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Minimal varieties of PI-algebras with graded involution ☆

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ABSTRACT

Let F be an algebraically closed field of characteristic zero and G a cyclic group of odd prime order. We consider the class of finite dimensional $(G, *)$-algebras, namely G-graded algebras endowed with graded involution $*$, and we characterize the varieties generated by algebras of this class which are minimal with respect to the $(G, *)$-exponent.

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1. Introduction

Let F be a field of characteristic zero and $F\langle X\rangle$ be the free associative algebra on a countable set X over F. One of the most interesting problems in PI-theory is that of finding numerical invariants allowing to classify the T-ideals of $F\langle X\rangle$, i.e. the ideals invariant under all endomorphisms of $F\langle X\rangle$. There is a well understood connection between T-ideals of $F\langle X\rangle$ and PI-algebras: every T-ideal is the ideal of polynomial identities satisfied by a given F-algebra. If A is an associative PI-algebra over F, i.e. an algebra satisfying a non-trivial polynomial identity, a very useful numerical invariant that can be attached to $Id(A)$, the T-ideal of identities of A, is given by the sequence of codimensions $\{c_n(A)\}_{n\geq 1}$. Such numerical sequence was introduced by Regev in [29] and measures in some sense the rate of growth of the multilinear polynomials lying in $Id(A)$. Regev proved that if A satisfies a non-trivial polynomial identity then its sequence of codimensions is exponentially bounded. Later Giambruno and Zaicev (see [17] and [18]) captured the exponential growth of this sequence showing that, the exponent of A,

$$\exp(A) := \lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

exists and is a non-negative integer. Afterwards the same approach has been applied to study the corresponding polynomial identities of algebras endowed with additional structures such as superalgebras, group graded algebras, algebras with involution, graded algebras with involutions. Besides their own interesting features, the main motivations for this type of work are the important information on fairly general issues that the additional structure and related objects may provide. For instance this is the case for the solution of the Specht problem due to Kemer (see [25]) in which $\mathbb{Z}_2$-gradings play a fundamental role.

In this paper we deal with $(G,*)$-algebras, namely associative algebras graded by a finite abelian group G and endowed with a graded involution $*$, and the growth of their $(G,*)$-codimensions. If A is a PI-algebra, it follows from the literature, for instance see [21, Lemma 10.1.3], that the sequence $\{c_n^{(G,*)}(A)\}_{n\geq 1}$ of $(G,*)$-codimensions is exponentially bounded, moreover, as in the ordinary case, the existence of the $(G,*)$-exponent has been confirmed in [22] in the finite-dimensional case. Obviously this statement applies to commutative graded algebras (which can be seen as $(G,*)$-graded algebras with trivial involution) and algebras with involutions (which are nothing but $(G,*)$-graded algebras endowed with trivial grading), but it was independently proved for PI-algebras graded by a finite group in [1] and for $*$-PI algebras with involution- in [16] without further restrictions. In particular, the last mentioned paper is based on the Representability Theorem as presented in [2], where a crucial role is just played by superinvolutions and graded involutions of a superalgebra. More precisely, any PI-algebra with involution is $*$-PI equivalent to the Grassmann envelope $G(A) = A\hat{\otimes}E$ of a finite dimensional superalgebra A equipped by some superinvolution $*$. Notice that the involution on $G(A)$ induced by $*$ and by the parity automorphism of E is indeed a graded involution. Superalgebras endowed with a superinvolution play a natural and relevant role in the Theory of Lie and Jordan superalgebras [24], [28] and have recently become a subject of increasing interest and extensive investigations in PI-Theory (see [23], [15], [9]).

In this context the previous results represent the most significant developments of the quantitative investigation of the corresponding polynomial identities satisfied by algebras in these classes. The important feature of the existence of the exponent is that it provides a measure of the growth of any variety: in particular, it allows to classify varieties on an integral scale whose steps are the minimal varieties of given exponent d, namely those varieties of exponent d such that every proper subvariety has exponent strictly less than d. A deep outcome in this direction is obtained in [20] with the characterization of minimal varieties of given exponent d. Previously the same authors in [19] proved that a variety of PI-algebras of finite basic rank or affine variety, i.e. generated by a finitely generated algebra, is minimal if it is generated by an upper block triangular matrix algebra. Along this line in [10] the minimal varieties of finite-dimensional algebras with involution were completely described. Sviridova proved that actually any affine $*$-variety can be generated by a finite dimensional $*$-algebra [31]. In [12] the authors characterized minimal varieties of $\mathbb{C}_p$-graded PI-algebras in the affine case, where $\mathbb{C}_p$ is a cyclic group of order a prime p. Recently the classification of minimal affine varieties of PI $*$-superalgebras was obtained in [13].

In this paper we consider the varieties of $(\mathbb{C}_p, *)$-algebras for an odd prime p: our goal is the classification of those which are minimal of a given exponent. It is proved that a variety of $(\mathbb{C}_p, *)$-algebras generated by a finite dimensional $(\mathbb{C}_p, *)$-algebra is minimal of fixed exponent if and only if it is generated by an upper block triangular matrix algebra equipped with a suitable elementary grading and graded involution (see Theorem 6).

2. Algebras with graded involution and polynomial identities

Let F be a field of characteristic zero and G a finite group. Throughout this paper all algebras are assumed to be associative and to have the same ground field F so we will normally drop the word associative in what follows.

An algebra A is a *G-graded algebra* if A can be written as a direct sum of vector spaces $A = \bigoplus_{g\in G} A_g$ such that $A_g A_h \subseteq A_{gh}$, for all $g, h \in G$. The subspaces A_g are called *homogeneous components* of A of degree g and an element $a \in A_g$ is called *homogeneous of degree* g, we denote its degree by $|a|_A = g$.

A subspace V of A is graded if $V = \bigoplus_{g\in G}(V \cap A_g)$. Similarly, we define graded subalgebras and ideals of A.

Recall that an *involution* $*$ on an algebra A is an antiautomorphism of A of order at most two. An algebra A endowed with an involution $*$ is called a *$*$-algebra*. In this case we write $A = A^+ \oplus A^-$ where $A^+ = \{a \in A \,|\, a^* = a\}$ and $A^- = \{a \in A \,|\, a^* = -a\}$ denote the subspaces of symmetric and skew-symmetric elements of A, respectively.

Given a G-graded algebra $A = \bigoplus_{g\in G} A_g$ endowed with an involution $*$, we say that $*$ is a *graded involution* if it preserves the homogeneous components of A, i.e. if $A_g^* \subseteq A_g$, for all $g \in G$. A G-graded algebra endowed with a graded involution is called $(G,*)$-*algebra*.

Moreover, for each $g \in G$, we denote by A_g^+ and A_g^- the subspaces of symmetric and skew-symmetric elements of A_g, respectively. We say that (g,μ) is the *type* of the homogeneous element a belonging to A_g^μ, where $\mu \in \{+,-\}$.

An isomorphism of algebras $\phi : A \to B$ from a $(G,*)$-algebra A to a $(G,\bar{*})$-algebra B is an isomorphism of $(G,*)$-algebras if $\phi(A_g) \subseteq B_g$, for all $g \in G$, and $\phi(a^*) = \phi(a)^{\bar{*}}$, for all $a \in A$. In this case, we say that A and B are isomorphic as $(G,*)$-algebras.

Let $F\langle X_G\rangle$ denote the free associative G-graded algebra on the countable set X_G over F. Here the set X_G decomposes as $X_G = \bigcup_{g\in G} X_g$ the union of disjoint countable sets $X_g = \{x_{1,g}, x_{2,g}, \ldots\}$, where $g \in G$ and the elements of X_g have degree g. The free algebra $F\langle X_G\rangle$ has a natural G-grading $F\langle X_G\rangle = \bigoplus_{g\in G} F_g$, where F_g is the subspace spanned by the monomials $x_{i_1,g_{j_1}} \cdots x_{i_t,g_{j_t}}$ of homogeneous degree $g = g_{j_1} \cdots g_{j_t}$.

Let us now consider the free algebra with involution $F\langle X_G, *\rangle$ which is a $(G,*)$-algebra if we assume also that, for each variable $x_{i,g}$, $x_{i,g}^*$ is homogeneouos of degree g. $F\langle X_G, *\rangle$ is said to be the *free* $(G,*)$-*algebra* over F.

If $A = \bigoplus_{g\in G} A_g$ is a $(G,*)$-algebra, an element $f = f(x_{1,g_1}, x_{1,g_1}^*, \ldots, x_{n,g_n}, x_{n,g_n}^*) \in F\langle X_G, *\rangle$ is a $(G,*)$-*polynomial identity* of A if $f(a_{1,g_1}, a_{1,g_1}^*, \ldots, a_{n,g_n}, a_{n,g_n}^*) = 0$ for all $a_{i,g_i} \in A_{g_i}$, $i = 1, \ldots, n$. In this case we write $f \equiv 0$ on A. The set of all $(G,*)$-polynomial identities satisfied by A

$$Id_G^*(A) = \{f \in F\langle X_G, *\rangle \mid f \equiv 0 \text{ on } A\}$$

is an ideal of $F\langle X_G, *\rangle$ called the *ideal of* $(G,*)$-*identities of* A. It is easy to show that $Id_G^*(A)$ is a T_G^*-*ideal* of $F\langle X_G, *\rangle$, i.e. a two-sided ideal of the free $(G,*)$-algebra invariant under all endomorphisms of $F\langle X_G, *\rangle$ that preserve the G-structure and commute with the graded involution $*$. Now, let

$$P_n^{(G,*)} = \{x_{\sigma(1),g_1}^{\gamma_1} \cdots x_{\sigma(n),g_n}^{\gamma_n} \mid \sigma \in S_n,\ \gamma_i \in \{1,*\},\ i = 1, \ldots, n\}$$

be the space of all multilinear $(G,*)$-polynomials of degree n. Since char F=0, standard Vandermonde arguments and linearization process prove that $Id_G^*(A)$ is completely determined by its multilinear polynomials, then the study of $Id_G^*(A)$ is equivalent to that of $P_n^{(G,*)} \cap Id_G^*(A)$ for all $n \geq 1$. As in the ordinary case (see [29]), one defines the *n-th* $(G,*)$-*codimension* of A as

$$c_n^{(G,*)}(A) = \dim_F \frac{P_n^{(G,*)}}{P_n^{(G,*)} \cap Id_G^*(A)}.$$

If A is a PI-algebra, i.e. it satisfies an ordinary polynomial identity, it can be easily proved that the relation between the ordinary codimensions $c_n(A)$ and the $(G,*)$-codimension of A is given by the inequalities

$$c_n(A) \leq c_n^{(G,*)}(A) \leq 2^n|G|^n c_n(A),$$

hence the sequence $\{c_n^{(G,*)}(A)\}_{n\geq 1}$ is exponentially bounded.

A $(G,*)$-algebra can be viewed as an algebra with a generalized $F[G \times \mathbb{Z}_2]$-action, where $G \times \mathbb{Z}_2$ acts on it by automorphisms and antiautomorphisms. If A is a finite dimensional $(G,*)$-algebra, Gordienko in [22] captured its exponential growth proving that the $(G,*)$-*exponent* of A

$$\exp_G^*(A) = \lim_{n\to\infty} \sqrt[n]{c_n^{(G,*)}(A)}$$

exists and it is a non-negative integer.

He actually provides an explicit formula to compute the $(G,*)$-exponent which is a natural generalization of that for the ordinary PI-exponent. If A is a finite dimensional $(G,*)$-algebra, since F is a field of characteristic zero, $Id_G^*(A) = Id_G^*(A \otimes_F L) \cap F\langle X_G, *\rangle$ for any extension field L of F then also the $(G,*)$-codimensions of A do not change upon extension of the base field. Hence we can assume that F is algebraically closed. By the generalization of the Wedderburn-Malcev Theorem, we can write

$$A = A_1 \oplus \cdots \oplus A_m + J,$$

where $A_1, \ldots, A_m$ are $(G,*)$-simple algebras, and $J = J(A)$ is the Jacobson radical of A which is a T_G^*-ideal. We recall that A is called $(G,*)$-*simple algebra* if $A^2 \neq \{0\}$ and it has no non-zero $(G,*)$-ideals.

We say that a subalgebra $A_{i_1} \oplus \cdots \oplus A_{i_k}$ of A, where $A_{i_1}, \ldots, A_{i_k}$ are distinct $(G,*)$-simple components, is *admissible* if for some permutation $(l_1, \ldots, l_k)$ of $(i_1, \ldots, i_k)$ we have that $A_{l_1} J \cdots J A_{l_k} \neq 0$. Moreover, if $A_{i_1} \oplus \cdots \oplus A_{i_k}$ is an admissible subalgebra of A then $A' = A_{i_1} \oplus \cdots \oplus A_{i_k} + J$ is called a *reduced* $(G,*)$-algebra.

The notion of admissible $(G,*)$-algebra is closely linked to that of $(G,*)$-exponent in fact, in [22], it was proved that $\exp_G^*(A) = d$ where d is the maximal dimension of an admissible subalgebra of A. It follows immediately that

Remark 1. If A is a $(G,*)$-simple algebra then $\exp_G^*(A) = \dim_F A$.

It is often more useful to study $(G,*)$-algebras up to PI-equivalence, then it is convenient to use the language of varieties. Let I be a T_G^*-ideal of $F\langle X_G, *\rangle$ and $\mathcal{V}$ the *variety of* $(G,*)$-*algebras* associated to I, i.e. the class of all $(G,*)$-algebras A such that I is contained in $Id_G^*(A)$. We put $I = Id_G^*(\mathcal{V})$. Since $Id_G^*(\mathcal{V}) = I = Id_G^*(A)$, for some $(G,*)$-algebra A, we say that the variety $\mathcal{V}$ is generated by A and we write $\mathcal{V} = \mathrm{var}_G^*(A)$, $\exp_G^*(\mathcal{V}) = \exp_G^*(A)$ if $\exp_G^*(A)$ exists. We call $\exp_G^*(\mathcal{V})$ the $(G,*)$-exponent of the variety $\mathcal{V}$.

To ensure the existence of the $(G,*)$-exponent, in this paper we consider only varieties generated by finite dimensional $(G,*)$-algebras and we introduce the following definition:

Definition 1. A variety $\mathcal{V}$ generated by a finite dimensional $(G,*)$-algebra is said to be **minimal** of $(G,*)$-exponent d if

$$\exp_G^*(\mathcal{V}) = d \text{ and } \exp_G^*(\mathcal{U}) < d$$

for every proper subvariety $\mathcal{U}$ of $\mathcal{V}$ generated by a finite dimensional $(G,*)$-algebra.

Our main purpose is to characterize the minimal varieties of $(G,*)$-algebras when G is a finite group of order an odd prime p (see Theorem 6).

We emphasize that our result would hold in its full generality in the context of affine varieties if a representability theorem for finitely generated $(G,*)$-algebras satisfying a polynomial identity were proved.

More precisely, every finitely generated $(G,*)$-algebra satisfying a polynomial identity would be equivalent to a finite-dimensional $(G,*)$-algebra (representability theorem). All subvarieties of an affine variety would themselves be affine, and consequently would be generated by a finite-dimensional $(G,*)$-algebra. This is precisely the case in which G has order 2 (see [15, Theorem 5.3]).

However, since the validity of this representability theorem has not yet been confirmed, in this paper we preferred to state our result only for varieties generated by finite-dimensional $(G,*)$-algebras as specified above and in Theorem 6.

3. Elementary gradings on matrix algebras and graded involutions

In this section, we focus our study on the graded involutions on matrix algebras with elementary gradings. Let us recall that the elementary gradings play a fundamental role in the classification of all G-gradings on matrix algebras over an algebraically closed field. Moreover, as a consequence of this classification, one has that when G is a finite cyclic group and the field F is algebraically closed then any G-grading over $M_n(F)$ is isomorphic to an elementary grading.

An explicit classification of graded involutions on matrix algebras was made by Bahturin and Zaicev in [8] (see also [7] and [4]). Here we present a different description of their result in the case of elementary gradings that we will use for the classification of finite dimensional $(G,*)$-simple algebras when $G = \mathbb{C}_p$ is finite of order an odd prime p.

We fix the notation throughout the paper: for any ring R and pair of positive integers s and t let $M_{s\times t}(R)$ denote the space of all matrices with s rows and t columns over R and set $M_s(R) = M_{s\times s}(R)$. If $R = F$, we simply write $M_{s\times t}$ and M_s instead of $M_{s\times t}(F)$ and $M_s(F)$, respectively. Setting from now on, for every positive integers m, n with $m \le n$, $[m,n] := \{m, m+1, \dots, n\}$, we denote by e_{ij} the usual (i,j)-matrix unit of M_n, $i,j \in [1,n]$. Also we indicate by I_n the $n \times n$ matrix identity of M_n and by J_n the matrix of M_n with all ones in the secondary diagonal and zeros in the other entries

$$J_n = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & & 1 & 0 \\ \vdots & & & & \vdots \\ 0 & 1 & & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix} \in M_n.$$

Let us recall that a G-grading on $M_n = \bigoplus_{g\in G}(M_n)_g$ is an elementary grading if there exists an n-tuple $(g_1,\dots,g_n) \in G^n$ such that the matrix units e_{ij}, $i,j \in [1,n]$, are homogeneous and $e_{ij} \in (M_n)_g$ if and only if $g = g_i^{-1}g_j$. In an equivalent manner we have the following

Definition 2. Let n be a positive integer, and let $\mathcal{E}_n := \{e_{ij} \mid i,j \in [1,n]\}$ be the canonical basis of matrix units of M_n. For any map $\alpha : [1,n] \to G$, we write $\alpha_i = \alpha(i)$ and define $|e_{ij}|_\alpha := \alpha_i^{-1}\alpha_j$. The resulting grading on M_n is called the **elementary grading induced by** α.

We write (M_n, α) to denote the matrix algebra equipped with an elementary grading induced by α. Also we write $(M_n, \alpha, *)$ when $*$ is a graded involution on (M_n, α).

In this section, if it is not explicitly stated, G is a finite abelian group. Before of proceeding, recall some considerations and results on elementary gradings and multilinear polynomials that we shall frequently use throughout the paper (see Section 2 of [11]). First of all we recall:

Proposition 1. *The algebras (M_n, α) and (M_n, β) are graded isomorphic if and only if there exist $h \in G$ and a permutation ϱ in the symmetric group S_n such that*

$$\beta_{\varrho(i)} = h\alpha_i \quad \textit{for all } i \in [1,n].$$

In this case the linear map

$$\tilde{\varrho} : (M_n, \alpha) \longrightarrow (M_n, \beta) \qquad e_{ij} \longmapsto e_{\varrho(i)\varrho(j)}$$

is the graded isomorphism and there exists a permutation-matrix $C \in M_n$ such that

$$\tilde{\varrho}(x) = CxC^{-1} \quad \textit{ for all } \quad x \in M_n.$$

At the light of the previous proposition we say that α and β are **equivalent** and we write $\alpha \sim \beta$ if and only if there exist $h \in G$ and $\varrho \in S_n$ such that $\beta_{\varrho(i)} = h\alpha_i$ for all $i \in [1,n]$.

Now, given $\alpha : [1,n] \longrightarrow G$ we denote by $\mathcal{I}_\alpha$ the image of α, that is $\mathcal{I}_\alpha = \alpha([1,n])$, we consider the corresponding weight map $w_\alpha : G \longrightarrow \mathbb{N}$ defined by

$$w_\alpha(g) = |\{i \mid 1 \le i \le n, \;\; \alpha(i) = g\}|$$

and we define **the invariance subgroup** of α by

$$H_\alpha := \{h \mid h \in G,\, w_\alpha(hg) = w_\alpha(g) \text{ for all } g \in G\}.$$

Clearly, α and β are equivalent if and only if there exists $h \in G$ such that $w_\beta(x) = w_\alpha(hx)$ for all $x \in G$. In this case we obtain $H_\alpha = H_\beta$ hence, given the matrix algebra $A = M_n$ endowed with an elementary G-grading, we can define the invariance subgroup of A by $H_A = H_\alpha$, where α is any map inducing the grading.

Now let us consider the polynomials Φ_A and Ψ_A introduced in [11], using a more appropriate notation. More precisely, if $\alpha : [1, n] \longrightarrow G$ let $\tilde{g} = (g_1, \ldots, g_r)$ be a fixed sequence of all the elements of $\mathcal{I}_\alpha$ in some order, we consider the derived sequence $(h_1, \ldots, h_{r-1})$ defined by $h_t = g_t^{-1} g_{t+1}$ for all $t = 1, \ldots, r-1$ and the corresponding weight vector $\hat{m} = (m_1, \ldots, m_r) = (w_\alpha(g_1), \ldots, w_\alpha(g_r))$. Now, as pointed out in [11], if $A = (M_n, \alpha)$ the definition of Φ_A depends only on the sequence $\tilde{g} = (g_1, \ldots, g_r)$ and α of course. In this paper we denote Φ_A as

$$\Phi_{\alpha,\tilde{g}} = \Phi_{\alpha,\tilde{g}}(y_1, \ldots, y_{2n-r}, z_1, \ldots z_{r-1}) =$$
$$\mathrm{St}_{n_1}(y_1, \ldots, y_{n_1}) z_1 \,\mathrm{St}_{n_2}(y_{n_1+1}, \ldots, y_{n_1+n_2}) z_2 \cdots z_{r-1} \,\mathrm{St}_{n_r}(y_{n_1+\cdots+n_{r-1}+1}, \ldots, y_{2n-r}),$$

where $n = m_1 + \cdots + m_r$ is the size of A, $n_i = 2m_i - 1$, for all $i = 1, \ldots, r$, $y_1, \ldots, y_{2n-r}$ are homogeneous variables of degree 1_G, whereas $z_1, \ldots, z_{r-1}$ are homogeneous variables of degree $h_1, \ldots, h_{r-1}$ respectively and $\mathrm{St}_t(x_1, \ldots, x_t) = \sum_{\sigma \in S_n} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(t)}$ denotes the *standard polynomial* of degree t.

