Pairs of nontrivial smooth solutions for nonlinear Neumann problems

Calogero Vetro[a]

[a] *Department of Mathematics and Computer Science, University of Palermo, Via Archirafi 34, 90123, Palermo, Italy*

Abstract

We consider a nonlinear Neumann problem driven by a nonhomogeneous differential operator with a reaction term that exhibits strong resonance at infinity. Using variational tools based on the critical point theory, we prove the existence of two nontrivial smooth solutions.

Keywords: Second deformation theorem, strong resonance, nonlinear regularity, C_c-condition
2010 MSC: 35J60, 35J65,

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following nonlinear Neumann problem:

$$-\operatorname{div} a(\nabla u(z)) = f(z, u(z)) \text{ in } \Omega, \quad \frac{\partial u}{\partial n_a} = 0 \text{ on } \partial\Omega. \tag{1}$$

In this problem $a : \mathbb{R}^N \to \mathbb{R}^N$ is a strictly monotone and continuous map which satisfies certain other regularity and growth properties listed in hypotheses $H(a)$ below. These conditions provide a general framework in which we can fit many differential operators of interest such as the p-Laplacian and the (p,q)-Laplacian which appears in many models of physical processes (see [13, 14, 15, 16] and the references therein). The reaction (source) term $f(z,x)$ is a Carathéodory function (that is, $z \to f(z,x)$ is measurable and $x \to f(z,x)$ is continuous). We assume that this term is strongly resonant with respect to the principal eigenvalue $\widehat{\lambda}_1 = 0$ of the Neumann p-Laplacian. By $\frac{\partial u}{\partial n_a}$ we denote the conormal derivative of u corresponding to the differential operator $\operatorname{div} a(\nabla u)$.

Strongly resonant problems were first considered by Landesman-Lazer [7], who coined the term "strong resonance". Further results were obtained later by Thews [18], Gonçalves-Miyagaki [6] (semilinear equations driven by the Laplacian) and by Bartolo-Benci-Fortunato [2], Filippakis-Gasiński-Papageorgiou [5] (nonlinear equations driven by the p-Laplacian). All these works deal with the Dirichlet problem and multiplicity results were proved in [5, 6]. Finally, we mention some recent related works focusing on the so-called double phase operators: Byun-Ryu-Shin [3], Papageorgiou-Rădulescu-Repovš [11], Ragusa-Tachikawa [17] (regularity results), Cencelj-Rădulescu-Repovš [4], Papageorgiou-Rădulescu-Repovš [10] (existence of solutions with variable growth conditions).

2. Preliminaries - Hypotheses

Let $\xi \in C^1(0, +\infty)$ with $\xi(t) > 0$ for all $t > 0$ and assume that

$$0 < \widehat{c} \le \frac{\xi'(t)t}{\xi(t)} \le c_0 \text{ and } c_1 t^{p-1} \le \xi(t) \le c_2[t^{s-1} + t^{p-1}] \text{ for all } t > 0, \text{ with } c_1, c_2 > 0,\ 1 \le s < p < +\infty. \tag{2}$$

Email address: calogero.vetro@unipa.it (Calogero Vetro)

The hypotheses on the map $a(\cdot)$ are:
$H(a)$: $a \in C^1(\mathbb{R}^N \setminus \{0\}, \mathbb{R}^N) \cap C(\mathbb{R}^N, \mathbb{R}^N)$ satisfies

(*i*) $|\nabla a(y)| \leq c_3 \dfrac{\xi(|y|)}{|y|}$ for some $c_3 > 0$, all $y \in \mathbb{R}^N \setminus \{0\}$;

(*ii*) $(\nabla a(y)v, v)_{\mathbb{R}^N} \geq \dfrac{\xi(|y|)}{|y|}|v|^2$ for all $y \in \mathbb{R}^N \setminus \{0\}$, all $v \in \mathbb{R}^N$.

Remark 1. These conditions are motivated by the nonlinear regularity theory of Lieberman [8].

These conditions on $a(\cdot)$ and (2) lead to the following properties.

