcal{H}_2(\bar{J}, \bar{I}) + \mathcal{H}_3^{(2)}(\bar{J}, \bar{I}, \varphi) + \mathcal{H}_4^{(1)}(\bar{I}) + \mathcal{H}_4^{(2)}(\bar{J}, \bar{I}, \varphi) \\ &\quad + \mathcal{H}_5^{(1)}(\bar{J}, \bar{I}, \varphi) + \mathcal{K}_1(\bar{J}, \bar{I}, \varphi) + \sum_{j=6}^{+\infty} \mathcal{K}_j^{(1)}(\bar{J}, \bar{I}, \varphi),\end{aligned}$$

where

$$\begin{aligned}\mathcal{K}_1(\bar{J}, \bar{I}, \varphi) &= -\frac{\eta^2}{r_*^2} \frac{2\bar{n}_*^2}{|\bar{\kappa}_*|^2} \bar{I}^2 + \frac{\eta^3}{r_*^2} \frac{3\sqrt{2}}{2} \frac{\bar{n}_* \bar{d}_{0,3}}{|\bar{\kappa}_*|^{3/2}} (1 - \cos(2\varphi)) \bar{I} \bar{J} \\ &\quad + \frac{\eta^3}{r_*^{9/4}} \left(\frac{6\bar{d}_{0,3}\bar{n}_*^2}{|\bar{\kappa}_*|^3} + \frac{12\sqrt{2}\bar{n}_*^2}{|\bar{\kappa}_*|^{5/2}} \right) \bar{I}^2 \bar{J}^{1/2} \sin \varphi \\ &\quad + \frac{\eta^3}{r_*^{5/2}} \left(\frac{2\sqrt{2}\bar{d}_{0,3}\bar{n}_*^3}{|\bar{\kappa}_*|^{9/2}} + \frac{12\bar{n}_*^3}{|\bar{\kappa}_*|^4} - \frac{2\bar{n}_*}{|\bar{\kappa}_*|^2} \right) \bar{I}^3,\end{aligned}$$

while $\mathcal{K}_j^{(1)}$, $j \geq 6$, are trigonometric polynomials in $\bar{I}$, $\sqrt{\bar{J}}$ of order greater than $\eta^3/r_*^{5/2}$. Then, we proceed with a sequence of normalization steps in order to remove the angle dependence from the terms in $\mathcal{H}^{(1)}$. We start by removing the term $\mathcal{H}_3^{(2)}$ (of order $\eta^3/r_*^{7/4}$) by looking for a function χ_2 such that

$$\mathcal{H}_3^{(2)} + \{\mathcal{H}_2, \chi_2\} = 0,$$

leading to

$$\chi_2(\bar{J}, \bar{I}, \varphi) = \frac{\eta}{r_*^{1/4}} \frac{\bar{d}_{0,3}}{12|\bar{\kappa}_*|} \bar{J}^{3/2} (\cos(3\varphi) - 9\cos \varphi).$$

The resulting Hamiltonian, given by $\mathcal{H}^{(2)} = \exp(L_{\chi_2})\mathcal{H}^{(1)}$, takes the form

$$\begin{aligned}\mathcal{H}^{(2)}(\bar{J}, \bar{I}, \varphi) &= \mathcal{H}_2(\bar{J}, \bar{I}) + \mathcal{H}_4^{(1)}(\bar{I}) + \mathcal{H}_4^{(2)}(\bar{J}, \bar{I}, \varphi) + \mathcal{H}_5^{(1)}(\bar{J}, \bar{I}, \varphi) \\ &\quad + \mathcal{K}_2(\bar{J}, \bar{I}, \varphi) + \sum_{j=6}^{+\infty} \mathcal{K}_j^{(2)}(\bar{J}, \bar{I}, \varphi),\end{aligned}$$

where it turns out that $\mathcal{K}_2 = \mathcal{K}_1$, while $\mathcal{K}_j^{(2)}$, $j \geq 6$, are trigonometric polynomials in $\bar{I}$, $\sqrt{\bar{J}}$ of the order greater than $\eta^3/r_*^{5/2}$. We need another normalization step to remove the terms of order η^3/r_*^2 , namely, the term $\mathcal{F}_3$ with $\mathcal{F}_3 = \{\mathcal{H}_3^{(2)}, \chi_1\} + \mathcal{H}_4^{(2)}$. Thus, we look for a function χ_3 such that

$$\tilde{\mathcal{F}}_3 + \{\mathcal{H}_2, \chi_3\} = 0,$$

where $\tilde{\mathcal{F}}_3 = \mathcal{F}_3 - \bar{\mathcal{F}}_3$ and $\bar{\mathcal{F}}_3$ is the average of $\mathcal{F}_3$ with respect to the angle φ . Afterwards, we set $\mathcal{H}^{(3)} = \exp(L_{\chi_3})\mathcal{H}^{(2)}$. By explicit computations, we obtain

$$\chi_3(\bar{J}, \bar{I}, \varphi) = -\frac{\eta}{r_*^{1/2}} \left(\frac{3\sqrt{2}\bar{d}_{0,3}\bar{n}_*}{4|\bar{\kappa}_*|^{5/2}} + \frac{3\bar{n}_*}{2|\bar{\kappa}_*|^2} \right) \sin(2\varphi) \bar{I} \bar{J},$$

and the resulting Hamiltonian takes the form

$$\begin{aligned}\mathcal{H}^{(3)}(\bar{J}, \bar{I}, \varphi) &= \mathcal{H}_2(\bar{J}, \bar{I}) + \mathcal{H}_4^{(1)}(\bar{I}) + \mathcal{H}_5^{(1)}(\bar{J}, \bar{I}, \varphi) \\ &+ \mathcal{K}_3(\bar{J}, \bar{I}, \varphi) + \sum_{j=6}^{+\infty} \mathcal{K}_j^{(3)}(\bar{J}, \bar{I}, \varphi),\end{aligned}$$

where

$$\begin{aligned}\mathcal{K}_3(\bar{J}, \bar{I}, \varphi) &= -\frac{\eta^2}{r_*^2} \frac{2\bar{n}_*^2}{|\bar{\kappa}_*|^2} \bar{I}^2 + \frac{\eta^3}{r_*^2} \left(\frac{3\sqrt{2}\bar{d}_{0,3}\bar{n}_*}{2|\bar{\kappa}_*|^{3/2}} + \frac{3\bar{n}_*}{\bar{\kappa}_*} \right) \bar{I}\bar{J} \\ &+ \frac{\eta^3}{r_*^{9/4}} \left(\frac{6\bar{d}_{0,3}\bar{n}_*^2}{|\bar{\kappa}_*|^3} + \frac{12\sqrt{2}\bar{n}_*^2}{|\bar{\kappa}_*|^{5/2}} \right) \bar{I}^2 \bar{J}^{1/2} \sin \varphi \\ &+ \frac{\eta^3}{r_*^{5/2}} \left(\frac{2\sqrt{2}\bar{d}_{0,3}\bar{n}_*^3}{|\bar{\kappa}_*|^{9/2}} + \frac{12\bar{n}_*^3}{|\bar{\kappa}_*|^4} - \frac{2\bar{n}_*}{|\bar{\kappa}_*|^2} \right) \bar{I}^3,\end{aligned}$$

while $\mathcal{K}_j^{(3)}$, $j \geq 6$, are trigonometric polynomials in $\bar{I}$, $\sqrt{\bar{J}}$ of order greater than $\eta^3/r_*^{5/2}$. We continue by removing the term of order $\eta^3/r_*^{9/4}$, namely the term $\mathcal{F}_4 = \mathcal{H}_5^{(1)} + \{\mathcal{H}_4^{(2)}, \chi_1\} + \frac{1}{2}\{\{\mathcal{H}_3^{(2)}, \chi_1\}, \chi_1\}$; we look for a function χ_4 such that

$$\mathcal{F}_4 + \{\mathcal{H}_2, \chi_4\} = 0,$$

since the average of $\mathcal{F}_4$ vanishes. Hence, we obtain:

$$\chi_4(\bar{J}, \bar{I}, \varphi) = \frac{\eta}{r_*^{3/4}} \left(-\frac{6\bar{d}_{0,3}\bar{n}_*^2}{|\bar{\kappa}_*|^4} - \frac{12\sqrt{2}\bar{n}_*^2}{|\bar{\kappa}_*|^{7/2}} + \frac{\sqrt{2}}{|\bar{\kappa}_*|^{3/2}} \right) \bar{I}^2 \sqrt{\bar{J}} \cos \varphi.$$

Setting $\mathcal{H}^{(4)} = \exp(L_{\chi_4})\mathcal{H}^{(3)}$, the new Hamiltonian is

$$\mathcal{H}^{(4)}(\bar{J}, \bar{I}, \varphi) = \mathcal{H}_2(\bar{J}, \bar{I}) + \mathcal{H}_4^{(1)}(\bar{I}) + \mathcal{K}_4(\bar{J}, \bar{I}) + \sum_{j=6}^{+\infty} \mathcal{K}_j^{(4)}(\bar{J}, \bar{I}, \varphi),$$

where

$$\mathcal{K}_4(\bar{J}, \bar{I}) = \mathcal{K}_3 - \left(\{\mathcal{H}_4^{(2)}, \chi_1\} + \frac{1}{2}\{\{\mathcal{H}_3^{(2)}, \chi_1\}, \chi_1\} \right)$$

and $\mathcal{K}_j^{(4)}$, $j \geq 6$, are trigonometric polynomials in $\bar{I}$, $\sqrt{\bar{J}}$ of order greater than $\eta^3/r_*^{5/2}$. Recalling condition (4.10), we define the new Hamiltonian $h^{\text{norm}} = h^{\text{norm}}(\bar{J}, \bar{I})$ as the angle-independent part of $\mathcal{H}^{(4)}$, which is given by:

$$\begin{aligned}h^{\text{norm}}(\bar{J}, \bar{I}) &= \eta \left(\frac{\bar{n}_*}{r_*^{3/2}} - \Omega_P \right) \bar{I} + \frac{\eta^2}{r_*^{3/2}} \left[|\bar{\kappa}_*| \bar{J} + \left(\frac{1}{2r_*^{1/2}} - \frac{2\bar{n}_*^2}{r_*^{1/2}|\bar{\kappa}_*|^2} \right) \bar{I}^2 \right] \\ &+ \frac{\eta^3}{r_*^2} \left[\left(\frac{3\sqrt{2}\bar{d}_{0,3}\bar{n}_*}{2|\bar{\kappa}_*|^{3/2}} + \frac{3\bar{n}_*}{|\bar{\kappa}_*|} \right) \bar{I}\bar{J} + \left(\frac{2\sqrt{2}\bar{d}_{0,3}\bar{n}_*^3}{r_*^{1/2}|\bar{\kappa}_*|^{9/2}} + \frac{12\bar{n}_*^3}{r_*^{1/2}|\bar{\kappa}_*|^4} - \frac{2\bar{n}_*}{r_*^{1/2}|\bar{\kappa}_*|^2} \right) \bar{I}^3 \right],\end{aligned}$$

which coincides with the expression (4.11). □

After the transformation described in proposition 14, we can write the Hamiltonian (3.21) as

$$\mathcal{H}^{\text{norm}}(\bar{J}, \bar{I}, \varphi, \theta) = h^{\text{norm}}(\bar{J}, \bar{I}) + f^{\text{norm}}(\bar{J}, \bar{I}, \varphi, \theta), \quad (4.13)$$

where h^{norm} is given explicitly in (4.11), while f^{norm} is the sum of the following quantities:

- (i) the sum $\sum_{j=6}^{+\infty} \mathcal{K}_j^{(4)}$, which is a trigonometric polynomial of order greater than $\eta^3/r_*^{5/2}$,
- (ii) the non-axisymmetric part $\lambda \tilde{F}_{ns}$, transformed by the rescalings in (4.6) and the transformations induced by the generating functions $\chi_1, \dots, \chi_4$.

Notice that, in view of proposition 21, whenever λ is of order higher than $\eta^3/r_*^{3/2}$, the term (ii), and thus f^{norm} , can be considered as a perturbation of the normal part h^{norm} . More precisely, we require

$$\lambda \leq \frac{\eta^{3+\nu}}{r_*^{3/2}}$$

for some $\nu > 0$ which, in view of proposition 21 and corollary 22, corresponds to require

$$\mathcal{O}_b \leq \frac{\eta^{3+\nu} r_*^{1/2}}{R^2}. \quad (4.14)$$

4.4. Verification of the multi-scale non-degeneracy conditions

In this section, we show that the multi-scale non-degeneracy conditions introduced in definition 10 and 11 hold.

Proposition 15. *Consider the Hamiltonian $\mathcal{H}^{\text{norm}}$ as in (4.13) with the condition (4.14) and with the integrable part h^{norm} as in (4.11). Let $\eta > 0$ and define*

$$h^{\text{norm}}(\bar{J}, \bar{I}) = h_0(\bar{I}) + \frac{\eta^2}{r_*^{3/2}} h_1(\bar{J}, \bar{I}),$$

where

$$\begin{aligned} h_0(\bar{I}) &= \eta \tilde{k}_{11} \bar{I}, \\ h_1(\bar{J}, \bar{I}) &= \tilde{k}_{21} \bar{J} + \frac{1}{r_*^{1/2}} \tilde{k}_{22} \bar{I}^2 + \frac{\eta}{r_*^{1/2}} \tilde{k}_{31} \bar{J} \bar{I} + \frac{\eta}{r_*} \tilde{k}_{32} \bar{I}^3 \end{aligned}$$

and

$$\Omega(\bar{J}, \bar{I}) = (\nabla_{\bar{I}} h_0, \nabla_{\bar{J}} h_1).$$

If $\Omega_p \neq n_*$ and $|\kappa_*|^2 \neq \frac{2\mathcal{G}M_p}{r_*^3}$, then $\text{Rank} \{(\Omega^T, \partial_{\bar{I}} \Omega^T, \partial_{\bar{J}} \Omega^T)\} = 2$ for every $(\bar{J}, \bar{I}) \in \mathbb{R}^2$, where T stands for transposed, showing that the function h^{norm} is non-degenerate in the sense of multi-scale KAM theorem according to [13].

Proof. By direct computations, we find

$$\begin{aligned} \Omega(\bar{J}, \bar{I}) &= (\eta \tilde{k}_{11}, \quad \tilde{k}_{21} + \frac{\eta}{r_*^{1/2}} \tilde{k}_{31} \bar{I}), \\ \partial_{\bar{I}} \Omega &= (0, \quad \frac{\eta}{r_*^{1/2}} \tilde{k}_{31}), \\ \partial_{\bar{J}} \Omega &= (0, \quad 0). \end{aligned}$$

Let $\mathcal{M}_1$ the matrix defined by

$$\mathcal{M}_1 = (\Omega^T, \partial_{\bar{I}} \Omega^T, \partial_{\bar{J}} \Omega^T) = \begin{pmatrix} \eta \tilde{k}_{11} & 0 & 0 \\ \tilde{k}_{21} + \frac{\eta}{r_*^{1/2}} \tilde{k}_{31} \bar{I} & \frac{\eta}{r_*^{1/2}} \tilde{k}_{31} & 0 \end{pmatrix}.$$

Then, we obtain

$$\text{Rank} \mathcal{M}_1 = 2 \iff \tilde{k}_{11} \tilde{k}_{31} \neq 0,$$

which is satisfied provided $\tilde{k}_{11} \neq 0$ and $\tilde{k}_{31} \neq 0$. At first order, from (3.7), (4.6), the condition $\tilde{k}_{11} \neq 0$ is equivalent to require $\Omega_p \neq n_*$, while the condition $\tilde{k}_{31} \neq 0$ is satisfied provided

$$|\kappa_*|^2 \neq \frac{2\mathcal{G}M_p}{r_*^3}. \quad (4.15)$$

□

Notice that, in view of (3.7), condition (4.15) is always satisfied at first order.