With the same notation, we consider the inverse order $\tilde{g}^* = (g_r, \ldots, g_1)$ of the elements of $\mathcal{I}_\alpha$, so the derived sequence becomes $(h_{r-1}^{-1}, \ldots, h_1^{-1})$, the associated vector $\hat{m}^*$ is $(m_r, \ldots, m_1)$ and the corresponding polynomial is $\Phi_{\alpha,\tilde{g}^*}(y'_1, \ldots, y'_{2n-r}, z'_1, \ldots, z'_{r-1})$. Here we use a set of homogeneous variables $\{y'_1, \ldots, y'_{2n-r}, z'_1, \ldots, z'_{r-1}\}$ disjoint from that of the variables of $\Phi_{\alpha,\tilde{g}}$, and, as above $y'_1, \ldots, y'_{2n-r}$ have degree 1_G while $z'_1, \ldots, z'_{r-1}$ have degree $h_{r-1}^{-1}, \ldots, h_1^{-1}$ respectively. We write

$$\tilde{\Phi}_{\alpha,\tilde{g}} = \Phi_{\alpha,\tilde{g}}(y_1, \ldots, y_{2n-r}, z_1, \ldots z_{r-1}) \Phi_{\alpha,\tilde{g}^*}(y'_1, \ldots, y'_{2n-r}, z'_1, \ldots, z'_{r-1}).$$

Clearly, for a fixed $i \in [1, n]$, there exists a natural way to determine an order $\overrightarrow{\alpha}_i$ of all the elements of $\mathcal{I}_\alpha$: simply by reading from left to right the sequence $(\alpha_i, \alpha_{i+1}, \ldots, \alpha_n, \alpha_1, \ldots, \alpha_{i-1})$ and writing the different elements that occur in it with their weight. In this way $g_1 = \alpha_i$ and $m_1 = w_\alpha(\alpha_i)$. Finally we denote by $\tilde{\Phi}_{\alpha,i}$ the corresponding polynomial, that is

$$\tilde{\Phi}_{\alpha,i} = \tilde{\Phi}_{\alpha,\overrightarrow{\alpha}_i}.$$

As an easy consequence of Lemma 2.2 of [11] and its proof we obtain:

Proposition 2. *Let $A = (M_n, \alpha)$, $i \in [1, n]$ and consider the polynomial $\tilde{\Phi}_{\alpha,i}$, then*

(a) $\tilde{\Phi}_{\alpha,i}$ *is a multilinear polynomial of* $F\langle X_G\rangle$ *of* G*-degree* 1_G*;*
(b) $\tilde{\Phi}_{\alpha,i}$ *is not a graded polynomial identity of* A*, more precisely there exists a suitable graded evaluation* $\tilde{\mu} : F\langle X_G\rangle \longrightarrow (M_n, \alpha)$ *at matrix units such that* $\tilde{\mu}(\tilde{\Phi}_{\alpha,i}) = e_{ii}$*;*
(c) *If* $B = (M_n, \beta)$ *and* $\tilde{\Phi}_{\alpha,i} \notin Id_G(B)$ *then* β *is equivalent to* $(\underbrace{g_1, \ldots, g_1}_{m_1}, \ldots\ldots, \underbrace{g_r, \ldots, g_r}_{m_r})$*. In particular* (M_n, α) *and* (M_n, β) *are isomorphic as* G*-graded algebras.*

Similarly we denote by $\Psi_{\alpha,h}$ the polynomial Ψ_A introduced in Lemma 2.5 of [11] since its definition actually depends only on the map $\alpha : [1, n] \longrightarrow G$, on the fixed sequence $\tilde{g} = (g_1, \ldots, g_r)$ of all the elements of $\mathcal{I}_\alpha$ in some order and on the choice of an index $h \in [1, n]$. More precisely,

$$\Psi_{\alpha,h} = \psi_1 \cdots \psi_r,$$

where, for every $j \in [1, r]$,

$$\psi_j = \sum_{\sigma \in S_{t_j}} (-1)^\sigma u^{(j)}_{\sigma(1)} v^{(j)}_1 u^{(j)}_{\sigma(2)} v^{(j)}_2 \cdots u^{(j)}_{\sigma(t_j)} v^{(j)}_{t_j}$$

with $t_j = m_h m_j$ and $\{u^{(1)}_1, \ldots, u^{(1)}_{t_1}\}, \ldots, \{u^{(r)}_1, \ldots, u^{(r)}_{t_r}\}$, $\{v^{(1)}_1, \ldots, v^{(1)}_{t_1}\}, \ldots, \{v^{(r)}_1, \ldots, v^{(r)}_{t_r}\}$ pairwise disjoint sets of homogeneous variables of degree $\deg(u^{(j)}_l) = g_h^{-1} g_j$ and $\deg(v^{(j)}_l) = g_j^{-1} g_h$, for any $l \in [1, t_j]$. Moreover, if α_h has maximal weight then for $\Psi_{\alpha,h}$ the same conclusions of that lemma hold. More specifically, we have:

Proposition 3. *Let* $A = (M_n, \alpha)$*,* $h \in [1, n]$ *and consider the polynomial* $\Psi_{\alpha,h}$*, then*

(a) $\Psi_{\alpha,h}$ *is a multilinear polynomial of* $F\langle X_G\rangle$ *of* G*-degree* 1_G*;*
(b) *If* α_h *has maximal weight, then* $\Psi_{\alpha,h}$ *is not a graded polynomial identity of* A*, and there exists a suitable graded evaluation* $\mu : F\langle X_G\rangle \longrightarrow (M_n, \alpha)$ *at matrix units such that* $\mu(\Psi_{\alpha,h}) = e_{hh}$*;*
(c) *If* $\bar{\mu}$ *is a graded evaluation then*

$$\bar{\mu}(\Psi_{\alpha,h}) \subseteq span_F\{e_{ij} \mid \alpha_i = \alpha_j = g\alpha_h,\ for\ some\ g \in H_\alpha\}.$$

Now we return to the study of graded involutions on matrix algebras, introducing an important class of involutions on M_n of transpose type.

Definition 3. Let n be a positive integer, and let $\gamma \in S_n$ be a permutation of $[1, n]$ such that $\gamma^2 = id$. We define an involution θ_γ on M_n by $e_{ij}^{\theta_\gamma} = e_{\gamma(j)\gamma(i)}$ for all $i, j \in [1, n]$.

Let $(g_1, \ldots, g_n) \in G^n$ be the n-tuple defining the G-grading on M_n, since G is abelian θ_γ is a graded involution on M_n if and only if

$$g_i g_{\gamma(i)} = g_j g_{\gamma(j)} \text{ for all } i, j \in [1, n]. \tag{1}$$

When $\gamma = \gamma_{n,q}$, with $2q \leq n$, is the permutation on $[1, n]$ defined by

$$\gamma_{n,q}(i) = \begin{cases} n + 1 - i, & \text{if } i \in [1, q] \cup [n + 1 - q, n] \\ i, & \text{if } i \in [q + 1, n - q], \end{cases}$$

we denote the corresponding involution θ_γ simply by $\theta_{n,q}$. Note that the number of fixed points of $\gamma_{n,q}$ is exactly $n-2q$. When $n-2q = 0$ or 1 we denote by γ_n and θ_n respectively the corresponding map and involution. In this case

$$e_{ij}^{\theta_n} = e_{n+1-j\, n+1-i} \text{ for all } i, j \in [1, n]$$

and θ_n is called the *reflection involution*.

When $q = 0$ then $\theta_{n,0}$ is the classical transpose involution defined by $e_{ij}^t = e_{ji}$. Finally, when $n = 2m$ is an even positive integer we denote by σ_n the *symplectic type involution* on M_n defined by:

$$e_{ij}^{\sigma_n} = \delta_{[1,m]}(i)\delta_{[1,m]}(j)e_{ij}^{\theta_n} = \delta_{[1,m]}(i)\delta_{[1,m]}(j)e_{n+1-j\, n+1-i} \text{ for all } i, j \in [1, 2m]$$

where

$$\delta_{[1,m]}(i) = \begin{cases} 1 & \text{if } i \in [1, m] \\ -1 & \text{otherwise.} \end{cases}$$

The description of the involution defined on any finite dimensional central simple algebra (see, for example, [30, Chapter 3]) is well known. Here we recall only the following result:

Proposition 4. *Let F be an algebraically closed field of characteristic zero, and let $*$ be an involution on M_n. Then either $*$ is of transpose type and*

$$(M_n, *) \cong (M_n, t) \cong (M_n, \theta_n)$$

or n is even, $*$ is of symplectic type and

$$(M_n, *) \cong (M_n, \sigma_n).$$

Once again, θ_n and σ_n are graded involutions on M_n with respect the elementary G-grading induced by $(g_1, \ldots, g_n)$ if and only if

$$g_i g_{n+1-i} = g_j g_{n+1-j} \text{ for all } i, j \in [1, n]$$

and n is even in the symplectic case.

Now we recall the duality between the involutions on the matrix algebra M_n and the bilinear forms on the vector space $V = F^n$. If

$$\varphi : V \times V \to F$$

is a non-degenerate bilinear form, for any $X \in \mathrm{End}(V), a, b \in V$, the relation

$$\varphi(Xa, b) = \varphi(a, X^* b),$$

defines an antiautomorphism $* : \mathrm{End}(V) \to \mathrm{End}(V)$. If B is a fixed basis of V then, in the matrix form, $*$ can be written as

$$X^* = (\Phi^{-1})X^t\Phi,$$

where Φ is the matrix of φ with respect to B and, as above, t is the transpose involution on M_n. This antiautomorphism is of order two if and only if φ is symmetric or skew-symmetric. Conversely, by Noether Skolem Theorem (see [26]) any involution $*$ of M_n is of the previous form with $\Phi^t = \pm\Phi$ and Φ is uniquely defined by $*$ up to a scalar factor. Clearly $\Phi = J_n$ when $* = \theta_n$ and $\Phi = \begin{pmatrix} 0 & J_m \\ -J_m & 0 \end{pmatrix}$ when $n = 2m$ and $* = \sigma_n$.

The relevance of the graded involutions defined above is expressed in the following result

Proposition 5. *Let $(M_n, \alpha, *)$ be a matrix algebra over an algebraically closed field of characteristic zero F, with elementary G-grading α and graded involution $*$ defined by a non-degenerate bilinear form φ.*

(a) *If φ is symmetric then, for some integer q such that $0 \leq 2q \leq n$, $(M_n, \alpha, *)$ is isomorphic as $(G, *)$-algebra to $(M_n, \tilde{g}, \theta_{n,q})$ where the elementary G-grading is defined by a n-tuple $\tilde{g} = (g_1, \dots, g_n) \in G^n$ satisfying*

$$g_i g_{\gamma_{n,q}(i)} = g_j g_{\gamma_{n,q}(j)} \text{ for all } i, j \in [1, n].$$

Moreover:

- *if $|G|$ is odd, then $(M_n, \alpha, *)$ is isomorphic as a $(G, *)$-algebra to $(M_n, \tilde{g}, \theta_n)$;*
- *if $|G| = 2$, that is M_n is a $*$-superalgebra, then either $(M_n, \alpha, *)$ is isomorphic to the $*$-superalgebra $(M_{q,q}, \theta_{2q})$ for some q such that $n = 2q$ or $(M_n, \alpha, *)$ is isomorphic to (M_{m_1,m_2}, t) for some $m_1 \geqslant m_2$ such that $n = m_1 + m_2$.*

(b) *If φ is skew-symmetric then $n = 2m$ is an even integer and $(M_n, \alpha, *)$ is isomorphic as $(G, *)$-algebra to $(M_n, \tilde{g}, \sigma_n)$ where the elementary G-grading is defined by the n-tuple $\tilde{g} = (g_1, \dots, g_n) \in G^n$ satisfying*

$$g_i g_{n+1-i} = g_j g_{n+1-j} \text{ for all } i, j \in [1, n].$$

Proof. Let $A = M_n = \bigoplus_{g\in G}(M_n)_g$ be a matrix algebra with an elementary G-grading induced by $\alpha : [1,n] \to G$. Let $\mathcal{I}_\alpha = \{h_1,\dots,h_k\} \subseteq G$ and $m_i = w_\alpha(h_i)$ the weight of h_i in α, for all $i \in [1,k]$. Recall that the elementary G-grading defined by α can be viewed as an induced grading on the algebra of all linear transformations of a G-graded vector space $V = V_1 \oplus \cdots \oplus V_k$ with $\deg V_i = {h_i}^{-1}$ and $\dim_F V_i = m_i$. Fixing any bases in V_i we obtain an elementary grading on M_n isomorphic to the initial one such that any matrix M is decomposed into k^2 blocks

$$M = \begin{pmatrix} M_{11} & \cdots & M_{1k} \\ \vdots & & \vdots \\ M_{k1} & \cdots & M_{kk} \end{pmatrix},$$

where M_{ij} is of order $m_i \times m_j$ and all matrix units of this block are of degree $h_i^{-1}h_j$ in the G-grading.

Clearly M is a homogeneous element of degree 1_G if and only if $M_{rs} = 0$ for all $r \neq s$. It follows that

$$A_{1_G} \cong M_{m_1} \oplus \cdots \oplus M_{m_k}$$

and A_{1_G} decomposes in the direct sum $\mathfrak{I}_1 \oplus \cdots \oplus \mathfrak{I}_k$ of its minimal two-sided ideals $\mathfrak{I}_1,\dots,\mathfrak{I}_k$, where $\mathfrak{I}_h \cong M_{m_h}$ for all $h = 1,\dots,k$.

Since $*$ is a graded involution of M_n, then it induces an involution on A_{1_G} and so $\mathfrak{I}_h^*$ is in $\{\mathfrak{I}_1,\dots,\mathfrak{I}_k\}$ for all $h \in [1,k]$. Then there exists a bijection $\eta : [1,k] \to [1,k]$ such that $\mathfrak{I}_j^* = \mathfrak{I}_{\eta(j)}$ and so $m_j = m_{\eta(j)}$. We observe that $\eta^2 = id$ and moreover, for all $r,s \in [1,k]$, we have

$$(\mathfrak{I}_r A_{1_G} \mathfrak{I}_s)^* = \mathfrak{I}_s^* A_{1_G}^* \mathfrak{I}_r^* = \mathfrak{I}_{\eta(s)} A_{1_G} \mathfrak{I}_{\eta(r)}.$$

Now $|\mathfrak{I}_r A_{1_G} \mathfrak{I}_s|_\alpha = h_r^{-1}h_s$, so we obtain that

$$h_r^{-1}h_s = |\mathfrak{I}_r A_{1_G} \mathfrak{I}_s|_\alpha = |(\mathfrak{I}_r A_{1_G} \mathfrak{I}_s)^*|_\alpha = |\mathfrak{I}_{\eta(s)} A_{1_G} \mathfrak{I}_{\eta(r)}|_\alpha = h_{\eta(s)}^{-1}h_{\eta(r)}$$

and so

$$h_r h_{\eta(r)} = h_s h_{\eta(s)} = c, \text{ for all } r,s \in [1,k]. \tag{2}$$

If $1_h = \sum_{i\in[1,m_h]} e_{ii}$ is the identity of $M_{m_h} \cong \mathfrak{I}_h$, then $1_h^* = 1_{\eta(h)}$.

As observed before the involution $*$ of M_n can be written as

$$X^* = (\Phi^{-1})X^t\Phi,$$

where Φ is the matrix of the non-degenerate bilinear form φ with respect a fixed basis B of V associated to $*$, $\Phi^t = \pm\Phi$, and t stands for the transpose involution on M_n. By the above equality we get

$$\Phi X^* = X^t \Phi$$

and in particular we have

$$\Phi 1_{\eta(h)} = 1_h \Phi,$$

for all $h \in [1, k]$. Now, by writing the matrix Φ in the form

$$\Phi = \Sigma_{r,s\in[1,k]} \bar{\Phi}_{rs} = \begin{pmatrix} \bar{\Phi}_{11} & \cdots & \bar{\Phi}_{1k} \\ \vdots & & \vdots \\ \bar{\Phi}_{k1} & \cdots & \bar{\Phi}_{kk} \end{pmatrix},$$

the previous formula becomes

$$\Sigma_{r\in[1,k]} \bar{\Phi}_{r\eta(h)} = \Sigma_{s\in[1,k]} \bar{\Phi}_{hs}$$

and we obtain

$$\bar{\Phi}_{hs} = 0, \quad \forall s \neq \eta(h).$$

That is in any column and in any row of Φ there exists exactly one non-zero block and the subspaces V_r and V_s are orthogonal to each other with respect to Φ for all r, s such that $s \neq \eta(r)$.

Now, we consider $\mathfrak{F} = \{r \in [1, k] \mid \eta(r) = r\}$ and $\mathfrak{S} = \{s \in [1, k] \mid \eta(s) \neq s\}$. Clearly $\mathfrak{S}$ splits in orbits of length 2 under the action of η and we write $\mathfrak{S} = \mathfrak{S}' \cup \eta(\mathfrak{S}')$, where $\mathfrak{S}'$ is a complete set of representatives of these orbits.

We put $W_r = V_r$ for all $r \in \mathfrak{F}$ and $W_r = V_r \oplus V_{\eta(r)}$ for all $r \in \mathfrak{S}'$. We remark that W_r is a graded subspace of V, φ induces on W_r a non-degenerate bilinear form and moreover W_r and W_s are orthogonal to each other with respect to Φ, for all $r, s \in \mathfrak{F} \cup \mathfrak{S}'$, $r \neq s$. Let Φ_r be the matrix of φ restricted to W_r.

If Φ is symmetric, then every Φ_r is symmetric and we can choose convenient basis of W_r such that, $\forall r \in \mathfrak{F}$, $\Phi_r = I_{m_r} \in M_{m_r}$ and, $\forall r \in \mathfrak{S}'$, $\Phi_r = \begin{pmatrix} 0 & J_{m_r} \\ J_{m_r} & 0 \end{pmatrix} \in M_{2m_r}$. Finally, we put $W_i = U_i$ for all $i \in \mathfrak{F}$. If $h = |\mathfrak{F}|$ and $2l = |\mathfrak{S}|$ with $l = |\mathfrak{S}'|$, then reordering the bases appropriately we can write

$$V = V_{r_1} \oplus \cdots \oplus V_{r_l} \oplus U_{s_1} \oplus \cdots \oplus U_{s_h} \oplus V_{\eta(r_l)} \oplus \cdots \oplus V_{\eta(r_1)}$$

and

$$\Phi = \begin{pmatrix} 0 & 0 & J_q \\ 0 & I_m & 0 \\ J_q & 0 & 0 \end{pmatrix}$$

where $m = \sum_{i\in\mathfrak{F}} m_i$, $q = \sum_{i\in\mathfrak{S}'} m_i$ and $n = m + 2q$. Clearly the involution on M_n is $\theta_{n,q}$ and the elementary grading on $M_n \cong \operatorname{End} V$ comes rearranging the given n-tuple $(\alpha(1), \ldots, \alpha(n))$ accordingly to this decomposition of V. By (2) it follows that the grading on M_n is the elementary one induced by $(g_1, \ldots, g_n) \in G^n$ satisfying $g_i g_{\gamma_{n,q}(i)} = g_j g_{\gamma_{n,q}(j)}$ for all $i, j \in [1, n]$.

Now, let us assume that G has odd order. If r, s are in $\mathfrak{F}$ then (2) implies that $h_r^2 = h_r h_{\eta(r)} = h_s h_{\eta(s)} = h_s^2$ and so $h_r = h_s$, that is $r = s$ and $|\mathfrak{F}| \leq 1$. If $|\mathfrak{F}| = 1$ then the previous decomposition of V becomes

$$V = V_{r_1} \oplus \cdots \oplus V_{r_l} \oplus U_s \oplus V_{\eta(r_l)} \oplus \cdots \oplus V_{\eta(r_1)}$$

for some $s \in [1, k]$ and U_s is the homogeneous component of V having degree h_s^{-1}. Hence, by considering an appropriate basis of U_s we can assume that $\Phi_s = J_{m_s}$ and so $\Phi = J_n$, that is $*$ is the graded involution θ_n.

If $|G| = 2$ then $k \leqslant 2$. If $\mathfrak{S} \neq \emptyset$ then $2 \leqslant |\mathfrak{S}| \leqslant k \leqslant 2$, hence $\mathfrak{S} = \{1, 2\}$ and $\mathfrak{F} = \emptyset$. The previous decomposition of V becomes $V = V_1 \oplus V_2$, where $\eta(1) = 2$, moreover one has $m = 0$, $q = m_1 = \dim V_1 = \dim V_2 = m_2$ and $n = 2q$. In this case the involution on M_n is the reflection involution θ_n and M_n is the superalgebra $M_{q,q}$.

When $\mathfrak{S} = \emptyset$ then $\mathfrak{F} = [1, k]$, $q = 0$ and the involution $\theta_{n,0}$ on M_n is the classical transpose t. If $k = 1$ the grading is the trivial one. If $k = 2$, the previous decomposition of V becomes $V = V_1 \oplus V_2$, where $\eta(i) = i$ for all $i = 1, 2$. Moreover one has $n = m_1 + m_2$ and so M_n is the superalgebra M_{m_1,m_2}, where we can assume $m_1 \geqslant m_2$.

Finally, when Φ is skew-symmetric then every Φ_r is skew-symmetric. For all $r \in \mathfrak{S}'$ we can choose basis of W_r such that $\Phi_r = \begin{pmatrix} 0 & J_{m_r} \\ -J_{m_r} & 0 \end{pmatrix} \in M_{2m_r}$.

If r belongs to $\mathfrak{F}$ then the subspace W_r is homogeneous of degree h_r^{-1} and its dimension m_r is an even integer $2n_r$. In this case we have $\Phi_r = \begin{pmatrix} 0 & J_{n_r} \\ -J_{n_r} & 0 \end{pmatrix} \in M_{2n_r}$ and we can write $W_r = U_r \oplus U_r'$, where $n_r = \dim U_r = \dim U_r'$.

By properly rearranging the bases, one obtains

$$V = V_{r_1} \oplus \cdots V_{r_l} \oplus U_{s_1} \oplus \cdots \oplus U_{s_h} \oplus U'_{s_h} \oplus \cdots \oplus U'_{s_1} \oplus V_{\eta(r_l)} \oplus \cdots \oplus V_{\eta(r_1)}$$

and

$$\Phi = \begin{pmatrix} 0 & J_m \\ -J_m & 0 \end{pmatrix}$$

where $m = \sum_{i\in\mathfrak{F}} n_i + \sum_{i\in\mathfrak{S}'} m_i$ and $n = 2m$. As above, this decomposition of V determines on M_n the elementary grading defined by the n-tuple $(g_1, \ldots, g_n)$ satisfying the condition $g_i g_{n+1-i} = g_j g_{n+1-j}$ for all $i, j \in [1, n]$ and we are done. □

As in the ordinary case, the dimensions of the subspaces A_g^μ identify the isomorphism class of the $(G,*)$-algebra of matrices $A=(M_n,\alpha,*)$ for any finite group G of odd order. More precisely, we have:

Proposition 6. *Let $A=(M_n,\alpha,*_A)$ and $B=(M_m,\beta,*_B)$ be matrix algebras with elementary G-gradings and graded involutions $*_A\in\{\theta_n,\sigma_n\}$ and $*_B\in\{\theta_m,\sigma_m\}$. If $|G|$ is odd the following conditions are equivalent:*

(*a*) *A and B are isomorphic as $(G,*)$-algebras;*
(*b*) *$\dim_F A_g^\mu=\dim_F B_g^\mu$, for all $g\in G$ and $\mu\in\{+,-\}$;*
(*c*) *$n=m$ and there exist $\varrho\in S_n$ and $h\in G$ such that*

$$\varrho\gamma_n=\gamma_n\varrho \quad \textit{and} \quad \beta_{\varrho(k)}=h\alpha_k$$

for all $k\in[1,n]$.

