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SINGULAR NEUMANN (p,q)-EQUATIONS

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Abstract. We consider a nonlinear parametric Neumann problem driven by the sum of a p-Laplacian and of a q-Laplacian and exhibiting in the reaction the competing effects of a singular term and of a resonant term. Using variational methods together with suitable truncation and comparison techniques, we show that for small values of the parameter the problem has at least two positive smooth solutions.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following parametric Neumann (p,q)-equation:

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u(z) - \Delta_q u(z) + \xi(z)u(z)^{p-1} = \lambda u(z)^{-\gamma} + f(z,u(z)) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \quad u \geq 0. \end{cases}$$

For $r \in (1,+\infty)$ by Δ_r we denote the r-Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u) \quad \text{for all } u \in W^{1,r}(\Omega).$$

In problem (P_λ) we have $1 < q < p < +\infty$, $\xi \in L^\infty(\Omega)$ with $\xi(z) \geq 0$ for a.a. $z \in \Omega$. In the reaction (right hand side), $\lambda > 0$ is a parameter and $\lambda u^{-\gamma}$ is a singular term with $0 < \gamma < 1$. The perturbation $f(z,x)$ is a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \to f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z,x)$ is continuous) and we assume that $f(z,\cdot)$ exhibits $(p-1)$-linear growth near $+\infty$ and it can be resonant with respect to the principal eigenvalue $\widehat{\lambda}_1 > 0$ of the differential operator $u \to -\Delta_p u + \xi(z)u^{p-1}$ with Neumann boundary condition. So, the reaction of problem (P_λ) exhibits the competing effects of singular and resonant terms. In the boundary condition $\dfrac{\partial u}{\partial n}$ denotes the outward normal derivative of u.

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We look for positive solutions. Using variational tools based on the critical point theory, together with truncation, perturbation and comparison techniques, we show that for $\lambda > 0$ small, problem (P_λ) has at least two positive smooth solutions.

Singular problems were studied primarily in the context of Dirichlet equations, driven by the Laplacian or p-Laplacian. In this direction we mention the works of Gasiński-Papageorgiou [5], Giacomoni-Schindler-Takáč [6], Hirano-Saccon-Shioji [7], Kyritsi-Papageorgiou [8], Lair-Shaker [9], Papageorgiou-Rădulescu [14], Papageorgiou-Rădulescu-Repovš [17], Papageorgiou-Smyrlis [18], Perera-Zhang [20], Sun-Wu-Long [22]. To the best of our knowledge there are no papers dealing with singular (p, q)-equations. We mention that (p, q)-equations arise in problems of mathematical physics (see Cherfils-Il′ yasov [3]) and recently attracted considerable interest. We mention the survey paper of Marano-Mosconi [11] and the recent work of Papageorgiou-Vetro [19] for equations with variable exponents. The interested reader can find additional references in them.

2. Mathematical Background, Hypotheses, Auxiliary Results

Let X be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Given $\varphi \in C^1(X, \mathbb{R})$, we say that φ satisfies the "Cerami condition" (the "C-condition" for short), if it has the following property:

"Every sequence $\{u_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and $(1 + \|u_n\|)\varphi'(u_n) \to 0$ in X^* as $n \to +\infty$, admits a strongly convergent subsequence".

This is a compactness-type condition on the functional $\varphi(\cdot)$ which compensates for the fact that the ambient space X need not be locally compact (being in general infinite dimensional). Using this condition one can prove a deformation theorem from which follows the minimax theory of the critical values of φ. One of the main results of this theory is the "mountain pass theorem" of Ambrosetti-Rabinowitz [2].

Theorem 1. *If $\varphi \in C^1(X, \mathbb{R})$ satisfies the C-condition, $u_0, u_1 \in X$, $0 < \rho < \|u_1 - u_0\|$, $\max\{\varphi(u_0), \varphi(u_1)\} < \inf\{\varphi(u) : \|u - u_0\| = \rho\} = m_\rho$ and $c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} \varphi(\gamma(t))$ with $\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$, then $c \geq m_\rho$ and c is a critical value of φ (that is, there exists $\widehat{u} \in X$ such that $\varphi(\widehat{u}) = c$, $\varphi'(\widehat{u}) = 0$).*

The Sobolev space $W^{1,p}(\Omega)$ and the Banach space $C^1(\overline{\Omega})$ are the main spaces in the analysis of problem (P_λ). By $\|\cdot\|$ we denote the norm of $W^{1,p}(\Omega)$ given by

$$\|u\| = \left[\|u\|_p^p + \|\nabla u\|_p^p\right]^{1/p} \quad \text{for all } u \in W^{1,p}(\Omega).$$

The Banach space $C^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone given by

$$C_+ = \{u \in C^1(\overline{\Omega}) \,:\, u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$D_+ = \{u \in C_+ \,:\, u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

In fact D_+ is also the interior of C_+ when $C^1(\overline{\Omega})$ is furnished with the relative $C(\overline{\Omega})$-norm topology.

For any $r \in (1,+\infty)$, let $A_r : W^{1,r}(\Omega) \to W^{1,r}(\Omega)^*$ be the nonlinear map defined by

$$\langle A_r(u), h\rangle = \int_\Omega |\nabla u|^{r-2}(\nabla u, \nabla h)_{\mathbb{R}^N} dz \quad \text{for all } u, h \in W^{1,r}(\Omega).$$

From Motreanu-Motreanu-Papageorgiou [13] (p. 40), we have:

Proposition 1. *The map $A_r : W^{1,r}(\Omega) \to W^{1,r}(\Omega)^*$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$ (that is, if $u_n \xrightarrow{w} u$ in $W^{1,r}(\Omega)$ and $\limsup\limits_{n\to+\infty}\langle A_r(u_n), u_n - u\rangle \leq 0$, then $u_n \to u$ in $W^{1,r}(\Omega)$).*

We impose the following conditions on the potential function:

$\underline{H(\xi)}$: $\xi \in L^\infty(\Omega)$, $\xi(z) \geq 0$ for a.a. $z \in \Omega$ and the inequality is strict on a set of positive measure.

Lemma 1. *If hypothesis $H(\xi)$ holds, then there exists $c_0 > 0$ such that*

$$c_0\|u\|^p \leq \|\nabla u\|_p^p + \int_\Omega \xi(z)|u|^p dz = \mu_p(u) \quad \textit{for all } u \in W^{1,p}(\Omega).$$

Consider the following nonlinear Neumann eigenvalue problem:

$$\begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \widehat{\lambda}|u(z)|^{p-2}u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{1}$$

From Motreanu-Motreanu-Papageorgiou [13], we know that this problem has a smallest eigenvalue $\widehat{\lambda}_1(p)$ which has the following properties:

- $\widehat{\lambda}_1(p)$ is isolated (that is, if $\widehat{\sigma}(p)$ denotes the spectrum of (1), then there exists $\varepsilon > 0$ such that $(\widehat{\lambda}_1(p), \widehat{\lambda}_1(p) + \varepsilon) \cap \widehat{\sigma}(p) = \emptyset$).
- $\widehat{\lambda}_1(p)$ is simple (that is, if $\widehat{u}$, $\widehat{v}$ are eigenfunctions corresponding to $\widehat{\lambda}_1(p)$ then $\widehat{u} = \theta\widehat{v}$ with $\theta \neq 0$).
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$$\widehat{\lambda}_1(p) = \inf\left\{\frac{\mu_p(u)}{\|u\|_p^p} : u \in W^{1,p}(\Omega), \quad u \neq 0\right\}. \tag{2}$$

On account of Lemma 1 and (2), we have that $\widehat{\lambda}_1(p) > 0$. The above properties imply that the elements of the eigenspace corresponding to $\widehat{\lambda}_1(p)$ do not change sign and belong to $C^1(\overline{\Omega})$ (nonlinear regularity theory, see for example, Gasiński-Papageorgiou [4] (pp. 737-738)). By $\widehat{u}(1,p)$ we denote the non-negative, L^p-normalized (that is, $\|\widehat{u}(1,p)\|_p = 1$) eigenfunction corresponding to $\widehat{\lambda}_1(p) > 0$. We know that $\widehat{u}(1,p) \in C_+$ and in fact from the nonlinear maximum principle (see, for example, Gasiński-Papageorgiou [4] (p. 738)), we have that $\widehat{u}(1,p) \in D_+$. The infimum in (2) is realized at $\widehat{u}(1,p)$. Every other eigenfunction $\widehat{u} \in C^1(\overline{\Omega})$ corresponding to an eigenvalue $\widehat{\lambda} \neq \widehat{\lambda}_1(p)$, is nodal (that is, $\widehat{u}$ is sign changing).

We will also use a weighted version of the eigenvalue problem (1). So, let $m \in L^\infty(\Omega)$, $m(z) \geq 0$ for a.a. $z \in \Omega$, $m \not\equiv 0$. We consider the following nonlinear eigenvalue problem:

$$\begin{cases} -\Delta_p u(z) + \xi(z)|u(z)|^{p-2}u(z) = \widetilde{\lambda} m(z)|u(z)|^{p-2}u(z) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

This problem too has a smallest eigenvalue $\widetilde{\lambda}_1(p,m) > 0$ which is isolated, simple and admits the following variational characterization

$$\widetilde{\lambda}_1(p,m) = \inf\left[\frac{\mu_p(u)}{\int_\Omega m(z)|u|^p dz} : u \in W^{1,p}(\Omega),\ u \neq 0\right]. \tag{3}$$

Also the corresponding positive L^p-normalized eigenfunction $\widetilde{u}_1(p,m)$ belongs in D_+ and realizes the infimum in (3). These properties and (3) lead to the following strict monotonicity property for the map $m \to \widetilde{\lambda}_1(p,m)$.