Proposition 16. Consider the Hamiltonian $\mathcal{H}^{norm}$ as in (4.13) with the condition (4.14) and with the integrable part h^{norm} as in (4.11). Define

$$\Omega(\bar{J}, \bar{I}) = \nabla_{(\bar{J}, \bar{I})} h_{norm}.$$

Then, $\text{Rank} \{(\Omega^T, \partial_{(\bar{J}, \bar{I})} \Omega)\} = 2$ for every $(\bar{J}, \bar{I}) \in \mathbb{R}^2$, where T stands for transposed, showing that the function h^{norm} is non-degenerate in the sense of multi-scale KAM theorem according to [25].

Proof. By direct computations, we find

$$\begin{aligned} \Omega(\bar{J}, \bar{I}) &= \left(\frac{\eta^2}{r_*^{3/2}} \tilde{k}_{21} + \frac{\eta^3}{r_*^2} \tilde{k}_{31} \bar{I}, \quad \eta \tilde{k}_{11} + 2 \frac{\eta^2}{r_*^2} \tilde{k}_{22} \bar{I} + \frac{\eta^3}{r_*^2} \tilde{k}_{31} \bar{J} + 3 \frac{\eta^3}{r_*^{5/2}} \tilde{k}_{32} \bar{I}^2 \right), \\ \partial_{(\bar{J}, \bar{I})} \Omega &= \begin{pmatrix} 0 & \frac{\eta^3}{r_*^2} \tilde{k}_{31} \\ \frac{\eta^3}{r_*^2} \tilde{k}_{31} & 2 \frac{\eta^2}{r_*^2} \tilde{k}_{22} + 6 \frac{\eta^3}{r_*^{5/2}} \tilde{k}_{32} \bar{I} \end{pmatrix} \end{aligned}$$

Define $S = \eta \tilde{k}_{11} + 2 \frac{\eta^2}{r_*^{3/2}} \tilde{k}_{22} \bar{I} + \frac{\eta^3}{r_*^2} \tilde{k}_{31} \bar{J} + 3 \frac{\eta^3}{r_*^{5/2}} \tilde{k}_{32} \bar{I}^2$, $Q = 2 \frac{\eta^2}{r_*^2} \tilde{k}_{22} + 6 \frac{\eta^3}{r_*^{5/2}} \tilde{k}_{32} \bar{I}$, and let the matrix $\mathcal{M}_2$ be defined as

$$\mathcal{M}_2 = (\Omega^T, \partial_{(\bar{J}, \bar{I})} \Omega) = \begin{pmatrix} \frac{\eta^2}{r_*^{3/2}} \tilde{k}_{21} + \frac{\eta^3}{r_*^2} \tilde{k}_{31} \bar{I} & 0 & \frac{\eta^3}{r_*^2} \tilde{k}_{31} \\ S & \frac{\eta^3}{r_*^2} \tilde{k}_{31} & Q \end{pmatrix}.$$

Then, $\text{Rank} \mathcal{M}_2 = 2$ if and only if

$$\begin{aligned} i) \quad & \left(\tilde{k}_{21} + \frac{\eta}{r_*^{1/2}} \tilde{k}_{31} \bar{I} \right) \tilde{k}_{31} \neq 0, \quad \text{or} \\ ii) \quad & \tilde{k}_{31} \neq 0, \quad \text{or} \\ iii) \quad & \left(\tilde{k}_{21} + \frac{\eta}{r_*^{1/2}} \tilde{k}_{31} \bar{I} \right) Q - \frac{\eta}{r_*^{1/2}} \tilde{k}_{31} S \neq 0. \end{aligned}$$

We immediately have that $i)$ and $ii)$ are satisfied for $\tilde{k}_{31} \neq 0$, namely $|\kappa_*|^2 \neq \frac{2GM_p}{r_*^3}$. Alternatively, we notice that to lowest order condition $iii)$ is equivalent to require $\Omega_p \neq 0$, which is always satisfied. $\square$

Remark 17. We observe that, in [13], the function h_1 appearing in proposition 15 does not depend on the small parameters η and $1/r_*$. This differs from our setting, in which we assume $\eta > 0$ and regard it as fixed in the definition of h_1 , as is commonly done in the context of explicit KAM estimates. We also observe that in [25] the perturbation is controlled by a parameter of the order of the square of the integrable part. Although this setting does not coincide exactly with ours, the aim of proposition 16 is merely to verify the multi-scale condition, rather than to provide a full implementation of the KAM theory developed in [25], which would require nontrivial adaptations to be applied to our model problem.

5. Analysis of three resonances in two test cases

In this section, we want to analyse the differences among several resonances, depending on the shape of the central body; we consider two test cases: the first one representing an almost spherical (AS) body, the second one representing the opposite case of a highly aspherical (HA) body. We consider three resonances: corotation (also denoted as 1 : 1), 1 : 2 and 1 : 3 resonances, whose choice is motivated by the fact that they represent, respectively, resonances at close, intermediate and far distances from the primary body (see section 5.2, compare with [7] for applications to real test cases). In table 1, we report the astronomical values of the parameters of the two test cases, chosen to be comparable with the values of small bodies in the Solar System (like dwarf planets or Centaurs). Besides the physical parameters (mass, semi-axes, reference radius—computed from equation (2.2)), the elongation and oblateness parameters (computed from equations (2.4) and (2.5)), we provide the rotation period of the central body and the resonant radii (see section 5.2) for corotation, 1 : 2 and 1 : 3 resonances. We notice that with the values in table 1, the coefficients $C_{2\ell, 2p}$, which define the potential, can be computed using the expressions given in equation (2.3).

Table 1. Parameter values of the two selected test cases of an almost spherical body (AS) and a highly aspherical object (HA); the table provides also the location of the three selected resonances.

	AS	HA
Rotation period T_{rot} (h)	8	8
Mass M_P (kg)	10^{21}	10^{21}
Semi-axes $a \times b \times c$ (km)	$1000 \times 980 \times 960$	$1000 \times 850 \times 750$
Reference radius R (km)	979	849
Elongation parameter	0.020 6336	0.192 486
Oblateness parameter	0.030 5336	0.207 226
Corotation radius (km)	1124.6	1148.08
1 : 2 resonance radius (km)	1776.73	1775.51
1 : 3 resonance radius (km)	2327.36	2323.11

We have checked that for AS and HA the multi-scale non-degeneracy conditions as stated in propositions 15 and 16 hold. In particular, we confirm that the Hamiltonian (4.13) is non-degenerate according to both the definitions 10 and 11 for the resonances 1 : 2, 1 : 3; for the corotation, it is non-degenerate only in the sense of definition 11.

In the following, we normalize the units of measure so that the biggest semi-axis a is the unit of length, the mass M_P is the unit of mass, and one hour is the unit of time. The gravitational constant $\mathcal{G}$ is expressed in these normalized units. We remark that all computations have been performed with 50 decimal digits to ensure enough accuracy.

5.1. Resonances in the epicyclic variables

The first task consists in defining the resonances using the epicyclic formulation; from equation (3.9) we have

$$\dot{\varphi} = \frac{\partial \mathcal{H}}{\partial J}, \quad \dot{\theta} = \frac{\partial \mathcal{H}}{\partial I}$$

and at lowest order it is

$$\dot{\varphi} = |\kappa_*|, \quad \dot{\theta} = n_* - \Omega_P;$$

Moreover, we observe that the angles (θ, φ) appear as combinations of the form $2i\theta + j\varphi$ with $i, j \in \mathbb{Z}$. Following definitions 5 and 6, we have that

- (i) a corotation or 1 : 1 resonance corresponds to $j = 0$ and any i ;
- (ii) a Lindblad resonance of order $m : (m - j)$ occurs when $j\kappa_* = m(n_* - \Omega_P)$ and it corresponds to $2i = -m$ and any j .

The resonant combination of the angles associated to a Lindblad resonance as in (ii) is $-m\theta + j\varphi$ with $m = -2i$. In particular,

- (A) $m = -1, j = 1$ gives $m : (m - j) = 1 : 2$; the resonant angle is $\theta + \varphi$ and its multiples;
- (B) $m = -1, j = 2$ gives $m : (m - j) = 1 : 3$; the resonant angle is $\theta + 2\varphi$ and its multiples.

Remark 18. We note that, at the corotation resonance, the expression $n_* = \Omega_P$ holds; the term of first order in I in Hamiltonian (3.9) vanishes; this is consistent with the fact that $\dot{\theta} = \partial \mathcal{H} / \partial I = 0$, namely $\theta = L - \Omega_P t$ is constant, meaning that the particle $\mathcal{P}$ is revolving with the same angular velocity of the body $\mathcal{E}$.

5.2. The resonant radius

As a first task, we compute the value of the radius corresponding to a resonance of order $p : q$. We will use two different methods, described in the following Definition.

Definition 19. The Keplerian resonant radius r_{pq}^{kep} is defined as

$$r_{pq}^{\text{kep}} = (\mathcal{G} M_P T_{pq}^2)^{1/3}, \quad (5.1)$$

where T_{pq} is the period of revolution of the particle $\mathcal{P}$ about $\mathcal{E}$, namely

$$T_{pq} = \frac{q}{p} \frac{1}{\Omega_P}.$$

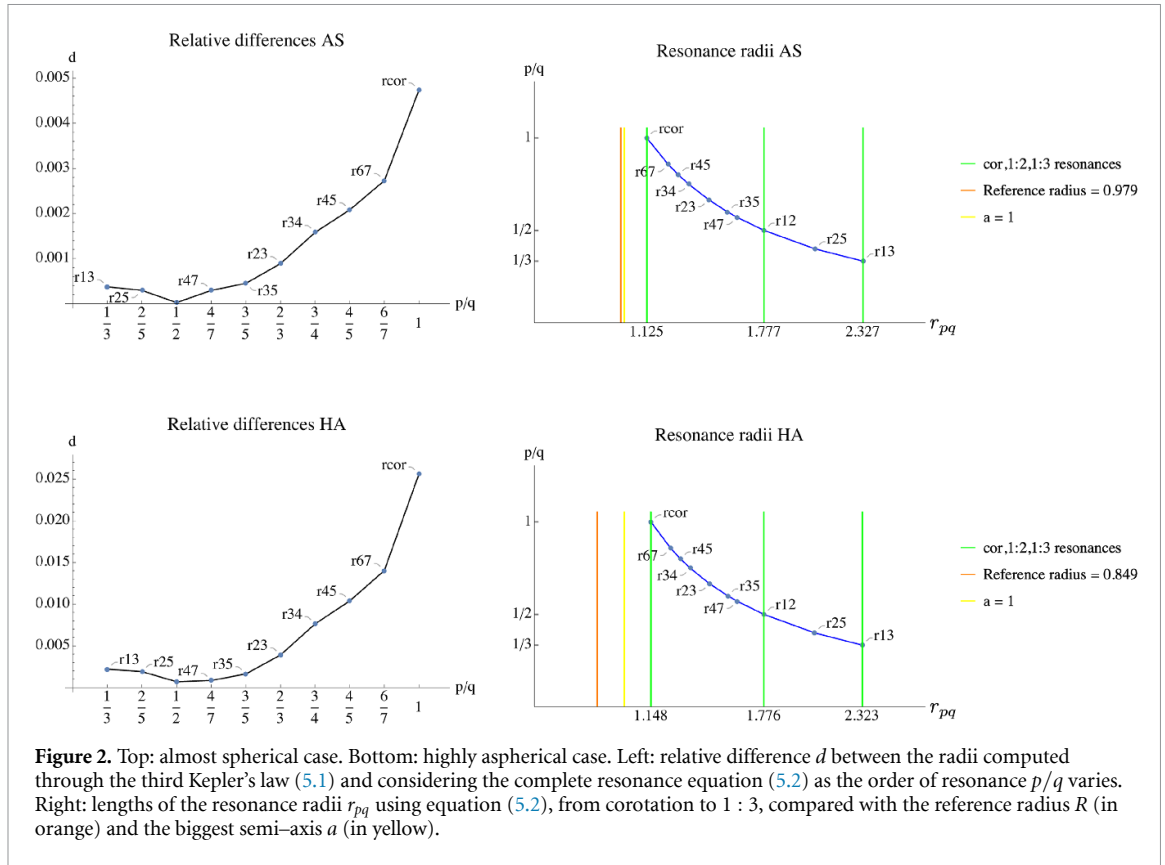


Figure 2. Top: almost spherical case. Bottom: highly aspherical case. Left: relative difference d between the radii computed through the third Kepler's law (5.1) and considering the complete resonance equation (5.2) as the order of resonance p/q varies. Right: lengths of the resonance radii r_{pq} using equation (5.2), from corotation to 1 : 3, compared with the reference radius R (in orange) and the biggest semi-axis a (in yellow).

Moreover, from equation (3.8), the resonant radius r_{pq} is defined as a solution of the equation

$$(p - q)\kappa(r_{pq}) = p(n(r_{pq}) - \Omega_p), \quad (5.2)$$

where $n(r)$, $\kappa(r)$ can be computed through equations (3.7).

While the definition of r_{pq}^{kep} takes advantage of Kepler's third law, considering then a spherical approximation of the central body $\mathcal{E}$, the method used to find r_{pq} , although heavier from a computational point of view, takes into account higher order harmonics of the gravitational potential of the ellipsoid $\mathcal{E}$. Note that r_{pq} is the analogous of the Lindblad radius given in definition 5 or the corotation radius given in definition 6. In figure 2, left panels, we plot the relative difference d between the radii computed by Kepler's law (5.1) and by solving equation (5.2), namely, the quantity

$$d = \frac{|r_{pq}^{\text{kep}} - r_{pq}|}{r_{pq}^{\text{kep}}},$$

considering several resonances ranging from 1 : 3 to corotation.

In figure 2, right panels, we plot the values of the resonant radii obtained from equation (5.2) for several resonances from 1 : 3 to corotation, compared with the reference radius R and the biggest semi-axes a in the two test cases (AS and HA).

We notice that the relative difference of the radii is much smaller in the AS case than in the HA case, as expected since the latter differs more from a spherical body. Moreover, the highest difference is found for the corotation resonance. Interestingly, in both cases the corotation radius is very close to the body (namely, to the semi-axis a), while the 1 : 3 resonant radius is the farthest.

5.3. Amplitudes of libration and occurrence of bifurcations

As an indication of the strength of the resonances, we compute their amplitude of libration. To this end, we proceed to reduce the Hamiltonian to the lowest terms of the expansion around the resonance; this leads to a one-dimensional pendulum Hamiltonian for which we can compute, using standard formulae (see, e.g. [15]), the amplitude around the stable equilibrium. We will limit the implementation of this computation to the corotation case.

A more accurate estimate is obtained following the dynamics induced by the whole Hamiltonian through a numerical integration and in computing the distance between the separatrices around the elliptic equilibria.

The results described in the following are obtained for the Hamiltonian (3.9) truncated up to $i = 5$ and $j = 8$; this means truncating the axisymmetric potential $U_s(r)$ given in equation (3.3) and the non-axisymmetric part $U_{ns}(r, \theta)$ given by equation (3.4) at order $\ell = 5$ and expanding the function $F_s(I, \rho)$, given in equation (3.17), and the function $F_{ns}(\rho, \theta)$, given in equation (3.18), in ρ up to order 16.