Proof. Clearly $(a)\Rightarrow(b)$.

$(b)\Rightarrow(c)$. Let us assume that $\dim_F A_g^\mu=\dim_F B_g^\mu$ for all $g\in G$ and $\mu\in\{+,-\}$. One has

$$\dim_F A=\dim_F B,\ \ \dim_F A^+=\dim_F B^+,\ \ \dim_F A^-=\dim_F B^-.$$

Hence it follows that $n=m$ and, as in the ordinary case, $*_A$ and $*_B$ are of the same type. More precisely, as above we write $\gamma=\gamma_n$, so $\gamma(i)=n+1-i$ for all $i\in[1,n]$ and we consider the *standard bases* of A and B given by the elements

$$p_{ij}=\begin{cases} e_{ij}+e_{ij}^*, & \text{if } i<\gamma(j)\\ e_{ij}, & \text{if } i=\gamma(j)\\ e_{ij}-e_{ij}^*, & \text{if } i>\gamma(j)\end{cases}$$

for all $i,j\in[1,n]$. Here $*$ denotes, depending on the considered case, the involution in A or that in B. Now, let $\Delta(A)$ and $\Delta(B)$ be the subspaces linearly generated by the matrix units $e_{1\gamma(1)},\dots,e_{n\gamma(n)}$ in A and B respectively. For all $g\in G$, $\mu\in\{+,-\}$ we have:

$$\dim_F \Delta(A)_g^\mu=\dim_F \Delta(B)_g^\mu \tag{3}$$

From the previous condition (3), if $\alpha=(g_1,\dots,g_n)$ and $\beta=(h_1,\dots,h_n)$ there exists $\varrho\in S_n$ such that $g_k^{-1}g_{\gamma(k)}=|e_{k\gamma(k)}|_A=|e_{\varrho(k)\gamma(\varrho(k))}|_B=h_{\varrho(k)}^{-1}h_{\gamma(\varrho(k))}$ and $\gamma(\varrho(k))=\varrho(\gamma(k))$ for all $k\in[1,n]$. By Proposition 5, there exist $a,b\in G$ such that $g_{\gamma(k)}=ag_k^{-1}$ and $h_{\gamma(k)}=bh_k^{-1}$, for all $k\in[1,n]$. Hence $ag_k^{-2}=g_k^{-1}g_{\gamma(k)}=h_{\varrho(k)}^{-1}h_{\gamma(\varrho(k))}=bh_{\varrho(k)}^{-2}$ and, since G is abelian of odd order, one has $h_{\varrho(k)}=hg_k$, for some $h\in G$.

$(c)\Rightarrow(a)$. If $*_A$ (and $*_B$) are of transpose type, then the linear map $\tilde\varrho: A\to B$ defined by $\tilde\varrho(e_{ij})=e_{\varrho(i)\varrho(j)}$ is an isomorphism of $(G,*)$-algebras. If $*_A$ (and $*_B$) are of symplectic type, then $n = 2q$ and we consider the linear map $\psi : A \to B$ defined by $\psi(e_{ij}) = \varepsilon(i)\varepsilon(j)e_{\varrho(i)\varrho(j)}$ where

$$\varepsilon(i) = \begin{cases} -1 & \text{if } i \in [1,q] \text{ and } \varrho(i) \notin [1,q] \\ 1 & \text{otherwise.} \end{cases}$$

Clearly $|\psi(a)|_B = |\tilde{\varrho}(a)|_B = |a|_A$ for all homogeneous element $a \in A$ and a straightforward calculation shows that ψ is indeed an isomorphism of $(G,*)$-algebras. □

Let us emphasize that the isomorphisms of the previous proposition are defined by

$$\tilde{\varrho}(x) = CxC^{-1} \quad \text{and} \quad \psi(x) = ExE^{-1} \tag{4}$$

where C is the permutation-matrix related to ϱ while E is a generalized permutation-matrix, obtained from C by changing 1 to -1 in the appropriate places. More precisely

$$C = \sum_{i=1}^{n} e_{\varrho(i)\,i} \quad \text{and} \quad E = \sum_{i=1}^{n} \varepsilon(i)e_{\varrho(i)\,i}.$$

Remark 2. If A and B are $(G,*)$-algebras then also $A \otimes B$ is in a natural way a $(G,*)$-algebra. More precisely

$$|a \otimes b|_{A\otimes B} = |a|_A|b|_B$$

for all homogeneous elements $a \in A$, $b \in B$ and the involution $*_{A\otimes B}$ is the map defined by tensorising the involutions of A and B, that is $*_{A\otimes B} = *_A \otimes *_B$ and

$$(a \otimes b)^{*_{A\otimes B}} = a^{*_A} \otimes b^{*_B}$$

for all $a \in A$ and $b \in B$.

When B is a matrix algebra we have:

Remark 3. Let $B = (M_n, \beta, *_B)$ be the $(G,*)$-algebra of $n \times n$-matrices endowed with the elementary G-grading defined by the map β. Since G is abelian, if A is a $(G,*)$-algebra then $A \otimes B = A \otimes M_n$ is isomorphic as $(G,*)$-algebra to $M_n(A)$ via the map

$$a \otimes e_{ij} \mapsto ae_{ij}$$

where the involution on $M_n(A)$ is the natural one, that is $(ae_{ij})^* = a^{*_A}e_{ij}^{*_B}$ and the grading is defined, as in [5], by

$$|ae_{ij}| = \beta(i)^{-1}|a|_A\beta(j).$$

Finally, it is straightforward to prove the following result:

Proposition 7. *Let m, n be positive integers and $\mu : [1, m]\times[1, n] \to [1, mn]$ be the bijection defined by $\mu(r, i) = m(i - 1) + r$ then:*

$$(M_m, \alpha, \theta_m) \otimes (M_n, \beta, \theta_n) \cong (M_{mn}, \alpha \otimes \beta, \theta_{mn})$$
$$(M_m, \alpha, \theta_m) \otimes (M_n, \beta, \sigma_n) \cong (M_{mn}, \alpha \otimes \beta, \sigma_{mn})$$

as $(G, *)$-algebras, where $\alpha \otimes \beta : [1, mn] \to G$ is defined by $(\alpha \otimes \beta)(h) = \alpha(r)\beta(i)$ for all $h = m(i - 1) + r \in [1, mn]$ and the isomorphism is given by

$$e_{rs} \otimes e_{ij} \mapsto e_{\mu(r,i)\mu(s,j)}$$

for all $r, s \in [1, m]$ and $i, j \in [1, n]$.

Let us recall that given a G-graded algebra A then the opposite algebra A^{op} has a natural G-grading defined by $A^{op}_g = A_g$, for all $g \in G$, because A and A^{op} have the same structure as vector space over the field F and G is abelian. Clearly $C = A \oplus A^{op}$ has an induced G-grading, given by $C_g = A_g \oplus A^{op}_g$ and in this case the exchange involution

$$(a, b)^{exc} = (b, a), \quad \text{for all} \quad (a, b) \in C$$

is a graded involution on $A \oplus A^{op}$.

When A is a matrix algebra with an elementary grading α we can realize $(A \oplus A^{op}, \alpha, exc)$ in two more convenient ways. More precisely:

Definition 4. If $\alpha : [1, n] \to G$ is any map, we present the map α as a G-word (or n-tuple) of lenght n, that is $\alpha := (\alpha_1, \dots, \alpha_n)$, then we define $\alpha^{-1} := (\alpha_n^{-1}, \dots, \alpha_1^{-1})$ and we consider the concatenation $\bar{\alpha} = (\alpha \mid \alpha^{-1})$ of the words α and α^{-1}, that is

$$\bar{\alpha} = (\alpha_1, \dots, \alpha_n, \alpha_n^{-1}, \dots, \alpha_1^{-1}).$$

We say that $\bar{\alpha}$ is the $*$-**closure** of α.

Clearly $\bar{\alpha}_i\bar{\alpha}_{2n+1-i} = 1_G$ for all $i \in [1, 2n]$ and so θ_{2n} is a graded involution on $(M_{2n}, \bar{\alpha})$. Now let $(M_n \oplus M_n, \bar{\alpha}, \theta_{2n})$ denote the $(G, *)$-subalgebra of $n \times n$ block diagonal matrices of $(M_{2n}, \bar{\alpha}, \theta_{2n})$ and let us consider the linear map $\psi_n : M_n \oplus M_n^{op} \longrightarrow M_{2n}$ defined by:

$$\psi_n(a, b) = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}^{\theta_{2n}} = \begin{pmatrix} a & 0 \\ 0 & b^{\theta_n} \end{pmatrix},$$

for all $(a, b) \in M_n \oplus M_n^{op}$. We obtain the following:

Lemma 1. *The map ψ_n induces an isomorphism as $(G,*)$-algebras from*

$$(M_n \oplus M_n^{op}, \alpha, exc) \quad to \quad (M_n \oplus M_n, \bar{\alpha}, \theta_{2n}).$$

On the other hand we can consider the graded algebra $(M_n, \alpha) \oplus (M_n, \alpha^{-1})$ and define a G-graded involution $\tilde{\vartheta}_n$ on it by

$$(a,b)^{\tilde{\vartheta}_n} = (b^{\theta_n}, a^{\theta_n}).$$

We denote this $(G,*)$-algebra by $(M_n \oplus M_n, \bar{\alpha}, \tilde{\vartheta}_n)$. Clearly, considering the map

$$(a,b) \mapsto (a, b^{\theta_n})$$

we obtain

Lemma 2. *The $(G,*)$-algebras $(M_n \oplus M_n^{op}, \alpha, exc)$ and $(M_n \oplus M_n, \bar{\alpha}, \tilde{\vartheta}_n)$ are isomorphic.*

We conclude with

Proposition 8. *Let $A = (M_n \oplus M_n^{op}, \alpha, exc)$ and $B = (M_n \oplus M_n^{op}, \beta, exc)$, then A and B are isomorphic as $(G,*)$-algebras if and only if, for all $h \in [1,n]$, the polynomial $\tilde{\Phi}_{\alpha,h}$ is not a graded polynomial identity of B. In this case, either $\beta \sim \alpha$ or $\beta \sim \alpha^{-1}$.*

Proof. We realize A and B as $(M_n \oplus M_n, \bar{\alpha}, \tilde{\vartheta}_n)$ and $(M_n \oplus M_n, \bar{\beta}, \tilde{\vartheta}_n)$ respectively. Clearly, by Proposition 2, there exists an evaluation μ_h of $\tilde{\Phi}_{\alpha,h}$ such that $\mu_h(\tilde{\Phi}_{\alpha,h}) = (e_{hh}, 0) \in A$. Hence the same results holds for $B \cong A$. Conversely, if $\tilde{\Phi}_{\alpha,h}$ is not a graded polynomial identity of B then either $\tilde{\Phi}_{\alpha,h} \notin Id_G((M_n, \beta))$ or $\tilde{\Phi}_{\alpha,h}$ does not belong to $Id_G((M_n, \beta^{-1}))$ and so either $\beta \sim \alpha$ or $\beta \sim \alpha^{-1}$.

In the first case, by Proposition 1, the map $\tilde{\varrho}$, defined by $\tilde{\varrho}(e_{ij}) = e_{\varrho(i)\varrho(j)}$, is a graded isomorphism from (M_n, α) to (M_n, β) for some $\varrho \in S_n$. Clearly the permutation $\varrho' = \gamma_n \varrho \gamma_n$ induces a graded isomorphism $\tilde{\varrho}'$ from (M_n, α^{-1}) to (M_n, β^{-1}) and so the map

$$(a,b) \mapsto (\tilde{\varrho}(a), \tilde{\varrho}'(b))$$

is the required isomorphism as $(G,*)$-algebras.

In the second case, the graded isomorphism $\tilde{\varrho}$ goes from (M_n, α^{-1}) to (M_n, β) and so the map

$$(a,b) \mapsto (\tilde{\varrho}(b), \tilde{\varrho}'(a))$$

is the required isomorphism as $(G,*)$-algebras. □

We remark that the corresponding $(G,*)$-isomorphisms from $(M_n \oplus M_n, \bar{\alpha}, \theta_{2n})$ to the algebra $(M_n \oplus M_n, \bar{\beta}, \theta_{2n})$ are given by conjugating with the $2n \times 2n$-matrices

$$\begin{pmatrix} C & 0 \\ 0 & C' \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & C \\ C' & 0 \end{pmatrix} \tag{5}$$

respectively in the first and in the second case. Here C is the permutation-matrix related to ϱ and C' is the matrix related to $\varrho^{\gamma_n} = \gamma_n \varrho \gamma_n$, that is $C' = J_n C J_n = (C^t)^{\theta_n} = (C^{-1})^{\theta_n}$.

4. A classification of finite dimensional simple $(\mathbb{C}_p, *)$-algebras

In this section we present a classification of finite dimensional $(\mathbb{C}_p, *)$-simple algebras over an algebraically closed field of characteristic zero, where p is an odd prime and $\mathbb{C}_p =< \varepsilon >$ is the cyclic group of order p. We start recalling that, when G is a finite abelian group and $\hat{G}$ denotes its dual group, there exists a well understood duality between G-gradings and $\hat{G}$-actions. Moreover if $*$ is an involution on a G-graded algebra A then $*$ is a graded involution if and only if $*$ commutes with the $\hat{G}$-action on A.

In this situation, the proof of next theorem can be easily derived from the literature (e.g., see [21]). It is a generalization of the classical Wedderburn and Wedderburn-Malcev Theorems.

Theorem 1. *Let G be a finite abelian group, A be a finite dimensional $(G,*)$-algebra over an algebraically closed field of characteristic zero and $J = J(A)$ the Jacobson radical of A. Then:*

- (a) *J is a $(G,*)$-ideal;*
- (b) *If A is a $(G,*)$-simple algebra, then either A is G-simple or $A = B \oplus B^*$ where B is a G-simple subalgebra of A;*
- (c) *If A is semisimple, then A is a finite direct sum of $(G,*)$-simple algebras;*
- (d) *$A = A_1 \oplus \cdots \oplus A_m + J$, where each algebra A_i, for all $i \in [1,m]$, is a $(G,*)$-simple algebra.*

The classification of finite dimensional simple $\mathbb{C}_p$-algebras is a particular case of a deep result due to Bahturin, Seghal and Zaicev [6] concerning the structure of finite dimensional graded algebras.

Let us consider the group algebra $F[\mathbb{C}_p]$, we can realize this algebra as a graded subalgebra of M_p with respect the elementary grading induced by $\tilde{\varepsilon} = (1_G, \varepsilon, \ldots, \varepsilon^{p-1})$. More precisely, given the cyclic permutation $\eta := (12 \ldots p)$ of the symmetric group S_p, let us denote by D the graded subalgebra of M_p generated, as a vector space, by the elements

$$c_{\varepsilon^{h-1}} := \sum_{i=0}^{p-1} e_{\eta^i(1)\eta^i(h)} \text{ for all } h \in [1,p], \tag{6}$$

so that the elements of D are:

$$a_1c_{1_G} + a_2c_\varepsilon + \cdots + a_pc_{\varepsilon^{p-1}} = \begin{pmatrix} a_1 & a_2 & \cdots & a_{p-1} & a_p \\ a_p & a_1 & \ddots & & a_{p-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_3 & & \ddots & \ddots & a_2 \\ a_2 & a_3 & \cdots & a_p & a_1 \end{pmatrix} \in M_p$$

where $a_1, \ldots, a_p \in F$. Clearly $D \cong F[\mathbb{C}_p]$ as graded algebras, moreover $M_n(D)$ has a natural $\mathbb{C}_p$-grading, induced by the natural grading on D and the trivial one on M_n, denoted by $\tilde{1}_G = (\underbrace{1_G, \ldots, 1_G}_{n})$. As above $M_n(D) \cong D \otimes M_n$ is a graded subalgebra of $M_{pn} \cong M_p \otimes M_n$ with respect the elementary grading induced by

$$\tilde{\varepsilon} \otimes \tilde{1}_G = (\underbrace{\underbrace{1_G, \varepsilon, \ldots, \varepsilon^{p-1}}, \ldots, \underbrace{1_G, \varepsilon, \ldots, \varepsilon^{p-1}}}_{n \ times}).$$

Hence we denote by $\tilde{\varepsilon}_n$ both this elementary grading on M_{pn} and the natural one in $M_n(D)$. We remark that the invariance subgroup $H_{\tilde{\varepsilon}_n}$ is the whole group $\mathbb{C}_p$ and $w_{\tilde{\varepsilon}_n}(g) = n$ for all $g \in \mathbb{C}_p$.

Now, as in [12, Proposition 3.1.], we have:

Proposition 9. *Let F be an algebraically closed field of characteristic zero and $\mathbb{C}_p = \langle \varepsilon \rangle$ a group of prime order p. If A is a finite dimensional $\mathbb{C}_p$-simple algebra, then it is isomorphic to one of the following $\mathbb{C}_p$-graded algebras:*

(a) (M_n, α)*, for some elementary grading* α*;*
(b) $(M_n(D), \tilde{\varepsilon}_n)$*.*

We now deal with the classification of graded involutions on $(M_n(D), \tilde{\varepsilon}_n)$. Notice that the F-linear maps on $M_n(D)$ defined by, for all $d \in D$,

$$(de_{ij})^{\bar{\theta}_n} = de_{ij}^{\theta_n}$$

or, when n is an even integer

$$(de_{ij})^{\bar{\sigma}_n} = de_{ij}^{\sigma_n},$$

are graded involutions on $(M_n(D), \tilde{\varepsilon}_n)$. More precisely they are induced on $M_n(D)$ by the graded involutions θ_{pn} or σ_{pn} defined in the whole matrix algebra $(M_{pn}, \tilde{\varepsilon}_n)$. It is not surprising to find that, up to isomorphisms, these are the only graded involutions defined on it. More precisely, we have:

Proposition 10. *If F is an algebraically closed field of characteristic zero then, for any graded involution on $(M_n(D), \tilde{\varepsilon}_n)$, either*

$$(M_n(D), \tilde{\varepsilon}_n, *) \cong (M_n(D), \tilde{\varepsilon}_n, \bar{\theta}_n)$$

or

$$(M_n(D), \tilde{\varepsilon}_n, *) \cong (M_n(D), \tilde{\varepsilon}_n, \bar{\sigma}_n)$$

and n is even.

Proof. Since D is abelian, the center of $M_n(D)$ is $A = \{dI_n \mid d \in D\}$ and so A is a $(\mathbb{C}_p, *)$ subalgebra of $M_n(D)$. On the other hand $B = M_n(Fc_{1_G})$ is the homogeneous component of degree 1_G of $(M_n(D), \tilde{\varepsilon}_n)$, and so $*$ induces an involution on it. Clearly

$$(M_n(D), \tilde{\varepsilon}_n, *) \cong (A \otimes B, \tilde{\varepsilon} \otimes \tilde{1}_G, *_A \otimes *_B)$$

where $*_A$ and $*_B$ denote the involutions induced by $*$ on A and B respectively. Moreover $(A, \tilde{\varepsilon}, *_A) \cong (D, \tilde{\varepsilon}, \diamond)$ and $(B, \tilde{1}_G, *_B) \cong (M_n, \tilde{1}_G, \natural)$ where $\diamond$ and $\natural$ are the involutions corresponding to $*_A$ and $*_B$ in the natural isomorphisms of G-graded algebras between A and D, B and M_n respectively. Since $*$ preserves the grading and the homogeneous components of D are 1-dimensional then for each homogeneous element d of D there exists $t \in F$ such that $d^\diamond = td$. Clearly, $t = \pm 1$ since $\diamond$ is an involution. Moreover $c_{1_G}^\diamond = 1_D^\diamond = c_{1_G}$ and so $(d^p)^\diamond = d^p$, for each homogeneous element d of D, because $d^p \in Fc_{1_G}$. It follows that $t^p = 1$ and so $t = 1$. Therefore $\diamond$ is the identity on D and $*$ is the identity on A. On the other hand, by Proposition 4, either $(M_n, \natural) \cong (M_n, \theta_n)$ or $(M_n, \natural) \cong (M_n, \sigma_n)$ and n is even. The result follows by previous Remarks 2, 3. □

Now we are able to classify the finite dimensional $(\mathbb{C}_p, *)$-simple algebras.

Theorem 2. *Let A be a finite dimensional $(\mathbb{C}_p, *)$-simple algebra over an algebraically closed field of characteristic zero. Then A is isomorphic to one of the following simple $(\mathbb{C}_p, *)$-algebras:*

(*a*) *(M_n, α, θ_n) or (M_n, α, σ_n) where the elementary grading is defined by $\alpha : [1, n] \to \mathbb{C}_p$ satisfying the condition $\alpha_i\alpha_{n+1-i} = \alpha_j\alpha_{n+1-j}$ for all $i, j \in [1, n]$;*
(*b*) *$(M_n(D), \tilde{\varepsilon}_n, \bar{\theta}_n)$ or $(M_n(D), \tilde{\varepsilon}_n, \bar{\sigma}_n)$;*
(*c*) *$(M_n \oplus M_n^{op}, \alpha, exc)$, with grading induced by the elementary grading defined on M_n by the map $\alpha : [1, n] \to \mathbb{C}_p$ and exchange involution;*

(*d*) $(M_n(D)\oplus M_n(D)^{op}, \tilde{\varepsilon}_n, exc)$, *with grading induced by the natural one on* $M_n(D)$ *and exchange involution.*

Proof. Let A be a finite dimensional $(\mathbb{C}_p,*)$-simple algebra. By Theorem 1, it follows that either A is $\mathbb{C}_p$-simple or $A = B\oplus B^*$, for some $\mathbb{C}_p$-simple subalgebra B of A.

Let assume that A is a $\mathbb{C}_p$-simple algebra. If A is simple then by item (a) of Proposition 9 and by Proposition 5 we have (a). If A is $\mathbb{C}_p$-simple but not simple, then by item (b) of Proposition 9 and by Proposition 10 we obtain (b).