Lemma 1. *If hypotheses $H(a)$ hold, then*

(*a*) $y \to a(y)$ *is continuous and maximal monotone;*

(*b*) $|a(y)| \leq c_4[|y|^{s-1} + |y|^{p-1}]$ *for all* $y \in \mathbb{R}^N$, *some* $c_4 > 0$;

(*c*) $(a(y), y)_{\mathbb{R}^N} \geq \dfrac{c_1}{p-1}|y|^p$ *for all* $y \in \mathbb{R}^N$.

If $G(y) = \int_0^1 (a(ty), y)_{\mathbb{R}^N} dt$ for all $y \in \mathbb{R}^N$, then $\nabla G(y) = a(y)$ and $G(\cdot)$ is convex. So, we have

$$G(y) \leq (a(y), y)_{\mathbb{R}^N} \quad \text{for all } y \in \mathbb{R}^N. \tag{3}$$

From Lemma 1 and (3), we obtain the following growth properties of $G(\cdot)$.

Corollary 1. $\dfrac{c_1}{p(p-1)}|y|^p \leq G(y) \leq c_5[1 + |y|^p]$ *for all* $y \in \mathbb{R}^N$, *some* $c_5 > 0$.

Example 1. The following maps $a(y)$ satisfy hypotheses $H(a)$:

$$\begin{aligned}
&a(y) = |y|^{p-2}y \quad 1 < p < +\infty \quad \text{(the p-Laplacian)},\\
&a(y) = |y|^{p-2}y + |y|^{q-2}y \quad 1 < q < p < +\infty \quad \text{(the (p,q)-Laplacian)},\\
&a(y) = [1 + |y|^2]^{\frac{p-2}{2}}y \quad 1 < p < +\infty \quad \text{(the modified capillary operator)}.
\end{aligned}$$

Consider the following nonlinear eigenvalue problem:

$$-\Delta_p u(z) = \widehat{\lambda}|u(z)|^{p-2}u(z) \text{ in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega. \tag{4}$$

This problem has a smallest eigenvalue $\widehat{\lambda}_1 = 0$ which is isolated and simple. Let $\widehat{u}_1 = \frac{1}{|\Omega|_N^{1/p}}$ ($|\cdot|_N$ being the Lebesgue measure on $\mathbb{R}^N$) be the normalized positive principal eigenfunction. Since $\widehat{\lambda}_1 = 0$ is isolated, we can define the second eigenvalue $\widehat{\lambda}_2 > 0$ of (4). If $\partial B_1^{L^p} = \{u \in L^p(\Omega) : \|u\|_p = 1\}$ and $M = W^{1,p}(\Omega) \cap \partial B_1^{L^p}$, then from Aizicovici-Papageorgiou-Staicu [1] (Proposition 2), we have

Lemma 2. $\widehat{\lambda}_2 = \inf\limits_{\widehat{\gamma} \in \widehat{\Gamma}} \max\limits_{-1 \leq t \leq 1} \|\nabla \widehat{\gamma}(t)\|_p^p$ *with* $\widehat{\Gamma} = \{\widehat{\gamma} \in C([-1, 1], M) : \widehat{\gamma}(-1) = -\widehat{u}_1, \widehat{\gamma}(1) = \widehat{u}_1\}$.

Let $V = \{u \in W^{1,p}(\Omega) : \int_\Omega u dz = 0\}$ and note that $W^{1,p}(\Omega) = \mathbb{R} \oplus V$. We define

$$\widehat{\lambda}_V = \inf\left[\frac{\|\nabla v\|_p^p}{\|v\|_p^p} \,:\, v \in V,\, v \neq 0\right]. \tag{5}$$

Lemma 3. $0 < \widehat{\lambda}_V \leq \widehat{\lambda}_2$.