This choice has been done taking into account that the expansion in ρ up to order 16 corresponds to an expansion of the Hamiltonian (3.9) in J up to the order 8 and in the eccentricity of the particle's orbit up to the order 16 as proved in proposition 9; in this case, the remainder of the complete Hamiltonian with respect to the truncated Hamiltonian turns out to be of the order of 10^{-6} for values of the eccentricities up to 0.45. Moreover, the truncation of the gravitational potential at order $\ell = 5$ provides an error of the highest terms still of the order of 10^{-6} . Notice that such errors vary according to the chosen resonance.

The first step to study the local behaviour of a particle in the vicinity of a resonance is to consider a suitable truncation of the Hamiltonian $\mathcal{H}(J, I, \varphi, \theta)$ as expressed in equation (3.9), which takes into account only the terms affected by the resonance itself. In general, this is done by taking r_* as the resonant radius, and keeping only the trigonometric terms depending on the resonant angular combination.

Proposition 20. *Let us consider a resonance of order $m : (m - j)$, and consider the corresponding resonant radius $r_{m, m-j}$, which is, according to definitions 5 and 6, solution of the equation (3.8). Let $r_* = r_{m, m-j}$; then, the resonant Hamiltonian associated to the resonance $m : (m - j)$ is given by*

$$\mathcal{H}_{\text{res}}^{m, m-j}(J, I, \varphi, \theta) = Z(I, J) + \sum_{i=1}^N f_i(I, J) \text{cs}(i(j\varphi - m\theta)), \quad (5.3)$$

where $Z(I, J)$, called normal part, collects all the non-trigonometric terms of $\mathcal{H}(J, I, \varphi, \theta)$, f_i are suitable polynomials in the actions I, J , cs stands for cosine or sine, depending on the parity of i and j , N depends on the maximal order of the expansion in θ and all the terms are evaluated at r_* .

Proof. The expression of $\mathcal{H}_{\text{res}}^{m, m-j}(J, I, \varphi, \theta)$ comes from equation (3.9). In the resonant Hamiltonian, only the terms corresponding to the resonant angles associated to the $m : (m - j)$ resonance, that is, multiples of $j\varphi - m\theta$, are taken into account. The polynomials $f_i(I, J)$ are the analogous of $\alpha_{2i, j}, \beta_{2i, j}$ in equation (3.25), taking into account the change of notation in the indices. The relation between the parity of the coefficient of φ and the presence of either a sine or a cosine in $\mathcal{H}_{\text{res}}^{m, m-j}$ can be deduced from the expressions of $\alpha_{2i, j}, \beta_{2i, j}$ in appendix A: whenever such coefficient is even, we have a cosine in the sum; on the other hand, when it is odd a sine appears in equation (5.3). $\square$

As we will see in the specific cases of the 1 : 2 and 1 : 3 resonances, with a suitable change of coordinates it is possible to reduce $\mathcal{H}_{\text{res}}^{m, m-j}$ to a 1 DOF Hamiltonian, depending only on the resonant angle and the conjugated action.

The Hamiltonian (5.3) properly defines the dynamics around the resonance of order $m : (m - j)$ provided that the remainder $\mathcal{R}^{m, m-j}$, defined as

$$\mathcal{R}^{m, m-j}(J, I, \varphi, \theta) = \mathcal{H}(J, I, \varphi, \theta) - \mathcal{H}_{\text{res}}^{m, m-j}(J, I, \varphi, \theta)$$

with $\mathcal{H}$ as in (3.9), is smaller than $\mathcal{H}_{\text{res}}^{m, m-j}$ using the sup norm $\|\cdot\|$ over a suitable domain. Thus, for each of the three analysed resonances, we chose the domain of the actions (J, I) such that $\|\mathcal{H}_{\text{res}}^{m, m-j}\| > \|\mathcal{R}^{m, m-j}\|$. Notice that the condition on the domain of the actions translates into a domain of the eccentricity of the orbit of the particle around the central body; as a consequence, the domain in the eccentricity varies according to the chosen resonance.

5.3.1. Corotation

For the corotation case, we write the Hamiltonian (5.3) as the sum of the normal part (depending only on the actions) computed in $r_* = r_{\text{cor}}$ and the part depending on the resonant angle θ . Evaluating the coefficients at r_{cor} obtained by solving equation (5.2), we get the Hamiltonian

$$\begin{aligned} \mathcal{H}_{\text{res}}^{\text{cor}}(J, I, \varphi, \theta) = & f_1(J)I + f_2(J)I^2 + f_3(J)\cos(2\theta) + f_4(J)\cos(4\theta) \\ & + f_5(J)\cos(6\theta) + f_6(J)\cos(8\theta) + f_7(J)\cos(10\theta), \end{aligned} \quad (5.4)$$

Table 2. Coefficients appearing in the corotation resonant Hamiltonian (5.6) for the sample cases AS and HA at $e = 10^{-3}$.

Coefficients	1 : 1 AS	1 : 1 HA
α_1	$2.4 \cdot 10^{-6}$	$2.6 \cdot 10^{-6}$
α_2	$3.9 \cdot 10^{-1}$	$3.8 \cdot 10^{-1}$
α_3	$-7.3 \cdot 10^{-3}$	$-5.2 \cdot 10^{-2}$
α_4	$-7.2 \cdot 10^{-5}$	$-3.6 \cdot 10^{-3}$
α_5	$-1.0 \cdot 10^{-6}$	$-3.5 \cdot 10^{-4}$
α_6	$-1.8 \cdot 10^{-8}$	$-4.0 \cdot 10^{-5}$
α_7	$-3.4 \cdot 10^{-10}$	$-4.6 \cdot 10^{-6}$

where $f_i(J)$, $i = 1, \dots, 7$, are polynomials in the J variable of degree 8, provided the actions J, I are small enough. Since the variable φ is cyclic, then J is constant, say J_0 . Thus, we get a Hamiltonian of the type

$$\mathcal{H}_{\text{res}}^{\text{cor}}(J, I, \varphi, \theta) = \mathcal{H}_{\text{res}}^{\text{cor}}(I, \theta; J_0)$$

with $f_i(J) = f_i(J_0)$ constant terms for $i = 1, \dots, 7$.

After fixing J_0 , we can integrate the equations of motion associated to the 1 DOF Hamiltonian (5.4). To fix J_0 , recall that from equations (3.20) at corotation one has

$$J_0 = \frac{p_\rho^2}{2|\kappa(r_{\text{cor}})|} + \frac{|\kappa(r_{\text{cor}})|\rho^2}{2}, \quad (5.5)$$

which depends on ρ and p_ρ . From standard Kepler's relations (see for example [9]), the value of J_0 is determined by taking $\rho = er_{\text{cor}}$ and $p_\rho = r_{\text{cor}} en_*$ (where we used $a = r_{\text{cor}}$, $n = n_*$).

Fixing, for example, $e = 10^{-3}$, from equation (5.5) we obtain J_0 which, in the two sample cases AS and HA, takes the values

$$J_{0,\text{AS}} = 9.93418 \cdot 10^{-7}, \quad J_{0,\text{HA}} = 1.03984 \cdot 10^{-6}.$$

At J_0 the Hamiltonian (5.4) has the form

$$\mathcal{H}_{\text{res}}^{\text{cor}}(I, \theta) = \alpha_1 I + \alpha_2 I^2 + \alpha_3 \cos(2\theta) + \alpha_4 \cos(4\theta) + \alpha_5 \cos(6\theta) + \alpha_6 \cos(8\theta) + \alpha_7 \cos(10\theta) \quad (5.6)$$

with the coefficients α_j , $j = 1, \dots, 7$, as in table 2 for the test cases AS and HA when $e = 10^{-3}$.

If we retain only the term in $\cos(2\theta)$, we obtain the Hamiltonian

$$\mathcal{H}(I, \theta) = \alpha_1 I + \alpha_2 I^2 + \alpha_3 \cos(2\theta);$$

it is straightforward to see that the semi-amplitude of the resonant island can be computed as (see [15])

$$\Delta I = \sqrt{\frac{-2\alpha_3}{\alpha_2}}. \quad (5.7)$$

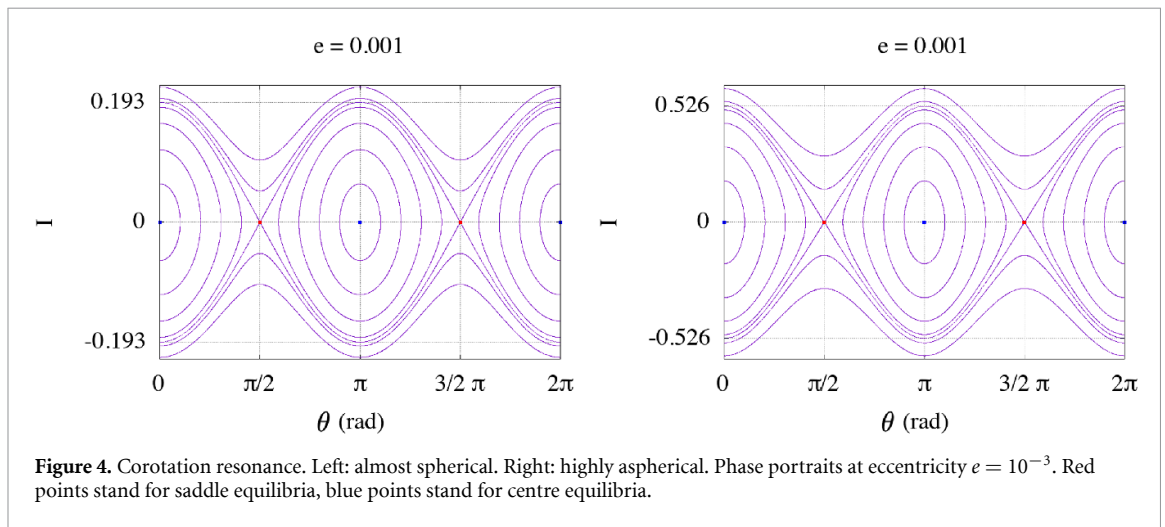
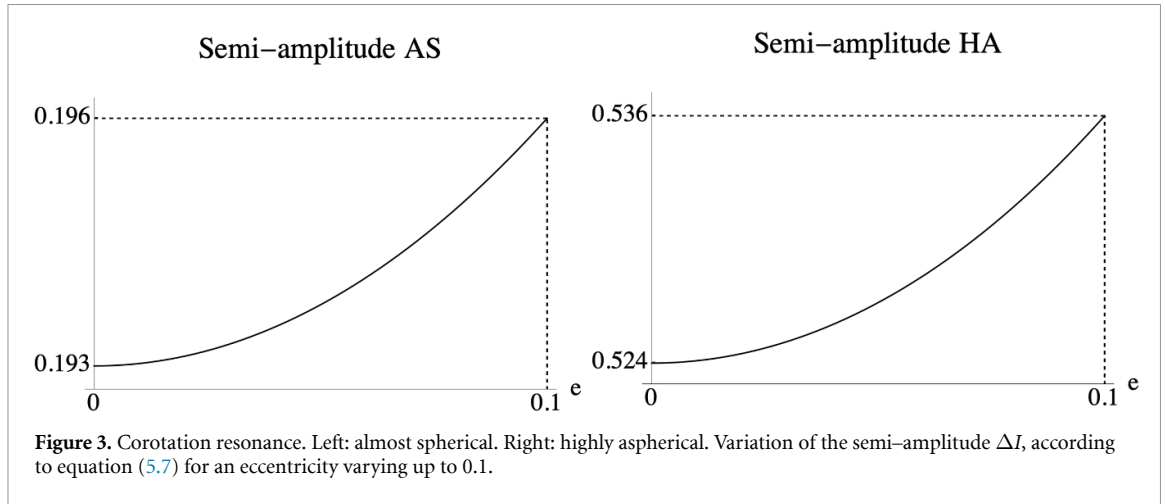
Using the values in table 2 for the two test cases, we obtain:

$$\Delta I_{\text{AS}} = 0.193, \quad \Delta I_{\text{HA}} = 0.524. \quad (5.8)$$

In figure 3, we plot the variation of the semi-amplitude ΔI as the eccentricity varies according to equation (5.7).

A more accurate computation, based on a numerical analysis, is obtained integrating the equations of motions associated to the Hamiltonian (5.6):

$$\begin{cases} \dot{\theta} = \frac{\partial \mathcal{H}_{\text{res}}^{\text{cor}}(I, \theta)}{\partial I} = \alpha_1 + 2\alpha_2 I \\ \dot{I} = -\frac{\partial \mathcal{H}_{\text{res}}^{\text{cor}}(I, \theta)}{\partial \theta} = 2\alpha_3 \sin(2\theta) + 4\alpha_4 \sin(4\theta) + 6\alpha_5 \sin(6\theta) \\ \quad + 8\alpha_6 \sin(8\theta) + 10\alpha_7 \sin(10\theta); \end{cases}$$



from the integration, we compute the distance between the separatrices and the elliptic equilibria, obtaining the values

$$\Delta I_{AS} = 0.193, \quad \Delta I_{HA} = 0.526,$$

which are in good agreement with the values in (5.8). In figure 4, we show the phase portraits of the Hamiltonian (5.6) for a small eccentricity, say $e = 10^{-3}$. The equilibria correspond to $\theta = 0, \pi/2, \pi, 3\pi/2$; the linear stability of the equilibria in $\theta = 0$ and $\theta = \pi$ is of centre type, while the equilibria in $\theta = \pi/2, 3\pi/2$ are saddles. We remark that the amplitudes of libration are greater in the HA case and as the eccentricity increases.

5.3.2. Resonance 1 : 2

We consider the Hamiltonian (5.3) taking into account the normal part computed in $r_* = r_{12}$ plus the part depending on the angles associated to the 1 : 2 resonance, which corresponds to take in equation (3.9) only the angle $\theta + \varphi$ and its multiples, according to section 5.1; we notice that the Hamiltonian (3.25) contains the resonant combination $2\theta + 2\varphi$ (instead of $\theta + \varphi$) and its multiples. Hence, in a suitable domain of the action variables (J, I) , which corresponds to take values of the eccentricity less or equal than 0.1, both for the sample cases AS and HA, we obtain the Hamiltonian

$$\begin{aligned} \mathcal{H}_{\text{res}}^{12}(J, I, \varphi, \theta) = & |\kappa_*|J + (n_* - \Omega_p)I + \frac{I^2}{2r_*} + f_1(J)I + f_2(J)I^2 \\ & + \sum_{i=1}^5 f_{i+2}(J) \cos(2i\theta + 2i\varphi), \end{aligned} \quad (5.9)$$

Table 3. Terms $|\alpha_i G_0^i|$ appearing in the 1 : 2 resonant Hamiltonian (5.12) for the sample cases AS and HA at $e = 10^{-3}$; the value G_0 is taken as $G_0 = -\frac{1}{2}J_0 + I_0$.