Proposition 2. *If $m_1, m_2 \in L^\infty(\Omega)$, $0 \leq m_1(z) \leq m_2(z)$ for a.a. $z \in \Omega$ and both inequalities are strict on sets of positive measure, then $\widetilde{\lambda}_1(p,m_2) < \widetilde{\lambda}_1(p,m_1)$.*

For $x \in \mathbb{R}$, we set $x^\pm = \max\{\pm x, 0\}$. Then for $u \in W^{1,p}(\Omega)$ we define $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W^{1,p}(\Omega), \quad u = u^+ - u^-, \quad |u| = u^+ + u^-.$$

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Given $u, v \in W^{1,p}(\Omega)$ with $u \leq v$, we set

$$[u,v] = \left\{y \in W^{1,p}(\Omega) : u(z) \leq y(z) \leq v(z) \quad \text{for a.a. } z \in \Omega\right\}.$$

By $\mathrm{int}_{C^1(\overline{\Omega})}[u,v]$ we denote the interior in the $C^1(\overline{\Omega})$-norm topology of the set $[u,v] \cap C^1(\overline{\Omega})$. Also, $[u) = \{y \in W^{1,p}(\Omega) : u(z) \leq y(z) \quad \text{for a.a. } z \in \Omega\}$.

The hypotheses on the perturbation term $f(z,x)$ are the following:

$\underline{H(f)}$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(i) for every $\rho > 0$, there exists $a_\rho \in L^\infty(\Omega)$ such that

$$|f(z,x)| \leq a_\rho(z) \text{ for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \rho;$$

(ii) $\widehat{\lambda}_1(p) \leq \liminf_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} \leq \limsup_{x\to+\infty} \frac{f(z,x)}{x^{p-1}} \leq c_0$ uniformly for a.a. $z \in \Omega$;

(iii) if $F(z,x) = \int_0^x f(z,s)ds$, then

$$\frac{f(z,x)x - pF(z,x)}{x^q} \to -\infty \text{ uniformly for a.a. } z \in \Omega \text{ as } x \to +\infty;$$

(iv) there exists $w \in D_+$ such that

$$A_p(w) + A_q(w) \geq 0 \text{ in } W^{1,p}(\Omega)^*, \quad \Delta_p w + \Delta_q w \in L^{p'}(\Omega) \quad \left(\frac{1}{p} + \frac{1}{p'} = 1\right),$$

$$w(z)^{-\gamma} + f(z,w(z)) \leq -c_w < 0 \text{ for a.a. } z \in \Omega;$$

(v) if $m_w = \min_{\overline{\Omega}} w > 0$ (recall $w \in D_+$, see (iv)), then there exists $\delta_0 \in (0, m_w)$ such that

$$0 < c_m \leq f(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 < m \leq x \leq \delta_0;$$

(vi) for every $\rho > 0$, there exists $\widehat{\xi}_\rho > 0$ such that for a.a. $z \in \Omega$ the function $x \to f(z,x)+\widehat{\xi}_\rho x^{p-1}$ is nondecreasing on $[0,\rho]$.

Remark 1. Since we are interested on positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$, without any loss of generality we may assume that

$$f(z,x) = 0 \quad \text{for a.a. } z \in \Omega, \text{ all } x \leq 0.$$

Hypotheses $H(f)\,(ii)$ allows for resonance to occur with respect to the principal eigenvalue $\widehat{\lambda}_1(p) > 0$. We shall see in the process of the proof that hypothesis $H(f)\,(iii)$ implies that this resonance occurs from the right of $\widehat{\lambda}_1(p) > 0$ in the sense that

$$\widehat{\lambda}_1(p)x^p - pF(z,x) \to -\infty \quad \text{uniformly for a.a } z \in \Omega, \text{ as } x \to +\infty.$$

This implies that the energy functional of the problem is indefinite and so the direct method of the calculus of variations can not be used directly on the energy functional of (P_λ). In hypothesis $H(f)\,(iv)$, $A_p(w) + A_q(w) \geq 0$ in $W^{1,p}(\Omega)^*$ means that

$$\langle A_p(w) + A_q(w), h\rangle \geq 0 \quad \text{ for all } h \in W^{1,p}(\Omega),\ h \geq 0.$$

If there exists $\widetilde{\eta} > 0$ such that

$$\widetilde{\eta}^{-\gamma} + f(z,\widetilde{\eta}) \leq -\widetilde{c} < 0 \quad \text{for a.a. } z \in \Omega,$$

then hypothesis $H(f)\,(iv)$ is satisfied. Hypothesis $H(f)\,(vi)$ is satisfied, if for example, for a.a. $z \in \Omega$ $f(z,\cdot)$ is differentiable on $(0,+\infty)$ and for every $\rho > 0$ there exists $\widehat{a}_\rho > 0$ such that

$$f'_x(z,x)x \geq -\widehat{a}_\rho x^{p-1} \quad \text{for a.a. } z \in \Omega, \text{ all } 0 \leq x \leq \rho.$$

Example 1. The following function satisfies hypotheses $H(f)$. For the sake of simplicity we drop the z-dependence:

$$f(x) = \begin{cases} x^{\tau-1} - 2x^{\theta-1} & \text{if } 0 \leq x \leq 1, \\ \widehat{\lambda}_1(p)x^{p-1} + x^{s-1} - (\widehat{\lambda}_1(p) + 2) & \text{if } 1 < x, \end{cases}$$

with $1 < \tau < s < p$, $\tau < \theta$.

3. A Purely Singular Problem

In this section we deal with the following purely singular Neumann (p,q)-equation:

$$(Au_\lambda) \qquad \begin{cases} -\Delta_p u(z) - \Delta_q u(z) + \xi(z)u(z)^{p-1} = \lambda u(z)^{-\gamma} & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \quad u \geq 0, \quad 0 < \gamma < 1. \end{cases}$$

Proposition 3. *If hypotheses $H(\xi)$ holds and $\lambda > 0$, then problem (Au_λ) admits a unique positive solution $\widetilde{u}_\lambda \in D_+$, $\lambda \to \widetilde{u}_\lambda$ is nondecreasing and $\|\widetilde{u}_\lambda\|_{C^1(\overline{\Omega})} \to 0$ as $\lambda \to 0^+$.*

Proof. Recall that $\mu_p : W^{1,p}(\Omega) \to \mathbb{R}$ is the C^1-functional defined by

$$\mu_p(u) = \|\nabla u\|_p^p + \int_\Omega \xi(z)|u|^p dz \quad \text{for all } u \in W^{1,p}(\Omega)$$

(see Lemma 1). Given $\varepsilon > 0$, we consider the C^1-functional $\psi_\lambda^\varepsilon : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\lambda^\varepsilon(u) = \frac{1}{p}\mu_p(u) + \frac{1}{q}\|\nabla u\|_q^q - \frac{\lambda}{1-\gamma}\int_\Omega \left[(u^+)^p + \varepsilon\right]^{\frac{1-\gamma}{p}} dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Using Lemma 1, we have

$$\psi_\lambda^\varepsilon(u) \geq c_0\|u\|^p - \frac{\lambda}{1-\gamma}\int_\Omega \left(u^+\right)^{1-\gamma} dz - \lambda c_1 \quad \text{for some } c_1 > 0, \text{ all } u \in W^{1,p}(\Omega),$$

$$\Rightarrow \quad \psi_\lambda^\varepsilon(\cdot) \text{ is coercive.}$$

Also using the Sobolev embedding theorem, we see that $\psi_\lambda^\varepsilon(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $u_\lambda^\varepsilon \in W^{1,p}(\Omega)$ such that

$$\psi_\lambda^\varepsilon(u_\lambda^\varepsilon) = \inf\left[\psi_\lambda^\varepsilon(u) : u \in W^{1,p}(\Omega)\right]. \tag{4}$$

Let $r \in (0,1)$. Then

$$\begin{aligned}\psi_\lambda^\varepsilon(r) &< \frac{r^p}{p}c_2 - \frac{\lambda r^{1-\gamma}}{1-\gamma}|\Omega|_N \quad \text{for some } c_2 > 0 \\ &< \frac{r^p}{p}c_2 + \frac{\lambda}{1-\gamma}\left[\varepsilon^{\frac{1-\gamma}{p}} - r^{1-\gamma}\right]|\Omega|_N.\end{aligned}$$

For $r > 2\varepsilon^{1/p}$ we have

$$\begin{aligned}&\frac{r^p}{p}c_2 + \frac{\lambda}{1-\gamma}\left[\varepsilon^{\frac{1-\gamma}{p}} - r^{1-\gamma}\right]|\Omega|_N \\ &< \frac{r^p}{p}c_2 - \frac{\lambda r^{1-\gamma}}{1-\gamma}\left[1 - \left(\frac{1}{2}\right)^{1-\gamma}\right]|\Omega|_N.\end{aligned}$$

Since $r \in (0,1)$ and $0 < 1-\gamma < 1 < p$, we can find $r_0 \in (0,1)$ small such that

$$\frac{r_0^p}{p}c_2 - \frac{\lambda r_0^{1-\gamma}}{1-\gamma}\left[1 - \left(\frac{1}{2}\right)^{1-\gamma}\right]|\Omega|_N < 0.$$

Therefore for $\varepsilon \in \left(0, \left(\frac{r_0}{2}\right)^p\right)$ we have

$$\begin{aligned}&\psi_\lambda^\varepsilon(r_0) < \psi_\lambda^\varepsilon(0) = -\frac{\lambda}{1-\gamma}\varepsilon^{\frac{1-\gamma}{p}}|\Omega|_N, \\ \Rightarrow \quad &\psi_\lambda^\varepsilon(u_\lambda^\varepsilon) < \psi_\lambda^\varepsilon(0) \quad \text{(see (4))}, \\ \Rightarrow \quad &u_\lambda^\varepsilon \neq 0.\end{aligned}$$

From (4) we have

$$(\psi_\lambda^\varepsilon)'(u_\lambda^\varepsilon) = 0,$$

$$\Rightarrow \quad \langle A_p(u_\lambda^\varepsilon), h\rangle + \langle A_q(u_\lambda^\varepsilon), h\rangle + \int_\Omega \xi(z)|u_\lambda^\varepsilon|^{p-2}u_\lambda^\varepsilon h dz$$

$$= \lambda \int_\Omega \left((u_\lambda^\varepsilon)^+\right)^{p-1}\left[\left((u_\lambda^\varepsilon)^+\right)^p + \varepsilon\right]^{\frac{1-(\gamma+p)}{p}} h dz \quad \text{for all } h \in W^{1,p}(\Omega). \tag{5}$$

