A MODEL OF CAPILLARY PHENOMENA IN $\mathbb{R}^N$ WITH SUB-CRITICAL GROWTH

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Abstract. This paper deals with the nonlinear Dirichlet problem of capillary phenomena involving an equation driven by the p-Laplacian-like differential operator in $\mathbb{R}^N$. We prove the existence of at least one nontrivial nonnegative weak solution, when the reaction term satisfies a sub-critical growth condition and the potential term has certain regularities. We apply the energy functional method and weaker compactness conditions.

1. Introduction

In this paper we study the following problem:

$$-\Delta_p^l u + \xi(x)|u|^{p-2}u = g(x,u),\ x \in \mathbb{R}^N,\ 1 < p < N. \tag{1}$$

In this problem $\Delta_p^l u$ denotes the p-Laplacian-like operator defined by

$$\Delta_p^l u := \operatorname{div}\left(\left(1 + \frac{|\nabla u|^p}{\sqrt{1+|\nabla u|^{2p}}}\right)|\nabla u|^{p-2}\nabla u\right).$$

The potential function $\xi(\cdot)$ is continuous, coercive (that is, $\xi(x) \to +\infty$ as $|x| \to +\infty$) and positive. In the reaction (right hand side of (1)), $g(x,z)$ is a Carathéodory function (that is, for all $z \in \mathbb{R}$, $x \to g(x,z)$ is measurable and for a.a. $x \in \mathbb{R}^N$, $z \to g(x,z)$ is continuous), and satisfies the following hypotheses:

(g_0) $g(x,z) = 0$ for $z \leq 0$ and $G(x,z) > 0$ if $z > 0$, where

$$G(x,z) := \int_0^z g(x,s)ds, \quad \text{for all } x \in \mathbb{R}^N,\ \text{all } z \in \mathbb{R};$$

(g_1) there exist $s \in]p,p^*[$, $k \in L^\infty(\mathbb{R}^N)_+ \cap L^{\frac{s}{s-1}}(\mathbb{R}^N)$ and two constants $c > 0$ and $\alpha \in]p,p^*[$ such that

$$|g(x,z)| \leq k(x) + c|z|^{\alpha-1} \quad \text{for all } (x,z) \in \mathbb{R}^N \times \mathbb{R},$$

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where $p^* = \dfrac{Np}{N-p}$ if $p < N$ and $p^* = +\infty$ if $p \geq N$;

(g_2) $\displaystyle\lim_{z\to+\infty} \frac{G(x,z)}{z^p} = +\infty$ uniformly in $x \in \mathbb{R}^N$;

(g_3) there exists $\beta \in L^1(\mathbb{R}^N)_+$ such that

$$\sigma(x,z) \leq \sigma(x,v) + \beta(x) \quad \text{for all } 0 < z < v,$$

where $\sigma(x,z) = g(x,z)z - 2pG(x,z)$;

(g_4) there exists $l \in L^\infty(\mathbb{R}^N)_+$ such that $\|l\|_{L^\infty(\mathbb{R}^N)} C_p^p < 1$, where C_p is the constant relative to the continuous embedding $W \hookrightarrow L^p(\mathbb{R}^N)$ (see Proposition 1) and

$$\limsup_{z\to 0^+} \frac{pG(x,z)}{z^p} \leq l(x) \quad \text{uniformly in } x \in \mathbb{R}^N.$$

Under these conditions, we show that problem (1) has at least one nontrivial nonnegative weak solution.

We present a characteristic example of function $g(x,z)$ satisfying the hypotheses $(g_0) - (g_4)$.

Example 1. *Let $g(x,z) = |z|^{p-2}z\log(1+|z|)$ for all $z \in \mathbb{R}$ with $z > 0$ and $g(x,z) = 0$ for all $z \in \mathbb{R}$ with $z \leq 0$. We drop the dependence on x for simplicity.*

We mention that recently differential equations driven by p-Laplacian-like operators attracted considerable interest (for instance, such a kind of operators is used to model the phenomenon of capillarity). Consequently, there have been various existence and multiplicity results for such equations. We mention the works of Chen-Luo [4], Papageorgiou-Rocha [7], Rodrigues [8], Vetro [10], Zhou [11] (Dirichlet problem), Afrouzi-Kirane-Shokooh [1], Shokooh [9] (Neumann problem).

2. Mathematical background

The main space in the analysis of problem (1) is the Sobolev space $W^{1,p}(\mathbb{R}^N)$. Recall that when the domain is the whole $\mathbb{R}^N$, the Sobolev embedding is not compact. This is a difficulty in our study (see, for example, Chaves-Ercole-Miyagaki [3]). Let $D^{1,p}(\mathbb{R}^N)$ denote the closure of $C_0^\infty(\mathbb{R}^N)$ with respect to the norm of $W^{1,p}(\mathbb{R}^N)$. Then $D^{1,p}(\mathbb{R}^N)$ is the reflexive Banach space

$$D^{1,p}(\mathbb{R}^N) := \left\{ u \in L^{p^*}(\mathbb{R}^N) : \frac{\partial u}{\partial x_i} \in L^p(\mathbb{R}^N),\ i = 1, \dots, N \right\}$$

(see Ben-Naoum-Troestler-Willem [2]). Also, its norm is equivalent to $\|\nabla(\cdot)\|$. We point out that $W^{1,p}(\mathbb{R}^N) \subsetneqq D^{1,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N)$.