Terms	1 : 2 AS	1 : 2 HA
$ \alpha_1 G_0 $	$4.5 \cdot 10^{-6}$	$4.2 \cdot 10^{-6}$
$ \alpha_2 G_0^2 $	$2.7 \cdot 10^{-6}$	$2.5 \cdot 10^{-6}$
$ \alpha_3 G_0^3 $	$7.2 \cdot 10^{-9}$	$6.4 \cdot 10^{-9}$
$ \alpha_4 G_0^4 $	$1.7 \cdot 10^{-11}$	$1.4 \cdot 10^{-11}$
$ \alpha_5 G_0^5 $	$3.8 \cdot 10^{-14}$	$3.1 \cdot 10^{-14}$
$ \alpha_6 G_0^6 $	$8.4 \cdot 10^{-17}$	$6.3 \cdot 10^{-17}$
$ \alpha_7 G_0^7 $	$1.8 \cdot 10^{-19}$	$1.2 \cdot 10^{-19}$
$ \alpha_8 G_0^8 $	$3.9 \cdot 10^{-22}$	$2.2 \cdot 10^{-22}$
$ \alpha_9 G_0^9 $	$4.8 \cdot 10^{-25}$	$3.8 \cdot 10^{-25}$
$ \alpha_{10} G_0^{10} $	$1.9 \cdot 10^{-28}$	$1.4 \cdot 10^{-28}$

where $\kappa_* = \kappa(r_{12})$, $n_* = n(r_{12})$ with r_{12} computed through equation (3.8). The terms $f_j(J)$, $j = 1, \dots, 7$, are polynomials of degree 8 in the variable J . To isolate the resonant angle, let us introduce the canonical transformation

$$\begin{cases} \psi = 2\varphi + 2\theta \\ \mu = 2\varphi + \theta \\ G = -\frac{1}{2}J + I \\ L = J - I \end{cases} \quad (5.10)$$

In these new variables, μ is cyclic, thus the action L is a constant, say $L = L_0$, and the resonant Hamiltonian becomes

$$\mathcal{H}_{\text{res}}^{12}(G, \psi; L_0) = \sum_{i=1}^{10} \gamma_i(L_0) G^i + \sum_{i=1}^5 \gamma_{i+10}(G; L_0) \cos(i\psi), \quad (5.11)$$

where the constants $\gamma_1, \dots, \gamma_{10}$ and the polynomials $\gamma_{11}, \dots, \gamma_{15}$ can be computed through equations (5.9) and (5.10). Among the possible choices of the initial condition, we adopt the following procedure to define an integral level $L = L_0$: let us fix the constant $L_0 = J_0 - I_0$ from the initial values J_0, I_0 . As in the corotation case, we can compute J_0 by setting suitable values of ρ and p_ρ ; fixing, for example, $e = 10^{-3}$, we get $\rho = 10^{-3} r_{12}$ and hence, from equation (5.5) (substituting the subscript cor by 12), we obtain

$$J_{0,\text{AS}} = 1.24321 \cdot 10^{-6}, \quad J_{0,\text{HA}} = 1.25848 \cdot 10^{-6}.$$

Moreover, by definition, we have $I = p_\theta - p_*$, where $p_\theta = n(r)r^2$ and $p_* = n_*r_*^2$ with $r_* = r_{12}$, $n_* = n(r_*)$. After computing p_θ in $r = r_{12} + \rho$, we get;

$$I_{0,\text{AS}} = 6.14415 \cdot 10^{-4}, \quad I_{0,\text{HA}} = 5.89698 \cdot 10^{-4},$$

from which we obtain

$$L_{0,\text{AS}} = -6.13172 \cdot 10^{-4}, \quad L_{0,\text{HA}} = -5.8844 \cdot 10^{-4},$$

that, substituted in equation (5.11) and omitting constant terms, give the 1 DOF Hamiltonian

$$\mathcal{H}_{\text{res}}^{12}(G, \psi) = \sum_{i=1}^{10} \alpha_i G^i + \sum_{i=1}^5 \delta_i(G) \cos(i\psi), \quad (5.12)$$

with the terms $|\alpha_i G^i|$, computed in $G = G_0$, given in table 3 and $\delta_1, \dots, \delta_5$ polynomials of degree 8 in G given in appendix C. We stress that the coefficients of the trigonometric terms in equation (5.12) depend on the action G .

The equilibrium points associated to equation (5.12), depending on the eccentricity, are given by the solutions of the system:

$$\begin{cases} \dot{\psi} = \frac{\partial \mathcal{H}_{\text{res}}^{12}(G, \psi)}{\partial G} = 0 \\ \dot{G} = -\frac{\partial \mathcal{H}_{\text{res}}^{12}(G, \psi)}{\partial \psi} = 0. \end{cases}$$

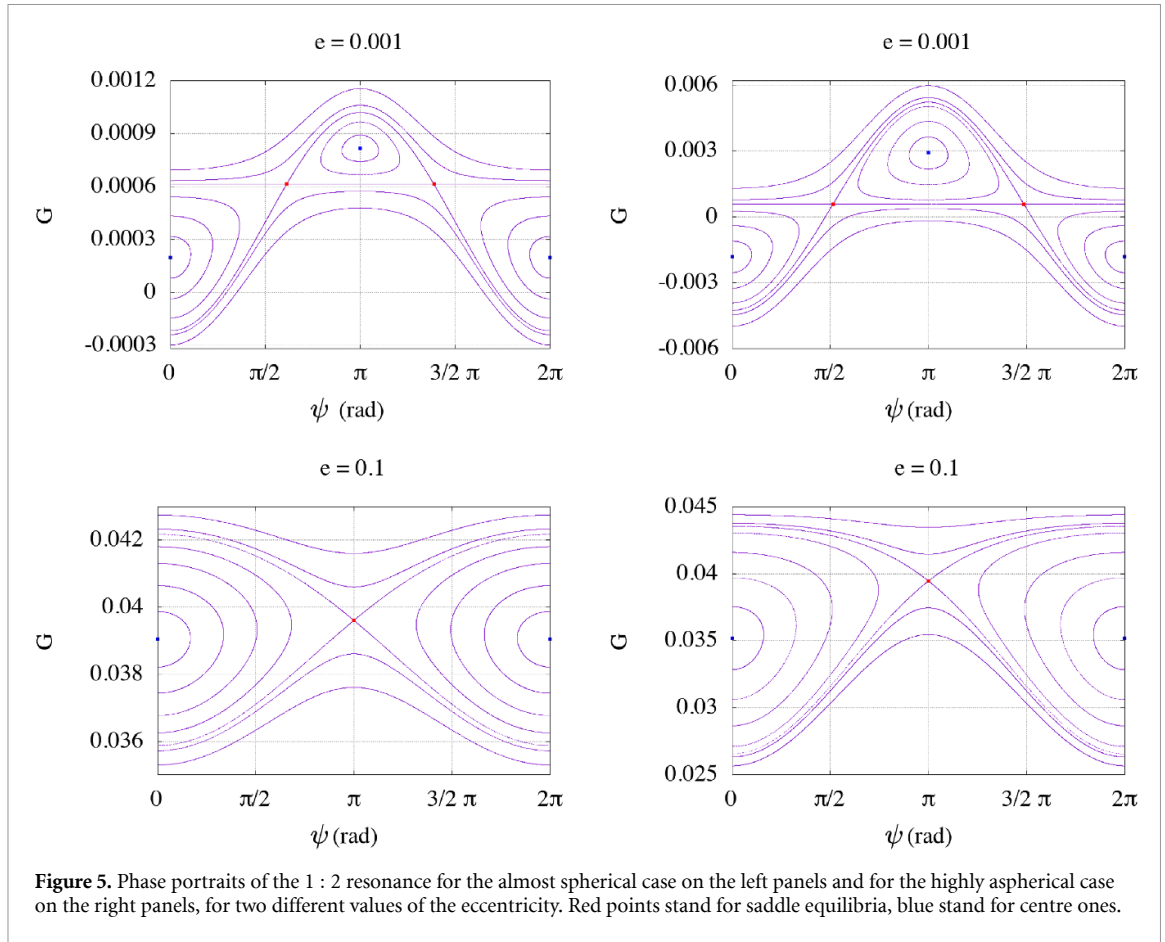


Figure 5. Phase portraits of the 1 : 2 resonance for the almost spherical case on the left panels and for the highly aspherical case on the right panels, for two different values of the eccentricity. Red points stand for saddle equilibria, blue stand for centre ones.

For $e = 10^{-3}$, the equilibria of AS are:

$$\begin{aligned} (\psi_1 = 0, G_1 = 1.97699 \cdot 10^{-4}), & \quad (\psi_2 = \pi, G_2 = 8.16673 \cdot 10^{-4}), \\ (\psi_3 = 1.92323, G_3 = 6.13172 \cdot 10^{-4}), & \quad (\psi_4 = 4.35995, G_4 = 6.13172 \cdot 10^{-4}) \end{aligned}$$

and for HA they are given by

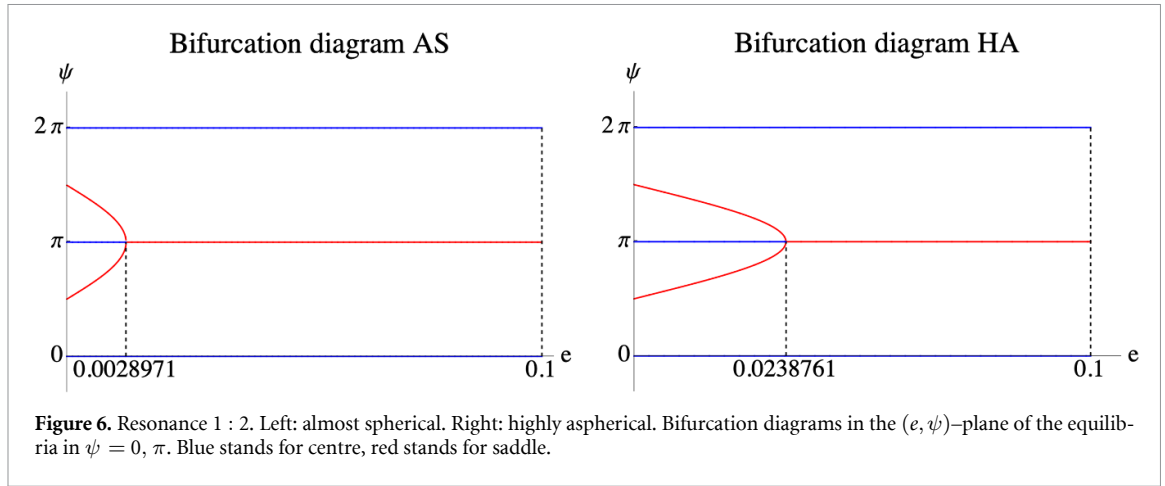
$$\begin{aligned} (\psi_1 = 0, G_1 = -1.81881 \cdot 10^{-3}), & \quad (\psi_2 = \pi, G_2 = 2.91528 \cdot 10^{-3}), \\ (\psi_3 = 1.61491, G_3 = 5.8844 \cdot 10^{-4}), & \quad (\psi_4 = 4.66827, G_4 = 5.8844 \cdot 10^{-4}). \end{aligned}$$

The equilibria (ψ_1, G_1) , (ψ_2, G_2) are elliptic points, while (ψ_3, G_3) and (ψ_4, G_4) are hyperbolic points. Computing the amplitude of the resonant islands (obtained by evaluating the distance between the separatrices), one gets

$$\begin{aligned} \Delta G_{AS}^{(1)} &= 8.31687 \cdot 10^{-4}, & \Delta G_{AS}^{(2)} &= 4.06825 \cdot 10^{-4}, \\ \Delta G_{HA}^{(1)} &= 4.83983 \cdot 10^{-3}, & \Delta G_{HA}^{(2)} &= 4.63121 \cdot 10^{-3}. \end{aligned}$$

These values fully agree with the results of the numerical integration of Hamilton's equations associated to equation (5.12) as shown in the phase portraits (top panels) of figure 5, on the left for the AS case and on the right for the HA case. We remark that in this case the computed values are not the semi-amplitudes as in the corotation case, but the amplitudes of the whole libration islands which are asymmetric with respect to the elliptic equilibria.

As already evident in figure 5 where the phase portraits are represented for two different values of the eccentricity, the 1 : 2 resonance presents different situations as the eccentricity increases. Concerning the bifurcations, we observe that Hamilton's equations associated to equation (5.12) admit trivial equilibria for $\psi = 0$ and $\psi = \pi$; the equilibrium in $\psi = 0$ is always stable, while the one in $\psi = \pi$ shows a change of stability. The equilibrium is stable at $e = 0$ and it becomes unstable at $e = 0.0028971$ for the AS case and at $e = 0.0238761$ for the HA case. Figure 6 provides a finer analysis, showing the occurrence of the pitchfork bifurcation for $\psi = \pi$ for the AS and the HA cases. For very small values of e , there are two secondary equilibria absorbed by $\psi = \pi$ as e increases.



5.3.3. Resonance 1 : 3

Let us consider the Hamiltonian (5.3) taking into account the normal part in $r_* = r_{13}$ and the part depending on the angles associated to the 1 : 3 resonance, corresponding to take in equation (3.9) the resonant combination $\theta + 2\varphi$ and its multiples (see section 5.1):

$$\begin{aligned} \mathcal{H}_{\text{res}}^{13}(J, I, \varphi, \theta) = & |\kappa_*|J + (n_* - \Omega_p)I + \frac{I^2}{2r_*} + f_1(J)I + f_2(J)I^2 \\ & + \sum_{i=1}^4 f_{i+2}(J) \cos(2i\theta + 4i\varphi), \end{aligned} \quad (5.13)$$

where the terms $f_j(J)$, $j = 1, \dots, 7$, are polynomials of degree 8 in the variable J .