In (5) we choose $h = -(u_\lambda^\varepsilon)^- \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} & \mu_p((u_\lambda^\varepsilon)^-) + \|\nabla (u_\lambda^\varepsilon)^-\|_q^q = 0, \\ \Rightarrow \quad & c_0\|(u_\lambda^\varepsilon)^-\|^p \le 0 \quad \text{(see Lemma 1)}, \\ \Rightarrow \quad & u_\lambda^\varepsilon \ge 0, \quad u_\lambda^\varepsilon \ne 0. \end{aligned}$$

From (5) it follows that

$$\begin{cases} -\Delta_p u_\lambda^\varepsilon(z) - \Delta_q u_\lambda^\varepsilon(z) + \xi(z)u_\lambda^\varepsilon(z)^{p-1} = \lambda u_\lambda^\varepsilon(z)^{p-1}\left[u_\lambda^\varepsilon(z)^p + \varepsilon\right]^{\frac{1-(\gamma+p)}{p}} \\ \hfill \text{for a.a. } z \in \Omega, \\ \dfrac{\partial u_\lambda^\varepsilon}{\partial n} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{6}$$

(see Papageorgiou-Rădulescu [15]). From (6) and Papageorgiou-Rădulescu [16], we have $u_\lambda^\varepsilon \in L^\infty(\Omega)$. Then the nonlinear regularity theory of Lieberman [10] implies that $u_\lambda^\varepsilon \in C_+ \setminus \{0\}$.

From (6) we have

$$\Delta_p u_\lambda^\varepsilon(z) + \Delta_q u_\lambda^\varepsilon(z) \le \|\xi\|_\infty u_\lambda^\varepsilon(z)^{p-1} \quad \text{for a.a. } z \in \Omega.$$

Then the strong maximum principle of Pucci-Serrin [21] (pp. 111, 120), implies that $u_\lambda^\varepsilon \in D_+$.

Claim: For every $\lambda > 0$, the set $\{u_\lambda^\varepsilon\}_{\varepsilon \in \left(0, \left(\frac{r_0}{2}\right)^p\right)} \subseteq W^{1,p}(\Omega)$ is bounded.

Arguing by contradiction, suppose that the Claim is not true. Then we can find $\{\varepsilon_n\}_{n\ge1} \subseteq \left(0, \left(\frac{r_0}{2}\right)^p\right)$ and $\{u_\lambda^n = u_\lambda^{\varepsilon_n}\}_{n\ge1} \subseteq D_+$ such that

$$\|u_\lambda^n\| \to +\infty \quad \text{as } n \to +\infty. \tag{7}$$

We set $y_\lambda^n = \dfrac{u_\lambda^n}{\|u_\lambda^n\|}$, $n \in \mathbb{N}$. Then

$$\|y_\lambda^n\| = 1 \quad \text{and} \quad y_\lambda^n \ge 0 \text{ for all } n \in \mathbb{N}. \tag{8}$$

From (5) we have

$$\langle \mu_p'(y_\lambda^n), h\rangle + \frac{1}{\|u_\lambda^n\|^{p-q}}\langle A_q(y_\lambda^n), h\rangle = \lambda \int_\Omega (y_\lambda^n)^{p-1}\left[(u_\lambda^n)^p + \varepsilon_n\right]^{\frac{1-(\gamma+p)}{p}} h dz \tag{9}$$

for all $h \in W^{1,p}(\Omega)$, all $n \in \mathbb{N}$. In (9) we choose $h = y_\lambda^n \in W^{1,p}(\Omega)$ and obtain

$$\mu_p(y_\lambda^n) + \frac{1}{\|u_\lambda^n\|^{p-q}}\|\nabla y_\lambda^n\|_q^q = \lambda \int_\Omega \frac{(y_\lambda^n)^p}{\left[(u_\lambda^n)^p + \varepsilon_n\right]^{\frac{p+\gamma-1}{p}}} dz \quad \text{for all } n \in \mathbb{N}. \tag{10}$$

From the first part of the proof we know

$$p\psi_\lambda^{\varepsilon_n}(u_\lambda^n) < 0 \quad \text{for all } n \in \mathbb{N},$$

$$\Rightarrow \quad \mu_p(y_\lambda^n) + \frac{p}{q\|u_\lambda^n\|^{p-q}}\|\nabla y_\lambda^n\|_q^q - \frac{\lambda p}{1-\gamma}\int_\Omega \frac{[(u_\lambda^n)^p + \varepsilon_n]^{\frac{1-\gamma}{p}}}{\|u_\lambda^n\|^p}dz < 0 \quad \text{for all } n \in \mathbb{N}. \tag{11}$$

From (10) and (11) and since $q < p$, we obtain

$$\begin{aligned} 0 &\le \int_\Omega \frac{(y_\lambda^n)^p}{[(u_\lambda^n)^p + \varepsilon_n]^{\frac{p+\gamma-1}{p}}}dz < \frac{p}{1-\gamma}\int_\Omega \frac{[(u_\lambda^n)^p + \varepsilon_n]^{\frac{1-\gamma}{p}}}{\|u_\lambda^n\|^p}dz \\ &\le \frac{p}{1-\gamma}\int_\Omega \frac{(u_\lambda^n)^{1-\gamma} + \varepsilon_n^{\frac{1-\gamma}{p}}}{\|u_\lambda^n\|^p}dz \to 0 \quad \text{as } n \to +\infty \text{ (see (7))}. \end{aligned}$$

We return to (10) and use this last convergence together with (7) and the fact that $q < p$. We obtain

$$\begin{aligned} &\mu_p(y_\lambda^n) \to 0 \quad \text{as } n \to +\infty, \\ \Rightarrow \quad & y_\lambda^n \to 0 \text{ in } W^{1,p}(\Omega) \quad \text{as } n \to +\infty \text{ (see Lemma 1)}. \end{aligned}$$

This contradicts (8). So, we have proved the Claim.

Next we consider a sequence $\{\varepsilon_n\}_{n\ge1} \subseteq \left(0, \left(\frac{r_0}{2}\right)^p\right)$ such that $\varepsilon_n \to 0^+$. On account of the Claim, we may assume that

$$u_\lambda^n = u_\lambda^{\varepsilon_n} \xrightarrow{w} \widetilde{u}_\lambda \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad u_\lambda^n = u_\lambda^{\varepsilon_n} \to \widetilde{u}_\lambda \text{ in } L^p(\Omega). \tag{12}$$

From (5) with $h = u_\lambda^n \in W^{1,p}(\Omega)$, we have

$$-\mu_p(u_\lambda^n) - \|\nabla u_\lambda^n\|_q^q + \int_\Omega \frac{\lambda(u_\lambda^n)^p}{[(u_\lambda^n)^p + \varepsilon_n]^{\frac{p+\gamma-1}{p}}}dz = 0 \quad \text{for all } n \in \mathbb{N}. \tag{13}$$

Also, from the first part of the proof, we know that

$$\begin{aligned} \mu_p(u_\lambda^n) + \frac{p}{q}\|\nabla u_\lambda^n\|_q^q - \frac{\lambda p}{1-\gamma}\int_\Omega [(u_\lambda^n)^p + \varepsilon_n]^{\frac{1-\gamma}{p}}\, dz \le -c_3 < 0 \\ \text{for some } c_3 > 0, \text{ all } n \in \mathbb{N}. \end{aligned} \tag{14}$$

Adding (13) and (14) and recalling that $q < p$, we obtain

$$\begin{aligned} 0 &\le \int_\Omega \frac{(u_\lambda^n)^p}{[(u_\lambda^n)^p + \varepsilon_n]^{\frac{p+\gamma-1}{p}}}dz \le -c_3 + \frac{p}{1-\gamma}\int_\Omega [(u_\lambda^n)^p + \varepsilon_n]^{\frac{1-\gamma}{p}}\, dz \\ &\le -c_3 + \frac{p}{1-\gamma}\int_\Omega \left[(u_\lambda^n)^{1-\gamma} + \varepsilon_n^{\frac{1-\gamma}{p}}\right] dz \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{15}$$

Suppose that $\widetilde{u}_\lambda \equiv 0$. Then from (12) we have

$$\begin{aligned} &\frac{p}{1-\gamma}\int_\Omega \left[(u_\lambda^n)^{1-\gamma} + \varepsilon_n^{\frac{1-\gamma}{p}}\right] dz \to 0 \quad \text{as } n \to +\infty, \\ \Rightarrow \quad & 0 \le \limsup_{n\to+\infty} \int_\Omega \frac{(u_\lambda^n)^p}{[(u_\lambda^n)^p + \varepsilon_n]^{\frac{p+\gamma-1}{p}}}dz \le -c_3 < 0 \quad \text{(see (15))}, \end{aligned}$$

a contradiction. Hence $\widetilde{u}_\lambda \neq 0$.