Now, let $A = B\oplus B^*$, where B is a $\mathbb{C}_p$-simple subalgebra of A. Let consider $C = B\oplus B^{op}$, with the natural $\mathbb{C}_p$-grading induced by the one existing on B and exchange involution $(a,b)^{exc} = (b,a)$. Clearly A and C are isomorphic as $(\mathbb{C}_p,*)$-algebras, because $(a+b^*)^* = b+a^*$ in $A = B\oplus B^*$. Then, by Proposition 9, we obtain the statements (c), (d) and the proof is complete. □

We remark that all the algebras in the statement of Theorem 2 can be defined even in the case in which the field F is not algebraically closed, in this case we say that A is a **classical** finite dimensional $(\mathbb{C}_p,*)$-simple algebra over the field F.

Now as a consequence of Proposition 6 we have:

Corollary 1. *Let A and B be classical finite dimensional $(\mathbb{C}_p,*)$-simple algebras, if $\dim_F A_g^\mu = \dim_F B_g^\mu$ for all $g\in\mathbb{C}_p$ and $\mu\in\{+,-\}$, then one of the following cases occurs*

(*a*) $A\cong B$ *as* $(\mathbb{C}_p,*)$-*algebras;*
(*b*) *there exist some maps* $\alpha,\beta : [1,n]\to\mathbb{C}_p$ *such that* $A = (M_n\oplus M_n^{op}, \alpha, exc)$ *and* $B = (M_n\oplus M_n^{op}, \beta, exc)$ *with exchange involutions and gradings induced by the elementary ones defined on* M_n *by* α *and* β *respectively.*

We conclude this section by noting that the canonical basis $\mathcal{E}_n := \{e_{ij}\mid i,j\in[1,n]\}$ of M_n is always homogeneous since the grading is elementary, and moreover it is **strongly multiplicative**, that is if b_1, b_2 are basis elements and $b_1b_2\neq 0$ then b_1b_2 is a basis element too. The same properties hold for the basis

$$\mathcal{D}_n := \{c_{\varepsilon^{h-1}}e_{ij}\mid h\in[1,p],\ i,j\in[1,n]\}$$

of $M_n(D)$ and for those of algebras $M_n\oplus M_n^{op}$ and $M_n(D)\oplus M_n(D)^{op}$, given respectively by $\tilde{\mathcal{E}}_n := \{(b,0)\mid b\in\mathcal{E}_n\}\cup\{(0,b)\mid b\in\mathcal{E}_n\}$ and $\tilde{\mathcal{D}}_n := \{(b,0)\mid b\in\mathcal{D}_n\}\cup\{(0,b)\mid b\in\mathcal{D}_n\}$.

If $\mathcal{B}$ is any of these canonical bases and $b\in\mathcal{B}$ then there exists $c\in\{1,-1\}$ such that cb^* belongs to $\mathcal{B}$. So we write $b^*\in\pm\mathcal{B}$ and we say that $\mathcal{B}$ is **almost $*$-invariant**.

5. The $(\mathbb{C}_p, *)$-algebra $UT^*_{\mathbb{C}_p}(A_1, \dots, A_m)$

In this section we will define the algebra $UT^*_{\mathbb{C}_p}(A_1, \dots, A_m)$ of upper block triangular matrices associated with a sequence $(A_1, \dots, A_m)$ of classical finite dimensional $(\mathbb{C}_p, *)$-simple algebras. To this end, some preliminary considerations are in order.

More precisely, if A is one of these algebras then, for some positive integer n, we assume that either $A = S$ or $A = S \oplus S^{op}$ where $S \in \{M_n, M_n(D)\}$ and $M_n(D)$ is the graded subalgebra of M_{np} defined in the previous section.

In this situation we say that S is the *constituent* of A and n is the size of A. Moreover, if $S = M_n$ we say that A has *extended size* n, and if $S = M_n(D)$ we say that A has *extended size* pn. We denote by s_A the extended size of A.

We remark that in the previous definition S is always a graded subalgebra of M_{s_A} with respect to the corresponding elementary grading $\alpha_S : [1, s_A] \to \mathbb{C}_p$.

We have:

Proposition 11. *If A is a classical finite dimensional $(\mathbb{C}_p, *)$-simple algebra, with constituent S, extended size $s = s_A$ and corresponding grading $\alpha = \alpha_S$ then A is isomorphic to a $(\mathbb{C}_p, *)$-subalgebra of $(M_s \oplus M_s, \bar{\alpha}, \theta_{2s})$ via the map φ_A defined by:*

$$(a, b) \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}^{\theta_{2s}} = \begin{pmatrix} a & 0 \\ 0 & b^{\theta_s} \end{pmatrix} \quad \textit{if} \quad A = S \oplus S^{op}$$

and

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a^* & 0 \\ 0 & 0 \end{pmatrix}^{\theta_{2s}} = \begin{pmatrix} a & 0 \\ 0 & a^{*\theta_s} \end{pmatrix} \quad \textit{if} \quad A = S.$$

Proof. If $A = (M_s \oplus M_s^{op}, \alpha, exc)$ then the map φ_A is precisely the map ψ_s considered in Lemma 1. If $A = (M_n(D) \oplus M_n(D)^{op}, \tilde{\varepsilon}_n, exc)$ then the map φ_A is the restriction of ψ_{pn}. Finally, we remark that we can consider any $(G, *)$-algebra A as a $(G, *)$-subalgebra of $(A \oplus A^{op}, exc)$ with the induced grading, via the $(G, *)$-embedding given by $a \mapsto (a, a^*)$ for all $a \in A$. So, by the first part of this proof, we are done. $\square$

If we look at M_{2s} as the tensor product $M_s \otimes M_2$, then we can write the matrix $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ as $a \otimes e_{11} + b \otimes e_{22}$ or, using the involution θ_2, as $a \otimes e_{11} + b \otimes e_{11}^{\theta_2}$, for all $a, b \in M_s$.

Generalizing this construction to a direct sum we obtain the following:

Proposition 12. *Let $A_1, \dots, A_m$ be classical finite dimensional $(\mathbb{C}_p, *)$-simple algebras with graded involutions $*_i$, constituents $S_i \subseteq M_{s_i}$ of extended size s_i, grading words α^i on M_{s_i}, and embeddings $\varphi_i : A_i \to R_i = M_{s_i} \oplus M_{s_i}$ respectively.*

Let $s := s_1 + \ldots + s_m$ *and* $\alpha := (\alpha^1|\ldots|\alpha^m)$ *be the concatenation of the words* $\alpha^1, \ldots, \alpha^m$*; consider* M_{2s} *endowed with the grading induced by* $\bar{\alpha}$*, the* $*$*-closure of* α*, and involution* θ_{2s}*. Finally, define* $\zeta : R_1 \oplus \cdots \oplus R_m \to M_{2s}$ *by*

$$\zeta\left(\begin{pmatrix} a_1 & 0 \\ 0 & b_1 \end{pmatrix}, \ldots, \begin{pmatrix} a_m & 0 \\ 0 & b_m \end{pmatrix}\right) := \sum_{i=1}^{m} a_i \otimes e_{ii} + \sum_{i=1}^{m} b_i \otimes e_{ii}^{\theta_{2m}}$$

Then the map φ*, defined by*

$$\varphi(u_1, \ldots, u_m) := \zeta(\varphi_1(u_1), \ldots, \varphi_m(u_m))$$

is a $*$*-embedding of* $A_1 \oplus \cdots \oplus A_m$ *in* $(M_{2s}, \bar{\alpha}, \theta_{2s})$*.*

Now, with the same notation of the previous proposition we have:

Definition 5. Let $A_1, \ldots, A_m$ be classical finite dimensional $(\mathbb{C}_p, *)$-simple algebras, and φ the $*$-embedding of $A_1 \oplus \cdots \oplus A_m$ in $(M_{2s}, \bar{\alpha}, \theta_{2s})$. For $i, j \in [1, m]$, let $V_{ij} := M_{s_i \times s_j}$ and in M_{2s} denote:

$$U_{ij} := V_{ij} \otimes e_{ij}, \quad \overline{U_{ij}} := U_{ij}^{\theta_{2s}} = V_{ji} \otimes e_{ij}^{\theta_{2m}} \quad \text{and} \quad V = \bigoplus_{\substack{i,j \in [1,m] \\ i<j}} \left(U_{ij} \oplus \overline{U_{ij}}\right),$$

then we define

$$\mathbf{A} := UT^*_{\mathbb{C}_p}(A_1, \ldots, A_m) = \varphi(A_1 \oplus \cdots \oplus A_m) \oplus V.$$

A direct verification shows that $UT^*_{\mathbb{C}_p}(A_1, \ldots, A_m)$ is actually a $(\mathbb{C}_p, *)$-subalgebra of M_{2s} enveloping $A_1 \oplus \cdots \oplus A_m$ and whose Jacobson radical coincides with V.

We observe that the definition of the $(\mathbb{C}_p, *)$-structure on $UT^*_{\mathbb{C}_p}(A_1, \ldots, A_m)$ depends heavily on the maps $\alpha^1, \ldots, \alpha^m$ that define the gradings on each of our classical algebras $A_1, \ldots, A_m$. In this paper, when appropriate, we will emphasize the role of these maps. An important case is the following

Definition 6. Given the sequence $A_1, \ldots, A_m$ of classical finite dimensional $(\mathbb{C}_p, *)$-simple algebras, with grading words $\alpha^1, \ldots, \alpha^m$ for any m-tuple $\mathbf{g} = (g_1, \ldots, g_m) \in \mathbb{C}_p^m$, we consider the $\mathbb{C}_p$-words $g_1\alpha^1, \ldots, g_m\alpha^m$ of lenght $s_1, \ldots, s_m$ respectively, defined by $(g_h\alpha^h)_t = g_h\alpha^h_t$ for all $t \in [1, s_h]$ and for all $h \in [1, m]$. We define the words

$$\alpha_{\mathbf{g}} := (g_1\alpha^1|\ldots|g_m\alpha^m) \quad \text{and its } *\text{-closure} \quad \bar{\alpha}_{\mathbf{g}}$$

and the corresponding subalgebra of $(M_{2s}, \bar{\alpha}_{\mathbf{g}}, \theta_{2s})$, that we denote by

$$\mathbf{A_g} := UT^*_{\mathbb{C}_p,\mathbf{g}}(A_1, \ldots, A_m)$$

distinguishing it from the algebra $\mathbf{A} = UT^*_{\mathbb{C}_p}(A_1, \dots, A_m)$ determined using the previous grading words $\alpha^1, \dots, \alpha^m$.

Once again, the h-th simple component of the maximal semisimple $(\mathbb{C}_p, *)$-subalgebra of $UT^*_{\mathbb{C}_p,\mathbf{g}}(A_1, \dots, A_m)$ is isomorphic to A_h. Moreover, if $\mathbf{g} = (g, \dots, g)$ for some $g \in \mathbb{C}_p$ then

$$UT^*_{\mathbb{C}_p,\mathbf{g}}(A_1, \dots, A_m) = UT^*_{\mathbb{C}_p}(A_1, \dots, A_m).$$

Now we establish some conditions on the presentations of $A_1, \dots, A_m$ which yield isomorphic $(\mathbb{C}_p, *)$-algebras.

Proposition 13. *Given the sequence $A_1, \dots, A_m$ of classical finite dimensional $(\mathbb{C}_p, *)$-simple algebras, with grading words $\alpha^1, \dots, \alpha^m$, for any $k \in [1, m]$ we have*

$$UT^*_{\mathbb{C}_p}(A_1, \dots, A_k, \dots, A_m) \cong UT^*_{\mathbb{C}_p}(A_1, \dots, B_k, \dots, A_m)$$

where:

(a) *If $A_k = (M_n, \alpha^k, *_k)$ we can permute appropriately the elements of the word α^k; that is, we consider $B_k = (M_n, \beta^k, *_k)$ and $\beta^k = \alpha^k \varrho^{-1}$ for some $\varrho \in S_n$ such that $\varrho\gamma_n = \gamma_n\varrho$.*

(b) *If $A_k = (M_n \oplus M_n^{op}, \alpha^k, exc)$ we can permute without any restriction the elements of the word α^k; that is, we consider $B_k = (M_n \oplus M_n^{op}, \beta^k, exc)$ and $\beta^k = \alpha^k \varrho^{-1}$ for any $\varrho \in S_n$.*

(c) *If $A_k = (M_n \oplus M_n^{op}, \alpha^k, exc)$ we can multiply the elements of the word α^k by any element of the invariance subgroup H_{α^k}; that is, we consider $B_k = (M_n \oplus M_n^{op}, \beta^k, exc)$ and $\beta^k = g\alpha^k = (g\alpha^k_1, \dots, g\alpha^k_n)$ for any $g \in H_{\alpha^k}$.*

(d) *If $A_k = (M_n(D), \tilde{\varepsilon}_n, \bar{*}_k)$ or $A_k = (M_n(D) \oplus M_n(D)^{op}, \tilde{\varepsilon}_n, exc)$ we can multiply the elements of the word $\tilde{\varepsilon}_n$ by any element of $\mathbb{C}_p$; that is, we consider respectively $B_k = (M_n(D), g\tilde{\varepsilon}_n, \bar{*}_k)$ or $B_k = (M_n(D) \oplus M_n(D)^{op}, g\tilde{\varepsilon}_n, exc)$ and*

$$g\tilde{\varepsilon}_n = (g, g\varepsilon, \dots, g\varepsilon^{p-1}, \dots, g, g\varepsilon, \dots, g\varepsilon^{p-1}).$$

Proof. For each case we describe the isomorphisms maps acting on the whole algebra $(M_{2s}, \bar{\alpha}, \theta_{2s})$ which induce the required isomorphisms between $UT^*_{\mathbb{C}_p}(A_1, \dots, A_k, \dots A_m)$ and $UT^*_{\mathbb{C}_p}(A_1, \dots, B_k, \dots, A_m)$. The isomorphisms are determined by conjugating with the matrix

$$I_{2s} + (P - I_{s_k}) \otimes e_{kk} + ((P^{-1})^{\theta_{s_k}} - I_{s_k}) \otimes e_{kk}^{\theta_{2m}},$$

where for each case we consider the corresponding matrix P as follows:

(a) In this case clearly $s_k = n$, moreover we have:

$$\text{if } *_k = \theta_n \text{ then } P = \sum_{i=1}^{n} e_{\varrho(i)i}, \text{ while if } *_k = \sigma_n \text{ then } P = \sum_{i=1}^{n} \varepsilon(i) e_{\varrho(i)i}$$

that is P is one of the matrices C and E considered in (4).

(b) In this case

$$P = \sum_{i=1}^{n} e_{\varrho(i)i}$$

is the matrix C considered in (5).

(c) In this case, since $g \in H_\alpha$, there exists $\varrho \in S_n$ such that $\beta^k = \alpha^k \varrho^{-1}$ and, as in the previous case, we choose

$$P = \sum_{i=1}^{n} e_{\varrho(i)i}$$

(d) In this case, $s_k = pn$ and there exists some power ϱ of the permutation $\eta = (12 \dots p)$ such that $g\tilde{\varepsilon} = (g, g\varepsilon, \dots, g\varepsilon^{p-1}) = \tilde{\varepsilon}\varrho^{-1}$. Therefore we can consider the matrix

$$P = \sum_{i=1}^{p} e_{\varrho(i)i} \otimes I_n \in M_p \otimes M_n. \quad \square$$

In the light of this result, it is appropriate to present the natural grading on the algebra $M_n(D)$ as induced by the map $g\tilde{\varepsilon}_n$, where g is any element of $\mathbb{C}_p$. Once again $H_{\tilde{\varepsilon}_n} = H_{g\tilde{\varepsilon}_n} = \mathbb{C}_p$ and any element of $\mathbb{C}_p$ has maximal weight. In the following, in analogy to the case of M_n, we will denote by α the map considered to define the given grading on $M_n(D)$. Clearly the same considerations apply for the algebra $M_n(D) \oplus M_n(D)^{op}$. Moreover, it makes sense to give in this paper the following

Definition 7. Let A be a classical finite dimensional $(\mathbb{C}_p, *)$-simple algebra with grading word α, we say that

· A is $(\mathbb{C}_p, *)$***-regular*** if, for some $n \geq 1$, A is one of the following algebras:

$$M_n(D), \quad M_n(D) \oplus M_n(D)^{op} \quad \text{or} \quad M_n \oplus M_n^{op} \text{ with } H_\alpha = \mathbb{C}_p,$$

· A is $(\mathbb{C}_p, *)$***-singular*** if, for some $n \geq 1$, either

$$A = M_n \quad \text{or} \quad A = M_n \oplus M_n^{op} \text{ with } H_\alpha = \{1_{\mathbb{C}_p}\}.$$

The last but not least instance of basic moves leading to isomorphic algebras is the following:

Lemma 3. *Given the sequence $A_1, \ldots, A_m$ of classical finite dimensional $(\mathbb{C}_p, *)$-simple algebras, with grading words $\alpha^1, \ldots, \alpha^m$, let $B_k = A_{m+1-k}$ with grading word $\beta^k = (\alpha^{m+1-k})^{-1}$ for all $k \in [1, m]$, then*

$$UT^*_{\mathbb{C}_p}(A_1, \ldots, A_m) \cong UT^*_{\mathbb{C}_p}(B_1, \ldots, B_m).$$

Proof. The result follows considering the $(\mathbb{C}_p, *)$-isomorphism $\tilde{\tau}$ between $(M_s \oplus M_s, \overline{\alpha}, \theta_{2s})$ and $(M_s \oplus M_s, \overline{\alpha^{-1}}, \theta_{2s})$, induced by the permutation $\tau \in S_{2s}$:

$$\tau(i) = \begin{cases} i + s & \text{if } i \in [1, s] \\ i - s & \text{if } i \in [s + 1, 2s]. \end{cases} \quad \square$$

Once again, let us make explicit a canonical basis of $\mathbf{A} := UT^*_{\mathbb{C}_p}(A_1, \ldots, A_m)$. Clearly, a basis of $J(\mathbf{A})$ is given by the following set

$$\mathfrak{B}_J := \bigcup_{i,j \in [1,m], i<j} (\{e_{uv} \otimes e_{ij} \mid u \in [1, s_i],\, v \in [1, s_j]\} \cup \{e_{vu} \otimes e_{ij}^{\theta_{2m}} \mid v \in [1, s_j],\, u \in [1, s_i]\}).$$

To obtain a canonical basis of $\mathbf{A}$ we consider, for each $k \in [1, m]$, the canonical basis $\mathcal{B}_k$ of the classical finite dimensional $(\mathbb{C}_p, *)$-simple algebra A_k, we denote by $\mathfrak{B}_k$ its image $\varphi(\mathcal{B}_k)$ in $\mathbf{A}$, and we add the elements of $\mathfrak{B}_J$. At the end, we get:

Definition 8. Let $\mathfrak{B}_{\mathbf{A}} := (\bigcup_{k \in [1,m]} \mathfrak{B}_k) \cup \mathfrak{B}_J$. This homogeneous basis of $\mathbf{A}$ is called the **canonical basis** of $\mathbf{A}$.

As in the previous section it is easy to show that $\mathfrak{B} = \mathfrak{B}_{\mathbf{A}}$ is strongly multiplicative and almost $*$-invariant, that is:

- $b_1 b_2 \in \mathfrak{B}$ for all $b_1, b_2 \in \mathfrak{B}$ such that $b_1 b_2 \neq 0$;
- $b^* \in \pm\mathfrak{B}$ for all $b \in \mathfrak{B}$.

Actually, any non-zero product of elements of $\mathfrak{B}$ is well behaved with respect to the action of permutations and the involution. More precisely, let A be a $(\mathbb{C}_p, *)$-algebra, $a_1, \ldots, a_r \in A$ and $a = a_1 \cdots a_r$. If $\sigma \in S_r$ and $\lambda : [1, r] \to \{1, *\}$ is any map we denote

$$a_\sigma^\lambda := a_{\sigma(1)}^{\lambda(1)} \ldots a_{\sigma(r)}^{\lambda(r)}.$$

We are interested to compute the product b_σ^λ when the elements $b_1, \ldots, b_r$ belong to the canonical basis $\mathfrak{B}$ of $\mathbf{A}$.

We start by considering first only elements belonging to a unique classical finite dimensional $(\mathbb{C}_p,*)$-simple algebra. In this case let us say that (a,b) is a diagonal element of $M_n \oplus M_n^{op}$, or $M_n(D) \oplus M_n^{op}(D)$, if and only if a and b are diagonal matrices in M_n or $M_n(D)$ respectively. Now, with this terminology we have:

Lemma 4. *Let A be a classical finite dimensional $(\mathbb{C}_p,*)$-simple algebra, $\mathcal{B}$ its canonical basis, $b_1,\dots,b_r \in \mathcal{B}$, $b := b_1\cdots b_r$, $\sigma \in S_r$ and $\lambda : [1,r] \to \{1,*\}$. If $b, b_\sigma^\lambda \neq 0$ then*

- *b is in $\mathcal{B}$ and b_σ^λ belongs to $\pm\mathcal{B}$;*
- *if b is a diagonal element, then b_σ^λ is diagonal, too;*
- *if b is not a diagonal element, then $b_\sigma^\lambda \in \{\pm b, \pm b^*\}$.*

Proof. If $A = (M_n, \tilde{g}, *)$, where $* = \theta_n, \sigma_n$ or $A = (M_n(D), \tilde{\varepsilon}_n, *)$, with $* = \bar{\theta}_n, \bar{\sigma}_n$ then the statement follows from Lemma 1 of [14].