As in the case of the 1 : 2 resonance, let us consider the canonical transformation

$$\begin{cases} \psi = 4\varphi + 2\theta \\ \mu = \varphi + \theta \\ G = \frac{1}{2}J - \frac{1}{2}I \\ L = -J + 2I. \end{cases} \quad (5.14)$$

In the new variables, μ is cyclic, hence $L = L_0$ is constant, and the resonant Hamiltonian takes the form

$$\mathcal{H}_{\text{res}}^{13}(G, \psi; L_0) = \sum_{i=1}^{10} \gamma_i(L_0) G^i + \sum_{i=1}^4 \gamma_{i+10}(G; L_0) \cos(i\psi) \quad (5.15)$$

where the constants $\gamma_1, \dots, \gamma_{10}$ and the polynomials $\gamma_{11}, \dots, \gamma_{14}$ can be computed through equations (5.13) and (5.14). Let us choose initial conditions to fix L_0 through $L_0 = -J_0 + 2I_0$; we can compute J_0 by setting suitable values for ρ and p_ρ , say $\rho = 10^{-3} r_{13}$ (corresponding to $e = 10^{-3}$). Hence, from equation (5.5) we obtain

$$J_{0,AS} = 1.42118 \cdot 10^{-6}, \quad J_{0,HA} = 1.42991 \cdot 10^{-6}.$$

Using the same procedure as in the 1 : 2 resonance, we get

$$I_{0,AS} = 7.05761 \cdot 10^{-4}, \quad I_{0,HA} = 6.8971 \cdot 10^{-4},$$

from which we obtain

$$L_{0,AS} = 1.4101 \cdot 10^{-3}, \quad L_{0,HA} = 1.37799 \cdot 10^{-3}.$$

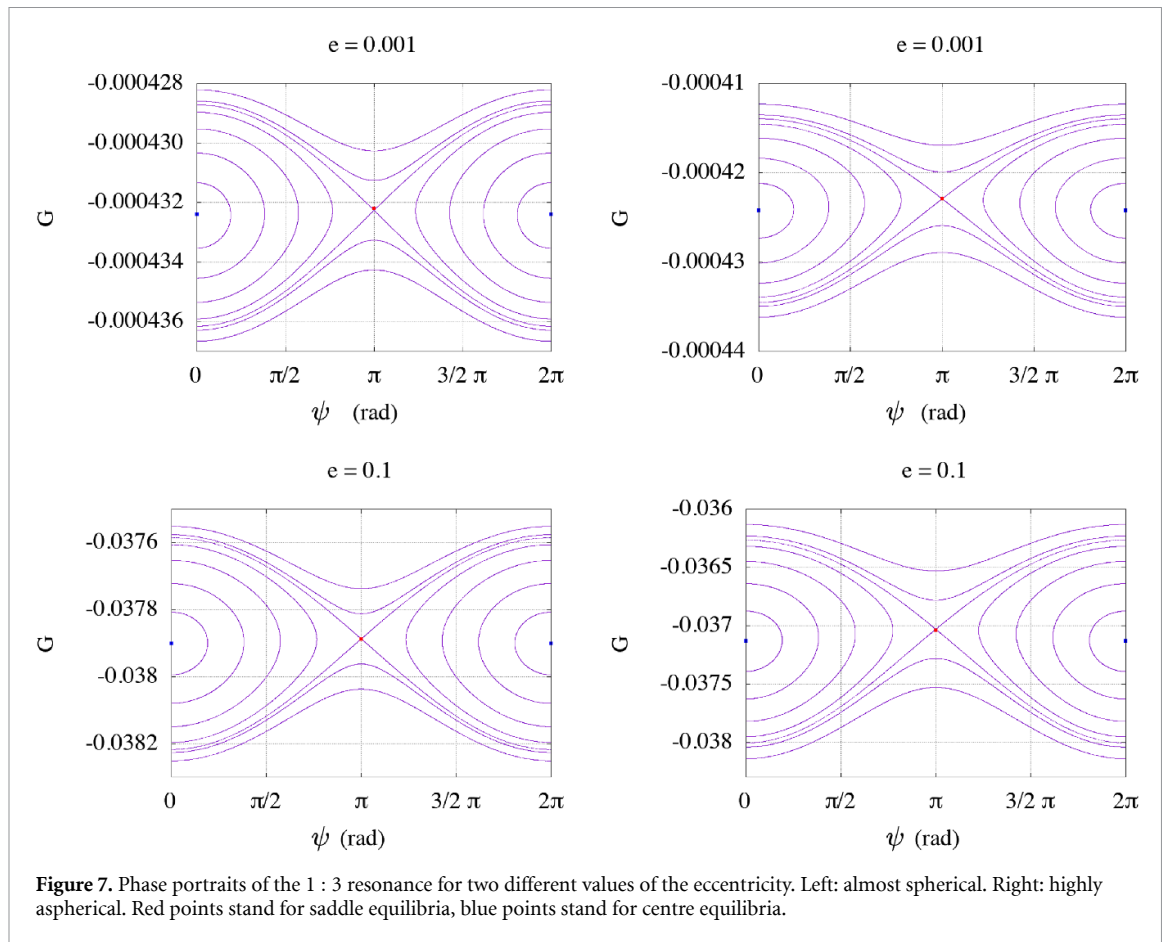
Substituting L_0 in equation (5.15), we arrive at the Hamiltonian

$$\mathcal{H}_{\text{res}}^{13}(G, \psi) = \sum_{i=1}^{10} \alpha_i G^i + \sum_{i=1}^4 \delta_i(G) \cos(i\psi), \quad (5.16)$$

where $\alpha_1, \dots, \alpha_{10}$ are constants and $\delta_1, \dots, \delta_4$ are polynomial of degree 8 in G given in appendix C; also in this case, the coefficients of the trigonometric terms depend on the action G . In table 4, we report the

Table 4. Coefficients appearing in the 1 : 3 resonant Hamiltonian (5.16) for the sample cases AS and HA, at $e = 10^{-3}$; G_0 is computed as $G_0 = 1/2J_0 - 1/2I_0$.

Coefficients	1 : 3 AS	1 : 3 HA
$ \alpha_1 G_0 $	$3.5 \cdot 10^{-6}$	$3.4 \cdot 10^{-6}$
$ \alpha_2 G_0^2 $	$1.4 \cdot 10^{-6}$	$1.4 \cdot 10^{-6}$
$ \alpha_3 G_0^3 $	$3.8 \cdot 10^{-9}$	$3.6 \cdot 10^{-9}$
$ \alpha_4 G_0^4 $	$9.1 \cdot 10^{-12}$	$8.3 \cdot 10^{-12}$
$ \alpha_5 G_0^5 $	$2.1 \cdot 10^{-14}$	$1.8 \cdot 10^{-14}$
$ \alpha_6 G_0^6 $	$4.6 \cdot 10^{-17}$	$3.9 \cdot 10^{-17}$
$ \alpha_7 G_0^7 $	$1.0 \cdot 10^{-19}$	$8.2 \cdot 10^{-20}$
$ \alpha_8 G_0^8 $	$2.1 \cdot 10^{-22}$	$1.7 \cdot 10^{-22}$
$ \alpha_9 G_0^9 $	$1.7 \cdot 10^{-25}$	$1.5 \cdot 10^{-25}$
$ \alpha_{10} G_0^{10} $	$3.7 \cdot 10^{-29}$	$3.2 \cdot 10^{-29}$



values of the terms $|\alpha_i G^i|$, computed in $G = G_0$, for $i = 1, \dots, 10$ and for $e = 10^{-3}$, in the two sample cases AS and HA.

In figure 7, we plot the phase portrait of the Hamiltonian (5.16) for two different values of the eccentricity. The equilibrium points are given by the solutions of the system

$$\begin{cases} \dot{\psi} = \frac{\partial \mathcal{H}_{\text{res}}^{13}(G, \psi)}{\partial G} = 0 \\ \dot{G} = -\frac{\partial \mathcal{H}_{\text{res}}^{13}(G, \psi)}{\partial \psi} = 0. \end{cases}$$

The equilibrium positions, for $e = 10^{-3}$, are:

$$(\psi_1 = 0, G_1 = -4.324 \cdot 10^{-4}), \quad (\psi_2 = \pi, G_2 = -4.323 \cdot 10^{-4})$$

for AS and

$$(\psi_1 = 0, G_1 = -4.243 \cdot 10^{-4}), \quad (\psi_2 = \pi, G_2 = -4.229 \cdot 10^{-4})$$

for HA. In both cases the first one is an elliptic point, while the second one is an hyperbolic point. For $e = 10^{-3}$, we compute the amplitude of the resonant islands around the elliptic solution (ψ_1, G_1) which is given by

$$\Delta G_{AS} = 7.437 \cdot 10^{-6}, \quad \Delta G_{HA} = 2.058 \cdot 10^{-5}$$

(compare with the numerical integrations shown in figure 7, top panels).

We stress that, for all values of the eccentricity, we find two equilibria associated to equation (5.15), corresponding to $\psi = 0$ and $\psi = \pi$, respectively centre and saddle. Remarkably, contrary to the 1 : 2 resonance, the 1 : 3 resonance does not experience bifurcations in the considered range of values of the eccentricity up to 0.3.

6. Conclusions

Once considered a classical topic, ring dynamics has experienced renewed interest due to the identification of ring systems surrounding minor celestial bodies. From a dynamical perspective, two aspects make the problem particularly interesting. Firstly, the irregular shapes of minor bodies need a precise characterization of the resulting gravitational potential. Secondly, the frequent occurrence of ring systems in a 1 : 3 resonance demands a thorough exploration of the underlying reasons for the selection of this specific resonance. In this study, we introduce the gravitational potential formulated in the epicyclic variables. This leads to a Hamiltonian function that exhibits the property of depending on variables evolving over different characteristic scales. Motivated by these considerations, the first part of this work is devoted to establishing that the Hamiltonian satisfies the multi-scale non-degeneracy condition. We start by showing that the Hamiltonian can be considered as a nearly-integrable system. With regard to the multi-scale condition, and after applying appropriate normalization procedures, we have verified that the assumptions formulated in [13, 25] are satisfied, thereby obtaining explicit bounds on the parameters that guarantee their validity. We consider this procedure to be useful for addressing a broad class of problems in Celestial Mechanics, particularly those governed by models that involve variables evolving on multiple, different scales.

To investigate the resonances, we average the resulting Hamiltonian over the fast variables to retain only the terms corresponding to specific resonances. In this procedure, we control the error induced by the expansion in polynomial series of the action variables, bearing in mind that we retain only a finite number of terms. This analysis is performed for the principal resonances, specifically the corotation, as well as the 1 : 2 and 1 : 3 resonances. Upon constructing the model, we undertake an analysis of the dynamical features of the resonances. Precisely, we determine the linear stability of the equilibria and we identify bifurcations as the parameters are varied. Specifically, bifurcation values are determined as the eccentricity of the particle within the ring varies over a reasonable range of values. This analysis enables us to draw conclusions regarding the different behaviour of the resonances. The corotation resonance is situated in close proximity to the central body, thereby experiencing substantial influence from the gravitational potential, particularly in instances where the body has an irregular shape. Besides, unlike the other resonances, the corotation resonance shows a very large amplitude of libration around the equilibria. The 1 : 2 resonance is situated at a greater distance from the central body, and presents bifurcations, even at low eccentricity values. This could potentially be a reason for the non-selection of the 1 : 2 resonance, as it experiences dynamical changes as the eccentricity varies, a scenario that may occur throughout the evolutionary history of the ring system. The 1 : 3 resonance is located even further from the central body, thereby resulting in a diminished interaction with the gravitational potential of the irregular body. Furthermore, the 1 : 3 resonance does not exhibit any bifurcation as the eccentricity varies and it has the smallest amplitude of libration, thereby providing enhanced stability relative to other resonances. Consequently, it is plausible to expect that the 1 : 3 resonance has been predominantly adopted by ring systems around minor bodies. Specific case studies, precisely Chariklo, Haumea and Quaoar, are investigated in [7] on the basis of the findings of the present work.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Explicit expression of the coefficients $c_{i,j}$, $V_{j,i}$, $\alpha_{i,j}$, $\beta_{i,j}$

In this appendix, we provide the expressions of the coefficients defined in section 3. We start with the coefficients $c_{i,j}$ for $i = 0, 1, 2$ and for j up to 4; we recall that the index i corresponds to the order of the variable I and it can assume only the values 0, 1, 2; the index j corresponds to the order of the variable ρ and it can assume any value from $3 - i$ to ∞ . The maximum value of j appearing in the Hamiltonian (3.16) corresponds to the order of truncation of the expansion in ρ . The expressions of the coefficients $c_{i,j}$ are the following:

$$\begin{aligned} c_{0,3} &= -\frac{P_{0,0} C_{0,0} \mathcal{G} M_P}{r_*^4} + \frac{4 P_{2,0} C_{2,0} \mathcal{G} M_P R^2}{r_*^6} + \frac{25 P_{4,0} C_{4,0} \mathcal{G} M_P R^4}{r_*^8} \\ &\quad + \frac{70 P_{6,0} C_{6,0} \mathcal{G} M_P R^6}{r_*^{10}} \\ c_{0,4} &= \frac{3 P_{0,0} C_{0,0} \mathcal{G} M_P}{2 r_*^5} - \frac{15 P_{2,0} C_{2,0} \mathcal{G} M_P R^2}{2 r_*^7} - \frac{115 P_{4,0} C_{4,0} \mathcal{G} M_P R^4}{2 r_*^9} \\ &\quad - \frac{385 P_{6,0} C_{6,0} \mathcal{G} M_P R^6}{2 r_*^{11}} \\ c_{1,2} &= \frac{3 n_*}{r_*^2} \quad c_{1,3} = -\frac{4 n_*}{r_*^3} \quad c_{1,4} = \frac{5 n_*}{r_*^4} \\ c_{2,1} &= -\frac{1}{r_*^3} \quad c_{2,2} = \frac{3}{2 r_*^4} \quad c_{2,3} = -\frac{2}{r_*^5} \quad c_{2,4} = \frac{5}{2 r_*^6}. \end{aligned}$$

Next, we provide the expressions of the coefficients $V_{j,i}$ appearing in equation (3.18), for j up to 4 and for i up to 3. We recall that the index j represents the order of the variable ρ thus, in principle, it varies from 0 to ∞ . The index i is half of the coefficient of the angle θ appearing in the expansion of the non-axisymmetric part U_{ns} . The expressions of the coefficients $V_{j,i}$ are the following:

$$\begin{aligned} V_{0,1} &= -\frac{P_{2,2} C_{2,2} \mathcal{G} M_P R^2}{r_*^3} - \frac{P_{4,2} C_{4,2} \mathcal{G} M_P R^4}{r_*^5} - \frac{P_{6,2} C_{6,2} \mathcal{G} M_P R^6}{r_*^7} \\ V_{0,2} &= -\frac{P_{4,4} C_{4,4} \mathcal{G} M_P R^4}{r_*^5} - \frac{P_{6,4} C_{6,4} \mathcal{G} M_P R^6}{r_*^7} \\ V_{0,3} &= -\frac{P_{6,6} C_{6,6} \mathcal{G} M_P R^6}{r_*^7} \\ V_{1,1} &= \frac{3 P_{2,2} C_{2,2} \mathcal{G} M_P R^2}{r_*^4} + \frac{5 P_{4,2} C_{4,2} \mathcal{G} M_P R^4}{r_*^6} + \frac{7 P_{6,2} C_{6,2} \mathcal{G} M_P R^6}{r_*^8} \\ V_{1,2} &= \frac{5 P_{4,4} C_{4,4} \mathcal{G} M_P R^4}{r_*^6} + \frac{7 P_{6,4} C_{6,4} \mathcal{G} M_P R^6}{r_*^8} \\ V_{1,3} &= \frac{7 P_{6,6} C_{6,6} \mathcal{G} M_P R^6}{r_*^8} \\ V_{2,1} &= -\frac{6 P_{2,2} C_{2,2} \mathcal{G} M_P R^2}{r_*^5} - \frac{15 P_{4,2} C_{4,2} \mathcal{G} M_P R^4}{r_*^7} - \frac{28 P_{6,2} C_{6,2} \mathcal{G} M_P R^6}{r_*^9} \\ V_{2,2} &= -\frac{15 P_{4,4} C_{4,4} \mathcal{G} M_P R^4}{r_*^7} - \frac{28 P_{6,4} C_{6,4} \mathcal{G} M_P R^6}{r_*^9} \\ V_{2,3} &= -\frac{28 P_{6,6} C_{6,6} \mathcal{G} M_P R^6}{r_*^9} \end{aligned}$$

$$\begin{aligned}
V_{3,1} &= \frac{10 P_{2,2} C_{2,2} \mathcal{G} M_P R^2}{r_*^6} + \frac{35 P_{4,2} C_{4,2} \mathcal{G} M_P R^4}{r_*^8} + \frac{84 P_{6,2} C_{6,2} \mathcal{G} M_P R^6}{r_*^{10}} \\
V_{3,2} &= \frac{35 P_{4,4} C_{4,4} \mathcal{G} M_P R^4}{r_*^8} + \frac{84 P_{6,4} C_{6,4} \mathcal{G} M_P R^6}{r_*^{10}} \\
V_{3,3} &= \frac{84 P_{6,6} C_{6,6} \mathcal{G} M_P R^6}{r_*^{10}} \\
V_{4,1} &= -\frac{15 P_{2,2} C_{2,2} \mathcal{G} M_P R^2}{r_*^7} - \frac{70 P_{4,2} C_{4,2} \mathcal{G} M_P R^4}{r_*^9} - \frac{210 P_{6,2} C_{6,2} \mathcal{G} M_P R^6}{r_*^{11}} \\
V_{4,2} &= -\frac{70 P_{4,4} C_{4,4} \mathcal{G} M_P R^4}{r_*^9} - \frac{210 P_{6,4} C_{6,4} \mathcal{G} M_P R^6}{r_*^{11}} \\
V_{4,3} &= -\frac{210 P_{6,6} C_{6,6} \mathcal{G} M_P R^6}{r_*^{11}}
\end{aligned}$$