Proof. Consider a sequence $\{v_n\}_{n\geq 1} \subseteq V$ such that $\|v_n\|_p = 1$ for all $n \in \mathbb{N}$ and

$$\|\nabla v_n\|_p^p \downarrow \widehat{\lambda}_V \quad \text{as } n \to +\infty.$$

Evidently $\{v_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded and so we may assume that $v_n \xrightarrow{w} v$ in $W^{1,p}(\Omega)$. Hence $\|v\|_p = 1$ and $v \in V$. From the weak lower semicontinuity of the norm functional we have $\|\nabla v\|_p^p \leq \widehat{\lambda}_V$, hence $\|\nabla v\|_p^p = \widehat{\lambda}_V$ (see (5)). If $\widehat{\lambda}_V = 0$, then $v = \eta \in \mathbb{R} \setminus \{0\}$, contradicting the fact that $v \in V$. Therefore $0 < \widehat{\lambda}_V$.

Next suppose that $\widehat{\lambda}_2 < \widehat{\lambda}_V$. Then according to Lemma 2, we can find $\widehat{\gamma} \in \widehat{\Gamma}$ such that

$$\|\nabla \widehat{\gamma}(t)\|_p^p < \widehat{\lambda}_V \quad \text{for all } t \in [-1, 1]. \tag{6}$$

Consider the map $k(t) = \int_\Omega \widehat{\gamma}(t)(z)dz$ for all $t \in [-1, 1]$. We see that $k(-1) < 0 < k(1)$ and $k(\cdot)$ is continuous. So, by Bolzano's theorem, we can find $t_0 \in (-1, 1)$ such that $k(t_0) = 0$. Then $\widehat{\gamma}(t_0) \in V$ and so $\widehat{\lambda}_V \leq \|\nabla \widehat{\gamma}(t_0)\|_p^p$, which contradicts (6). Therefore $\widehat{\lambda}_V \leq \widehat{\lambda}_2$. $\square$

Remark 2. If $p = 2$, then $\widehat{\lambda}_V = \widehat{\lambda}_2$.

The hypotheses on the reaction $f(z, x)$ are the following:

H: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ for a.a. $z \in \Omega$ and

- (i) $|f(z, x)| \leq a(z)[1 + |x|^{r-1}]$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$, with $a \in L^\infty(\Omega)$, $p < r < p^*$;
- (ii) if $F(z, x) = \int_0^x f(z, s)ds$, then there exist functions $F_\pm \in L^\infty(\Omega)$ such that $\int_\Omega F_\pm(z)dz \leq 0$, $\lim\limits_{x\to\pm\infty} F(z, x) = F_\pm(z)$ uniformly for a.a. $z \in \Omega$;
- (iii) $F(z, x) \leq \dfrac{c_1}{p(p-1)} \widehat{\lambda}_V |x|^p$ for a.a. $z \in \Omega$, all $x \in \mathbb{R}$;
- (iv) there exists a function $\vartheta \in L^\infty(\Omega)$ such that $\vartheta(z) \geq 0$ for a.a. $z \in \Omega$, $\vartheta \not\equiv 0$, $\liminf\limits_{x\to 0} \dfrac{pF(z, x)}{|x|^p} \geq \vartheta(z)$ uniformly for a.a. $z \in \Omega$.

Remark 3. In the special case of the p-Laplacian, hypothesis $H\,(ii)$ implies that at $\pm\infty$ we have strong resonance with respect to $\widehat{\lambda}_1 = 0$, while at $x = 0$ we have nonuniform nonresonance with respect to $\widehat{\lambda}_1 = 0$ (see $H\,(iv)$).

3. Pair of Nontrivial Solutions

Let $\varphi : W^{1,p}(\Omega) \to \mathbb{R}$ be the energy Euler functional for problem (1) defined by

$$\varphi(u) = \int_\Omega G(\nabla u)dz - \int_\Omega F(z, u)dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

We have $\varphi \in C^1(W^{1,p}(\Omega))$. We recall that $\varphi \in C^1(W^{1,p}(\Omega))$ satisfies the C_c-condition if any sequence $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ such that $\varphi(u_n) \to c \in \mathbb{R}$ and $(1 + \|u_n\|)\varphi'(u_n) \to 0$ in $W^{1,p}(\Omega)^*$ as $n \to +\infty$ has a convergent subsequence. It is well-known that strongly resonant problems are characterized by partial loss of compactness. This is evident in the next proposition.