From (12) and by passing to a subsequence if necessary, we can say that there exists $\vartheta_\lambda \in L^p(\Omega)$, $\vartheta_\lambda(z) \geq 1$ for a.a. $z \in \Omega$, such that

$$\begin{gathered} 0 \leq u_\lambda^n(z) \leq \vartheta_\lambda(z) \quad \text{for a.a. } z \in \Omega, \text{ all } n \in \mathbb{N}, \\ u_\lambda^n(z) \to \widetilde{u}_\lambda(z) \quad \text{for a.a. } z \in \Omega, \text{ as } n \to +\infty. \end{gathered} \tag{16}$$

We set

$$\Omega_1^n = \{z \in \Omega : (u_\lambda^n - \widetilde{u}_\lambda)(z) > 0\} \quad \text{and} \quad \Omega_2^n = \{z \in \Omega : (u_\lambda^n - \widetilde{u}_\lambda)(z) < 0\}.$$

We have

$$\begin{aligned} &\int_\Omega \frac{(u_\lambda^n)^{p-1}}{[(u_\lambda^n)^p + \varepsilon_n]^{\frac{p+\gamma-1}{p}}}(u_\lambda^n - \widetilde{u}_\lambda)dz \\ &= \int_{\Omega_1^n} \frac{(u_\lambda^n)^{p-1}}{[(u_\lambda^n)^p + \varepsilon_n]^{\frac{p+\gamma-1}{p}}}(u_\lambda^n - \widetilde{u}_\lambda)dz + \int_{\Omega_2^n} \frac{(u_\lambda^n)^{p-1}}{[(u_\lambda^n)^p + \varepsilon_n]^{\frac{p+\gamma-1}{p}}}(u_\lambda^n - \widetilde{u}_\lambda)dz \\ &\leq \int_{\Omega_1^n} \frac{u_\lambda^n - \widetilde{u}_\lambda}{(u_\lambda^n)^\gamma}dz + \int_{\Omega_2^n} \frac{1}{2\vartheta_\lambda^\gamma}\left(\frac{u_\lambda^n}{\vartheta_\lambda}\right)^{p-1}(u_\lambda^n - \widetilde{u}_\lambda)dz \end{aligned} \tag{17}$$

for all $n \in \mathbb{N}$ (see (16) and recall that $\vartheta_\lambda(z) \geq 1$ a.e.). From (16) we have

$$0 \leq \widetilde{u}_\lambda \leq \vartheta_\lambda \quad \text{and} \quad 0 \leq u_\lambda^n \leq \vartheta_\lambda \quad \text{for all } n \in \mathbb{N}. \tag{18}$$

So, we have

$$\begin{aligned} &\int_{\Omega_1^n} \frac{u_\lambda^n - \widetilde{u}_\lambda}{(u_\lambda^n)^\gamma}dz = \int_{\Omega_1^n} \left[(u_\lambda^n)^{1-\gamma} - \widetilde{u}_\lambda(u_\lambda^n)^{-\gamma}\right]dz \quad \text{for all } n \in \mathbb{N}, \\ \Rightarrow \quad &\lim_{n\to+\infty} \int_{\Omega_1^n} \frac{u_\lambda^n - \widetilde{u}_\lambda}{(u_\lambda^n)^\gamma}dz = 0 \quad \text{(see (16), (18))}. \end{aligned} \tag{19}$$

In addition we have

$$\int_{\Omega_2^n} \frac{1}{2\vartheta_\lambda^\gamma}\left(\frac{u_\lambda^n}{\vartheta_\lambda}\right)^{p-1}(u_\lambda^n - \widetilde{u}_\lambda)dz \to 0 \quad \text{as } n \to +\infty \text{ (see (16))}. \tag{20}$$

Returning to (17), passing to the limit as $n \to +\infty$ and using (19) and (20), we obtain

$$\lim_{n\to+\infty} \int_\Omega \frac{(u_\lambda^n)^{p-1}}{[(u_\lambda^n)^p + \varepsilon_n]^{\frac{p+\gamma-1}{p}}}(u_\lambda^n - \widetilde{u}_\lambda)dz = 0. \tag{21}$$

In (5) we choose $h = u_\lambda^n - \widetilde{u}_\lambda \in W^{1,p}(\Omega)$, pass to the limit as $n \to +\infty$ and use (12) and (21). Then

$$\begin{aligned} &\limsup_{n\to+\infty} \left[\langle A_p(u_\lambda^n), u_\lambda^n - \widetilde{u}_\lambda\rangle + \langle A_q(u_\lambda^n), u_\lambda^n - \widetilde{u}_\lambda\rangle\right] \leq 0, \\ \Rightarrow \quad &\limsup_{n\to+\infty} \left[\langle A_p(u_\lambda^n), u_\lambda^n - \widetilde{u}_\lambda\rangle + \langle A_q(\widetilde{u}_\lambda), u_\lambda^n - \widetilde{u}_\lambda\rangle\right] \leq 0, \end{aligned}$$

$$\text{(since } A_q(\cdot) \text{ is monotone)}$$

$$\Rightarrow \quad \limsup_{n\to+\infty}\langle A_p(u_\lambda^n), u_\lambda^n - \widetilde{u}_\lambda\rangle \leq 0 \quad \text{(see (12))},$$

$$\Rightarrow \quad u_\lambda^n \to \widetilde{u}_\lambda \text{ in } W^{1,p}(\Omega), \quad \widetilde{u}_\lambda \geq 0, \quad \widetilde{u}_\lambda \neq 0 \quad \text{(see Proposition 1)}. \tag{22}$$

We may assume that $\|u_\lambda^n\|_\infty < 1$, since on $\{u_\lambda^n \geq 1\}$ we can use the dominated convergence theorem

$$\begin{aligned}
&\left|\int_\Omega \frac{(u_\lambda^n)^{p-1}}{[(u_\lambda^n)^p + \varepsilon_n]^{\frac{p+\gamma-1}{p}}} h dz\right| \\
&\leq \int_\Omega \frac{(u_\lambda^n)^{p-1}}{[(u_\lambda^n)^p + \varepsilon_n]^{\frac{p+\gamma-1}{p}}} |h| dz \\
&\leq \int_\Omega \frac{(u_\lambda^n)^{p-1}}{[(u_\lambda^n)^p + \varepsilon_n]^{\frac{p+\gamma-1}{pp'}}} |h| dz \\
&\leq \left[\int_\Omega \frac{(u_\lambda^n)^{p}}{[(u_\lambda^n)^p + \varepsilon_n]^{\frac{p+\gamma-1}{p}}} dz\right]^{1/p'} \|h\|_p \quad \text{(by Hölder's inequality)} \\
&\leq \left[\int_\Omega (u_\lambda^n)^{1-\gamma} dz\right]^{1/p'} \|h\|_p \\
&\leq c_4 \|u_\lambda^n\|^{1-\gamma}\|h\|_p \quad \text{for some } c_4 > 0, \text{ all } n \in \mathbb{N} \text{ big}, \\
&\leq c_5 \|h\|_p \quad \text{for some } c_5 > 0, \text{ all } h \in W^{1,p}(\Omega), \text{ all } n \in \mathbb{N},
\end{aligned}$$

$$\Rightarrow \quad \left\{\frac{(u_\lambda^n)^{p-1}}{[(u_\lambda^n)^p + \varepsilon_n]^{\frac{p+\gamma-1}{p}}}\right\}_{n\geq 1} \subseteq L^{p'}(\Omega) \quad \text{is bounded } \left(\frac{1}{p} + \frac{1}{p'} = 1\right).$$

In addition, we have

$$\frac{(u_\lambda^n)^{p-1}}{[(u_\lambda^n)^p + \varepsilon_n]^{\frac{p+\gamma-1}{p}}} \to \widetilde{u}_\lambda^{-\gamma} \quad \text{for a.a. } z \in \Omega \text{ as } n \to +\infty \text{ (see (16))}.$$

Then, we have

$$\frac{(u_\lambda^n)^{p-1}}{[(u_\lambda^n)^p + \varepsilon_n]^{\frac{p+\gamma-1}{p}}} \xrightarrow{w} \widetilde{u}_\lambda^{-\gamma} \quad \text{in } L^{p'}(\Omega). \tag{23}$$

Therefore if in (5) we pass to the limit as $n \to +\infty$ and use (22) and (23), then we obtain

$$\langle A_p(\widetilde{u}_\lambda), h\rangle + \langle A_q(\widetilde{u}_\lambda), h\rangle + \int_\Omega \xi(z)\widetilde{u}_\lambda^{p-1} h dz = \lambda \int_\Omega \widetilde{u}_\lambda^{-\gamma} h dz \quad \text{for all } h \in W^{1,p}(\Omega). \tag{24}$$

In (24) first we choose $h = \dfrac{1}{[\widetilde{u}_\lambda^p + \varepsilon]^{\frac{p-1}{p}}} \in W^{1,p}(\Omega)$ $(\varepsilon > 0)$. Then

$$\int_\Omega \frac{\widetilde{u}_\lambda^{-\gamma}}{[\widetilde{u}_\lambda^p + \varepsilon]^{\frac{p-1}{p}}} dz \leq \int_\Omega \frac{\xi(z)\widetilde{u}_\lambda^{p-1}}{[\widetilde{u}_\lambda^p + \varepsilon]^{\frac{p-1}{p}}} dz.$$

Let $\varepsilon \to 0^+$ and use Fatou's lemma. Then

$$\int_\Omega \frac{1}{\widetilde{u}_\lambda^{p+\gamma-1}} dz \leq \|\xi\|_\infty |\Omega|_N \quad \text{(see hypothesis } H(\xi)). \tag{25}$$

Next in (24) we choose $h = \dfrac{1}{[\widetilde{u}_\lambda^p + \varepsilon]^{\frac{2(p-1)+\gamma}{p}}} \in W^{1,p}(\Omega)$. Then arguing as above, we obtain

$$\begin{aligned}
&\int_\Omega \frac{\widetilde{u}_\lambda^{-\gamma}}{\widetilde{u}_\lambda^{2(p-1)+\gamma}} dz = \int_\Omega \frac{1}{\widetilde{u}_\lambda^{2(p+\gamma-1)}} dz \\
&\leq \int_\Omega \xi(z) \frac{1}{\widetilde{u}_\lambda^{p+\gamma-1}} dz \leq \|\xi\|_\infty^2 |\Omega|_N \quad \text{(see (25))}.
\end{aligned}$$

Continuing this way, we obtain

$$\int_\Omega \frac{1}{\widetilde{u}_\lambda^{k(p+\gamma-1)}} dz \leq \|\xi\|_\infty^k |\Omega|_N \quad \text{for all } k \in \mathbb{N}.$$

From this we infer that

$$\begin{aligned}
&\widetilde{u}_\lambda^{-(p+\gamma-1)} \in L^\tau(\Omega) \quad \text{for all } \tau \geq 1, \\
&\limsup_{\tau \to +\infty} \|\widetilde{u}_\lambda^{-(p+\gamma-1)}\|_\tau < +\infty.
\end{aligned}$$

Then we have that $\widetilde{u}_\lambda^{-(p+\gamma-1)} \in L^\infty(\Omega)$. Note that

$$\widetilde{u}_\lambda^{-\gamma} = \widetilde{u}_\lambda^{-(p+\gamma-1)} \widetilde{u}_\lambda^{p-1}.$$

So, from (24) and Papageorgiou-Rădulescu [16], we infer that $\widetilde{u}_\lambda \in L^\infty(\Omega)$. Then the nonlinear regularity theory of Lieberman [10] implies that $\widetilde{u}_\lambda \in C_+ \setminus \{0\}$.