Let us now write explicitly the coefficient $\alpha_{2i,j}, \beta_{2i,j}$ appearing in equation (3.25). Note that the index $2i$ is the coefficient of the angle θ , while the index j is the coefficient of the angle φ . In this appendix, $|i|$ varies from 0 to 3 and j from 0 to 4. We report the expressions only for the non-zero coefficients.

$$\begin{aligned}
\alpha_{-6,0} &= -\frac{315 P_{6,6} C_{6,6} \mathcal{G} M_P R^6 J^2}{\kappa_*^2 r_*^{11}} - \frac{28 P_{6,6} C_{6,6} \mathcal{G} M_P R^6 J}{|\kappa_*| r_*^9} - \frac{P_{6,6} C_{6,6} \mathcal{G} M_P R^6}{r_*^7} \\
\beta_{-6,1} &= \frac{63\sqrt{2} P_{6,6} C_{6,6} \mathcal{G} M_P R^6 J^{3/2}}{|\kappa_*|^{3/2} r_*^{10}} + \frac{7 P_{6,6} C_{6,6} \mathcal{G} M_P R^6 J^{1/2}}{\sqrt{2} |\kappa_*|^{1/2} r_*^8} \\
\alpha_{-6,2} &= \frac{210 P_{6,6} C_{6,6} \mathcal{G} M_P R^6 J^2}{\kappa_*^2 r_*^{11}} + \frac{14 P_{6,6} C_{6,6} \mathcal{G} M_P R^6 J}{|\kappa_*| r_*^9} \\
\beta_{-6,3} &= -\frac{21\sqrt{2} P_{6,6} C_{6,6} \mathcal{G} M_P R^6 J^{3/2}}{|\kappa_*|^{3/2} r_*^{10}} \\
\alpha_{-6,4} &= -\frac{105 P_{6,6} C_{6,6} \mathcal{G} M_P R^6 J^2}{2 \kappa_*^2 r_*^{11}} \\
\beta_{-6,4} &= 0 \\
\alpha_{-4,0} &= -\frac{315 P_{6,4} C_{6,4} \mathcal{G} M_P R^6 J^2}{\kappa_*^2 r_*^{11}} - \frac{105 P_{4,4} C_{4,4} \mathcal{G} M_P R^4 J^2}{\kappa_*^2 r_*^9} - \frac{28 P_{6,4} C_{6,4} \mathcal{G} M_P R^6 J}{|\kappa_*| r_*^9} \\
&\quad - \frac{15 P_{4,4} C_{4,4} \mathcal{G} M_P R^4 J}{|\kappa_*| r_*^7} - \frac{P_{6,4} C_{6,4} \mathcal{G} M_P R^6}{r_*^7} - \frac{P_{4,4} C_{4,4} \mathcal{G} M_P R^4}{r_*^5} \\
\beta_{-4,1} &= \frac{63\sqrt{2} P_{6,4} C_{6,4} \mathcal{G} M_P R^6 J^{3/2}}{|\kappa_*|^{3/2} r_*^{10}} + \frac{105 P_{4,4} C_{4,4} \mathcal{G} M_P R^4 J^{3/2}}{2\sqrt{2} |\kappa_*|^{3/2} r_*^8} + \frac{7 P_{6,4} C_{6,4} \mathcal{G} M_P R^6 J^{1/2}}{\sqrt{2} |\kappa_*|^{1/2} r_*^8} \\
&\quad + \frac{5 P_{4,4} C_{4,4} \mathcal{G} M_P R^4 J^{1/2}}{\sqrt{2} |\kappa_*|^{1/2} r_*^6} \\
\alpha_{-4,2} &= \frac{210 P_{6,4} C_{6,4} \mathcal{G} M_P R^6 J^2}{\kappa_*^2 r_*^{11}} + \frac{70 P_{4,4} C_{4,4} \mathcal{G} M_P R^4 J^2}{\kappa_*^2 r_*^9} + \frac{14 P_{6,4} C_{6,4} \mathcal{G} M_P R^6 J}{|\kappa_*| r_*^9} \\
&\quad + \frac{15 P_{4,4} C_{4,4} \mathcal{G} M_P R^4 J}{2 |\kappa_*| r_*^7} \\
\beta_{-4,3} &= -\frac{21\sqrt{2} P_{6,4} C_{6,4} \mathcal{G} M_P R^6 J^{3/2}}{|\kappa_*|^{3/2} r_*^{10}} - \frac{35 P_{4,4} C_{4,4} \mathcal{G} M_P R^4 J^{3/2}}{2\sqrt{2} |\kappa_*|^{3/2} r_*^8} \\
\alpha_{-4,4} &= -\frac{105 P_{6,4} C_{6,4} \mathcal{G} M_P R^6 J^2}{2 \kappa_*^2 r_*^{11}} - \frac{35 P_{4,4} C_{4,4} \mathcal{G} M_P R^4 J^2}{2 \kappa_*^2 r_*^9} \\
\alpha_{-2,0} &= -\frac{315 P_{6,2} C_{6,2} \mathcal{G} M_P R^6 J^2}{\kappa_*^2 r_*^{11}} - \frac{105 P_{4,2} C_{4,2} \mathcal{G} M_P R^4 J^2}{\kappa_*^2 r_*^9} - \frac{45 P_{2,2} C_{2,2} \mathcal{G} M_P R^2 J^2}{2 \kappa_*^2 r_*^7} \\
&\quad - \frac{28 P_{6,2} C_{6,2} \mathcal{G} M_P R^6 J}{|\kappa_*| r_*^9} - \frac{15 P_{4,2} C_{4,2} \mathcal{G} M_P R^4 J}{|\kappa_*| r_*^7} - \frac{6 P_{2,2} C_{2,2} \mathcal{G} M_P R^2 J}{|\kappa_*| r_*^5} \\
&\quad - \frac{P_{6,2} C_{6,2} \mathcal{G} M_P R^6}{r_*^7} - \frac{P_{4,2} C_{4,2} \mathcal{G} M_P R^4}{r_*^5} - \frac{P_{2,2} C_{2,2} \mathcal{G} M_P R^2}{r_*^3} \\
\beta_{-2,0} &= 0 \\
\alpha_{-2,1} &= 0
\end{aligned}$$

$$\begin{aligned}
\beta_{-2,1} &= \frac{63\sqrt{2}P_{6,2}C_{6,2}\mathcal{G}M_P R^6 J^{3/2}}{|\kappa_*|^{3/2}r_*^{10}} + \frac{105P_{4,2}C_{4,2}\mathcal{G}M_P R^4 J^{3/2}}{2\sqrt{2}|\kappa_*|^{3/2}r_*^8} + \frac{7P_{6,2}C_{6,2}\mathcal{G}M_P R^6 J^{1/2}}{\sqrt{2}|\kappa_*|^{1/2}r_*^8} \\
&\quad + \frac{15P_{2,2}C_{2,2}\mathcal{G}M_P R^2 J^{3/2}}{\sqrt{2}|\kappa_*|^{3/2}r_*^6} + \frac{5P_{4,2}C_{4,2}\mathcal{G}M_P R^4 J^{1/2}}{\sqrt{2}|\kappa_*|^{1/2}r_*^6} + \frac{3P_{2,2}C_{2,2}\mathcal{G}M_P R^2 J^{1/2}}{\sqrt{2}|\kappa_*|^{1/2}r_*^4} \\
\alpha_{-2,2} &= \frac{210P_{6,2}C_{6,2}\mathcal{G}M_P R^6 J^2}{\kappa_*^2 r_*^{11}} + \frac{70P_{4,2}C_{4,2}\mathcal{G}M_P R^4 J^2}{\kappa_*^2 r_*^9} + \frac{14P_{6,2}C_{6,2}\mathcal{G}M_P R^6 J}{|\kappa_*| r_*^9} \\
&\quad + \frac{15P_{2,2}C_{2,2}\mathcal{G}M_P R^2 J^2}{\kappa_*^2 r_*^7} + \frac{15P_{4,2}C_{4,2}\mathcal{G}M_P R^4 J}{2|\kappa_*| r_*^7} + \frac{3P_{2,2}C_{2,2}\mathcal{G}M_P R^2 J}{|\kappa_*| r_*^5} \\
\beta_{-2,3} &= -\frac{21\sqrt{2}P_{6,2}C_{6,2}\mathcal{G}M_P R^6 J^{3/2}}{|\kappa_*|^{3/2}r_*^{10}} - \frac{35P_{4,2}C_{4,2}\mathcal{G}M_P R^4 J^{3/2}}{2\sqrt{2}|\kappa_*|^{3/2}r_*^8} - \frac{5P_{2,2}C_{2,2}\mathcal{G}M_P R^2 J^{3/2}}{\sqrt{2}|\kappa_*|^{3/2}r_*^6} \\
\alpha_{-2,4} &= -\frac{105P_{6,2}C_{6,2}\mathcal{G}M_P R^6 J^2}{2\kappa_*^2 r_*^{11}} - \frac{35P_{4,2}C_{4,2}\mathcal{G}M_P R^4 J^2}{2\kappa_*^2 r_*^9} - \frac{15P_{2,2}C_{2,2}\mathcal{G}M_P R^2 J^2}{4\kappa_*^2 r_*^7} \\
\beta_{0,1} &= \frac{105\sqrt{2}P_{6,0}C_{6,0}\mathcal{G}M_P R^6 J^{3/2}}{|\kappa_*|^{3/2}r_*^{10}} + \frac{75P_{4,0}C_{4,0}\mathcal{G}M_P R^4 J^{3/2}}{\sqrt{2}|\kappa_*|^{3/2}r_*^8} + \frac{6\sqrt{2}P_{2,0}C_{2,0}\mathcal{G}M_P R^2 J^{3/2}}{|\kappa_*|^{3/2}r_*^6} \\
&\quad - \frac{3P_{0,0}C_{0,0}\mathcal{G}M_P J^{3/2}}{\sqrt{2}|\kappa_*|^{3/2}r_*^4} - \frac{3\sqrt{2}I^2 J^{3/2}}{|\kappa_*|^{3/2}r_*^5} - \frac{6\sqrt{2}n_* I J^{3/2}}{|\kappa_*|^{3/2}r_*^3} - \frac{\sqrt{2}I^2 J^{1/2}}{|\kappa_*|^{1/2}r_*^3} - \frac{2\sqrt{2}n_* I J^{1/2}}{|\kappa_*|^{1/2}r_*} \\
\alpha_{0,2} &= \frac{385P_{6,0}C_{6,0}\mathcal{G}M_P R^6 J^2}{\kappa_*^2 r_*^{11}} + \frac{115P_{4,0}C_{4,0}\mathcal{G}M_P R^4 J^2}{\kappa_*^2 r_*^9} + \frac{15P_{2,0}C_{2,0}\mathcal{G}M_P R^2 J^2}{\kappa_*^2 r_*^7} \\
&\quad - \frac{3P_{0,0}C_{0,0}\mathcal{G}M_P J^2}{\kappa_*^2 r_*^5} - \frac{5I^2 J^2}{\kappa_*^2 r_*^6} - \frac{10n_* I J^2}{\kappa_*^2 r_*^4} - \frac{3I^2 J}{2|\kappa_*| r_*^4} - \frac{3n_* I J}{|\kappa_*| r_*^2} \\
\beta_{0,3} &= -\frac{35\sqrt{2}P_{6,0}C_{6,0}\mathcal{G}M_P R^6 J^{3/2}}{|\kappa_*|^{3/2}r_*^{10}} - \frac{25P_{4,0}C_{4,0}\mathcal{G}M_P R^4 J^{3/2}}{\sqrt{2}|\kappa_*|^{3/2}r_*^8} - \frac{2\sqrt{2}P_{2,0}C_{2,0}\mathcal{G}M_P R^2 J^{3/2}}{|\kappa_*|^{3/2}r_*^6} \\
&\quad + \frac{P_{0,0}C_{0,0}\mathcal{G}M_P J^{3/2}}{\sqrt{2}|\kappa_*|^{3/2}r_*^4} + \frac{\sqrt{2}I^2 J^{3/2}}{|\kappa_*|^{3/2}r_*^5} + \frac{2\sqrt{2}n_* I J^{3/2}}{|\kappa_*|^{3/2}r_*^3} \\
\alpha_{0,4} &= -\frac{385P_{6,0}C_{6,0}\mathcal{G}M_P R^6 J^2}{4\kappa_*^2 r_*^{11}} - \frac{115P_{4,0}C_{4,0}\mathcal{G}M_P R^4 J^2}{4\kappa_*^2 r_*^9} - \frac{15P_{2,0}C_{2,0}\mathcal{G}M_P R^2 J^2}{4\kappa_*^2 r_*^7} \\
&\quad + \frac{3P_{0,0}C_{0,0}\mathcal{G}M_P J^2}{4\kappa_*^2 r_*^5} + \frac{5I^2 J^2}{4\kappa_*^2 r_*^6} + \frac{5n_* I J^2}{2\kappa_*^2 r_*^4} \\
\alpha_{2,0} &= -\frac{315P_{6,2}C_{6,2}\mathcal{G}M_P R^6 J^2}{\kappa_*^2 r_*^{11}} - \frac{105P_{4,2}C_{4,2}\mathcal{G}M_P R^4 J^2}{\kappa_*^2 r_*^9} - \frac{45P_{2,2}C_{2,2}\mathcal{G}M_P R^2 J^2}{2\kappa_*^2 r_*^7} \\
&\quad - \frac{28P_{6,2}C_{6,2}\mathcal{G}M_P R^6 J}{|\kappa_*| r_*^9} - \frac{15P_{4,2}C_{4,2}\mathcal{G}M_P R^4 J}{|\kappa_*| r_*^7} - \frac{6P_{2,2}C_{2,2}\mathcal{G}M_P R^2 J}{|\kappa_*| r_*^5} \\
&\quad - \frac{P_{6,2}C_{6,2}\mathcal{G}M_P R^6}{r_*^7} - \frac{P_{4,2}C_{4,2}\mathcal{G}M_P R^4}{r_*^5} - \frac{P_{2,2}C_{2,2}\mathcal{G}M_P R^2}{r_*^3} \\
\beta_{2,1} &= \frac{63\sqrt{2}P_{6,2}C_{6,2}\mathcal{G}M_P R^6 J^{3/2}}{|\kappa_*|^{3/2}r_*^{10}} + \frac{105P_{4,2}C_{4,2}\mathcal{G}M_P R^4 J^{3/2}}{2\sqrt{2}|\kappa_*|^{3/2}r_*^8} + \frac{7P_{6,2}C_{6,2}\mathcal{G}M_P R^6 J^{1/2}}{\sqrt{2}|\kappa_*|^{1/2}r_*^8} \\
&\quad + \frac{15P_{2,2}C_{2,2}\mathcal{G}M_P R^2 J^{3/2}}{\sqrt{2}|\kappa_*|^{3/2}r_*^6} + \frac{5P_{4,2}C_{4,2}\mathcal{G}M_P R^4 J^{1/2}}{\sqrt{2}|\kappa_*|^{1/2}r_*^6} + \frac{3P_{2,2}C_{2,2}\mathcal{G}M_P R^2 J^{1/2}}{\sqrt{2}|\kappa_*|^{1/2}r_*^4} \\
\alpha_{2,2} &= \frac{210P_{6,2}C_{6,2}\mathcal{G}M_P R^6 J^2}{\kappa_*^2 r_*^{11}} + \frac{70P_{4,2}C_{4,2}\mathcal{G}M_P R^4 J^2}{\kappa_*^2 r_*^9} + \frac{14P_{6,2}C_{6,2}\mathcal{G}M_P R^6 J}{|\kappa_*| r_*^9} \\
&\quad + \frac{15P_{2,2}C_{2,2}\mathcal{G}M_P R^2 J^2}{\kappa_*^2 r_*^7} + \frac{15P_{4,2}C_{4,2}\mathcal{G}M_P R^4 J}{2|\kappa_*| r_*^7} + \frac{3P_{2,2}C_{2,2}\mathcal{G}M_P R^2 J}{|\kappa_*| r_*^5} \\
\beta_{2,3} &= -\frac{21\sqrt{2}P_{6,2}C_{6,2}\mathcal{G}M_P R^6 J^{3/2}}{|\kappa_*|^{3/2}r_*^{10}} - \frac{35P_{4,2}C_{4,2}\mathcal{G}M_P R^4 J^{3/2}}{2\sqrt{2}|\kappa_*|^{3/2}r_*^8} - \frac{5P_{2,2}C_{2,2}\mathcal{G}M_P R^2 J^{3/2}}{\sqrt{2}|\kappa_*|^{3/2}r_*^6} \\
\alpha_{2,4} &= -\frac{105P_{6,2}C_{6,2}\mathcal{G}M_P R^6 J^2}{2\kappa_*^2 r_*^{11}} - \frac{35P_{4,2}C_{4,2}\mathcal{G}M_P R^4 J^2}{2\kappa_*^2 r_*^9} - \frac{15P_{2,2}C_{2,2}\mathcal{G}M_P R^2 J^2}{4\kappa_*^2 r_*^7} \\
\alpha_{4,0} &= -\frac{315P_{6,4}C_{6,4}\mathcal{G}M_P R^6 J^2}{\kappa_*^2 r_*^{11}} - \frac{105P_{4,4}C_{4,4}\mathcal{G}M_P R^4 J^2}{\kappa_*^2 r_*^9} - \frac{28P_{6,4}C_{6,4}\mathcal{G}M_P R^6 J}{|\kappa_*| r_*^9}
\end{aligned}$$