Proposition 1. *If hypotheses $H(a)$, H hold and $c < \min\{-\int_\Omega F_+dz, -\int_\Omega F_-dz\}$, then φ satisfies the C_c-condition.*

Proof. Consider a sequence $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ such that

$$\varphi(u_n) \to c \in \mathbb{R} \text{ and } (1 + \|u_n\|)\varphi'(u_n) \to 0 \text{ in } W^{1,p}(\Omega)^* \text{ as } n \to +\infty. \tag{7}$$

We will show that $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. Arguing by contradiction, suppose that $\|u_n\| \to +\infty$ and let $y_n = \frac{u_n}{\|u_n\|}$, $n \in \mathbb{N}$. Then $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W^{1,p}(\Omega).$$

From (7) we have

$$\begin{aligned} &\varphi(u_n) \leq M_1 \quad \text{for some } M_1 > 0, \text{ all } n \in \mathbb{N}, \\ \Rightarrow \quad &\frac{c_1}{p(p-1)}\|\nabla y_n\|_p^p - \int_\Omega \frac{F(z,u_n)}{\|u_n\|^p}dz \leq \frac{M_1}{\|u_n\|^p} \quad \text{for all } n \in \mathbb{N}. \end{aligned} \tag{8}$$

On account of hypothesis $H\,(ii)$, we can find $M_2 > 0$ such that

$$F(z,x) \leq \max\{|F_+(z)|, |F_-(z)|\} + 1 \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \geq M_2. \tag{9}$$

Combining (9) with hypothesis $H\,(i)$ we see that

$$F(z,x) \leq \widehat{a}(z) \quad \text{for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ with } \widehat{a} \in L^\infty(\Omega).$$

Using (9), Fatou's lemma and (8), we have

$$\begin{aligned} &\|\nabla y_n\|_p \to 0, \\ \Rightarrow \quad & y_n \to y = \eta \in \mathbb{R} \setminus \{0\} \text{ in } W^{1,p}(\Omega). \end{aligned}$$

We may assume $\eta > 0$ (the reasoning is similar if $\eta < 0$). We have $u_n(z) \to +\infty$ for a.a. $z \in \Omega$ and so $F(z, u_n(z)) \to F_+(z)$ for a.a. $z \in \Omega$ (see $H\,(ii)$). Then by the dominated convergence we have

$$\int_\Omega F(z,u_n)dz \to \int_\Omega F_+(z)dz. \tag{10}$$

We have

$$\begin{aligned} & -\int_\Omega F(z,u_n)dz \leq \varphi(u_n) \quad \text{for all } n \in \mathbb{N}, \\ \Rightarrow \quad & -\int_\Omega F_+(z)dz \leq c \quad \text{(see (10) and (7))}, \end{aligned}$$

which contradicts the choice of $c \in \mathbb{R}$. Therefore $\{u_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega). \tag{11}$$

From (7) we have

$$\left| \int_\Omega (a(\nabla u_n), \nabla h)_{\mathbb{R}^N} dz - \int_\Omega f(z,u_n)hdz \right| \leq \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \quad \text{for all } h \in W^{1,p}(\Omega) \text{ with } \varepsilon_n \to 0^+. \tag{12}$$

If in (12) we choose $h = u_n - u$, pass to the limit as $n \to +\infty$ and use (11). Then we have $\lim\limits_{n\to+\infty} \int_\Omega (a(\nabla u_n), \nabla u_n - \nabla u)_{\mathbb{R}^N} dz = 0$ and so $u_n \to u$ in $W^{1,p}(\Omega)$ (see Proposition 2.7 of Papageorgiou-Rădulescu [9]). We conclude that φ satisfies the C_c-condition. $\square$

Now we are ready for the multiplicity theorem.