If $A = (M_n \oplus M_n^{op}, \alpha, exc)$ or $A = (M_n(D) \oplus M_n(D)^{op}, \tilde{\varepsilon}_n, exc)$, $b \neq 0_A$ implies that $b_1,\dots,b_r$ belong to the same component of A. Moreover, since $b_\sigma^\lambda \neq 0_A$, we have that λ is a constant map, that is either $\lambda(i) = 1$ or $\lambda(i) = *$ for all $i \in [1,r]$. In the former case, $b_\sigma^\lambda = b_\sigma$ and the statement follows from the previous result about products of matrix units [27, Lemma 1]; in the latter case, instead, $b_\sigma^\lambda = b_{\sigma(1)}^* \dots b_{\sigma(r)}^* = (b_{\sigma(r)} \dots b_{\sigma(1)})^*$ and again the statement follows. □

Similarly, in the general case we have:

Lemma 5. *Let $\mathfrak{B}$ be the canonical basis of $\mathbf{A}$, consider $b_1,\dots,b_r \in \mathfrak{B}$ and let $b = b_1\cdots b_r$. If $b \neq 0_{\mathbf{A}}$ then either $b \in \mathfrak{B}_k$ for some $k \in [1,m]$ or $b \in \mathfrak{B}_J$. Moreover, for any $\sigma \in S_r$ and $\lambda : [1,r] \to \{1,*\}$ such that $b_\sigma^\lambda \neq 0$, we have:*

- *If $b \in \mathfrak{B}_k$, then $b_i \in \mathfrak{B}_k$ for all $i \in [1,r]$, and $b_\sigma^\lambda \in \pm\mathfrak{B}_k$;*
- *If $b \in \mathfrak{B}_J$ then $b_\sigma^\lambda \in \{\pm b, \pm b^*\}$.*

6. $(\mathbb{C}_p,*)$-varieties and algebras $UT^*_{\mathbb{C}_p}(A_1,\dots,A_m)$

Our goal in this section is to prove that any minimal variety (see Definition 1) of $(\mathbb{C}_p,*)$-algebras is generated by some of the finite dimensional $(\mathbb{C}_p,*)$-algebras $UT^*_{\mathbb{C}_p}(A_1,\dots,A_m)$ introduced in the previous section. For this purpose let us consider the following

Definition 9. Let A be a finite dimensional $(\mathbb{C}_p,*)$-algebra over an algebraically closed field F. We say that A is **triangular** if either it is $(\mathbb{C}_p,*)$-simple or $A = A_1 \oplus \cdots \oplus A_m + J(A)$, with $A_1,\dots,A_m$ $(\mathbb{C}_p,*)$-simple algebras, $m \geq 2$, and there exist homogeneous elements $w_{1,2},\dots,w_{m-1,m} \in J(A)$ and minimal homogeneous idempotents $e_i \in A_i$, for each $i \in [1,m]$, such that

(a) $w_{1,2}\cdots w_{m-1,m}\neq 0_A$;
(b) $e_i w_{i,i+1}=w_{i,i+1}=w_{i,i+1}e_{i+1}$ for all $i\in[1,m-1]$.
(c) $J(A)=I(A)\oplus I(A)^*$ where $I(A)$ is the ordinary two-sided ideal generated by $w_{1,2},\dots,w_{m-1,m}$.

In this situation the elements $w_{i,i+1}$'s are called the homogeneous connectors of the simple components of A and $J(A)$ is generated as an ordinary two-sided ideal by the elements $w_{1,2},w_{1,2}^*,\dots,w_{m-1,m},w_{m-1,m}^*$.

We observe that the order of the classical finite dimensional $(\mathbb{C}_p,*)$-simple algebras $A_1,\dots,A_m$ in the semisimple part of a triangular $(\mathbb{C}_p,*)$-algebra, $\bar{A}$, is important. For this reason, in the rest of this paper, when we write $\bar{A}=A_1\oplus\cdots\oplus A_m$ then we assume that $A_1J\cdots JA_m\neq 0_A$.

If A is a triangular $(\mathbb{C}_p,*)$-algebra, then remembering the connection between the notions of admissible $(\mathbb{C}_p,*)$-algebra and $(\mathbb{C}_p,*)$-exponent we get $\exp^*_{\mathbb{C}_p}(A)=\dim_F\bar{A}$.

Moreover using exactly the same arguments of the proof of Lemma 4.2 in [13], which deals with the case $p=2$, we obtain the following:

Proposition 14. *Let $\mathcal{V}=\mathrm{var}^*_{\mathbb{C}_p}(A)$ be the variety generated by a finite dimensional $(\mathbb{C}_p,*)$-algebra A over an algebraically closed field F, then there exists a triangular $(\mathbb{C}_p,*)$-algebra B in $\mathcal{V}$ such that $\exp^*_{\mathbb{C}_p}(A)=\exp^*_{\mathbb{C}_p}(B)$. In particular, if $\mathcal{V}$ is minimal, then one has $\mathcal{V}=\mathrm{var}^*_{\mathbb{C}_p}(B)$.*

The next theorem highlights the close link between triangular $(\mathbb{C}_p,*)$-algebras and algebras of type $\mathbf{A_g}$.

Theorem 3. *Let A be a triangular $(\mathbb{C}_p,*)$-algebra with semisimple part $\bar{A}=A_1\oplus\cdots\oplus A_m$. Then there exists $\tilde{\mathbf{g}}\in\mathbb{C}_p^m$ such that $\mathbf{A}_{\tilde{\mathbf{g}}}=UT^*_{\mathbb{C}_p,\tilde{\mathbf{g}}}(A_1,\dots,A_m)$ belongs to $\mathrm{var}^*_{\mathbb{C}_p}(A)$. In particular, if $\mathrm{var}^*_{\mathbb{C}_p}(A)$ is minimal, then $\mathrm{var}^*_{\mathbb{C}_p}(A)=\mathrm{var}^*_{\mathbb{C}_p}(\mathbf{A}_{\tilde{\mathbf{g}}})$.*

Proof. First of all, as it is allowed to do, we assume that, for all $k\in[1,m]$, the idempotent e_k of the k-th $(G,*)$-simple component A_k of the triangular algebra A corresponds to the matrix unit $e_{j_kj_k}$ of M_{n_k} in the case in which A_k is isomorphic to $(M_{n_k},\alpha^k,*_k)$ and to the elements $c_{1_G}e_{j_kj_k}$, $(e_{j_kj_k},0)$, $(c_{1_G}e_{j_kj_k},0)$ in the case in which A_k is isomorphic respectively to $M_{n_k}(D)$, $M_{n_k}\oplus M_{n_k}^{op}$, $M_{n_k}(D)\oplus M_{n_k}(D)^{op}$ with the corresponding gradings and involutions.
We remark that $c_{1_G}e_{j_kj_k}=\sum_{l=1}^{p}e_{p(j_k-1)+l\,p(j_k-1)+l}$ when we realize $M_n(D)$ as subalgebra of M_{pn}. Hence, for all $k\in[1,m]$, we put

$$i_k=\begin{cases} j_k & \text{if } A_k\cong M_{n_k} \text{ or } A_k\cong M_{n_k}\oplus M_{n_k}^{op}\\ p(j_k-1)+1 & \text{if } A_k\cong M_{n_k}(D) \text{ or } A_k\cong M_{n_k}(D)\oplus M_{n_k}(D)^{op}.\end{cases}$$

Now we define $\tilde{\mathbf{g}}=(g_1,\dots,g_m)$ where

$$g_1 = 1_G \text{ and } g_k := \alpha^k(i_k)^{-1}\alpha^1(i_1)|w_{1,2}\cdots w_{k-1,k}|_A, \text{ for all } k \in [2,m]$$

if $1 < m$ and $w_{1,2}, \ldots, w_{m-1,m}$ are the homogeneous connectors of the simple components of A. Our goal is to prove that $\mathbf{A}_{\tilde{\mathbf{g}}} \in \mathrm{var}^*_{\mathbb{C}_p}(A)$. To this end, we work by induction on the number m of the $(\mathbb{C}_p, *)$-simple components of A.

If $m = 1$, then $\mathbf{A}_{\tilde{\mathbf{g}}} = UT^*_{\mathbb{C}_p,\tilde{\mathbf{g}}}(A_1) \cong A_1 = A$ and the statement is clearly true. Hence, we assume that $m \geq 2$ and remark that $|w_{k,k+1}|_A = |e_{i_k i_{k+1}} \otimes e_{k\,k+1}|_{\mathbf{A}_{\tilde{\mathbf{g}}}}$ for all $k \in [1, m-1]$. Now let

$$f = f(x_1, \ldots, x_n, x_1^*, \ldots, x_n^*) \in F\langle X_G, *\rangle - Id^*_{\mathbb{C}_p}(\mathbf{A}_{\tilde{\mathbf{g}}}).$$

We want to show that f is not a $(\mathbb{C}_p, *)$-polynomial identity for A. Since F is a field of characteristic zero, we can assume that f is multilinear. Hence, there exist elements $b_1, \ldots, b_n$ of the canonical basis $\mathfrak{B}$ of $\mathbf{A}_{\tilde{\mathbf{g}}}$ such that $|b_i|_{\mathbf{A}_{\tilde{\mathbf{g}}}} = |x_i|_{F\langle X_G,*\rangle}$ for each $i \in [1, n]$ and $f(b_1, \ldots, b_n, b_1^*, \ldots, b_n^*) \neq 0_{\mathbf{A}_{\tilde{\mathbf{g}}}}$. Since $J(\mathbf{A}_{\tilde{\mathbf{g}}})$ is nilpotent of index m, if t is the number of the b_k's which are in $J(\mathbf{A}_{\tilde{\mathbf{g}}})$ then $t \leq m-1$. Let $b_1, \ldots, b_t$ be such elements, hence $b_1 = e_{u_1 v_1} \otimes e_{h_1 k_1}, \ldots, b_t = e_{u_t v_t} \otimes e_{h_t k_t}$ with $h_l < k_l$ for all $l \leq t$ and $b_r = e_{u_r v_r} \otimes e_{k_r k_r}$ for all $r > t$. Assume first that $t < m-1$. Then there exists an index $\bar{h} \in [1, m-1]$ such that none among of the elements $b_1, \ldots, b_n$ is in $\bigoplus_{j>\bar{h}}(U_{\bar{h},j} \oplus \overline{U_{\bar{h},j}})$. Now, we have to distinguish two cases:

- if there exists $l \in [1, n]$ such that $b_l \in U_{r,\bar{h}} \oplus \overline{U_{r,\bar{h}}}$, for some $r \leqslant \bar{h}$, then all elements b_k's are in the subalgebra

$$\bigoplus_{i\in[1,\bar{h}]} \varphi(A_i) \oplus \bigoplus_{1\leq i<j\leq \bar{h}} (U_{ij} \oplus \overline{U_{ij}}),$$

which is a $(\mathbb{C}_p, *)$-algebra isomorphic to $\mathbf{A}'_{\mathbf{g}'} := UT^*_{\mathbb{C}_p,\mathbf{g}'}(A_1, \ldots, A_{\bar{h}})$, where $\mathbf{g}' = (g_1, \ldots, g_{\bar{h}})$, corresponding to the triangular $(\mathbb{C}_p, *)$-subalgebra A' of A with $(\mathbb{C}_p, *)$-simple components $A_1, \ldots, A_{\bar{h}}$ and connectors $w_{1,2}, \ldots, w_{\bar{h}-1,\bar{h}}$;

- the basis elements $b_1, \ldots, b_n$ are in

$$\bigoplus_{i\in[1,m], i\neq\bar{h}} \varphi(A_i) \oplus \bigoplus_{\substack{1\leq i<j\leq m \\ i\neq\bar{h}\neq j}} \left(U_{ij} \oplus \overline{U_{ij}}\right),$$

which is a $(\mathbb{C}_p, *)$-algebra isomorphic to $\mathbf{A}'_{\mathbf{g}'} := UT^*_{\mathbb{C}_p,\mathbf{g}'}(A_1, \ldots, A_{\bar{h}-1}, A_{\bar{h}+1}, \ldots, A_m)$, for $\mathbf{g}' = (g_1, \ldots, g_{\bar{h}-1}, g_{\bar{h}+1}, \ldots, g_m)$, corresponding to the triangular $(\mathbb{C}_p, *)$-subalgebra A' of A with $m-1$ simple components $A_1, \ldots, A_{\bar{h}-1}, A_{\bar{h}+1}, \ldots, A_m$ and connector elements $w_{1,2}, \ldots, w_{\bar{h}-2,\bar{h}-1}, (w_{\bar{h}-1,\bar{h}} w_{\bar{h},\bar{h}+1}), w_{\bar{h}+1,\bar{h}+2}, \ldots, w_{m-1,m}$.

In both cases $f \notin Id^*_{\mathbb{C}_p}(\mathbf{A}'_{\mathbf{g}'})$. By inductive hypothesis, we conclude that f does not belong to $Id^*_{\mathbb{C}_p}(A)$.

Now, we suppose that $t = m-1$. Let us define $e := \sum_{i=1}^{s} e_{ii} \in M_{2s}$ and $\overline{e} := e^{\theta_{2s}}$. Hence e is an homogeneous idempotent of degree 1_G and $1_{M_{2s}} = e + \overline{e}$. Let $\pi^{\uparrow} : M_{2s} \to$

M_{2s} be the $\mathbb{C}_p$-graded homomorphism defined by $\pi^{\uparrow}(a) := eae$ and denote $\pi^{\downarrow}$ the similar map induced by $\overline{e}$. Then

$$f(b_1, \ldots, b_n, b_1^*, \ldots, b_n^*) = \pi^{\uparrow}(f(b_1, \ldots, b_n, b_1^*, \ldots, b_n^*)) \\ + \pi^{\downarrow}(f(b_1, \ldots, b_n, b_1^*, \ldots, b_n^*)) \neq 0_{\mathbf{A}_{\tilde{\mathbf{g}}}}$$

implies that at least one summand is not zero. We can suppose $\pi^{\uparrow}(f(b_1, \ldots, b_n, b_1^*, \ldots, b_n^*)) \neq 0_{\mathbf{A}_{\tilde{\mathbf{g}}}}$. Then we confine ourselves to analyze only the monomials of f whose image by $\pi^{\uparrow}$ is not zero. By appropriately replacing the variables of f, we can also assume that $x_1 x_2 \cdots x_n$ is one of these monomials, that is $b = b_1 b_2 \cdots b_n \neq 0_{\mathbf{A}_{\tilde{\mathbf{g}}}}$. Hence, there exist $t_1, \ldots, t_{m-1} \in [1, n]$ such that $b_{t_1} = e_{u_1 v_2} \otimes e_{12}, \ldots, b_{t_{m-1}} = e_{u_{m-1} v_m} \otimes e_{m-1m}$, where, if s_i is the extended size of A_i, $u_i \in [1, s_i]$ and $v_{i+1} \in [1, s_{i+1}]$, for all $i \in [1, m-1]$. Moreover each $b_i \in \{b_1, \ldots, b_n\} \backslash \{b_{t_1}, \ldots, b_{t_{m-1}}\}$ is in the diagonal blocks of $\mathbf{A}_{\tilde{\mathbf{g}}}$.

Hence, by Lemma 5,

$$\pi^{\uparrow}(f(b_1, \ldots, b_n, b_1^*, \ldots, b_n^*)) = \beta e_{uv} \otimes e_{1m}$$

for some $0 \neq \beta \in F$, $u \in [1, s_1]$ and $v \in [1, s_m]$. Now, for every $k \in [1, m]$ and pairs of indices $i, j \in [1, s_k]$, we denote by $\mathbf{e}_k(i, j)$ the only element of the basis $\mathfrak{B}_k$ having the matrix unit $e_{ij} \otimes e_{kk}$ of U_{kk} as its component, so that the idempotents $e_1, e_2, \ldots, e_m$ of the triangular algebra A correspond respectively to $\mathbf{e}_1(i_1, i_1), \mathbf{e}_2(i_2, i_2), \ldots, \mathbf{e}_m(i_m, i_m)$. We set $b_0 := \mathbf{e}_1(1, u)$, $b_{n+1} := \mathbf{e}_m(v, 1)$ and let us consider the polynomial $f' := x_0 f x_{n+1}$ with new graded variables x_0, x_{n+1} of degree $|b_0|_{\mathbf{A}_{\tilde{\mathbf{g}}}}$ and $|b_{n+1}|_{\mathbf{A}_{\tilde{\mathbf{g}}}}$ respectively. Then

$$b_0 \pi^{\uparrow}(f(b_1, \ldots, b_n, b_1^*, \ldots, b_n^*)) b_{n+1} = \mathbf{e}_1(1, u)(\beta e_{uv} \otimes e_{1m}) \mathbf{e}_m(v, 1) = \beta(e_{11} \otimes e_{1m}).$$

For all $k \in [1, m-1]$, let $c_k \in A_k$ and $d_{k+1} \in A_{k+1}$ be the elements corresponding to $\mathbf{e}_k(u_k, i_k)$ and $\mathbf{e}_{k+1}(i_{k+1}, v_{k+1})$ in the isomorphisms between the simple components of A and the diagonal blocks of $\mathbf{A}_{\tilde{\mathbf{g}}}$ respectively, then set $a_{t_k} := c_k w_{k,k+1} d_{k+1}$ where $w_{k,k+1}$ is a connector of A. From the equality

$$e_{u_k v_{k+1}} \otimes e_{kk+1} = \mathbf{e}_k(u_k, i_k)(e_{i_k i_{k+1}} \otimes e_{kk+1}) \mathbf{e}_{k+1}(i_{k+1}, v_{k+1}),$$

it follows

$$|a_{t_k}|_A = |c_k|_A |w_{k,k+1}|_A |d_{k+1}|_A = |\mathbf{e}_k(u_k, i_k)|_{\mathbf{A}_{\tilde{\mathbf{g}}}} |e_{i_k i_{k+1}} \otimes e_{k\,k+1}|_{\mathbf{A}_{\tilde{\mathbf{g}}}} |\mathbf{e}_{k+1}(i_{k+1}, v_{k+1})|_{\mathbf{A}_{\tilde{\mathbf{g}}}} = \\ |e_{u_k i_k} \otimes e_{kk}|_{\bar{\alpha}_{\mathbf{g}}} |e_{i_k i_{k+1}} \otimes e_{k\,k+1}|_{\bar{\alpha}_{\mathbf{g}}} |e_{i_{k+1} v_{k+1}} \otimes e_{k+1k+1}|_{\bar{\alpha}_{\mathbf{g}}} = |e_{u_k v_{k+1}} \otimes e_{kk+1}|_{\mathbf{A}_{\tilde{\mathbf{g}}}} = |b_{t_k}|_{\mathbf{A}_{\tilde{\mathbf{g}}}}.$$

Setting $t_0 := 0$ and $t_m := n+1$, we can conclude as above that the elements b_i belong to $U_{k,k} \bigoplus \overline{U_{kk}}$ for every $t_{k-1} + 1 \leq i \leq t_k - 1$. Let $a_i \in A_k$ be the element corresponding to b_i, $d_1 := a_0$ that corresponding to b_0 in A_1 and $c_m := a_{n+1}$ that corresponding to b_{n+1} in A_m. Finally, we consider the evaluation

$$a_0 f(a_1, \ldots, a_n, a_1^*, \ldots, a_n^*) a_{n+1}.$$

Clearly it is in $J(A)$ and we will prove that it is non-zero.

Recall that $J(A) = I(A) \oplus I(A)^*$, where I is the ordinary two-sided ideal generated by the connectors $w_{1,2}, \ldots, w_{m-1,m}$, so that we can write $a = \pi_A^{\uparrow}(a) + \pi_A^{\downarrow}(a)$ for all $a \in J(A)$. Here $\pi_A^{\uparrow}$ and $\pi_A^{\downarrow}$ are the natural projections on $I(A)$ and $I(A)^*$ respectively.

First, let $\pi_A^{\uparrow}(a_0 a_{\sigma(1)}^{\lambda_1} \cdots a_{\sigma(n)}^{\lambda_n} a_{n+1})$ be a non-zero element of $I(A)$, then $\sigma(t_k) = t_k$ and $\lambda_{t_k} = 1$, for all $k \in [1, m-1]$. Moreover, for every $q \in [1, m]$, the inequality $t_{q-1} < l < t_q$ implies that $t_{q-1} < \sigma(l) < t_q$ since $a_{\sigma(l)}^{\lambda_l} \in A_q$. In this case we obtain:

$$a_0 a_{\sigma(1)}^{\lambda_1} \cdots a_{\sigma(n)}^{\lambda_n} a_{n+1} = d_1 a_{\sigma(1)}^{\lambda_1} \cdots a_{\sigma(n)}^{\lambda_n} c_m =$$

$$d_1 a_{\sigma(1)}^{\lambda_1} \cdots a_{\sigma(t_1-1)}^{\lambda_{t_1-1}} c_1 w_{1,2} d_2 a_{\sigma(t_1+1)}^{\lambda_{t_1+1}} \cdots a_{\sigma(t_2-1)}^{\lambda_{t_2-1}} c_2 w_{2,3} \cdots w_{m-1,m} d_m a_{\sigma(t_{m-1}+1)}^{\lambda_{t_{m-1}+1}} \cdots a_{\sigma(n)}^{\lambda_n} c_m \neq 0_A$$

that provides the following equivalent statement:

$$d_k a_{\sigma(t_{k-1}+1)}^{\lambda_{t_{k-1}+1}} \cdots a_{\sigma(t_k-1)}^{\lambda_{t_k-1}} c_k \neq 0_A, \ \ \forall k \in [1, m]$$

and so

$$\mathbf{e}_k(i_k, v_k) b_{\sigma(t_{k-1}+1)}^{\lambda_{t_{k-1}+1}} \cdots b_{\sigma(t_k-1)}^{\lambda_{t_k-1}} \mathbf{e}_k(u_k, i_k) \neq 0_{\mathbf{A}_{\tilde{\mathbf{g}}}} \ \ \forall k \in [1, m],$$

where $v_1 = u$ and $u_m = v$. By Lemma 5, one has that, for all $k \in [1, m]$, the element $b_{\sigma(t_{k-1}+1)}^{\lambda_{t_{k-1}+1}} \cdots b_{\sigma(t_k-1)}^{\lambda_{t_k-1}}$ is in $\pm\mathfrak{B}_k$. Moreover, since $|b_{\sigma(t_{k-1}+1)}^{\lambda_{t_{k-1}+1}} \cdots b_{\sigma(t_k-1)}^{\lambda_{t_k-1}}|_{\mathbf{A}_{\tilde{\mathbf{g}}}} = |\mathbf{e}_k(v_k, u_k)|_{\mathbf{A}_{\tilde{\mathbf{g}}}}$, the same Lemma 5 implies that, for each $k \in [1, m]$, there exists $c_{\sigma,\lambda}^{(k)} \in \{-1_F, 1_F\}$ such that

$$b_{\sigma(t_{k-1}+1)}^{\lambda_{t_{k-1}+1}} \cdots b_{\sigma(t_k-1)}^{\lambda_{t_k-1}} = c_{\sigma,\lambda}^{(k)} \mathbf{e}_k(v_k, u_k).$$

Hence, for a such pair (σ, λ) we obtain:

$$\mathbf{e}_1(1, v_1) b_{\sigma(1)}^{\lambda_1} \cdots b_{\sigma(t_1-1)}^{\lambda_{t_1-1}} (e_{u_1 v_2} \otimes e_{12}) \cdots (e_{u_{m-1} v_m} \otimes e_{m-1m}) b_{\sigma(t_{m-1}+1)}^{\lambda_{t_{m-1}+1}} \cdots b_{\sigma(n)}^{\lambda_n} \mathbf{e}_m(u_m, 1) = \Big(\prod_{k=1}^{m} c_{\sigma,\lambda}^{(k)}\Big)(e_{1m} \otimes e_{11})$$

and so $\pi^{\uparrow}(b_{\sigma(1)}^{\lambda_1} \cdots b_{\sigma(n)}^{\lambda_n}) \neq 0_{\mathbf{A}_{\tilde{\mathbf{g}}}}$.