From (24) we have

$$\begin{cases} -\Delta_p \widetilde{u}_\lambda(z) - \Delta_q \widetilde{u}_\lambda(z) + \xi(z)\widetilde{u}_\lambda(z)^{p-1} = \lambda \widetilde{u}_\lambda(z)^{-\gamma} & \text{for a.a. } z \in \Omega, \\ \dfrac{\partial \widetilde{u}_\lambda}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{26}$$

From (26) as before via the nonlinear strong maximum principle of Pucci-Serrin [21], we have that $\widetilde{u}_\lambda \in D_+$.

Next we show that this solution is unique. To this end, let $\widetilde{v}_\lambda$ be another positive solution of (Au_λ). Again we can show that $\widetilde{v}_\lambda \in D_+$. We have

$$0 \leq \langle A_p(\widetilde{u}_\lambda) - A_p(\widetilde{v}_\lambda), \widetilde{u}_\lambda - \widetilde{v}_\lambda \rangle + \langle A_q(\widetilde{u}_\lambda) - A_q(\widetilde{v}_\lambda), \widetilde{u}_\lambda - \widetilde{v}_\lambda \rangle$$

$$
\begin{aligned}
&+\int_\Omega \xi(z)(\widetilde{u}_\lambda^{p-1}-\widetilde{v}_\lambda^{p-1})(\widetilde{u}_\lambda-\widetilde{v}_\lambda)dz=\lambda\int_\Omega[\widetilde{u}_\lambda^{-\gamma}-\widetilde{v}_\lambda^{-\gamma}](\widetilde{u}_\lambda-\widetilde{v}_\lambda)dz\leq 0,\\
\Rightarrow\quad &\widetilde{u}_\lambda=\widetilde{v}_\lambda.
\end{aligned}
$$

This proves the uniqueness of the positive solution of (Au_λ).

Next let $0<\tau<\eta$. We have

$$
\begin{aligned}
&0\leq\langle A_p(\widetilde{u}_\tau)-A_p(\widetilde{u}_\eta),(\widetilde{u}_\tau-\widetilde{u}_\eta)^+\rangle+\langle A_q(\widetilde{u}_\tau)-A_q(\widetilde{u}_\eta),(\widetilde{u}_\tau-\widetilde{u}_\eta)^+\rangle\\
&+\int_\Omega\xi(z)(\widetilde{u}_\tau^{p-1}-\widetilde{u}_\eta^{p-1})(\widetilde{u}_\tau-\widetilde{u}_\eta)^+dz=\int_\Omega[\tau\widetilde{u}_\tau^{-\gamma}-\eta\widetilde{u}_\eta^{-\gamma}](\widetilde{u}_\tau-\widetilde{u}_\eta)^+dz\\
&\leq\int_\Omega\tau[\widetilde{u}_\tau^{-\gamma}-\widetilde{u}_\eta^{-\gamma}](\widetilde{u}_\tau-\widetilde{u}_\eta)^+dz\leq 0,\\
\Rightarrow\quad &\widetilde{u}_\tau\leq\widetilde{u}_\eta.
\end{aligned}
$$

Therefore the map $\lambda\to\widetilde{u}_\lambda$ is nondecreasing from $(0,+\infty)$ into D_+. In (24) we choose $h=\widetilde{u}_\lambda\in W^{1,p}(\Omega)$. Then

$$
\begin{aligned}
&\|\nabla\widetilde{u}_\lambda\|_p^p+\int_\Omega\xi(z)\widetilde{u}_\lambda^p dz\leq\lambda\int_\Omega\widetilde{u}_\lambda^{1-\gamma}dz,\\
\Rightarrow\quad &c_0\|\widetilde{u}_\lambda\|^p\leq\lambda c_6\|\widetilde{u}_\lambda\|^{1-\gamma}\quad\text{for some } c_6,\\
\Rightarrow\quad &\{u_\lambda\}_{\lambda\in(0,1]}\subseteq W^{1,p}(\Omega)\text{ bounded and }\|\widetilde{u}_\lambda\|\to 0\text{ as }\lambda\to 0^+. \qquad (27)
\end{aligned}
$$

Then (27) and the nonlinear regularity theory of Lieberman [10], imply that we can find $\alpha\in(0,1)$ and $c_7>0$ such that

$$\widetilde{u}_\lambda\in C^{1,\alpha}(\overline{\Omega})\quad\text{and}\quad\|\widetilde{u}_\lambda\|_{C^{1,\alpha}(\overline{\Omega})}\leq c_7\quad\text{for all }\lambda\in(0,1].$$

Exploiting the compact embedding of $C^{1,\alpha}(\overline{\Omega})$ into $C^1(\overline{\Omega})$ and using (27), we conclude that $\widetilde{u}_\lambda\to 0$ in $C^1(\overline{\Omega})$ as $\lambda\to 0^+$. $\square$

4. Multiple Positive Solutions

On account of Proposition 3, we can find $\lambda_0>0$ such that

$$\widetilde{u}_\lambda(z)\in(0,\delta_0]\quad\text{for all }z\in\overline{\Omega},\text{ all }\lambda\in(0,\lambda_0], \qquad (28)$$

with $\delta_0>0$ as postulated by hypothesis $H(f)\,(v)$.

Using (28), suitable truncation and comparison techniques and the direct method of the calculus of variations, we can produce a positive solution for (P_λ) when $\lambda\in(0,\lambda_0]$.

Proposition 4. *If hypotheses $H(\xi)$, $H(f)$ hold and $\lambda\in(0,\lambda_0]$, then problem (P_λ) admits a positive solution $u_0\in D_+$ such that $u_0\in\mathrm{int}_{C^1(\overline{\Omega})}[\widetilde{u}_\lambda,w]$.*

Proof. Using $\widetilde{u}_\lambda \in D_+$ from Proposition 3 and $w \in D_+$ from hypothesis $H(f)\,(iv)$, we introduce the following truncation of the reaction in problem (P_λ):

$$g_\lambda(z,x) = \begin{cases} \lambda \widetilde{u}_\lambda(z)^{-\gamma} + f(z,\widetilde{u}_\lambda(z)) & \text{if } x < \widetilde{u}_\lambda(z), \\ \lambda x^{-\gamma} + f(z,x) & \text{if } \widetilde{u}_\lambda(z) \le x \le w(z), \\ \lambda w(z)^{-\gamma} + f(z,w(z)) & \text{if } w(z) < x. \end{cases} \tag{29}$$

Evidently this is a Carathéodory function. We set $G_\lambda(z,x) = \int_0^x g_\lambda(z,s)ds$ and consider the functional $\widehat{\psi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\widehat{\psi}_\lambda(u) = \frac{1}{p}\mu_p(u) + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega G_\lambda(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

Since $\widetilde{u}_\lambda, w \in D_+$, we see that $\widehat{\psi}_\lambda \in C^1(W^{1,p}(\Omega),\mathbb{R})$.

From (29) and Lemma 1, it follows that $\widehat{\psi}_\lambda$ is coercive. Also using the Sobolev embedding theorem, we see that $\widehat{\psi}_\lambda$ is sequentially weakly lower semicontinuous. Therefore by the Weierstrass-Tonelli theorem, we know that there exists $u_0 \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} & \widehat{\psi}_\lambda(u_0) = \inf\left[\widehat{\psi}_\lambda(u) : u \in W^{1,p}(\Omega)\right], \\ \Rightarrow\ & \widehat{\psi}'_\lambda(u_0) = 0, \end{aligned}$$

$$\Rightarrow\ \langle A_p(u_0),h\rangle + \langle A_q(u_0),h\rangle + \int_\Omega \xi(z)|u_0|^{p-2}u_0 h dz = \int_\Omega g_\lambda(z,u_0)hdz \text{ for all } h \in W^{1,p}(\Omega). \tag{30}$$

In (30), first we choose $h = (\widetilde{u}_\lambda - u_0)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} & \langle A_p(u_0),(\widetilde{u}_\lambda - u_0)^+\rangle + \langle A_q(u_0),(\widetilde{u}_\lambda - u_0)^+\rangle + \int_\Omega \xi(z)|u_0|^{p-2}u_0(\widetilde{u}_\lambda - u_0)^+dz \\ & = \int_\Omega \left[\lambda \widetilde{u}_\lambda^{-\gamma} + f(z,\widetilde{u}_\lambda)\right](\widetilde{u}_\lambda - u_0)^+dz \quad \text{(see (29))} \\ & \ge \int_\Omega \lambda \widetilde{u}_\lambda^{-\gamma}(\widetilde{u}_\lambda - u_0)^+dz \quad \text{(see (28) and hypothesis } H(f)\,(v)) \\ & = \langle A_p(\widetilde{u}_\lambda),(\widetilde{u}_\lambda - u_0)^+\rangle + \langle A_q(\widetilde{u}_\lambda),(\widetilde{u}_\lambda - u_0)^+\rangle + \int_\Omega \xi(z)\widetilde{u}_\lambda^{p-1}(\widetilde{u}_\lambda - u_0)^+dz, \\ \Rightarrow\ & \langle A_p(\widetilde{u}_\lambda) - A_p(u_0),(\widetilde{u}_\lambda - u_0)^+\rangle + \langle A_q(\widetilde{u}_\lambda) - A_q(u_0),(\widetilde{u}_\lambda - u_0)^+\rangle \\ & + \int_\Omega \xi(z)\left[\widetilde{u}_\lambda^{p-1} - |u_0|^{p-2}u_0\right](\widetilde{u}_\lambda - u_0)^+dz \le 0 \\ \Rightarrow\ & \widetilde{u}_\lambda \le u_0. \end{aligned}$$