Theorem 1. *If hypotheses $H(a)$, H hold, then problem* (1) *has at least two nontrivial smooth solutions $u_0, \widehat{u} \in C^1(\overline{\Omega})$, $u_0 \neq \widehat{u}$.*

Proof. On account of hypothesis $H\,(iv)$, given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$\frac{1}{p}[\vartheta(z) - \varepsilon]|x|^p \leq F(z,x) \quad \text{for a.a. } z \in \Omega, \text{ all } |x| \leq \delta.$$

If $\eta \in \mathbb{R}$ with $|\eta| \leq \delta$, then

$$\begin{aligned} &\varphi(\eta) = -\int_\Omega F(z,\eta)dz \leq \frac{|\eta|^p}{p}\int_\Omega [\varepsilon - \vartheta(z)]dz < 0 \quad \text{for } \varepsilon \in \left(0, \frac{\|\vartheta\|_1}{|\Omega|_N}\right), \\ \Rightarrow\quad &\inf[\varphi(u) \,:\, u \in W^{1,p}(\Omega)] = \widehat{m} < 0 = \varphi(0). \end{aligned}$$

Since $\widehat{m} < 0 \leq \min\{-\int_\Omega F_+ dz, -\int_\Omega F_- dz\}$, from Proposition 1 it follows that φ satisfies the $C_{\widehat{m}}$-condition. By Proposition 5.1.8, p. 302, of Papageorgiou-Rădulescu-Repovš [12], we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} &\varphi(u_0) = \widehat{m} < 0 = \varphi(0), \\ \Rightarrow\quad & u_0 \neq 0, \quad u_0 \in K_\varphi = \{u \in W^{1,p}(\Omega) \,:\, \varphi'(u) = 0\}. \end{aligned}$$

The regularity theory of Lieberman [8] implies that $K_\varphi \subseteq C^1(\overline{\Omega})$. So, $u_0 \in C^1(\overline{\Omega}) \setminus \{0\}$ and is a solution of (1).

If $v \in V$, then using Corollary 1 and $H\,(iii)$, we have

$$\begin{aligned} &\varphi(v) \geq \frac{c_1}{p(p-1)}\left[\|\nabla v\|_p^p - \widehat{\lambda}_V \|u\|_p^p\right] \geq 0 \quad (\text{see (5)}), \\ \Rightarrow\quad &\inf_V \varphi = 0. \end{aligned} \tag{13}$$

Also, from the previous arguments, we have that there exists $r > 0$ such that $\widetilde{m}_r = \sup\limits_{\overline{B}_r \cap \mathbb{R}} \varphi < 0$ ($B_r = \{u \in W^{1,p}(\Omega) \,:\, \|u\| < r\}$).

We introduce the following set of continuous paths in $W^{1,p}(\Omega)$

$$\Gamma = \left\{\gamma \in C(\overline{B}_r \cap \mathbb{R}, W^{1,p}(\Omega)) \,:\, \gamma\Big|_{\partial B_r \cap \mathbb{R}} = \mathrm{id}\right\}.$$

Suppose that $K_\varphi = \{0, u_0\}$ and consider the deformation $h(t,u)$ postulated by the second deformation theorem with $a = \widetilde{m}_r < 0 = b$ (see Papageorgiou-Rădulescu-Repovš [12], Theorem 5.3.12, p. 386). On account of Proposition 1, φ satisfies the C_c-condition for all $c \in [\widetilde{m}_r, 0)$. We consider the map $\widehat{\gamma} : \overline{B}_r \cap \mathbb{R} \to W^{1,p}(\Omega)$ defined by

$$\widehat{\gamma}(u) = \begin{cases} u_0 & \text{if } \|u\| \leq \frac{r}{2}, \\ h\left(\frac{2(r - \|u\|)}{r}, \frac{ru}{\|u\|}\right) & \text{if } \frac{r}{2} < \|u\|. \end{cases}$$

If $\|u\| = \frac{r}{2}$, then $h(1, 2u) = u_0$ and so $\widehat{\gamma}(\cdot)$ is continuous. Also, for $u \in \partial B_r \cap \mathbb{R}$, we have $h(0,u) = u$, hence $\widehat{\gamma} \in \Gamma$. Since the deformation h is φ-decreasing (see Papageorgiou-Rădulescu-Repovš [12], p. 386), it follows that

$$\varphi(\widehat{\gamma}(u)) < 0 \quad \text{for all } u \in \overline{B}_r \cap \mathbb{R}. \tag{14}$$