Conversely, if $\pi^{\uparrow}(b_{\sigma(1)}^{\lambda_1} \cdots b_{\sigma(n)}^{\lambda_n}) \neq 0_{\mathbf{A}_{\tilde{\mathbf{g}}}}$ then, in accordance with what was stated above, one has that $\sigma(t_k) = t_k$ and $\lambda_k = 1$ for all $k \in [1, m-1]$ and, for every $q \in [1, m]$, the inequality $t_{q-1} < l < t_q$ implies that $t_{q-1} < \sigma(l) < t_q$. Hence, for all $k \in [1, m]$

$$b_{\sigma(t_{k-1}+1)}^{\lambda_{t_{k-1}+1}} \cdots b_{\sigma(t_k-1)}^{\lambda_{t_k-1}} = c_{\sigma,\lambda}^{(k)} \mathbf{e}_k(v_k, u_k)$$

from which it follows that

$$\mathbf{e}_k(i_k, v_k) b_{\sigma(t_{k-1}+1)}^{\lambda_{t_{k-1}+1}} \cdots b_{\sigma(t_k-1)}^{\lambda_{t_k-1}} \mathbf{e}_k(u_k, i_k) = c_{\sigma,\lambda}^{(k)} \mathbf{e}_k(i_k, i_k).$$

Since e_k is the minimal homogeneous idempotent of A_k corresponding to $\mathbf{e}_k(i_k, i_k)$, we have that

$$d_k a_{\sigma(t_{k-1}+1)}^{\lambda_{t_{k-1}+1}} \cdots a_{\sigma(t_k-1)}^{\lambda_{t_k-1}} c_k = c_{\sigma,\lambda}^{(k)} e_k$$

for all $k \in [1, m]$ and

$$d_1 a_{\sigma(1)} \cdots a_{\sigma(n)} c_m = \Big(\prod_{k=1}^{m} c_{\sigma,\lambda}^{(k)}\Big) e_1 w_{1,2} e_2 w_{2,3} \cdots e_{m-1} w_{m-1,m} e_m =$$

$$\Big(\prod_{k=1}^{m} c_{\sigma,\lambda}^{(k)}\Big) w_{1,2} w_{2,3} \cdots w_{m-1,m} \neq 0_A.$$

Then, we obtain

$$\pi_A^{\uparrow}(a_0 f(a_1, \ldots, a_n, a_1^*, \ldots, a_n^*) a_{n+1}) = \beta w_{1,2} w_{2,3} \cdots w_{m-1,m} \neq 0_A.$$

Hence f is not a $(\mathbb{C}_p, *)$-identity for A and we can conclude that $\mathbf{A}_{\tilde{\mathbf{g}}}$ belongs to $\mathrm{var}_{\mathbb{C}_p}^*(A)$. Since $\exp_{\mathbb{C}_p}^*(A) = \exp_{\mathbb{C}_p}^*(\mathbf{A}_{\tilde{\mathbf{g}}})$, if $\mathrm{var}_{\mathbb{C}_p}^*(A)$ is minimal, then $\mathrm{var}_{\mathbb{C}_p}^*(A) = \mathrm{var}_{\mathbb{C}_p}^*(\mathbf{A}_{\tilde{\mathbf{g}}})$ and the theorem is proved. □

Now we can present the main result of this section about $(\mathbb{C}_p, *)$-varieties generated by finite dimensional algebras.

Theorem 4. *Let $\mathcal{V}$ be a variety of $(\mathbb{C}_p, *)$-algebras generated by a finite dimensional $(\mathbb{C}_p, *)$-algebra with $\exp_{\mathbb{C}_p}^*(\mathcal{V}) = d$, then there exists a sequence $A_1, \ldots, A_m$ of classical finite dimensional $(\mathbb{C}_p, *)$-simple algebras such that $\mathbf{A} = UT_{\mathbb{C}_p}^*(A_1, \ldots, A_m)$ belongs to $\mathcal{V}$ and $\dim_F(A_1 \oplus \cdots \oplus A_m) = d$.*

In particular, if $\mathcal{V}$ is minimal then $\mathcal{V} = \mathrm{var}_{\mathbb{C}_p}^*(UT_{\mathbb{C}_p}^*(A_1, \ldots, A_m))$.

Proof. Let $\bar{F}$ be the algebraic closure of F and consider the variety $\bar{\mathcal{V}}$ determined by the ideal of $\bar{F}\langle X_G, *\rangle$ generated by $Id_G^*(\mathcal{V})$. $\bar{\mathcal{V}}$ is generated by some $(\mathbb{C}_p, *)$-algebra of finite dimension over $\bar{F}$. Then, using Proposition 14 and Theorem 3, it follows that $\bar{\mathcal{V}}$ contains $UT_{\mathbb{C}_p}^*(\bar{A}_1, \ldots, \bar{A}_m)$, for suitable $(\mathbb{C}_p, *)$-simple algebras $\bar{A}_1, \ldots, \bar{A}_m$ with $\dim_{\bar{F}}(\bar{A}_1 \oplus \cdots \oplus \bar{A}_m) = d$. In light of Theorem 2 and Definition 5, we can determine a sequence $A_1, \ldots, A_m$ of classical finite dimensional $(\mathbb{C}_p, *)$-simple algebras over F such that $\bar{A}_k \cong A_k \otimes_F \bar{F}$ and so $UT_{\mathbb{C}_p}^*(\bar{A}_1, \ldots, \bar{A}_m) \cong UT_{\mathbb{C}_p}^*(A_1, \ldots, A_m) \otimes_F \bar{F}$. It follows that $UT_{\mathbb{C}_p}^*(A_1, \ldots, A_m) \in \mathcal{V}$ and clearly $d = \dim_F(A_1 \oplus \cdots \oplus A_m)$, as desidered.

Finally, if $\mathcal{V}$ is minimal then we obtain $\mathcal{V} = \mathrm{var}^*_{\mathbb{C}_p}(UT^*_{\mathbb{C}_p}(A_1, \ldots, A_m))$ since $\exp^*_{\mathbb{C}_p}(\mathcal{V}) = d = \exp^*_{\mathbb{C}_p}(UT^*_{\mathbb{C}_p}(A_1, \ldots, A_m))$. □

7. Alternating polynomials for $UT^*_{\mathbb{C}_p}(A_1, \ldots, A_m)$

Throughout this section we will assume that $(A_1, \ldots, A_m)$ is an m-tuple of classical finite dimensional $(\mathbb{C}_p, *)$-simple algebras and $\mathbf{A} = UT^*_{\mathbb{C}_p}(A_1, \ldots, A_m)$. Our purpose is to construct a family of $(\mathbb{C}_p, *)$-polynomials alternating in suitable subsets of variables which are not $(\mathbb{C}_p, *)$-identities for the algebra $\mathbf{A}$.

For each classical finite dimensional $(\mathbb{C}_p, *)$-simple algebra A first we will describe a basis $L(A)$ contained in $\bigcup_{(g,\mu)} A^\mu_g$, where $g \in \mathbb{C}_p$ and $\mu \in \{+, -\}$, which we call the standard basis of A. Next we will construct appropriate non-zero products of all the elements of that standard basis.

Case I. Let $A = (M_n, \alpha, *)$, where $* = \theta_n, \sigma_n$ is the transpose or the symplectic type involution and $\alpha = (g_1, \ldots, g_n) \in \mathbb{C}_p^n$ satisfies the condition $g_i g_{\gamma(i)} = g_j g_{\gamma(j)}$ for all $i, j \in [1, n]$. In this case, the standard basis $\{\, p_{ij} \mid i, j \in [1, n]\}$ of A introduced in the proof of Proposition 6 is the required one, $L(A)$. Let us recall that, for all $i, j \in [1, n]$,

$$p_{ij} = \begin{cases} e_{ij} + e^*_{ij}, & \text{if } i < \gamma(j) \\ e_{ij}, & \text{if } i = \gamma(j) \\ e_{ij} - e^*_{ij}, & \text{if } i > \gamma(j) \end{cases} \tag{7}$$

To construct the standard total product of the elements of this basis, we observe that one obtains the matrix unit e_{11} as a product

$$\mathcal{C}_1 \mathcal{C}_2 \cdots \mathcal{C}_n = e_{11}(e_{12}e_{22}e_{21}) \cdots (e_{1n}e_{n2}e_{2n} \cdots e_{n\,n-1}e_{n-1\,n}e_{nn}e_{n1})$$

of the n^2 matrix units of M_n. Let us fix such a product, which we denote by π_{1,M_n}. The first standard total product of A, $\pi_{1,A}$, is obtained from π_{1,M_n} replacing the matrix units e_{ij} appearing there with p_{ij}.

If $i = \gamma(j)$ then $e_{uv}p_{ij} = e_{uv}e_{ij} = \delta_{vi}e_{uj}$ where δ_{vi} is the Kronecker delta. If $i \neq \gamma(j)$, then $p_{ij} = e_{ij} + ce_{\gamma(j)\gamma(i)}$ for some $c = \pm 1_F$ and so $e_{uv}p_{ij} = \delta_{vi}e_{uj} + c\delta_{v\gamma(j)}e_{u\gamma(i)}$ and only one of the two addends can be nonzero. Since in the product $\pi_{1,A}$ exactly n factors are given by the matrices $e_{1n}, e_{2\,n-1}, \ldots, e_{n1}$, in the expansion of $\pi_{1,A}$ we can have at most one non-zero addend. Clearly, by our positions, π_{1,M_n} appears in the expansion of $\pi_{1,A}$ and so $\pi_{1,A} = \pi_{1,M_n}$. Hence

$$\pi_{1,A} = \pi_{1,(M_n,\alpha,*)} = e_{11}.$$

Similarly, if $n > 1$ we can consider the product π_{n,M_n} given by $\pi_{n,M_n} = e_{n1}\mathcal{C}_1 \cdots \mathcal{C}_{n-1}\mathcal{C}'_{n-1}$ where $\mathcal{C}'_{n-1} = e_{1n}e_{n2}e_{2n} \cdots e_{n\,n-1}e_{n-1\,n}e_{nn}$ and the corresponding total product $\pi_{n,A}$ obtained from π_{n,M_n} replacing the matrix units e_{ij} appearing there with p_{ij}. As above we have:

$$\pi_{n,A} = \pi_{n,(M_n,\alpha,*)} = e_{nn}.$$

Finally, if $1 < h < n$, we consider the product $\pi_{h,M_n} = \hat{\mathcal{C}}_h\mathcal{C}_1 \cdots \mathcal{C}_{h-1}\mathcal{C}_{h+1} \cdots \mathcal{C}_n e_{1h}$, where $\hat{\mathcal{C}}_h = e_{h2}e_{2h} \cdots e_{h\,h-1}e_{h-1\,h}e_{hh}e_{h1}$, and the corresponding total product $\pi_{h,A}$. Clearly, we obtain:

$$\pi_{h,A} = \pi_{h,(M_n,\alpha,*)} = e_{hh}.$$

Case II. Let $A = (M_n(D), \alpha, *)$, where $\alpha = g\tilde{\varepsilon}_n$, for some $g \in \mathbb{C}_p$ and $* = \bar{\theta}_n, \bar{\sigma}_n$ is the graded involution defined on $M_n(D)$ by

$$(de_{ij})^{\bar{\theta}_n} = de_{ij}^{\theta_n} \quad \text{and} \quad (de_{ij})^{\bar{\sigma}_n} = de_{ij}^{\sigma_n}.$$

In this case the standard basis of A is $L(A) = \{c_{\varepsilon^{l-1}}p_{ij} \mid l \in [1, p],\ i, j \in [1, n]\}$, where the p_{ij} are the elements of M_n defined above (7), and the $c_{\varepsilon^{l-1}}$ are the elements of the natural basis of D given in equation (6).

The h-th standard total product is obtained by multiplying first all the elements $c_{1_G}p_{ij}$ in the same order used in Case I for the elements p_{ij}, then all elements of degree ε and so on. The final result is

$$\pi_{h,A} = \pi_{h,(M_n(D),\alpha,*)} = (c_{1_G}e_{hh})(c_{\varepsilon}^{n^2}e_{hh}) \cdots (c_{\varepsilon^{p-1}}^{n^2}e_{hh}) = c_{1_G}e_{hh}.$$

Case III. Let $A = (M_n \oplus M_n^{op}, \alpha, exc)$ and let us define for all $i, j \in [1, n]$,

$$q_{ij} := (e_{ij}, e_{ij}) \quad \text{and} \quad \bar{q}_{ij} := (e_{ij}, -e_{ij}). \tag{8}$$

In this case the standard basis of A is $L(A) = \{q_{ij},\ \bar{q}_{ij} \mid i, j \in [1, n]\}$.

To construct the h-th standard total product, as made in Case I, we start from the product $\pi_{h,M_n} = e_{hh}$. Hence replace the matrix units e_{ij} appearing there with q_{ij}. After that, multiply this product with the product of all the elements of this basis involving the $\bar{q}_{ij}$'s in the same order used for the q_{ij}'s. With these positions one has that

$$\pi_{h,A} = \pi_{h,(M_n \oplus M_n^{op},\alpha,exc)} = \begin{cases} \bar{q}_{11} = (e_{11}, -e_{11}) & \text{if } n = 1, \\ (e_{hh}, 0) & \text{otherwise.} \end{cases}$$

Case IV. Let $A = (M_n(D) \oplus M_n(D)^{op}, \alpha, exc)$, where as above $\alpha = g\tilde{\varepsilon}_n$ for some $g \in \mathbb{C}_p$. In this case the standard basis of A is $L(A) = \{c_{\varepsilon^{l-1}}q_{ij}, c_{\varepsilon^{l-1}}\bar{q}_{ij} \mid h \in [1, p],\ i, j \in [1, n]\}$, where the q_{ij}'s and the $\bar{q}_{ij}$'s are the elements of $M_n \oplus M_n^{op}$, defined above (8).

The standard total product is obtained by multiplying first all the elements $c_{1_G}q_{ij}$ and $c_{1_G}\bar{q}_{ij}$ in the same order used in Case III for the elements q_{ij} and $\bar{q}_{ij}$, then all elements of degree ε and so on. The final outcome is:

$$\pi_{h,A} = \pi_{h,(M_n(D)\oplus M_n(D)^{op},\alpha,exc)} = \begin{cases} c_{1_G}\bar{q}_{11} = (c_{1_G}e_{11}, -c_{1_G}e_{11}) & \text{if } n = 1, \\ (c_{1_G}e_{hh}, 0) & \text{otherwise.} \end{cases}$$

Now, as in the case of ordinary G-graded algebras [3], we insert in the standard total product $\pi_{h,A}$ suitable homogeneous minimal idempotents of A that border each of the elements of $L(A)$. More precisely, we denote by $\tilde{\pi}_{h,A}$ this new product:

Case I for each element p_{ij} of $L(A)$ we insert the idempotent e_{ii} on the left and we conclude by multiplying on the right by the idempotent e_{hh}, as a final result we obtain

$$\tilde{\pi}_{h,A} = e_{hh};$$

Case II for each element $c_{\varepsilon^{l-1}}p_{ij}$ we insert on the left $c_{1_G}e_{ii}$ and we conclude by multiplying on the right by $c_{1_G}e_{hh}$ so that:

$$\tilde{\pi}_{h,A} = c_{1_G}e_{hh};$$

Case III for each of the elements q_{ij} and $\bar{q}_{ij}$ we insert on the left $(e_{ii}, 0)$ and we conclude by multiplying on the right by $(e_{hh}, 0)$, we have

$$\tilde{\pi}_{h,A} = (e_{hh}, 0);$$

Case IV for each of the elements $c_{\varepsilon^{l-1}}q_{ij}$ and $c_{\varepsilon^{l-1}}\bar{q}_{ij}$ we insert on the left $(c_{1_G}e_{ii}, 0)$ and we conclude by multiplying on the right by $(c_{1_G}e_{hh}, 0)$ and we obtain

$$\tilde{\pi}_{h,A} = (c_{1_G}e_{hh}, 0).$$

We note that the free algebra $F\langle X_G, *\rangle$ is also freely generated by the elements

$$x^+_{i,g} = x_{i,g} + x^*_{i,g} \quad \text{and} \quad x^-_{i,g} = x_{i,g} - x^*_{i,g},$$

where $g \in G$ and $i = 1, 2, \ldots$. In this paper we will refer to them as variables of type (g, μ) with $g \in G$ and $\mu \in \{+, -\}$. For any classical finite dimensional $(\mathbb{C}_p, *)$-simple algebra A, we denote by $\tilde{m}_{h,A}$ the multilinear monomial built by replacing in the enriched standard total product $\tilde{\pi}_{h,A}$ each element of type (g, μ) with a variable of the same type in $F\langle X_G, *\rangle$ and the minimal homogeneous idempotents with $\dim_F A + 1$ pairwise different variables of degree 1_G.

Clearly, the set of the variables in $\tilde{m}_{h,A}$ does not depend on the choice of h. When appropriate, we will denote by S_A the set of variables of $\tilde{m}_{h,A}$ corresponding to the elements of the standard basis $L(A)$ and with Y_A the set of variables of degree 1_G corresponding to the idempotents introduced to pass from $\pi_{h,A}$ to $\tilde{\pi}_{h,A}$. We will say that S_A is the set of *designed* variables and Y_A is the set of *controlling* variables. Clearly S_A is a disjoint union of the subsets $S_{A,g,\mu}$ given by the variables of the same type (g, μ).

Definition 10. Let A be a classical finite dimensional $(\mathbb{C}_p,*)$-simple algebra. We denote by $f_{h,A}$ the $(\mathbb{C}_p,*)$-graded polynomial obtained by alternating in $\tilde{m}_{h,A}$ the variables of $S_{A,g,\mu}$, for all $g \in \mathbb{C}_p$ and $\mu \in \{+,-\}$.

Proposition 15. *Let A be a classical finite dimensional $(\mathbb{C}_p,*)$-simple algebra of size n. For each $h \in [1,n]$ the polynomial $f_{h,A}$ is not a $(\mathbb{C}_p,*)$-graded polynomial identity for A. More precisely there exists a $(\mathbb{C}_p,*)$-evaluation $e_{h,A}$ such that*

$$e_{h,A}(f_{h,A}) = \tilde{\pi}_{h,A}.$$

Proof. The desired evaluation $e_{h,A}$ is determined by replacing each variable of $\tilde{m}_{h,A}$ with the corresponding element of A appearing in the enriched standard total product of A, $\tilde{\pi}_{h,A}$. Clearly in each monomial of $f_{h,A}$ the variables in Y_A appear in the same order as in $\tilde{m}_{h,A}$ and these variables delimit all those of S_A. Therefore, in order to not get zero, the evaluation of each variable in S_A with elements of $L(A)$ is uniquely determined by those of the bordering variables and by its type. Since we do not alternate variables of different type, we deduce that $\tilde{m}_{h,A}$ is the unique summand of $f_{h,A}$ which is not zero under the evaluation by $e_{h,A}$ and so the proof is completed. □

Clearly the polynomials $f_{h,A}$'s are related to the dimension of the components A_g^μ of the $(\mathbb{C}_p,*)$-algebra A.

More precisely $|S_{A,g,\mu}| = \dim_F A_g^\mu$, hence if $f_{h,A} \notin Id_G^*(B)$ then $\dim_F B_g^\mu \geq \dim_F A_g^\mu$ for all $g \in \mathbb{C}_p$ and $\mu \in \{+,-\}$. Now we define some other polynomials closer to the structure of the given $(\mathbb{C}_p,*)$-algebra, using the polynomials $\tilde{\Phi}_{\alpha,h}$ and $\Psi_{\alpha,h}$ introduced in Section 3.

Definition 11. Let A be a classical finite dimensional $(\mathbb{C}_p,*)$-simple algebra of size n and let ν be a positive integer. For each $h \in [1,n]$ we consider ν copies of $f_{h,A}$ in pairwise disjoint sets of variables and we denote by $f_{h,A}^{(i)}$ the i-th copy of $f_{h,A}$. Then we define

$$\bar{f}_{h,A}^{\nu} = f_{h,A}^{(1)} \cdots f_{h,A}^{(\nu)}$$

moreover we consider:

$$\tilde{f}_{h,A}^{\nu} = \begin{cases} \bar{f}_{h,A}^{\nu}\Psi_{\alpha,h} & \text{if } A = (M_n,\alpha,*) \\ \bar{f}_{h,A}^{\nu} & \text{if } A = (M_n(D),\alpha,*) \\ \bar{f}_{h,A}^{\nu}\tilde{\Phi}_{\alpha,h}\Psi_{\alpha,h} & \text{if } A = (M_n \oplus M_n^{op},\alpha,exc) \\ \bar{f}_{h,A}^{\nu} & \text{if } A = (M_n(D) \oplus M_n(D)^{op},\alpha,exc) \end{cases}$$

By previous Propositions 2, 3 and 15 we obtain:

Proposition 16. *Let A be a classical finite dimensional $(\mathbb{C}_p,*)$-simple algebra of size n and $h \in [1,n]$. If $\alpha(h)$ has maximal weight then the polynomial $\tilde{f}_{h,A}^{\nu}$ is not a $(\mathbb{C}_p,*)$-*

graded polynomial identity for A. More precisely there exists a $(\mathbb{C}_p, *)$*-evaluation* $\tilde{e}_{h,A}$ *on the bases* $L(A)$ *and* $\mathcal{B}$ *of A such that*

$$\tilde{e}_{h,A}(\tilde{f}^{\nu}_{h,A}) = \tilde{\pi}_{h,A}.$$

Later in the paper when we consider the polynomial $\tilde{f}^{\nu}_{h,A}$ we will tacitly assume that $\alpha(h)$ has maximal weight. Let us emphasize that this condition on $\alpha(h)$ is always satisfied, for all $h \in [1, n]$ when the classical finite dimensional $(\mathbb{C}_p, *)$-simple algebra is $M_n(D)$ or $M_n(D) \oplus M_n(D)^{op}$. Moreover, we have:

Lemma 6. *Let* $A = (M_n, \alpha, *)$ *be a classical finite dimensional* $(\mathbb{C}_p, *)$*-simple algebra. If* $\bar{\eta}$ *is a* $(\mathbb{C}_p, *)$*-evaluation on A, then*

$$\bar{\eta}(\tilde{f}^{\nu}_{h,A}) \subseteq span_F\{e_{ij} \mid \alpha_i = \alpha_j = \alpha_h\}.$$

Proof. By Definition 11, we have that

$$\tilde{f}^{\nu}_{h,A} = \bar{f}^{\nu}_{h,A}\Psi_{\alpha,h}.$$

Let us consider the invariance subgroup H_α. If $H_\alpha = \{1_{\mathbb{C}_p}\}$, since, by Proposition 3, $\bar{\eta}(\Psi_{\alpha,h}) \subseteq span_F\{e_{ij} \mid \alpha_i = \alpha_j = \alpha_h\}$ and $\bar{f}^{\nu}_{h,A}$ has homogeneous degree $1_{\mathbb{C}_p}$, then we are done. Let us now $H_\alpha \neq \{1_{\mathbb{C}_p}\}$, then $H_\alpha = \mathbb{C}_p$ and, $\forall g \in \mathbb{C}_p$, $w_\alpha(g) = w_\alpha(1_{\mathbb{C}_p}) = m$, for some $m \geq 1$. It follows that each element of $\mathbb{C}_p$ appears exactly m times in α and $n = mp$. Let $G = \mathbb{C}_p$, since $|G|$ is odd, we can assume that $\alpha(\gamma(i)) = \alpha(i)^{-1}$, $\forall i \in [1, n]$. Moreover, by Proposition 6, we can suppose that

$$\alpha = (\underbrace{\varepsilon, \dots, \varepsilon}_{m}, \dots, \underbrace{\varepsilon^r, \dots, \varepsilon^r}_{m}, \underbrace{1_G, \dots, 1_G}_{m}, \underbrace{\varepsilon^{r+1}, \dots, \varepsilon^{r+1}}_{m}, \dots, \underbrace{\varepsilon^{p-1}, \dots, \varepsilon^{p-1}}_{m})$$

where $p = 2r + 1$.