Next in (30), we choose $h = (u_0 - w)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned} & \langle A_p(u_0),(u_0 - w)^+\rangle + \langle A_q(u_0),(u_0 - w)^+\rangle + \int_\Omega \xi(z)u_0^{p-1}(u_0 - w)^+dz \\ & = \int_\Omega \left[\lambda w^{-\gamma} + f(z,w)\right](u_0 - w)^+dz \quad \text{(see (29))} \end{aligned}$$

$$
\begin{aligned}
&\leq \langle A_p(w),(u_0-w)^+\rangle + \langle A_q(w),(u_0-w)^+\rangle + \int_\Omega \xi(z)w^{p-1}(u_0-w)^+dz \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(see hypothesis } H(f)\,(iv)), \\
\Rightarrow\quad &\langle A_p(u_0)-A_p(w),(u_0-w)^+\rangle + \langle A_q(u_0)-A_q(w),(u_0-w)^+\rangle \\
&+\int_\Omega \xi(z)\left[u_0^{p-1}-w^{p-1}\right](u_0-w)^+dz \leq 0 \\
\Rightarrow\quad &u_0 \leq w.
\end{aligned}
$$

So, we have proved that

$$u_0 \in [\widetilde{u}_\lambda, w]. \tag{31}$$

From (29), (30) and (31), we obtain

$$
\begin{cases}
-\Delta_p u_0(z) - \Delta_q u_0(z) + \xi(z)u_0(z)^{p-1} = \lambda u_0(z)^{-\gamma} + f(z,u_0(z)) \text{ for a.a. } z \in \Omega, \\
\dfrac{\partial u_0}{\partial n} = 0 \quad \text{on } \partial\Omega,
\end{cases} \tag{32}
$$

(see Papageorgiou-Rădulescu [15]). From (32) and Papageorgiou-Rădulescu [16], it follows that $u_0 \in L^\infty(\Omega)$. Then the nonlinear regularity theory of Lieberman [10] implies that $u_0 \in [\widetilde{u}_\lambda, w] \cap D_+$.

Let $\widetilde{\rho}_\lambda = \min_{\overline{\Omega}} \widetilde{u}_\lambda > 0$ (recall that $\widetilde{u}_\lambda \in D_+$). Also, let $\rho = \|w\|_{C^1(\overline{\Omega})}$ and let $\widehat{\xi}_\rho > 0$ be as postulated by hypothesis $H(f)\,(vi)$. We can always increase $\widehat{\xi}_\rho > 0$ if necessary, in order to guarantee that the function $x \to \lambda x^{-\gamma} + f(z,x) + \widehat{\xi}_\rho x^{p-1}$ is nondecreasing on $[\widetilde{\rho}_\lambda, \rho]$.

Let $\delta > 0$ and set $\widetilde{u}_\lambda^\delta = \widetilde{u}_\lambda + \delta \in D_+$. Then

$$
\begin{aligned}
&-\Delta_p \widetilde{u}_\lambda^\delta - \Delta_q \widetilde{u}_\lambda^\delta + (\xi(z)+\widehat{\xi}_\rho)(\widetilde{u}_\lambda^\delta)^{p-1} \\
&\leq -\Delta_p \widetilde{u}_\lambda - \Delta_q \widetilde{u}_\lambda + (\xi(z)+\widehat{\xi}_\rho)\widetilde{u}_\lambda^{p-1} + \sigma(\delta) \quad \text{with } \sigma(\delta)\to 0^+ \text{ as } \delta \to 0^+ \\
&\leq \lambda \widetilde{u}_\lambda^{-\gamma} + f(z,\widetilde{u}_\lambda) + \widehat{\xi}_\rho \widetilde{u}_\lambda^{p-1} \quad \text{for } \delta > 0 \text{ small so that } \sigma(\delta) \leq c_{\widetilde{\rho}_\lambda} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(see hypothesis } H(f)\,(v) \text{ and (28))} \\
&\leq \lambda u_0^{-\gamma} + f(z,u_0) + \widehat{\xi}_\rho u_0^{p-1} \quad (\text{since } \widetilde{u}_\lambda \leq u_0) \\
&= -\Delta_p u_0 - \Delta_q u_0 + (\xi(z)+\widehat{\xi}_\rho)u_0^{p-1} \quad \text{for a.a. } z \in \Omega,
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow\quad &\widetilde{u}_\lambda^\delta \leq u_0 \quad \text{for all } \delta > 0 \text{ small,} \\
\Rightarrow\quad &u_0 - \widetilde{u}_\lambda \in D_+. \qquad (33)
\end{aligned}
$$

Similarly, we set $u_0^\delta = u_0 + \delta \in D_+$, for $\delta > 0$. We have

$$
\begin{aligned}
&-\Delta_p u_0^\delta - \Delta_q u_0^\delta + (\xi(z)+\widehat{\xi}_\rho)(u_0^\delta)^{p-1} \\
&\leq -\Delta_p u_0 - \Delta_q u_0 + (\xi(z)+\widehat{\xi}_\rho)u_0^{p-1} + \widetilde{\sigma}(\delta) \quad \text{with } \widetilde{\sigma}(\delta)\to 0^+ \text{ as } \delta \to 0^+ \\
&= \lambda u_0^{-\gamma} + f(z,u_0) + \widehat{\xi}_\rho u_0^{p-1} + \widetilde{\sigma}(\delta) \\
&\leq \lambda w^{-\gamma} + f(z,w) + \widehat{\xi}_\rho w^{p-1} + \widetilde{\sigma}(\delta) \quad (\text{since } u_0 \leq w)
\end{aligned}
$$

$$
\begin{aligned}
&\le -c_w + \widehat{\xi}_\rho w^{p-1} + \widetilde{\sigma}(\delta) \\
&\le -\Delta_p w - \Delta_q w + (\xi(z) + \widehat{\xi}_\rho) w^{p-1} \quad \text{for } \delta > 0 \text{ small so that } \widetilde{\sigma}(\delta) \le c_w \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(see hypothesis } H(f)\,(v)),
\end{aligned}
$$

$$
\Rightarrow \quad u_0^\delta \le w \quad \text{for } \delta > 0 \text{ small},
$$

$$
\Rightarrow \quad w - u_0 \in D_+. \tag{34}
$$

From (33) and (34), we conclude that $u_0 \in \operatorname{int}_{C^1(\overline{\Omega})}[\widetilde{u}_\lambda, w]$. ☐

Using $u_0 \in D_+$ from Proposition 4 and by employing variational tools (in particular Theorem 1), we can establish the existence of a second positive solution for problem (P_λ) when $\lambda \in (0, \lambda_0)$.

Proposition 5. *If hypotheses $H(\xi)$, $H(f)$ hold and $\lambda \in (0, \lambda_0)$, then problem (P_λ) has a second positive solution $\widehat{u} \in D_+$, $\widehat{u} \neq u_0$, $\widetilde{u}_\lambda \le \widehat{u}$.*

Proof. We consider the following truncation of the reaction for problem (P_λ):

$$
\widehat{f}_\lambda(z,x) = \begin{cases} \lambda \widetilde{u}_\lambda(z)^{-\gamma} + f(z, \widetilde{u}_\lambda(z)) & \text{if } x \le \widetilde{u}_\lambda(z), \\ \lambda x^{-\gamma} + f(z,x) & \text{if } \widetilde{u}_\lambda(z) < x. \end{cases} \tag{35}
$$

Evidently this is a Carathéodory function. We set $\widehat{F}_\lambda(z,x) = \int_0^x \widehat{f}_\lambda(z,s)ds$ and introduce the functional $\widehat{\varphi}_\lambda : W^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$
\widehat{\varphi}_\lambda(u) = \frac{1}{p}\mu_p(u) + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega \widehat{F}_\lambda(z,u)dz \quad \text{for all } u \in W^{1,p}(\Omega).
$$

Since $\widetilde{u}_\lambda \in D_+$, we see that $\widehat{\varphi}_\lambda \in C^1(W^{1,p}(\Omega), \mathbb{R})$.

Let $\widehat{\psi}_\lambda \in C^1(W^{1,p}(\Omega), \mathbb{R})$ be as in the proof of Proposition 4. From (29) and (35), we see that

$$
\widehat{\varphi}_\lambda\Big|_{[\widetilde{u}_\lambda, w]} = \widehat{\psi}_\lambda\Big|_{[\widetilde{u}_\lambda, w]}. \tag{36}
$$

Recall that

$$
u_0 \in \operatorname{int}_{C^1(\overline{\Omega})}[\widetilde{u}_\lambda, w] \text{ and } u_0 \text{ is a minimizer of } \widehat{\psi}_\lambda \tag{37}
$$

(see Proposition 4 and its proof).

From (36) and (37) it follows that

$$
u_0 \in D_+ \text{ is a local } C^1(\overline{\Omega})\text{-minimizer of } \widehat{\varphi}_\lambda,
$$

$$
\Rightarrow \quad u_0 \in D_+ \text{ is a local } W^{1,p}(\Omega)\text{-minimizer of } \widehat{\varphi}_\lambda \tag{38}
$$

(see Papageorgiou-Rădulescu [16], Proposition 8).

Let $K_{\widehat{\varphi}_\lambda} = \{u \in W^{1,p}(\Omega) : \widehat{\varphi}_\lambda'(u) = 0\}$ (the critical set of $\widehat{\varphi}_\lambda$) and

$$
[\widetilde{u}_\lambda) = \{u \in W^{1,p}(\Omega) : \widetilde{u}_\lambda(z) \le u(z) \text{ for a.a. } z \in \Omega\}.
$$

Claim 1: $K_{\widehat{\varphi}_\lambda} \subseteq [\widetilde{u}_\lambda) \cap D_+$.