$$\begin{aligned}
& -\frac{15 P_{4,4} C_{4,4} \mathcal{G} M_P R^4 J}{|\kappa_*| r_*^7} - \frac{P_{6,4} C_{6,4} \mathcal{G} M_P R^6}{r_*^7} - \frac{P_{4,4} C_{4,4} \mathcal{G} M_P R^4}{r_*^5} \\
\beta_{4,1} &= \frac{63\sqrt{2} P_{6,4} C_{6,4} \mathcal{G} M_P R^6 J^{3/2}}{|\kappa_*|^{3/2} r_*^{10}} + \frac{105 P_{4,4} C_{4,4} \mathcal{G} M_P R^4 J^{3/2}}{2\sqrt{2} |\kappa_*|^{3/2} r_*^8} + \frac{7 P_{6,4} C_{6,4} \mathcal{G} M_P R^6 J^{1/2}}{\sqrt{2} |\kappa_*|^{1/2} r_*^8} \\
& + \frac{5 P_{4,4} C_{4,4} \mathcal{G} M_P R^4 J^{1/2}}{\sqrt{2} |\kappa_*|^{1/2} r_*^6} \\
\alpha_{4,2} &= \frac{210 P_{6,4} C_{6,4} \mathcal{G} M_P R^6 J^2}{\kappa_*^2 r_*^{11}} + \frac{70 P_{4,4} C_{4,4} \mathcal{G} M_P R^4 J^2}{\kappa_*^2 r_*^9} + \frac{14 P_{6,4} C_{6,4} \mathcal{G} M_P R^6 J}{|\kappa_*| r_*^9} \\
& + \frac{15 P_{4,4} C_{4,4} \mathcal{G} M_P R^4 J}{2 |\kappa_*| r_*^7} \\
\beta_{4,3} &= -\frac{21\sqrt{2} P_{6,4} C_{6,4} \mathcal{G} M_P R^6 J^{3/2}}{|\kappa_*|^{3/2} r_*^{10}} - \frac{35 P_{4,4} C_{4,4} \mathcal{G} M_P R^4 J^{3/2}}{2\sqrt{2} |\kappa_*|^{3/2} r_*^8} \\
\alpha_{4,4} &= -\frac{105 P_{6,4} C_{6,4} \mathcal{G} M_P R^6 J^2}{2 \kappa_*^2 r_*^{11}} - \frac{35 P_{4,4} C_{4,4} \mathcal{G} M_P R^4 J^2}{2 \kappa_*^2 r_*^9} \\
\alpha_{6,0} &= -\frac{315 P_{6,6} C_{6,6} \mathcal{G} M_P R^6 J^2}{\kappa_*^2 r_*^{11}} - \frac{28 P_{6,6} C_{6,6} \mathcal{G} M_P R^6 J}{|\kappa_*| r_*^9} - \frac{P_{6,6} C_{6,6} \mathcal{G} M_P R^6}{r_*^7} \\
\beta_{6,1} &= \frac{63\sqrt{2} P_{6,6} C_{6,6} \mathcal{G} M_P R^6 J^{3/2}}{|\kappa_*|^{3/2} r_*^{10}} + \frac{7 P_{6,6} C_{6,6} \mathcal{G} M_P R^6 J^{1/2}}{\sqrt{2} |\kappa_*|^{1/2} r_*^8} \\
\alpha_{6,2} &= \frac{210 P_{6,6} C_{6,6} \mathcal{G} M_P R^6 J^2}{\kappa_*^2 r_*^{11}} + \frac{14 P_{6,6} C_{6,6} \mathcal{G} M_P R^6 J}{|\kappa_*| r_*^9} \\
\beta_{6,3} &= -\frac{21\sqrt{2} P_{6,6} C_{6,6} \mathcal{G} M_P R^6 J^{3/2}}{|\kappa_*|^{3/2} r_*^{10}} \\
\alpha_{6,4} &= -\frac{105 P_{6,6} C_{6,6} \mathcal{G} M_P R^6 J^2}{2 \kappa_*^2 r_*^{11}} .
\end{aligned}$$

Appendix B. Comparison between U_s and U_{ns}

In this appendix, we compare the sizes of U_s , defined in (3.3), and U_{ns} , defined in (3.4), showing that $|U_s| > |U_{ns}|$. We specify that both quantities depend on several small parameters, namely, the oblateness $\mathcal{O}_b$ and elongation $\mathcal{E}_\ell$ (see (2.4) and (2.5)), as well as the ratio R/r between the reference length, defined in (2.2), and the distance of the particle from the centre of the ellipsoid. To proceed, we introduce the quantities

$$\varepsilon = \frac{a^2 - b^2}{a^2 + b^2 - 2c^2} = \frac{1}{2} \frac{\mathcal{E}_\ell}{\mathcal{O}_b}, \quad \beta = a^2 + b^2 - 2c^2. \quad (\text{B.1})$$

From the relations $0 < c < b < a \leq 1$ and $c > a/\sqrt{2}$ (see section 2), one has that $\varepsilon, \beta < 1$; in the following, we will assume $\varepsilon \leq 1/2$, noticing that both test cases AS and HA (see section 5) satisfy such inequality. The potentials U_s and U_{ns} depend on the coefficients $C_{2\ell,2p}$ (see (2.3)) and the associated Legendre polynomials $P_{2\ell,2p}$ (see (2.8)), $\ell \geq 0$ and $p \in \{0, \dots, \ell\}$, which are such that

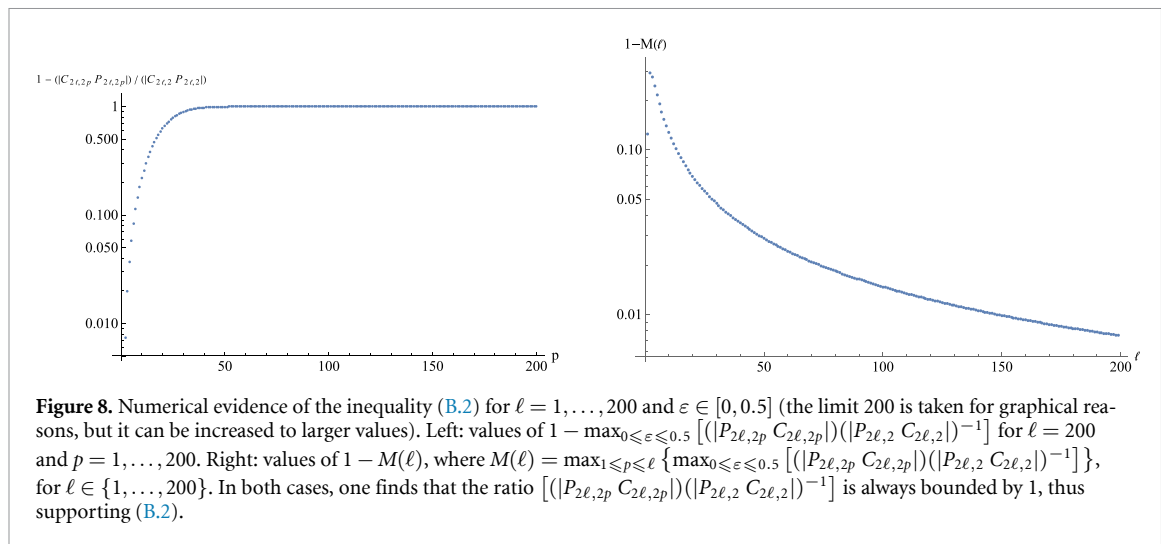
$$\begin{aligned}
|C_{2\ell,2p}| &= \frac{3}{R^{2\ell}} \frac{\ell!(2\ell-2p)!}{2^{2p}(2\ell+3)(2\ell+1)!} |2 - \delta_{0,p}| \sum_{k=0}^{\lfloor \frac{\ell-p}{2} \rfloor} \frac{\varepsilon^{p+2k} \beta^\ell}{2^{\ell-p-2k} 16^k (\ell-p-2k)!(p+k)!k!} \\
|P_{2\ell,2p}(0)| &= \frac{(2\ell+2p)!}{2^{2\ell}(\ell+p)!(\ell-p)!} .
\end{aligned}$$

From now on, we will assume that the following inequality, based on strong numerical evidence (see figure 8) holds for every $\ell \geq 1$ and every $p \in \{1, \dots, \ell\}$:

$$|P_{2\ell,2p}| |C_{2\ell,2p}| \leq |P_{2\ell,2}| |C_{2\ell,2}|. \quad (\text{B.2})$$

From the above equations, and observing that $r \geq 1$, one finds by direct computation

$$|U_{ns}| \leq \frac{\mathcal{G} M_P}{r} \sum_{\ell=1}^{\infty} a_\ell \quad (\text{B.3})$$



with

$$a_\ell = \frac{3}{2^{3\ell}} \frac{\ell}{2\ell+3} \frac{(2\ell-2)!}{(\ell-1)!} S_\ell, \quad S_\ell = \sum_{k=0}^{\lfloor \frac{\ell-1}{2} \rfloor} \frac{1}{2^{4k}(\ell-1-2k)!(k+1)!k!}.$$

To estimate the sum in (B.3), we compute directly the first 19 terms of the sum, and then provide a suitable bound for a_ℓ , $\ell \geq 20$: in particular, we find

$$\sum_{\ell=1}^{19} a_\ell \leq 0.12881$$

As for the cases $\ell \geq 20$, we observe that $\lfloor (\ell-1)/2 \rfloor$ is always bounded from above by $\ell/2$, so that

$$\begin{aligned} S_\ell &= \frac{1}{(\ell-1)!} + \frac{1}{2^5(\ell-3)!} + \sum_{k=2}^{\lfloor \frac{\ell-1}{2} \rfloor} \frac{1}{2^{4k}(\ell-1-2k)!(k+1)!k!} \\ &\leq \frac{1}{(\ell-1)!} + \frac{1}{2^5(\ell-3)!} + \frac{\ell}{2^{10} \cdot 3 (\ell-5)!}; \end{aligned}$$

thus, for $\ell \geq 20$ one has

$$a_\ell \leq \frac{3}{2^{3\ell}} \frac{\ell}{2\ell+3} \binom{2\ell-2}{\ell-1} \left[1 + \frac{(\ell-1)!}{2^5(\ell-3)!} + \frac{\ell!}{2^{10} \cdot 3 (\ell-5)!} \right].$$

The binomial term can be estimated by using Stirling formula in the version by Robbins ([28]), holding for any $n > 0$:

$$\sqrt{2\pi n} \left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12n+1}\right) \leq n! \leq \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12n}\right), \quad (\text{B.4})$$

from which we obtain:

$$\binom{2\ell-2}{\ell-1} \leq \frac{4^{\ell-1}}{\sqrt{\pi(\ell-1)}}.$$

On the other hand, since the function

$$f(\ell) = \left(10^{-2} - \frac{1}{2^{10} \cdot 3}\right) \ell^5 - \frac{\ell^2}{2^5} - 1$$

is strictly positive for every $\ell \geq 20$, one gets

$$1 + \frac{(\ell-1)!}{2^5(\ell-3)!} + \frac{\ell!}{2^{10} \cdot 3 (\ell-5)!} \leq 10^{-2} \ell^5.$$

We can then estimate the infinite sum $\sum_{\ell=20}^{\infty} a_{\ell}$ as

$$\sum_{\ell=20}^{\infty} a_{\ell} \leq 10^{-2} \frac{3}{16} \sqrt{\frac{5}{\pi}} \sum_{\ell=20}^{\infty} \frac{\ell^{9/2}}{2^{\ell}} \leq 10^{-2} \frac{3}{16} \sqrt{\frac{5}{\pi}} \int_{19}^{\infty} \frac{x^{9/2}}{2^x} dx \leq 0.00537,$$

where we used that, for every $\ell \geq 20$, $\ell(2\ell+3)^{-1} < 1/2$ and $\ell-1 > 4\ell/5$.

In conclusion, we have that

$$|U_{ns}| \leq \frac{\mathcal{G}M_P}{r} \left(\sum_{\ell=1}^{19} a_{\ell} + \sum_{\ell=20}^{\infty} a_{\ell} \right) \leq 0.5 \frac{\mathcal{G}M_P}{r};$$

from the definition of U_s , this implies that

$$|U_s| > |U_{ns}|,$$

whenever $r \geq 1$ and $a/\sqrt{2} < c < b < a \leq 1$. Last inequality is due to the fact that $|U_s|$ is the sum of positive quantities, precisely $(R/r)^{2\ell}$ and the positive products $C_{2\ell,0}P_{2\ell,0}(0)$, since $C_{2\ell,0}$ and $P_{2\ell,0}(0)$ are both negative for ℓ odd and both positive for ℓ even.

B.1. Estimates on the order of U_{ns}

In this section we estimate the size of U_{ns} , which will be used to bound the axisymmetric term in section 4. In particular, we will prove the following result.