Clearly, it is enough to consider the case $\nu = 1$. Since the polynomials $\tilde{f}_{h,A}$'s are multilinear we can assume that $\bar{\eta}$ evaluates the designed variables with elements of the standard basis $L(A)$ and the controlling variables with elements e_{ij} of the canonical basis $\mathcal{E}_n$ of degree 1_G in A, i.e. $\alpha_i = \alpha_j$.

As in the proof of the Proposition 15 there is a unique summand of $f_{h,A}$ which is non zero under this evaluation and we can assume that it is $\tilde{m}_{h,A}$, so

$$\bar{\eta}(f_{h,A}) = \bar{\eta}(\tilde{m}_{h,A}) \neq 0.$$

Let observe that every element p_{ij} of $L(A)$ appears once and only once in the evaluation $\bar{\eta}$. If $i = \gamma(j)$, then $p_{i\gamma(i)} = e_{i\gamma(i)}$ while if $i \neq \gamma(j)$, then $p_{i\gamma(i)}$ has two summands in the canonical basis $\mathcal{E}_n$, e_{ij} and $e_{\gamma(j)\gamma(i)}$, of the same degree but only one contributes to the calculation of $\bar{\eta}(\tilde{m}_{h,A}) \neq 0$. So, for each designed variable x of $\tilde{m}_{h,A}$, $\bar{x}$ will be the unique element of the canonical basis $\mathcal{E}_n$ such that $\bar{\eta}(x) \neq 0$ and which contributes to the non-zero product $\bar{\eta}(\tilde{m}_{h,A})$. Therefore $\bar{x} = e_{ab}$ for appropriate indices a, $b \in [1, n]$, and, in this case, we write $a = \lambda(\bar{x})$, $b = \rho(\bar{x})$, $x \rightsquigarrow \bar{x}$ and we say that "e_{ab} appears in $\bar{\eta}$"or "$\bar{\eta}$ contains e_{ab}"in the position x. If $b = \gamma(a)$, then e_{ab} appears once and only once in $\bar{\eta}$. If $b \neq \gamma(a)$, then at least one of the elements e_{ab}, $e_{\gamma(a)\gamma(b)}$ appears in $\bar{\eta}$ and only one of the following pairs (e_{ab}, e_{ab}), $(e_{ab}, e_{\gamma(a)\gamma(b)})$, $(e_{\gamma(a)\gamma(b)}, e_{\gamma(a)\gamma(b)})$ has both components appearing in different positions x_1 and x_2. In fact, the components $\bar{x}_1$ and $\bar{x}_2$ correspond to designed variables of the same degree but of opposite signature: one is symmetric and the other one skew-symmetric.

Suppose now that $h = 1$, then

$$\tilde{m}_{h,A} = \tilde{m}_{1,A} = y_0\mathcal{C}^1 y_1\mathcal{C}^2 \cdots \mathcal{C}^n y_n,$$

where, for all $i = 0, \dots, n$, y_i has degree 1_G and, for all $k \in [1, n]$, $\mathcal{C}^k$ are built by replacing in $\mathcal{C}_k$ each element of type (g, μ) with a variable of the same type in $F\langle X_G, *\rangle$, so they have $2k-1$ designed variables and $2k-2$ controlling variables. If x is a designed variable of $\mathcal{C}^k$, then we write $x \in S_A \cap \mathcal{C}^k$. In particular $\mathcal{C}_1 = z_1^1$ is a symmetric variable of degree 1_G. If $k = 2$, then the variables in $S_A \cap \mathcal{C}_2$ are z_1^2, w^2, z_2^2. If $k \geq 3$, then the elements of $S_A \cap \mathcal{C}^k$ are in the following order

$$z_1^k, u_1^k, v_1^k, \dots, u_{k-2}^k, v_{k-2}^k, w^k, z_2^k$$

with $|z_2^k| = |z_1^k|^{-1} = |e_{k1}|$, $|w^k| = |e_{kk}| = 1_G$, $|u_i^k| = |v_i^k|^{-1} = |e_{ki+1}|$, $\forall i \in [1, k-2]$. Since $\mathcal{C}^k$ has degree 1_G, then $z_1^1 \rightsquigarrow \bar{z}_1^1 = e_{a_1 b_1}$ with $\alpha(a_1) = \alpha(b_1)$ and $\bar{\eta}(\mathcal{C}^k) = e_{a_k d_k}$ with $\alpha(a_k) = \alpha(d_k)$, $\forall k \geq 2$. Moreover $\bar{\eta}(\tilde{m}_{h,A}) \neq 0$ and so $\alpha(a_1) = \alpha(b_1) = \alpha(a_k) = \alpha(d_k)$, $\forall k \geq 2$, then there exists $\bar{g} \in G$ such that $\bar{g} = \alpha(a_1)$. We want to prove that $\bar{g} = \alpha(1)$. We observe that

$$\bar{g} = \alpha(\lambda(\bar{z}_1^1)) = \alpha(\rho(\bar{z}_1^1)) = \alpha(\lambda(\bar{z}_1^k)) = \alpha(\rho(\bar{z}_2^k)), \ \ \forall k \geq 2.$$

Similarly, for all $k \in [2, n]$, there exists $g_k \in G$ such that

$$g_k = \alpha(\rho(\bar{z}_1^k)) = \alpha(\lambda(\bar{u}_i^k)) = \alpha(\rho(\bar{v}_i^k)) = \alpha(\lambda(\bar{w}^k)) = \alpha(\rho(\bar{w}^k)) = \alpha(\lambda(\bar{z}_2^k)).$$

Since

$$g_k^{-1}\bar{g} = \alpha^{-1}(\lambda(\bar{z}_2^k))\alpha(\rho(\bar{z}_2^k)) = |z_2^k| = \alpha(k)^{-1}\alpha(1),$$

we obtain that

$$g_k = \bar{g}\,\alpha(k)\alpha(1)^{-1}.$$

For every $g \in G$, let $O_g = \{i \in [1, n] \,|\, \alpha(i) = g\}$, then $\lambda(\bar{u}_i^k) = \rho(\bar{v}_i^k) \in O_{g_k}$ where $g_k = \alpha(\rho(\bar{z}_1^k))$. Also $\lambda(\bar{w}^k) = \rho(\bar{w}^k) \in O_{g_k}$.

In particular both indices of the elements of the canonical basis $\mathcal{E}_n$, determined by the designed variables of degree 1_G, appearing in $\mathcal{C}^k$ belong to O_{g_k}.

Clearly $g_k = g_h$ if and only if $\alpha(k) = \alpha(h)$, $\forall k, h \in [1, n]$. Then, for all $k \in [1, n]$, O_{g_k} is identified by $O_{\alpha(k)}$. From our hypothesis, we have

$$O_\varepsilon = [1, m], \dots, O_{\varepsilon^r} = [(r-1)m + 1, rm], O_{1_G} = [rm + 1, (r+1)m],$$
$$O_{\varepsilon^{r+1}} = [(r+1)m + 1, (r+2)m], \dots, O_{\varepsilon^{p-1}} = [(p-1)m + 1, pm].$$

Then, for all $i \in [1, r]$ and for all $k \in O_{\varepsilon^i}$, the designed variables of degree 1_G in $\mathcal{C}^k$ are all symmetric while, for all $i \in [r+1, p-1]$ and for all $k \in O_{\varepsilon^i}$, the designed variables of degree 1_G in $\mathcal{C}^k$ are all skew-symmetric.

Thus, for a fixed $\varepsilon^i \neq 1_G$ and for all $k \in O_{\varepsilon^i}$, the elements of the basis $\mathcal{E}_n$ determined by these variables are pairwise distinct. Their number, for $k \in O_{\varepsilon^i}$, is 1, 3, 5, $\dots$, $2m - 1$ hence they constitute the canonical basis of one of the simple blocks decomposing A_{1_G}. More precisely, O_{ε^i} identifies the block $U_i =< e_{ab} \,|\, \alpha(a) = \alpha(b) = \bar{g}\alpha(1)^{-1}\varepsilon^i >\cong M_m$, for all $i \in [1, p-1]$.

If $m > 1$, then there exists $e_{ab} \in U_i$ such that $e_{ab}^* = \pm e_{\gamma(a)\gamma(b)}$ and $a \neq \gamma(b)$. If $e_{\gamma(a)\gamma(b)} \in U_i$, then $e_{ab} + e_{ab}^*$ and $e_{ab} - e_{ab}^*$ should be evaluations of designed variables of the same type contrary to the fact that one element is symmetric and the other one is skew-symmetric. Therefore $e_{ab}^* \notin U_i$ and so U_i is not $*$-invariant. It follows that the $p-1$ blocks $U_1, \dots U_{p-1}$ identified by O_{ε^i}, for all $i \in [1, p-1]$, correspond two by two under the action of the involution $*$. In the decomposition of A_{1_G} only one block U is stable under $*$. Let $e_{ab} \in U$ then, for what as been said, $\alpha(a) = \alpha(b) = \bar{g}\alpha(1)^{-1}$. Since $\alpha = (\varepsilon, \dots, \varepsilon, \dots, \varepsilon^r, \dots, \varepsilon^r, 1_G, \dots, 1_G, \varepsilon^{r+1}, \dots, \varepsilon^{r+1}, \dots, \varepsilon^{p-1}, \dots, \varepsilon^{p-1})$ we have $\alpha(a) = \alpha(b) = 1_G$, then $\bar{g} = \alpha(1)$ as desired.

Let now $m = 1$. In this case $n = p$ and $\alpha : [1, n] \to G$ is bijective and $* = \theta_p$. Let us consider the monomial $\mathcal{C}^p$ whose designed variables are in the following order

$$z_1^p, u_1^p, v_1^p, \dots, u_{p-2}^p, v_{p-2}^p, w^p, z_2^p$$

with degree

$$|e_{1p}|, |e_{p2}|, |e_{2p}|, \dots |e_{pp-1}|, |e_{p-1p}|, |e_{pp}|, |e_{p1}|$$

respectively, and only z_1^p and z_2^p are symmetric.

Since α is bijective there exist $j, k \in [1, n]$ such that $\alpha(j) = \bar{g}$ and $\alpha(k) = g_p$. Then $z_1^p \rightsquigarrow e_{jk} = \bar{z}_1^p$, $z_2^p \rightsquigarrow e_{kj} = \bar{z}_2^p$, $u_i^p \rightsquigarrow e_{kc_i} = \bar{u}_i^p$, $v_i^p \rightsquigarrow e_{c_ik} = \bar{v}_i^p$, $w^p \rightsquigarrow e_{kk} = \bar{w}^p$. As the variables $u_1^p, \dots, u_{p-2}^p, w^p, z_2^p$ are both of different degrees one of them must be evaluated in $e_{k\gamma(k)}$. Considering that $e_{k\gamma(k)}$ is symmetric then $z_2^p \rightsquigarrow e_{k\gamma(k)}$, so $\gamma(k) = j$ and $|e_{k\gamma(k)}| = |z_2^p| = |e_{p1}|$ that is $\alpha(k)^{-1}\alpha(\gamma(k)) = \alpha(p)^{-1}\alpha(1)$. This implies that $\alpha(k)^{-2} = (\varepsilon^{p-1})^{-1}\varepsilon = \varepsilon^2$ and $\alpha(k) = \varepsilon^{-1}$. Hence $\bar{g} = \alpha(j) = \alpha(\gamma(k)) = \alpha(k)^{-1} = \varepsilon = \alpha(1)$ and we are done.

Suppose now that $h = n$ and let

$$\bar{\eta}(f_{n,A}) = \bar{\eta}(\tilde{m}_{n,A}) = \pm e_{ij}.$$

It follows that $\alpha(i) = \alpha(j)$ and we want to prove that $\alpha(i) = \alpha(n) = \varepsilon^{-1}$. Notice that

$$\tilde{m}_{n,A} = y_0 z_2^n y_1 \mathcal{C}^1 \cdots y_{n-1} \mathcal{C}^{n-1} y_n \mathcal{C}^n y_{n+1}.$$

Hence $z_2^n \rightsquigarrow e_{ab}$, $y_0 \rightsquigarrow e_{ia}$, $y_{n+1} \rightsquigarrow e_{kj}$ with $\alpha(i) = \alpha(a)$, $\alpha(k) = \alpha(j)$, $\alpha(a)^{-1}\alpha(b) = |z_2^n| = |e_{n1}| = \alpha(n)^{-1}\alpha(1) = \varepsilon^2$. The evaluation $\hat{\eta}$ obtained by $\bar{\eta}$ replacing $\bar{\eta}(y_{n+1})$ with $\hat{\eta}(y_{n+1}) = e_{ki}$ is admissible and we have

$$\hat{\eta}(f_{1,A}) = \hat{\eta}(\tilde{m}_{1,A}) = e_{bb}.$$

By the first part of the proof we obtain $\alpha(b) = \alpha(1) = \varepsilon$, then $\alpha(a)^{-1}\varepsilon = \varepsilon^2$ and so $\alpha(a) = \varepsilon^{-1} = \alpha(n)$.

If $1 < h < n$ the result is obtained as before. $\square$

Now, as at the beginning of the proof of Lemma 6, by Proposition 3 it follows that:

Lemma 7. *Let $A = (M_n \oplus M_n^{op}, \alpha, *)$ be classical finite dimensional $(\mathbb{C}_p, *)$-simple algebra and let $\pi^\uparrow$ be the projection on the first component defined by $\pi^\uparrow(a, b) = (a, 0)$. If $H_\alpha = \{1_{\mathbb{C}_p}\}$ and $\bar{\eta}$ is a $(\mathbb{C}_p, *)$-evaluation on A, then*

$$\pi^\uparrow(\bar{\eta}(\tilde{f}^\nu_{h,A})) \subseteq span_F\{(e_{ij}, 0) \,|\, \alpha_i = \alpha_j = \alpha_h\}.$$

Notice that the algebras considered in the previous lemmas are exactly the $(\mathbb{C}_p, *)$-singular algebras of Definition 7. We return to a more general situation, considering:

Proposition 17. *Let $(A, \alpha, *_A)$ and $(B, \beta, *_B)$ be classical finite dimensional $(\mathbb{C}_p, *)$-simple algebras of same dimension. If the polynomial $\tilde{f}^\nu_{h,A}$ is not a $(\mathbb{C}_p, *)$-graded polynomial identity for B then A and B are isomorphic as $(\mathbb{C}_p, *)$-algebras.*

Proof. Clearly $f_{h,A} \notin Id^*_G(B)$, hence $\dim_F B^\mu_g \geq \dim_F A^\mu_g$ for all $g \in \mathbb{C}_p$ and $\mu \in \{+, -\}$. Since A and B have the same dimension it follows that $\dim_F B^\mu_g = \dim_F A^\mu_g$ and, by Corollary 1, either A and B are isomorphic as $(\mathbb{C}_p, *)$-algebras or $A = (M_n \oplus M_n^{op}, \alpha, exc)$ and $B = (M_n \oplus M_n^{op}, \beta, exc)$. In this situation the thesis follows from Proposition 8, since B does not satisfy the polynomial $\tilde{\Phi}_{\alpha,h}$. $\square$

We conclude this section by extending the previous definitions and results to the case when we consider the algebra $\mathbf{A} = UT^*_{\mathbb{C}_p}(A_1, \ldots, A_m)$, where $m > 1$ and A_k is a classical finite dimensional $(\mathbb{C}_p, *)$-simple algebra of size n_k with grading defined by the map α^k, for each $k \in [1, m]$.

We start by considering, for each $k \in [1, m]$, the standard basis $L(A_k)$ of the $(\mathbb{C}_p, *)$-simple algebra A_k. We denote by $\mathfrak{L}_{k,k}(\mathbf{A})$ its image $\varphi(L(A_k))$ in $\mathbf{A}$. Moreover, for all $k, l \in [1, m]$ with $k < l$, we consider also the standard bases $\mathfrak{L}_{k,l}(\mathbf{A})$ of the subspaces $U_{k\,l} \oplus \overline{U_{k\,l}}$ of $J(\mathbf{A})$ consisting of the elements $(e_{ij} \otimes e_{kl}) \pm (e_{ij} \otimes e_{kl})^{\theta_{2s}}$, where $i \in [1, s_k]$ and $j \in [1, s_l]$. At the end, we get:

Definition 12. The standard basis of $\mathbf{A} = UT^*_{\mathbb{C}_p}(A_1, \ldots, A_m)$ is

$$\mathfrak{L}(\mathbf{A}) := \bigcup_{k,l \in [1,m];\, k \leq l} \mathfrak{L}_{k,l}(\mathbf{A})$$

Next we have:

Definition 13. Let $\nu \geq m$, $u_1, u_2, \ldots, u_{m-1}$ be homogeneous symmetric variables of type $(|u_i|, +)$ and $(h_1, h_2, \ldots, h_m)$ be a sequence of positive integers, such that $h_k \in [1, n_k]$, then we define:

$$f^{\nu}_{h_1,\ldots,h_m,u_1,\ldots,u_{m-1},A} = \tilde{f}^{\nu}_{h_1,A_1} u_1 \tilde{f}^{\nu}_{h_2,A_2} u_2 \cdots \tilde{f}^{\nu}_{h_{m-1},A_{m-1}} u_{m-1} \tilde{f}^{\nu}_{h_m,A_m}$$

Here, we assume that the sets of variables involved in the polynomials $\tilde{f}^{\nu}_{h_k,A_k}$ and also $\{u_1, \ldots, u_{m-1}\}$ are pairwise disjoint. Clearly the polynomial $f^{(i)}_{h_k,A_k}$ and the corresponding monomial $\tilde{m}^{(i)}_{h_k,A_k}$ have the same set of variables. For $i \in [1, \nu]$, let U_i be the set of all designed variables occurring in $f^{(i)}_{h_1,A_1}, \ldots, f^{(i)}_{h_m,A_m}$, together with u_i when $i < m$. Then each set U_i is the disjoint union of its subsets $U_{i,g,\mu}$, given by the variables of the same type (g, μ). Set

$$d^{\mathbf{A}}_k := \dim_F A_k, \quad d^{\mathbf{A}}_{k,g,\mu} := \dim_F (A_k)^{\mu}_g,$$

and

$$d^{\mathbf{A}}_{ss,g,\mu} := \dim_F (A_1 \oplus \ldots \oplus A_m)^{\mu}_g = \sum_{k \in [1,m]} d^{\mathbf{A}}_{k,g,\mu}.$$

In this way one has:

$$|U_{i,g,\mu}| = \begin{cases} 1 + d^{\mathbf{A}}_{ss,g,\mu} & \text{if } g = |u_i|,\ \mu = + \text{ and } i \in [1, m-1] \\ d^{\mathbf{A}}_{ss,g,\mu} & \text{otherwise} \end{cases} \tag{9}$$

In a similar way, we denote by Y_i the set of all controlling variables occurring in $f^{(i)}_{h_1,A_1}, \ldots, f^{(i)}_{h_m,A_m}$. We conclude with the following:

Definition 14. Let $\mathbf{A} = UT^*_{\mathbb{C}_p}(A_1, \ldots, A_m)$, if $m > 1$ we denote by

$$\tilde{f}^{\nu}_{h_1,\ldots,h_m,u_1,\ldots,u_{m-1},A}$$

the $(\mathbb{C}_p, *)$-graded polynomial obtained by alternating in $f^{\nu}_{h_1,\ldots,h_m,u_1,\ldots,u_{m-1},A}$ the variables of the set $U_{i,g,\mu}$, for each $i \in [1,\nu]$, $g \in \mathbb{C}_p$ and $\mu \in \{+,-\}$.

Now we choose suitable $\mathbb{C}_p$-degrees for each of the variables $u_1, \ldots, u_{m-1}$. More precisely, similarly to what was considered in the proof of Theorem 3, we define:

$$i_k = \begin{cases} h_k & \text{if } A_k = M_{n_k} \text{ or } A_k = M_{n_k} \oplus M^{op}_{n_k} \\ p(h_k - 1) + 1 & \text{if } A_k = M_{n_k}(D) \text{ or } A_k = M_{n_k}(D) \oplus M_{n_k}(D)^{op}, \end{cases}$$

and, for $k = 1, \ldots, m-1$,

$$|u_k|_{F\langle X_G,*\rangle} = g_k = \alpha^k(i_k)^{-1}\alpha^{k+1}(i_{k+1})$$

therefore the variable u_k and the element $e_{i_k i_{k+1}} \otimes e_{k\,k+1}$ of $J(\mathbf{A})$ have the same $\mathbb{C}_p$-degree.