Let $u \in K_{\widehat{\varphi}_\lambda}$. We have

$$
\widehat{\varphi}_\lambda'(u) = 0,
$$

$$\Rightarrow \quad \langle A_p(u), h\rangle + \langle A_q(u), h\rangle + \int_\Omega \xi(z)|u|^{p-2}uhdz = \int_\Omega \widehat{f}_\lambda(z,u)hdz \quad \text{for all } h \in W^{1,p}(\Omega). \tag{39}$$

In (39) we choose $h = (\widetilde{u}_\lambda - u)^+ \in W^{1,p}(\Omega)$. Then

$$\begin{aligned}
&\langle A_p(u), (\widetilde{u}_\lambda - u)^+\rangle + \langle A_q(u), (\widetilde{u}_\lambda - u)^+\rangle + \int_\Omega \xi(z)|u|^{p-2}(\widetilde{u}_\lambda - u)^+dz \\
&= \int_\Omega [\lambda \widetilde{u}_\lambda^{-\gamma} + f(z,\widetilde{u}_\lambda)](\widetilde{u}_\lambda - u)^+dz \quad \text{(see (35))} \\
&\geq \int_\Omega \lambda \widetilde{u}_\lambda^{-\gamma}(\widetilde{u}_\lambda - u)^+dz \quad \text{(see (28) and hypothesis } H(f)\,(v)) \\
&= \langle A_p(\widetilde{u}_\lambda), (\widetilde{u}_\lambda - u)^+\rangle + \langle A_q(\widetilde{u}_\lambda), (\widetilde{u}_\lambda - u)^+\rangle + \int_\Omega \xi(z)\widetilde{u}_\lambda^{p-1}(\widetilde{u}_\lambda - u)^+dz \\
&\qquad \text{(see Proposition 3)}, \\
\Rightarrow \quad &\langle A_p(\widetilde{u}_\lambda) - A_p(u), (\widetilde{u}_\lambda - u)^+\rangle + \langle A_q(\widetilde{u}_\lambda) - A_q(u), (\widetilde{u}_\lambda - u)^+\rangle \\
&\qquad + \int_\Omega \xi(z)\left[\widetilde{u}_\lambda^{p-1} - |u|^{p-2}u\right](\widetilde{u}_\lambda - u)^+dz \leq 0, \\
\Rightarrow \quad &\widetilde{u}_\lambda \leq u.
\end{aligned}$$

Then (39) becomes

$$\begin{aligned}
&\langle A_p(u), h\rangle + \langle A_q(u), h\rangle + \int_\Omega \xi(z)u^{p-1}hdz = \int_\Omega [\lambda u^{-\gamma} + f(z,u)]hdz \\
&\qquad \text{for all } h \in W^{1,p}(\Omega), \\
\Rightarrow \quad &\begin{cases} -\Delta_p u(z) - \Delta_q u(z) + \xi(z)u(z)^{p-1} = \lambda u(z)^{-\gamma} + f(z,u(z)) & \text{for a.a. } z \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \end{cases} \\
\Rightarrow \quad &u \in D_+ \quad \text{(by the nonlinear regularity theory)}.
\end{aligned}$$

Therefore we conclude that $K_{\widehat{\varphi}_\lambda} \subseteq [\widetilde{u}_\lambda) \cap D_+$. This proves Claim 1.

On account of Claim 1, we may assume that

$$K_{\widehat{\varphi}_\lambda} \text{ is finite.} \tag{40}$$

Otherwise we already have an infinity of positive solutions of (P_λ) and so we are done. Then from (38) and (40) we infer that there exists $\rho \in (0,1)$ small such that

$$\widehat{\varphi}_\lambda(u_0) < \inf\left[\widehat{\varphi}_\lambda(u) : \|u - u_0\| = \rho\right] = \widehat{m}_\rho \tag{41}$$

(see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 29).

Hypothesis $H(f)\,(iii)$ implies that given any $\eta > 0$, we can find $M_1 = M_1(\eta) > 0$ such that

$$f(z,x)x - pF(z,x) \leq -\eta x^q \quad \text{for a.a. } z \in \Omega, \text{ all } x \geq M_1. \tag{42}$$

We have

$$\begin{aligned}\frac{d}{dx}\left[\frac{F(z,x)}{x^p}\right] &= \frac{f(z,x)x^p - px^{p-1}F(z,x)}{x^{2p}}\\ &= \frac{f(z,x)x - pF(z,x)}{x^{p+1}}\\ &\leq -\frac{\eta}{x^{p-q+1}} \quad \text{for a.a. } z\in\Omega, \text{ all } x\geq M_1 \text{ (see (42))},\end{aligned}$$

$$\Rightarrow \quad \frac{F(z,x)}{x^p} - \frac{F(z,y)}{y^p} \leq \frac{\eta}{p-q}\left[\frac{1}{x^{p-q}} - \frac{1}{y^{p-q}}\right] \quad \text{for a.a. } z\in\Omega, \text{ all } x\geq y\geq M_1. \tag{43}$$

Note that hypothesis $H(f)\,(ii)$ implies that

$$\frac{\widehat{\lambda}_1(p)}{p} \leq \liminf_{x\to+\infty}\frac{F(z,x)}{x^p} \leq \limsup_{x\to+\infty}\frac{F(z,x)}{x^p} \leq \frac{c_0}{p} \quad \text{uniformly for a.a. } z\in\Omega. \tag{44}$$

So, if in (43) we let $x\to+\infty$ and use (44) and the fact that $-\frac{\eta}{p-q} < -\frac{\eta}{p}$ then we obtain

$$\frac{\widehat{\lambda}_1(p)y^p - pF(z,y)}{y^q} \leq -\eta \quad \text{for a.a. } z\in\Omega, \text{ all } x\geq M_1,$$

$$\Rightarrow \quad \lim_{y\to+\infty}\frac{\widehat{\lambda}_1(p)y^p - pF(z,y)}{y^q} = -\infty \quad \text{uniformly for a.a. } z\in\Omega. \tag{45}$$

Since $\widehat{u}(1,p)\in D_+$, we can find $t\geq 1$ big such that

$$\widetilde{u}_\lambda \leq t\,\widehat{u}(1,p)$$

(see also Proposition 2.1 of Marano-Papageorgiou [12]). We have

$$\begin{aligned}\widehat{\varphi}_\lambda(t\,\widehat{u}(1,p)) &\leq \frac{t^p}{p}\widehat{\lambda}_1(p)\|\widehat{u}(1,p)\|_p^p + \frac{t^q}{q}\|\nabla\widehat{u}(1,p)\|_q^q - \int_\Omega F(z,t\,\widehat{u}(1,p))dz + c_8\\ &\qquad\qquad \text{for some } c_8>0 \text{ (see (35))},\end{aligned}$$

$$\Rightarrow \quad p\widehat{\varphi}_\lambda(t\,\widehat{u}(1,p)) \leq \int_\Omega\left[\widehat{\lambda}_1(p)(t\,\widehat{u}(1,p))^p - pF(z,t\,\widehat{u}(1,p))\right]dz + \frac{t^qp}{q}\|\nabla\widehat{u}(1,p)\|_q^q + pc_8,$$

$$\Rightarrow \quad \frac{p\widehat{\varphi}_\lambda(t\,\widehat{u}(1,p))}{t^q} \leq \int_\Omega \frac{\widehat{\lambda}_1(p)(t\,\widehat{u}(1,p))^p - pF(z,t\,\widehat{u}(1,p))}{t^q\widehat{u}(1,p)^q}\widehat{u}(1,p)^q dz + \frac{p}{q}\|\nabla\widehat{u}(1,p)\|_q^q + \frac{pc_8}{t^q},$$

$$\Rightarrow \quad \frac{p\widehat{\varphi}_\lambda(t\,\widehat{u}(1,p))}{t^q} \to -\infty \quad \text{as } t\to+\infty \quad \text{(see (45))},$$

$$\Rightarrow \quad \widehat{\varphi}_\lambda(t\,\widehat{u}(1,p)) \to -\infty \quad \text{as } t\to+\infty. \tag{46}$$

Claim 2: $\widehat{\varphi}_\lambda$ satisfies the C-condition.

Let $\{u_n\}_{n\geq1}\subseteq W^{1,p}(\Omega)$ be a sequence such that

$$|\widehat{\varphi}_\lambda(u_n)| \leq M_2 \quad \text{for some } M_2>0, \text{ all } n\in\mathbb{N}, \tag{47}$$

$$(1+\|u_n\|)\widehat{\varphi}'_\lambda(u_n) \to 0 \quad \text{in } W^{1,p}(\Omega)^* \text{ as } n\to+\infty. \tag{48}$$

From (48) we have

$$\left|\langle A_p(u_n),h\rangle+\langle A_q(u_n),h\rangle+\int_\Omega \xi(z)|u_n|^{p-2}u_nhdz-\int_\Omega \widehat{f}_\lambda(z,u_n)hdz\right|\leq\frac{\varepsilon_n\|h\|}{1+\|u_n\|}\tag{49}$$
$$\text{for all } h\in W^{1,p}(\Omega) \text{ with } \varepsilon_n\to 0^+.$$

In (49) we choose $h=-u_n^-\in W^{1,p}(\Omega)$. Using (35) we obtain

$$\|\nabla u_n^-\|_p^p+\int_\Omega \xi(z)(u_n^-)^pdz\leq\int_\Omega\left[\widetilde{u}_\lambda^{-\gamma}+f(z,\widetilde{u}_\lambda)\right](-u_n^-)dz+\varepsilon_n,$$
$$\Rightarrow\quad c_0\|u_n^-\|^p\leq c_9\|u_n^-\|\quad\text{for some } c_9>0, \text{ all } n\in\mathbb{N}$$
$$\text{(see Lemma 1, hypothesis } H(f)\,(i) \text{ and recall that } \widetilde{u}_\lambda\in D_+),$$
$$\Rightarrow\quad \{u_n^-\}_{n\geq 1}\subseteq W^{1,p}(\Omega) \text{ is bounded.}\tag{50}$$