Proposition 21. *Let a, b, c be the semiaxes of the central body, satisfying $1 \geq a > b > c > a/\sqrt{2}$, and let $\rho = r - r_*$ be the distance of the particle from the exact resonance, which is determined as the solution of (3.8). Assuming that the following inequality is satisfied for some constant $K \geq 7$:*

$$K\mathcal{O}_b \left(\frac{R}{r_*} \right)^2 < 1,$$

then we can bound $U_{ns} = U_{ns}(r, \theta)$ as

$$|U_{ns}(r_* + \rho, \theta)| \leq \mathcal{O}_b \frac{R^2}{r_*^3} W(\rho), \quad (\text{B.5})$$

where $W(\rho)$ is a suitable bounded function.

Proof. Let us start by observing that, since $\mathcal{E} \leq \mathcal{O}_b$, then

$$\left| (a^2 - b^2)^{p+2k} \left(c^2 - \frac{1}{2}(a^2 + b^2) \right)^{\ell-p-2k} \right| \leq 2^{\ell} R^{2\ell} \mathcal{O}_b^{\ell},$$

so that, from (2.3),

$$|C_{2\ell,2p}| \leq 6 \frac{\ell!(2\ell-2p)!}{2^{2p}(2\ell+3)(2\ell+1)!} \sum_{k=0}^{\lfloor \frac{\ell-p}{2} \rfloor} \frac{2^{\ell} \mathcal{O}_b^{\ell}}{16^k (\ell-p-2k)!(p+k)!k!}.$$

As a consequence, from (3.4), one has

$$|U_{ns}| \leq \frac{\mathcal{G}M_P}{r} \sum_{\ell=1}^{\infty} \left(\frac{R}{r} \right)^{2\ell} \mathcal{O}_b^{\ell} \sum_{p=1}^{\ell} \sum_{k=0}^{\lfloor \frac{\ell-p}{2} \rfloor} d_{\ell pk}, \quad (\text{B.6})$$

where

$$d_{\ell pk} = 6 \frac{\ell!(2\ell-2p)!(2\ell+2p)!}{(2\ell+3)(2\ell+1)!(\ell+p)!(\ell-p)!} \frac{2^{\ell}}{2^{2p+2\ell+4k}(\ell-p-2k)!(p+k)!k!},$$

which gives

$$d_{\ell pk} \leq 6 \frac{\ell!(2\ell-2p)!(2\ell+2p)!}{(2\ell)!(\ell+p)!(\ell-p)!(\ell-p-2k)!(p+k)!k!}.$$

From (B.4) we have that, for any $n \in \mathbb{N}$,

$$B\sqrt{n}\left(\frac{n}{e}\right)^n \leq n! \leq A\sqrt{n}\left(\frac{n}{e}\right)^n, \quad B := \sqrt{2\pi}, \quad A := \sqrt{2\pi}e^{\frac{1}{12}}.$$

Using the above relations, one has that

$$d_{\ell pk} \leq c_1 \frac{\ell^\ell (2\ell - 2p)^{2\ell - 2p} (2\ell + 2p)^{2\ell + 2p}}{(2\ell)^{2\ell} (\ell + p)^{\ell + p} (\ell - p)^{\ell - p} (\ell - p - 2k)^{\ell - p - 2k} (p + k)^{p + k} k^k},$$

with $c_1 = 6\sqrt{2}A^3/B^6 = 6\sqrt{2}(2\pi)^{-3/2}e^{1/4}$.

For short, we introduce x and y such that

$$x = \frac{p}{\ell} \in [0, 1], \quad y = \frac{k}{\ell} \in \left[0, \frac{1-x}{2}\right];$$

then, we have

$$d_{\ell pk} \leq c_1 \left(4 \frac{(1-x)^{1-x}(1+x)^{1+x}}{(1-x-2y)^{1-x-2y}(x+y)^{x+y}y^y} \right)^\ell := c_1 (F(x, y))^\ell,$$

where the function $F(x, y)$ is considered in the compact domain

$$D = \left\{ (x, y) : x \in [0, 1], y \in \left[0, \frac{1-x}{2}\right] \right\},$$

within which F reaches its maximum value $F_{\max}$. Hence, we obtain $d_{\ell pk} \leq c_1 F_{\max}^\ell$ and using (B.6), one finds

$$|U_{ns}| \leq c_1 \frac{\mathcal{G}M_p}{r} \sum_{\ell=1}^{\infty} \left(\frac{R}{r}\right)^{2\ell} \ell^2 \mathcal{O}_b^\ell F_{\max}^\ell,$$

since the terms of the two sums are at most ℓ^2 . Consider now the function

$$f(\ell) = \left(\ell^{\frac{2}{3}} F_{\max}\right)^\ell;$$

by direct computation, one finds that

$$f(\ell) \leq K^\ell, \quad K = \left(\max_{\ell \geq 1} \ell^{\frac{2}{3}}\right) F_{\max}.$$

We finally get

$$|U_{ns}| \leq c_1 \frac{\mathcal{G}M_p}{r} \sum_{\ell=1}^{\infty} \left[K \left(\frac{R}{r}\right)^2 \mathcal{O}_b \right]^\ell.$$

If $K\mathcal{O}_b(R/r)^2 < 1$, then the series converges, providing

$$|U_{ns}| \leq c_1 \mathcal{G}M_p \mathcal{O}_b \frac{R^2}{r^3} \frac{K}{1 - K\mathcal{O}_b\left(\frac{R}{r}\right)^2}.$$

Substituting $r = r_* + \rho$, one gets

$$|U_{ns}| \leq c_1 \mathcal{G}M_p \mathcal{O}_b \frac{R^2}{r_*^3} W(\rho),$$

where

$$W(\rho) = \left(1 + \frac{\rho}{r_*}\right)^{-3} \frac{K}{1 - K\mathcal{O}_b\left(\frac{R}{r_*}\right)^2 \left(1 + \frac{\rho}{r_*}\right)^{-2}}.$$

Since $1 + \rho/r_* \geq 1$, we have

$$W(\rho) \leq \frac{K}{1 - K\mathcal{O}_b\left(\frac{R}{r_*}\right)^2},$$

which is bounded, provided

$$K\mathcal{O}_b\left(\frac{R}{r_*}\right)^2 < 1.$$

□

We notice that, by numerical computation, one can take

$$F_{\max} = 3.2382, \quad K = 7.$$

Equation (B.5) will be used in section 4 to express the non-axisymmetric contribution of the particle's Hamiltonian as a perturbation of the axisymmetric one, after suitable normalization steps. In particular, with reference to (4.13), we need that $|U_{ns}|$ is of order higher than $\eta^3/r_*^{5/2}$, where η is a parameter of the size of ρ . For this reason, it is useful to state the following result, which is a direct consequence of proposition 21.

Corollary 22. *With reference to (B.5), if $\mathcal{O}_b \leq \rho^{3+\alpha}(r_*^{1/2}/R^2)$ with $\alpha > 0$, then*

$$|U_{ns}(r_* + \rho, \theta)| \leq \frac{\rho^{3+\alpha}}{r_*^{5/2}} W(\rho),$$

where $W(\rho)$ is a suitable bounded function.

Appendix C. Values of the coefficients associated to the 1 : 2 and 1 : 3 resonances

The polynomials $\delta_1, \dots, \delta_5$ for the resonance 1 : 2 (see equation (5.12)) for the AS case are given by the following expressions:

$$\begin{aligned} \delta_1(G) &= -2.8 \cdot 10^{-6} + 4.5 \cdot 10^{-3} G + 3.7 \cdot 10^{-2} G^2 + 2.1 \cdot 10^{-1} G^3 \\ &\quad + 1.1 G^4 + 4.8 G^5 + 21 G^6 + 88 G^7 + 3.7 \cdot 10^2 G^8, \\ \delta_2(G) &= -6.3 \cdot 10^{-11} + 2 \cdot 10^{-7} G - 1.6 \cdot 10^{-4} G^2 - 2.4 \cdot 10^{-3} G^3 \\ &\quad - 2.2 \cdot 10^{-2} G^4 - 1.6 \cdot 10^{-1} G^5 - 9.8 \cdot 10^{-1} G^6 - 5.5 G^7 - 30 G^8, \\ \delta_3(G) &= -2.4 \cdot 10^{-15} + 1.2 \cdot 10^{-11} G - 1.9 \cdot 10^{-8} G^2 + 9.9 \cdot 10^{-6} G^3 \\ &\quad + 2.1 \cdot 10^{-4} G^4 + 2.7 \cdot 10^{-3} G^5 + 2.5 \cdot 10^{-2} G^6 \\ &\quad + 1.9 \cdot 10^{-1} G^7 + 1.4 G^8, \\ \delta_4(G) &= -1.1 \cdot 10^{-19} + 7.4 \cdot 10^{-16} G - 1.8 \cdot 10^{-12} G^2 + 1.9 \cdot 10^{-9} G^3 \\ &\quad - 7.6 \cdot 10^{-7} G^4 - 2.2 \cdot 10^{-5} G^5 - 3.4 \cdot 10^{-4} G^6 \\ &\quad - 3.9 \cdot 10^{-3} G^7 - 3.8 \cdot 10^{-2} G^8, \\ \delta_5(G) &= -6.1 \cdot 10^{-24} + 4.9 \cdot 10^{-20} G - 1.6 \cdot 10^{-16} G^2 + 2.6 \cdot 10^{-13} G^3 \\ &\quad - 2.1 \cdot 10^{-10} G^4 + 6.3 \cdot 10^{-8} G^5 + 2.2 \cdot 10^{-6} G^6 \\ &\quad + 4.2 \cdot 10^{-5} G^7 + 6.1 \cdot 10^{-4} G^8, \end{aligned}$$

while for the HA case, they are given by

$$\begin{aligned} \delta_1(G) &= -2 \cdot 10^{-5} + 3.4 \cdot 10^{-2} G + 3.1 \cdot 10^{-1} G^2 + 2.0 G^3 \\ &\quad + 11 G^4 + 61 G^5 + 3.3 \cdot 10^2 G^6 + 1.7 \cdot 10^3 G^7 + 9.5 \cdot 10^3 G^8, \\ \delta_2(G) &= -3.4 \cdot 10^{-9} + 1.1 \cdot 10^{-5} G - 9.5 \cdot 10^{-3} G^2 - 1.5 \cdot 10^{-1} G^3 \\ &\quad - 1.5 G^4 - 12 G^5 - 87 G^6 - 5.7 \cdot 10^2 G^7 - 3.7 \cdot 10^3 G^8, \\ \delta_3(G) &= -9.0 \cdot 10^{-13} + 4.6 \cdot 10^{-9} G - 7.7 \cdot 10^{-6} G^2 + 4.2 \cdot 10^{-3} G^3 \\ &\quad + 9.8 \cdot 10^{-2} G^4 + 1.3 G^5 + 13 G^6 + 1.1 \cdot 10^2 G^7 + 9.1 \cdot 10^2 G^8, \\ \delta_4(G) &= -2.9 \cdot 10^{-16} + 2.0 \cdot 10^{-12} G - 5.0 \cdot 10^{-9} G^2 + 5.6 \cdot 10^{-6} G^3 \\ &\quad - 2.3 \cdot 10^{-3} G^4 - 6.8 \cdot 10^{-2} G^5 - 1.1 G^6 - 13 G^7 - 1.4 \cdot 10^2 G^8, \\ \delta_5(G) &= -9.1 \cdot 10^{-20} + 7.7 \cdot 10^{-16} G - 2.6 \cdot 10^{-12} G^2 + 4.4 \cdot 10^{-9} G^3 \\ &\quad - 3.6 \cdot 10^{-6} G^4 + 1.2 \cdot 10^{-3} G^5 + 4.2 \cdot 10^{-2} G^6 + 8.0 \cdot 10^{-1} G^7 + 12 G^8. \end{aligned}$$

The polynomials $\delta_1, \dots, \delta_4$ for the 1 : 3 resonance (see equation (5.16)) for the AS case are given by

$$\begin{aligned}\delta_1(G) &= -1.6 \cdot 10^{-9} - 8.8 \cdot 10^{-6} G - 1.3 \cdot 10^{-2} G^2 - 2 \cdot 10^{-1} G^3 \\ &\quad - 2.2 G^4 - 20 G^5 - 1.6 \cdot 10^2 G^6 - 1.2 \cdot 10^3 G^7 - 9.0 \cdot 10^3 G^8, \\ \delta_2(G) &= -3.0 \cdot 10^{-17} - 3.4 \cdot 10^{-13} G - 1.4 \cdot 10^{-9} G^2 - 2.7 \cdot 10^{-6} G^3 \\ &\quad - 2.0 \cdot 10^{-3} G^4 - 5.7 \cdot 10^{-2} G^5 - 9.7 \cdot 10^{-1} G^6 - 13 G^7 - 1.4 \cdot 10^2 G^8, \\ \delta_3(G) &= -9.3 \cdot 10^{-25} - 1.6 \cdot 10^{-20} G - 1.1 \cdot 10^{-16} G^2 - 4.3 \cdot 10^{-13} G^3 \\ &\quad - 9.3 \cdot 10^{-10} G^4 - 1.1 \cdot 10^{-6} G^5 - 5.3 \cdot 10^{-4} G^6 - 2.1 \cdot 10^{-2} G^7 - 4.7 \cdot 10^{-1} G^8, \\ \delta_4(G) &= -3.7 \cdot 10^{-32} - 8.4 \cdot 10^{-28} G - 8.3 \cdot 10^{-24} G^2 - 4.7 \cdot 10^{-20} G^3 \\ &\quad - 1.7 \cdot 10^{-16} G^4 - 3.8 \cdot 10^{-13} G^5 - 5.4 \cdot 10^{-10} G^6 \\ &\quad - 4.4 \cdot 10^{-7} G^7 - 1.5 \cdot 10^{-4} G^8,\end{aligned}$$

and for the HA case, they take the following expression

$$\begin{aligned}\delta_1(G) &= -1.1 \cdot 10^{-8} - 6.7 \cdot 10^{-5} G - 9.8 \cdot 10^{-2} G^2 - 1.7 G^3 \\ &\quad - 19 G^4 - 1.9 \cdot 10^2 G^5 - 1.7 \cdot 10^3 G^6 - 1.5 \cdot 10^4 G^7 - 1.3 \cdot 10^5 G^8, \\ \delta_2(G) &= -1.6 \cdot 10^{-15} - 1.9 \cdot 10^{-11} G - 8.2 \cdot 10^{-8} G^2 - 1.6 \cdot 10^{-4} G^3 \\ &\quad - 1.2 \cdot 10^{-1} G^4 - 3.6 G^5 - 66 G^6 - 9.5 \cdot 10^2 G^7 - 1.1 \cdot 10^4 G^8, \\ \delta_3(G) &= -3.7 \cdot 10^{-22} - 6.4 \cdot 10^{-18} G - 4.6 \cdot 10^{-14} G^2 - 1.8 \cdot 10^{-10} G^3 \\ &\quad - 4.0 \cdot 10^{-7} G^4 - 4.7 \cdot 10^{-4} G^5 - 2.4 \cdot 10^{-1} G^6 - 10 G^7 - 2.4 \cdot 10^2 G^8, \\ \delta_4(G) &= -9.6 \cdot 10^{-29} - 2.2 \cdot 10^{-24} G - 2.3 \cdot 10^{-20} G^2 - 1.3 \cdot 10^{-16} G^3 \\ &\quad - 4.8 \cdot 10^{-13} G^4 - 1.1 \cdot 10^{-9} G^5 - 1.6 \cdot 10^{-6} G^6 - 1.3 \cdot 10^{-3} G^7 - 4.9 \cdot 10^{-1} G^8.\end{aligned}$$

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