Proposition 18. *Let $\mathbf{A} = UT^*_{\mathbb{C}_p}(A_1, \ldots, A_m)$ with $m > 1$. If the variables $u_1, \ldots, u_{m-1}$ have the $\mathbb{C}_p$-degrees defined before, then the polynomial $\tilde{f}^{\nu}_{h_1,\ldots,h_m,u_1,\ldots,u_{m-1},A}$ is not a $(\mathbb{C}_p, *)$-graded polynomial identity for $\mathbf{A}$.*

Proof. Let us recall that, by our assumption, $\alpha^k(h_k)$ has maximal weight for all $k \in [1, m]$. Hence, by Proposition 16, for each $k \in [1, m]$ there exists a $(\mathbb{C}_p, *)$-evaluation $\tilde{e}_{h_k,A_k}$ on the bases $L(A_k)$ and $\mathcal{B}_k$ of A_k such that

$$\tilde{e}_{h_k,A_k}(\tilde{f}^{\nu}_{h_k,A_k}) = \tilde{\pi}_{h_k,A_k}.$$

By the $*$-embedding φ we regard these elements as belonging to the algebra $\mathbf{A}$, which in its turn is, by construction, a $(\mathbb{C}_p, *)$-subalgebra of $(M_{2s}, \bar{\alpha}, \theta_{2s})$. Clearly we obtain

$$\varphi(\tilde{\pi}_{h_1,A_1})(e_{i_1 i_2} \otimes e_{12}) \cdots (e_{i_{m-1} i_m} \otimes e_{m-1\,m})\varphi(\tilde{\pi}_{h_m,A_m}) = e_{i_1 i_m} \otimes e_{1m}$$

where the indexes $i_1, \ldots, i_m$ are those after the Definition 14. We can glue these evaluations $\tilde{e}_{h_1,A_1} \ldots, \tilde{e}_{h_m,A_m}$ and extend the resulting evaluation by replacing the variables $u_1, \ldots, u_{m-1}$ with the elements $e_{i_1 i_2} \otimes e_{12} + (e_{i_1 i_2} \otimes e_{12})^{\theta_{2s}}, \ldots, (e_{i_{m-1} i_m} \otimes e_{m-1\,m}) + (e_{i_{m-1} i_m} \otimes e_{m-1\,m})^{\theta_{2s}}$ which have the same type $(g_1, +), \ldots, (g_{m-1}, +)$ as them. Now we easily obtain the thesis. □

In the rest of the paper, given the algebra $\mathbf{A} = UT^*_{\mathbb{C}_p}(A_1, \ldots, A_m)$ we will denote by $\tilde{f}^{\nu}_{\mathbf{A}}$ the polynomial considered in Proposition 16, if $m = 1$, and in Proposition 18, if $m > 1$, respectively.

8. Main results

According to Theorem 4, we have that any minimal variety (see Definition 1) of $(\mathbb{C}_p, *)$-algebras of $(\mathbb{C}_p, *)$-exponent d is generated by a suitable upper block triangular matrix algebra $UT^*_{\mathbb{C}_p}(A_1, \ldots, A_m)$, where $A_1, \ldots, A_m$ are classical finite dimensional $(\mathbb{C}_p, *)$-simple algebras such that $\dim_F(A_1 \oplus \cdots \oplus A_m) = d$. To provide a characterization of these varieties we will prove that $UT^*_{\mathbb{C}_p}(A_1, \ldots, A_m)$ generates a minimal variety for all sequences $A_1, \ldots, A_m$ of such $(\mathbb{C}_p, *)$-simple algebras.

Let us start with:

Proposition 19. *Let $\mathbf{A} = UT^*_{\mathbb{C}_p}(A_1, \ldots, A_m)$ and $\mathbf{B} = UT^*_{\mathbb{C}_p}(B_1, \ldots, B_n)$ with the same $(\mathbb{C}_p, *)$-exponent and let $\nu := m+n-1$. If $\tilde{f}^\nu_{\mathbf{A}}$ is not a $(\mathbb{C}_p, *)$-graded polynomial identity for $\mathbf{B}$, then $\mathbf{A}$ and $\mathbf{B}$ are isomorphic as $(\mathbb{C}_p, *)$-algebras.*

Proof. We start by proving that:

(a) $d^{\mathbf{A}}_{ss,g,\mu} = d^{\mathbf{B}}_{ss,g,\mu}$ for every $g \in \mathbb{C}_p$ and $\mu \in \{+, -\}$;
(b) $n = m$;
(c) either $d^{\mathbf{A}}_k = d^{\mathbf{B}}_k$ for all $k \in [1, m]$ or $d^{\mathbf{A}}_k = d^{\mathbf{B}}_{m+1-k}$ for all $k \in [1, m]$.

Since $\tilde{f}^\nu_{\mathbf{A}} \notin \mathrm{Id}^*_{\mathbb{C}_p}(\mathbf{B})$, there exists a non-zero $(\mathbb{C}_p, *)$-graded evaluation $e_{\mathbf{B}}$ of it. The polynomial $\tilde{f}^\nu_{\mathbf{A}}$ is multilinear then we evaluate the designed variables on the elements of the standard basis $\mathfrak{L}(\mathbf{B})$ and we evaluate all the remaining variables on the elements of the canonical basis $\mathfrak{B}_{\mathbf{B}}$.

Since the Jacobson radical $J(\mathbf{B})$ is nilpotent of index n there exists $l \in [m, m+n-1]$ such that all the variables of U_l and Y_l must be evaluated only by elements of the semisimple subalgebra $\overline{B} = B_1 \oplus \cdots \oplus B_n$ of $\mathbf{B}$.

As, for every $g \in \mathbb{C}_p$ and $\mu \in \{+, -\}$, $\tilde{f}^\nu_{\mathbf{A}}$ alternates in the set $U_{l,g,\mu}$, we have that $|U_{l,g,\mu}| \leq d^{\mathbf{B}}_{ss,g,\mu}$, and by (9) we obtain

$$d^{\mathbf{A}}_{ss,g,\mu} = |U_{l,g,\mu}| \leq d^{\mathbf{B}}_{ss,g,\mu} \qquad \text{for all } g \in \mathbb{C}_p \text{ and } \mu \in \{+, -\}.$$

Then

$$\exp^*_{\mathbb{C}_p}(\mathbf{A}) = \sum_{\substack{g \in \mathbb{C}_p \\ \mu \in \{+,-\}}} d^{\mathbf{A}}_{ss,g,\mu} \leq \sum_{\substack{g \in \mathbb{C}_p \\ \mu \in \{+,-\}}} d^{\mathbf{B}}_{ss,g,\mu} = \exp^*_{\mathbb{C}_p}(\mathbf{B}) = \exp^*_{\mathbb{C}_p}(\mathbf{A})$$

and, consequently, we get $d^{\mathbf{A}}_{ss,g,\mu} = d^{\mathbf{B}}_{ss,g,\mu}$ for every $g \in \mathbb{C}_p$ and $\mu \in \{+, -\}$ which proves item (a).

As a consequence, the evaluation $e_{\mathbf{B}}$ involves for the variables of U_l all and only the elements of the standard basis of $\overline{B}$ and each one exactly once. Since $\tilde{f}^\nu_{\mathbf{A}}$ is a sum of terms $\chi(f^{\nu}_{h_1,\dots,h_m,u_1,\dots,u_{m-1},A})$, for χ ranging in the appropriate group of permutations, we may assume that

$$e_{\mathbf{B}}(f^{\nu}_{h_1,\dots,h_m,u_1,\dots,u_{m-1},A}) \neq 0_{\mathbf{B}}.$$

Since $B_iB_j = 0_{\mathbf{B}}$ for all $i \neq j \in [1,n]$, then, for each $k \in [1,m]$, the polynomial $f^{(l)}_{h_k,A_k}$ must be evaluated in a unique block of $\overline{B}$ and so the elements of $\mathfrak{L}_{i,i}(\mathbf{B})$ and $\mathfrak{L}_{j,j}(\mathbf{B})$ must appear in polynomials corresponding to different indexes $k_i, k_j \in [1,m]$. Therefore, $n \leq m$.

If $m = 1$, then $n = 1$. In this case $\mathbf{A}$ and $\mathbf{B}$ are classical finite dimensional $(\mathbb{C}_p,*)$-simple algebras having the same dimension. Therefore, by Proposition 17, $\mathbf{A}$ and $\mathbf{B}$ are isomorphic as $(\mathbb{C}_p,*)$-algebras since $\tilde{f}^{\nu}_{h_1,A_1} = \tilde{f}^{\nu}_{\mathbf{A}} \notin \mathrm{Id}^*_{\mathbb{C}_p}(\mathbf{B})$.

Thus, assume that $m > 1$. For every $j \in [1,m-1]$, $\tilde{f}^{\nu}_{\mathbf{A}}$ alternates in the set $U_{j,|u_j|,+}$ and, by (9), we have

$$|U_{j,|u_j|,+}| = 1 + d^{\mathbf{A}}_{ss,|u_j|,+} = 1 + d^{\mathbf{B}}_{ss,|u_j|,+}.$$

Therefore in the evaluation $e_{\mathbf{B}}$ we must use at least $m-1$ elements of $J(\mathbf{B})$. This implies that $m-1 < n$ and so $m \leq n$. Thus $m = n$ and the item (b) is proved also in this case.

We remark that each of these $m-1$ elements of $J(\mathbf{B})$ must appear as the evaluation via $e_{\mathbf{B}}$ of one and only one designed variable in the sets $U_1,\dots,U_{m-1}$. All other variables of the polynomial $\tilde{f}^{\nu}_{\mathbf{A}}$ are necessarily evaluated in elements of $\overline{B}$. Moreover, we observe that, $B_1J(\mathbf{B})B_2\cdots B_{m-1}J(\mathbf{B})B_m \neq 0_{\mathbf{B}}$ and $B_mJ(\mathbf{B})B_{m-1}\cdots B_2J(\mathbf{B})B_1 \neq 0_{\mathbf{B}}$, whereas we obtain $0_{\mathbf{B}}$ for every rearrangement of the B_j's into the above sequences. This means that either the polynomial $f^{(l)}_{h_k,A_k}$ is evaluated in B_k for all $k \in [1,m]$ or it is evaluated in B_{m-k+1} for all $k = 1,\dots,m$. The first case is equivalent to say that

$$\pi^{\uparrow}(e_{\mathbf{B}}(f^{\nu}_{h_1,\dots,h_m,u_1,\dots,u_{m-1},A})) \neq 0_{\mathbf{B}},$$

the second case is equivalent to request

$$\pi^{\downarrow}(e_{\mathbf{B}}(f^{\nu}_{h_1,\dots,h_m,u_1,\dots,u_{m-1},A})) \neq 0_{\mathbf{B}},$$

where $\pi^{\uparrow}$ and $\pi^{\downarrow}$ are the $\mathbb{C}_p$-graded homomorphisms of $(M_{2s},\bar{\beta})$ defined as in the proof of Theorem 3 for $(M_{2s},\bar{\alpha})$.

Let us assume first that $\pi^{\uparrow}(e_{\mathbf{B}}(f^{\nu}_{h_1,\dots,h_m,u_1,\dots,u_{m-1},A})) \neq 0_{\mathbf{B}}$. Hence

$$0_{\mathbf{B}} \neq \pi^{\uparrow}(e_{\mathbf{B}}(\tilde{f}^{\nu}_{h_1,A_1}))\pi^{\uparrow}(e_{\mathbf{B}}(u_1))\cdots\pi^{\uparrow}(e_{\mathbf{B}}(u_{m-1}))\pi^{\uparrow}(e_{\mathbf{B}}(\tilde{f}^{\nu}_{h_m,A_m})). \tag{10}$$

Because, via $e_{\mathbf{B}}$, the polynomial $f^{(l)}_{h_k,A_k}$ is evaluated in B_k, the whole polynomial $\tilde{f}^{\nu}_{h_k,A_k}$ is evaluated in the same block B_k and the elements $\pi^{\uparrow}(e_{\mathbf{B}}(u_1)),\dots,\pi^{\uparrow}(e_{\mathbf{B}}(u_{m-1})$ lie respectively in the subspaces $U_{1\,2},\dots U_{m-1\,m}$ of $\mathbf{B}$. Clearly the above polynomials are not identities of B_k for all $k \in [1, m]$. As first result we have $d^{\mathbf{A}}_{k,g,\mu} \leq d^{\mathbf{B}}_{k,g,\mu}$ for all $k \in [1, m]$, $g \in \mathbb{C}_p$ and $\mu \in \{+, -\}$; which implies $d^{\mathbf{A}}_k \leq d^{\mathbf{B}}_k$. Once again we obtain $d^{\mathbf{A}}_k = d^{\mathbf{B}}_k$ for all $k \in [1, m]$ because

$$\exp^*_{\mathbb{C}_p}(\mathbf{A}) = \sum_{k=1}^{m} d^{\mathbf{A}}_k \leq \sum_{k=1}^{m} d^{\mathbf{B}}_k = \exp^*_{\mathbb{C}_p}(\mathbf{B}) = \exp^*_{\mathbb{C}_p}(\mathbf{A}).$$

In the second case, the polynomial $f^{(l)}_{h_k,A_k}$ is evaluated in B_{m+1-k} for each $k \in [1, m]$ and so, as above, $d^{\mathbf{A}}_k = d^{\mathbf{B}}_{m+1-k}$ which completes the proof of item (c).

Now we continue the proof with the analysis of the first case.

Let us denote $w_k = \pi^{\uparrow}(e_{\mathbf{B}}(u_k))$, for all $k \in [1, m]$. As we said above, $w_k \in U_{k\,k+1}$ and so $w_k = (e_{r_k c_k} \otimes e_{k\,k+1})$ for suitable indexes $r_k \in [1, s_k]$ and $c_k \in [1, s_{k+1}]$. Moreover:

$$\dim_F A_k = \dim_F B_k \quad \text{and} \quad \tilde{f}^{\nu}_{h_k,A_k} \notin \mathrm{Id}^*_{\mathbb{C}_p}(B_k), \quad \text{for all} \quad k \in [1, m].$$

Hence, by Proposition 17, the diagonal blocks $(A_k, \alpha^k, *_{A_k})$ and $(B_k, \beta^k, *_{B_k})$ are isomorphic as $(\mathbb{C}_p, *)$-algebras. From the results on isomorphic classical finite dimensional $(\mathbb{C}_p, *)$-simple algebras contained in the previous sections, specifically Proposition 6, 8 and Corollary 1, it follows that, for each $k \in [1, m]$, there exist $g_k \in \mathbb{C}_p$ and $\varrho_k \in S_{n_k}$ such that

$$\beta^k = g_k \alpha^k \varrho_k^{-1}.$$

Moreover:

$\varrho_k \gamma_{n_k} = \gamma_{n_k} \varrho_k$ if $A_k = (M_{n_k}, \alpha^k, *_{A_k})$,

$\varrho_k = 1_{S_{n_k}}$ if $A_k = M_{n_k}(D)$ or $A_k = M_{n_k}(D) \oplus M_{n_k}(D)^{op}$.

We remark that, when $A_k = M_{n_k} \oplus M^{op}_{n_k}$, the previous conclusion follows from the proof of Proposition 8, because $\pi^{\uparrow}(e_{\mathbf{B}}(\tilde{f}^{\nu}_{h_k,A_k})) \neq 0$ and so $\pi^{\uparrow}(e_{\mathbf{B}}(\tilde{\Phi}_{\alpha^k,h_k})) \neq 0$ too. As final result, by Proposition 13, it follows that

$$\mathbf{B} \cong UT^*_{\mathbb{C}_p,\mathbf{g}}(A_1, \ldots, A_m)$$

where $\mathbf{g} = (g_1, \ldots, g_m) \in \mathbb{C}_p^m$. Hence we assume that $\mathbf{B} = UT^*_{\mathbb{C}_p,\mathbf{g}}(A_1, \ldots, A_m)$.

Let us consider the following subsets of $[1, m]$:

$$\mathfrak{S} = \{\, k \mid k \in [1, m] \,\text{and}\, A_k \text{ is } *\text{-singular}\}$$

and

$$\mathfrak{R} = \{\, k \mid k \in [1, m] \,\text{and}\, A_k \text{ is } *\text{-regular}\}.$$

Clearly $\mathfrak{S} \cup \mathfrak{R} = [1, m]$ and $\mathfrak{S} \cap \mathfrak{R} = \emptyset$.

If $\mathfrak{S} = \emptyset$ then, by Definition 7 and Proposition 13, by multiplying for all $k \in [1, m]$ the elements of the word $g_k\alpha^k$ by the element g_k^{-1} we obtain a finite dimensional $(\mathbb{C}_p, *)$-algebra isomorphic to $\mathbf{B}$. In this way

$$\mathbf{B} = UT^*_{\mathbb{C}_p,\mathbf{g}}(A_1, \ldots, A_m) \cong UT^*_{\mathbb{C}_p,\mathbf{1}_{\mathbb{C}_p}}(A_1, \ldots, A_m) = \mathbf{A}.$$

Similarly, when $\mathfrak{S} = \{q\}$ has only one element, let $\mathbf{g}_q = (g_q, \ldots, g_q) \in \mathbb{C}_p^m$. In this case, for all $k \neq q$, we multiply the word $g_k\alpha^k$ by $g_q g_k^{-1}$ and we obtain:

$$\mathbf{B} = UT^*_{\mathbb{C}_p,\mathbf{g}}(A_1, \ldots, A_m) \cong UT^*_{\mathbb{C}_p,\mathbf{g}_q}(A_1, \ldots, A_m) = \mathbf{A}.$$

Now, let us assume that $t, v \in \mathfrak{S}$ with $t < v$. In this case we consider the factor of $f^{\nu}_{h_1,\ldots,h_m,u_1,\ldots,u_{m-1},A}$ defined by:

$$f^{\nu}_{[t,v],A} = \tilde{f}^{\nu}_{h_t,A_t} u_t \cdots u_{v-1} \tilde{f}^{\nu}_{h_v,A_v}.$$

The $\mathbb{C}_p$-degree of this polynomial is $\alpha^t(i_t)^{-1}\alpha^v(i_v)$ and moreover, since

$$0_{\mathbf{B}} \neq \pi^{\uparrow}(e_{\mathbf{B}}(f^{\nu}_{[t,v],A})) = \pi^{\uparrow}(e_{\mathbf{B}}(\tilde{f}^{\nu}_{h_t,A_t}))(e_{r_t c_t} \otimes e_{t\,t+1}) \cdots (e_{r_{v-1} c_{v-1}} \otimes e_{v-1\,v})\pi^{\uparrow}(e_{\mathbf{B}}(\tilde{f}^{\nu}_{h_v,A_v})),$$

we have

$$\alpha^t(i_t)^{-1}\alpha^v(i_v) = \alpha^t(r_t)^{-1} g_t^{-1} g_v \alpha^v(c_{v-1}).$$

Since $t, v \in \mathfrak{S}$ by Definitions 7 and 14, Lemmas 6 and 7, it follows $i_t = h_t,\ i_v = h_v$ and moreover

$$\alpha^t(r_t) = \alpha^t(i_t),\ \alpha^v(c_{v-1}) = \alpha^v(i_v), \quad \text{that is} \quad g_t = g_v.$$

Let us indicate by c the common value of g_k for all $k \in \mathfrak{S}$ and let $\mathbf{c} = (c, \ldots, c) \in \mathbb{C}_p^m$. Once again, for all $k \in \mathfrak{R}$, we multiply the word $g_k\alpha^k$ by $c g_k^{-1}$ and, by Proposition 13, we have:

$$\mathbf{B} = UT^*_{\mathbb{C}_p,\mathbf{g}}(A_1, \ldots, A_m) \cong UT^*_{\mathbb{C}_p,\mathbf{c}}(A_1, \ldots, A_m) = \mathbf{A},$$

concluding the proof of our result in the case $\pi^{\uparrow}(e_{\mathbf{B}}(f^{\nu}_{h_1,\ldots,h_m,u_1,\ldots,u_{m-1},A})) \neq 0_{\mathbf{B}}$.

Finally, when $\pi^{\downarrow}(e_{\mathbf{B}}(f^{\nu}_{h_1,\ldots,h_m,u_1,\ldots,u_{m-1},A})) \neq 0_{\mathbf{B}}$, the result follows by Lemma 3 and the arguments considered above. □

Remark 4. The isomorphisms of the previous proposition are given by a finite sequence of basic moves.

As an immediate consequence of the previous proposition we obtain the following

Theorem 5. *Let* $\mathbf{A} = UT^*_{\mathbb{C}_p}(A_1, \dots, A_m)$ *and* $\mathbf{B} = UT^*_{\mathbb{C}_p}(B_1, \dots, B_n)$*, then*

$$\mathbf{A} \cong \mathbf{B} \Leftrightarrow Id^*_{\mathbb{C}_p}(\mathbf{A}) = Id^*_{\mathbb{C}_p}(\mathbf{B}).$$

Now, we conclude with the main result of this paper

Theorem 6. *Let* $\mathcal{V}$ *be a variety of* $(\mathbb{C}_p, *)$*-algebras generated by a finite dimensional* $(\mathbb{C}_p, *)$*-algebra. Hence,* $\mathcal{V}$ *is minimal of exponent* d *if and only if* $\mathcal{V} = \mathrm{var}^*_{\mathbb{C}_p}(\mathbf{A})$ *where* $\mathbf{A} = UT^*_{\mathbb{C}_p}(A_1, \dots, A_m)$ *with* $A_1, \dots, A_m$ *classical finite dimensional* $(\mathbb{C}_p, *)$*-simple algebras satisfying* $dim_F(A_1 \oplus \dots \oplus A_m) = d$.

Proof. Let $A_1, \dots, A_m$ be a sequence of classical finite dimensional $(\mathbb{C}_p, *)$-simple algebras satisfying $\dim_F(A_1 \oplus \dots \oplus A_m) = d$ and $\mathbf{A} = UT^*_{\mathbb{C}_p}(A_1, \dots, A_m)$. We consider $\mathcal{U}^*_{\mathbb{C}_p}$ a subvariety of $\mathrm{var}^*_{\mathbb{C}_p}(\mathbf{A})$ generated by a finite dimensional $(\mathbb{C}_p, *)$-algebra such that $\exp^*_{\mathbb{C}_p}(\mathbf{A}) = \exp^*_{\mathbb{C}_p}(\mathcal{U}^*_{\mathbb{C}_p})$. By Theorem 4, we get that there exist classical finite dimensional $(\mathbb{C}_p, *)$-simple algebras $B_1, \dots, B_n$ such that, $\mathbf{B} = UT^*_{\mathbb{C}_p}(B_1, \dots, B_n)$, $\mathrm{Id}^*_{\mathbb{C}_p}(\mathcal{U}^*_{\mathbb{C}_p}) \subseteq \mathrm{Id}^*_{\mathbb{C}_p}(\mathbf{B})$ and $\exp^*_{\mathbb{C}_p}(\mathcal{U}^*_{\mathbb{C}_p}) = \exp^*_{\mathbb{C}_p}(B)$. Since $\tilde{f}^{\nu}_{\mathbf{B}}$ is not a $(\mathbb{C}_p, *)$-graded polynomial identity for $\mathbf{A}$, from Proposition 19, we can conclude that $\mathbf{A}$ and $\mathbf{B}$ are isomorphic as $(\mathbb{C}_p, *)$-algebras. Consequently, $\mathrm{Id}^*_{\mathbb{C}_p}(\mathbf{A}) = \mathrm{Id}^*_{\mathbb{C}_p}(\mathbf{B})$ and so $\mathrm{var}^*_{\mathbb{C}_p}(\mathbf{A}) = \mathcal{U}^*_{\mathbb{C}_p}$.

Conversely the result follows from Theorem 4. □

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

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