We will show that $\{u_n^+\}_{n\geq1}\subseteq W^{1,p}(\Omega)$ is bounded too. To establish this, we argue by contradiction. So, suppose that at least for a subsequence, we have

$$\|u_n^+\|\to+\infty\quad\text{as } n\to+\infty.\tag{51}$$

We set $y_n=\dfrac{u_n^+}{\|u_n^+\|}$, $n\in\mathbb{N}$. Then $\|y_n\|=1$, $y_n\geq 0$ for all $n\in\mathbb{N}$ and so we may assume that

$$y_n\xrightarrow{w}y \text{ in } W^{1,p}(\Omega)\quad\text{and}\quad y_n\to y \text{ in } L^p(\Omega),\ y\geq 0.\tag{52}$$

From (49) and (50), we have

$$\left|\langle A_p(u_n^+),h\rangle+\langle A_q(u_n^+),h\rangle+\int_\Omega \xi(z)(u_n^+)^{p-1}hdz-\int_\Omega \widehat{f}_\lambda(z,u_n^+)hdz\right|$$
$$\leq c_{10}\|h\|\quad\text{for some } c_{10}>0, \text{ all } h\in W^{1,p}(\Omega), \text{ all } n\in\mathbb{N},$$
$$\Rightarrow\quad\left|\langle A_p(y_n),h\rangle+\frac{1}{\|u_n^+\|^{p-q}}\langle A_q(y_n),h\rangle+\int_\Omega \xi(z)y^{p-1}hdz-\int_\Omega\frac{\widehat{f}_\lambda(z,u_n^+)}{\|u_n^+\|^{p-1}}hdz\right|\tag{53}$$
$$\leq\frac{c_{10}\|h\|}{\|u_n^+\|^{p-1}}\quad\text{for all } h\in W^{1,p}(\Omega), \text{ all } n\in\mathbb{N}.$$

Since $\widetilde{u}_\lambda\in D_+$, using (35) and hypothesis $H(f)\,(i),(ii)$ we see that

$$\left\{\frac{\widehat{f}_\lambda(\cdot,u_n^+(\cdot))}{\|u_n^+\|^{p-1}}\right\}_{n\geq1}\subseteq L^{p'}(\Omega) \text{ is bounded.}\tag{54}$$

So, by passing to a subsequence if necessary and using hypothesis $H(f)\,(iii)$ we have that

$$\frac{\widehat{f}_\lambda(\cdot,u_n^+(\cdot))}{\|u_n^+\|^{p-1}}\xrightarrow{w}\vartheta_0(z)y^{p-1} \text{ in } L^{p'}(\Omega),\ \text{with } \widehat{\lambda}_1(p)\leq\vartheta_0(z)\leq c_0 \text{ for a.a. } z\in\Omega\tag{55}$$

(see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 16). In (53) we choose $h=y_n-y\in W^{1,p}(\Omega)$, pass to the limit as $n\to+\infty$ and use (51), (52), (54) and the fact that $q<p$. Then

$$\lim\langle A_p(y_n),y_n-y\rangle=0,$$

$$\Rightarrow \quad y_n \to y \ \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 1)}, \ \|y\| = 1, \ y \geq 0. \tag{56}$$

If in (53) we pass to the limit as $n \to +\infty$ and use (51), (55), (56) and the fact that $q < p$, we obtain

$$\langle A_p(y), h\rangle + \int_\Omega \xi(z) y^{p-1} h dz = \int_\Omega \vartheta_0(z) y^{p-1} h dz \quad \text{for all } h \in W^{1,p}(\Omega),$$

$$\Rightarrow \quad -\Delta_p y(z) + \xi(z) y(z)^{p-1} = \vartheta_0(z) y(z)^{p-1} \quad \text{for a.a. } z \in \Omega, \ \frac{\partial y}{\partial n} = 0 \text{ on } \partial\Omega. \tag{57}$$

Recall that

$$\widehat{\lambda}_1(p) \leq \vartheta_0(z) \leq c_0 \quad \text{for a.a. } z \in \Omega \text{ (see (55))}.$$

If $\vartheta_0 \not\equiv \widehat{\lambda}_1(p)$, then using Proposition 2, we have

$$\widetilde{\lambda}_1(p, \vartheta_0) < \widetilde{\lambda}_1(p, \widehat{\lambda}_1(p)) = 1,$$
$$\Rightarrow \quad y \text{ must be nodal (see (57))},$$

a contradiction to (56).

If $\vartheta_0(z) = \widehat{\lambda}_1(p)$ for a.a. $z \in \Omega$, then from (57) we see that

$$\begin{aligned}
& y = \eta\, \widehat{u}(1,p) \quad \text{for some } \eta > 0, \\
\Rightarrow \quad & y \in D_+ \text{ and so } y(z) > 0 \text{ for all } z \in \overline{\Omega}, \\
\Rightarrow \quad & u_n^+(z) \to +\infty \quad \text{for all } z \in \overline{\Omega}, \\
\Rightarrow \quad & \frac{f(z, u_n^+(z)) u_n^+(z) - pF(z, u_n^+(z))}{u_n^+(z)^q} \to -\infty \quad \text{for a.a. } z \in \Omega,
\end{aligned}$$

$$\Rightarrow \quad \int_\Omega \frac{f(z, u_n^+) u_n^+ - pF(z, u_n^+)}{\|u_n^+\|^q} dz \to -\infty \quad \text{(by Fatou's lemma)}. \tag{58}$$

In (49) we choose $h = u_n^+ \in W^{1,p}(\Omega)$. Then

$$-\|\nabla u_n^+\|_p^p - \|\nabla u_n^+\|_q^q - \int_\Omega \xi(z)(u_n^+)^p dz + \int_\Omega \widehat{f}_\lambda(z, u_n^+) u_n^+ dz \geq -\varepsilon_n \ \text{ for all } n \in \mathbb{N}. \tag{59}$$

From (47) and (50) it follows that

$$\|\nabla u_n^+\|_p^p + \frac{p}{q}\|\nabla u_n^+\|_q^q + \int_\Omega \xi(z)(u_n^+)^p dz - \int_\Omega p\widehat{F}_\lambda(z, u_n^+) dz \geq -M_3 \tag{60}$$
$$\text{for some } M_3 > 0, \text{ all } n \in \mathbb{N}.$$

We add (59) and (60). Then

$$\left(\frac{p}{q} - 1\right) \|\nabla u_n^+\|_q^q + \int_\Omega \left[\widehat{f}_\lambda(z, u_n^+) u_n^+ - p\widehat{F}_\lambda(z, u_n^+)\right] dz \geq -M_4$$
$$\text{for some } M_4 > 0, \text{ all } n \in \mathbb{N},$$
$$\Rightarrow \quad \left(\frac{p}{q} - 1\right) \|\nabla u_n^+\|_q^q + \int_\Omega \left[f(z, u_n^+) u_n^+ - pF(z, u_n^+)\right] dz \geq -M_5$$
$$\text{for some } M_5 > 0, \text{ all } n \in \mathbb{N} \text{ (see (35) and recall } \widetilde{u}_\lambda \in D_+),$$

$$\Rightarrow \quad \left(\frac{p}{q}-1\right)\widetilde{c}+\int_\Omega \frac{[f(z,u_n^+)u_n^+ - pF(z,u_n^+)]}{\|u_n^+\|^q}dz \geq -\frac{M_5}{\|u_n^+\|^q}$$
$$\text{for all } n\in\mathbb{N}, \text{ some } \widetilde{c}>0,$$

$$\Rightarrow \quad \liminf_{n\to+\infty}\int_\Omega \frac{[f(z,u_n^+)u_n^+ - pF(z,u_n^+)]}{\|u_n^+\|^q}dz \geq -\frac{p-q}{q}\widetilde{c}. \tag{61}$$

Comparing (58) and (61), we have a contradiction. This proves that

$$\begin{aligned} &\{u_n^+\}_{n\geq 1}\subseteq W^{1,p}(\Omega) \text{ is bounded,}\\ \Rightarrow \quad &\{u_n\}_{n\geq 1}\subseteq W^{1,p}(\Omega) \text{ is bounded (see (50)).}\end{aligned}$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \quad \text{and} \quad u_n\to u \text{ in } L^p(\Omega) \quad \text{as } n\to+\infty. \tag{62}$$

Evidently

$$\left\{\widehat{f}_\lambda(\cdot,u_n(\cdot))\right\}_{n\geq 1}\subseteq L^{p'}(\Omega) \text{ is bounded (see (35)).} \tag{63}$$

So, if in (49) we choose $h=u_n-u\in W^{1,p}(\Omega)$, pass to the limit as $n\to+\infty$ and we use (62) and (63), then

$$\begin{aligned} &\lim_{n\to+\infty}[\langle A_p(u_n),u_n-u\rangle+\langle A_q(u_n),u_n-u\rangle]=0,\\ \Rightarrow \quad & u_n\to u \text{ in } W^{1,p}(\Omega) \text{ (see Proposition 1 and the proof of Proposition 3)}\\ \Rightarrow \quad & \widehat{\varphi}_\lambda \text{ satisfies the } C\text{-condition.}\end{aligned}$$

This proves Claim 2.

From (41), (46) and Claim 2, we see that we can apply Theorem 1 (the mountain pass theorem) and find $\widehat{u}\in W^{1,p}(\Omega)$ such that

$$\begin{aligned} &\widehat{u}\in K_{\widehat{\varphi}_\lambda}\subseteq [\widetilde{u}_\lambda)\cap D_+ \quad \text{(see Claim 1),}\\ &\widehat{\varphi}_\lambda(u_0)<\widehat{m}_\rho\leq\widehat{\varphi}_\lambda(\widehat{u}) \quad \text{(see (41)).}\end{aligned} \tag{64}$$

From (64) and (35), we infer that $\widehat{u}\in D_+$ is the second positive solution of (P_λ), $0<\lambda\leq\lambda_0$. $\square$

Summarizing, we can state the following multiplicity result for problem (P_λ).

Theorem 2. *If hypotheses $H(\xi)$, $H(f)$ hold, then there exists $\lambda_0>0$ such that for all $\lambda\in(0,\lambda_0]$ problem (P_λ) has at least two positive solutions u_0, $\widehat{u}\in D_+$, $u_0\neq\widehat{u}$.*

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