From Papageorgiou-Rădulescu-Repovš [12] (p. 327), we know that $\{\partial B_r \cap \mathbb{R}, \overline{B}_r \cap \mathbb{R}\}$ link with V. Hence we have

$$\widehat{\gamma}(\overline{B}_r \cap \mathbb{R}) \cap V \neq \emptyset. \tag{15}$$

Combining (13), (14), (15) we have a contradiction. So, there exists $\widehat{u} \in K_\varphi \subseteq C^1(\overline{\Omega})$, $\widehat{u} \notin \{0, u_0\}$. This is the second nontrivial smooth solution of (1). ☐

Acknowledgment: The author wishes to thank the two knowledgeable referees for their corrections and remarks.

References

References

[1] S. Aizicovici, N.S. Papageorgiou, V. Staicu, The spectrum and an index formula for the Neumann p-Laplacian and multiple solutions for problems with a crossing nonlinearity, Discrete Contin. Dyn. Syst. 25 (2009) 431–456.

[2] P. Bartolo, V. Benci, D. Fortunato, Abstract critical point theorems and applications to some nonlinear problems with strong resonance at infinity, Nonlinear Anal. 7 (1983) 981–1012.

[3] S.-S. Byun, S. Ryu, P. Shin, Calderón-Zygmund estimates for ω-minimizers of double phase variational problems, Appl. Math. Lett. 86 (2018) 256–263.

[4] M. Cencelj, V.D. Rădulescu, D.D. Repovš, Double phase problems with variable growth, Nonlinear Anal. 177 (2018) 270–287.

[5] M.E. Filippakis, L. Gasiński, N.S. Papageorgiou, Quasilinear hemivariational inequalities with strong resonance at infinity, Nonlinear Anal. 56 (2004) 331–345.

[6] J.V.A, Gonçalves, O. Miyagaki, Multiple nontrivial solutions of semilinear strongly resonant elliptic equations, Nonlinear Anal. 19 (1992) 43–52.

[7] E.M. Landesman, A.C. Lazer, Nonlinear perturbations of linear elliptic boundary value problems at resonance, J. Math. Mech. 19 (1970) 609–623.

[8] G.M. Lieberman, The natural generalization of the natural conditions of Ladyzhenskaya and Ural′tseva for elliptic equations, Comm. Partial Differential Equations 16 (1991) 311–361.

[9] N.S. Papageorgiou, V.D. Rădulescu, Nonlinear nonhomogeneous Robin problems with superlinear reaction term, Adv. Nonlinear Stud. 16 (2016) 737–764.

[10] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, Double-phase problems with reaction of arbitrary growth, Z. Angew. Math. Phys. 69:108 (2018) 1–21.

[11] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, Double-phase problems and a discontinuity property of the spectrum, Proc. Amer. Math. Soc. 147 (2019) 2899–2910.

[12] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, Nonlinear Analysis - Theory and Methods, Springer, Switzerland (2019).

[13] N.S. Papageorgiou, V.D. Rădulescu, D.D. Repovš, $(p, 2)$-equations asymmetric at both zero and infinity, Adv. Nonlinear Anal. 7 (2018) 327–351.

[14] N.S. Papageorgiou, C. Vetro, F. Vetro, Positive solutions for singular $(p, 2)$-equations, Z. Angew. Math. Phys. 70:72 (2019) 1–10.

[15] N.S. Papageorgiou, C. Vetro, F. Vetro, Multiple solutions with sign information for a $(p, 2)$-equation with combined nonlinearities, Nonlinear Anal. (2019), https://doi.org:10.1016/j.na.2019.111716

[16] N.S. Papageorgiou, C. Zhang, Noncoercive resonant $(p, 2)$-equations with concave terms, Adv. Nonlinear Anal. 9 (2020) 228–249.

[17] M.A. Ragusa, A. Tachikawa, Regularity for minimizers for functionals of double phase with variable exponents, Adv. Nonlinear Anal. 9 (2020) 710–728.

[18] K. Thews, Non-trivial solutions of elliptic equations at resonance, Proc. Roy. Soc. Edinburgh Sect. A 85 (1980) 119–129.