Consider the reflexive Banach space

$$W := \left\{ u \in D^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} \xi(x)|u|^p dx < +\infty \right\}$$

equipped with the norm

$$\|u\| := \left(\int_{\mathbb{R}^N} (|\nabla u|^p + \xi(x)|u|^p dx \right)^{\frac{1}{p}}.$$

The coercivity of the function ξ implies the continuity of the embedding $W \hookrightarrow L^s(\mathbb{R}^N)$ if $p \leq s \leq p^*$, and its compactness if $p \leq s < p^*$. In view of these considerations, we are able to prove that the energy functional associated with nonnegative solutions of (1) in W satisfies both the (C_c)-condition and a mountain pass geometry (see Motreanu-Motreanu-Papageorgiou [6]). In so doing, we do not impose the Ambrosetti-Rabinowitz condition (see also Li-Yang [5], where a more restricted version of the quasimonotonicity condition is first introduced). Note that the function of Example 1 does not satisfy the Ambrosetti-Rabinowitz condition.

We recall that a weak solution of problem (1) is a function $u \in W$ such that

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \nabla v dx + \int_{\mathbb{R}^N} \frac{|\nabla u|^{2p-2} \nabla u}{\sqrt{1 + |\nabla u|^{2p}}} \nabla v dx + \int_{\mathbb{R}^N} \xi(x)|u|^{p-2} uv dx = \int_{\mathbb{R}^N} g(x,u) v \, dx,$$

for all $v \in W$.

Let X be a Banach space and X^* its topological dual. Consider the C^1-functional $I : W \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{p} \int_{\mathbb{R}^N} [|\nabla u|^p + \xi(x)|u|^p] \, dx + \frac{1}{p} \int_{\mathbb{R}^N} \left[\sqrt{1 + |\nabla u|^{2p}} - 1 \right] dx - \int_{\mathbb{R}^N} G(x,u) dx$$

for all $u \in W$. Let $I' : W \to W^*$ be such that

$$\langle I'(u), v \rangle = \int_{\mathbb{R}^N} \left[|\nabla u|^{p-2} \nabla u \nabla v + \xi(x)|u|^{p-2} uv \right] dx + \int_{\mathbb{R}^N} \frac{|\nabla u|^{2p-2} \nabla u}{\sqrt{1 + |\nabla u|^{2p}}} \nabla v dx - \int_{\mathbb{R}^N} g(x,u) v dx$$

for all $u, v \in W$.

Also, we have

$$I(u) - \frac{1}{2p}\langle I'(u), u\rangle = \frac{1}{p}\|u\|^p + \frac{1}{p}\int_{\mathbb{R}^N}\left[\sqrt{1+|\nabla u|^{2p}} - 1\right]dx \tag{2}$$
$$- \int_{\mathbb{R}^N} G(x,u)dx - \frac{1}{2p}\|u\|^p - \frac{1}{2p}\int_{\mathbb{R}^N}\frac{|\nabla u|^{2p}}{\sqrt{1+|\nabla u|^{2p}}}dx$$
$$+ \frac{1}{2p}\int_{\mathbb{R}^N} g(x,u)u dx$$
$$= \frac{1}{2p}\|u\|^p + \frac{1}{p}\int_{\mathbb{R}^N}\left[\sqrt{1+|\nabla u|^{2p}} - 1 - \frac{1}{2}\frac{|\nabla u|^{2p}}{\sqrt{1+|\nabla u|^{2p}}}\right]dx$$
$$+ \frac{1}{2p}\int_{\mathbb{R}^N}\sigma(x,u)dx.$$

From Chaves-Ercole-Miyagaki [3] (Proposition 2), we have:

Proposition 1. *$(W, \|\cdot\|)$ is a reflexive (uniformly convex) Banach space and the embedding $W \hookrightarrow L^s(\mathbb{R}^N)$ is continuous, whenever $p \le s \le p^*$, and compact, whenever $p \le s < p^*$.*

Remark 1. Let $J : W \to \mathbb{R}$ be the linear functional defined by

$$\langle J(u), v\rangle := \int_{\mathbb{R}^N}\left[|\nabla u|^{p-2}\nabla u \nabla v + \xi(x)|u|^{p-2}uv\right]dx,$$

for all $v \in W$. We stress that J is bounded with

$$\left|\int_{\mathbb{R}^N}[|\nabla u|^{p-2}\nabla u\nabla v + \xi(x)|u|^{p-2}uv]dx\right|$$
$$\le 2\|u\|^{p-1}\|v\| \quad \text{(by Hölder inequality).}$$

We recall the following compactness condition.

Definition 1. Let X be a real Banach space and $I \in C^1(X,\mathbb{R})$. We say that I satisfies the (C_c)-condition if any sequence $\{u_n\} \subset X$ such that $I(u_n) \to c$ and $(1 + \|u_n\|)I'(u_n) \to 0$ in X^* as $n \to +\infty$ has a convergent subsequence.

From Motreanu-Motreanu-Papageorgiou [6], we know that the Mountain Pass Theorem remains true under (C_c)-condition. We recall the Mountain Pass Theorem (see Theorem 5.40, p.118, Motreanu-Motreanu-Papageorgiou [6]).

Theorem 1. *If $I \in C^1(X,\mathbb{R})$ satisfies the (C_c)-condition, there exist $u_0, u_1 \in X$ and $\rho > 0$ such that*

$$\|u_1 - u_0\| > \rho, \quad \max\{I(u_0), I(u_1)\} < \inf\{\|u - u_0\| = \rho\} = m_\rho,$$

and $c = \inf_{\gamma\in\Gamma} \max_{0\le t\le 1} I(\gamma(t))$ *with* $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}$*, then* $c \geq m_\rho$ *and* c *is a critical value of* I *(that is, there exists* $\widehat{u} \in X$ *such that* $I'(\widehat{u}) = 0$ *and* $I(\widehat{u}) = c$*).*

3. Weak solutions

In this section, we use the (C_c)-condition and the mountain pass geometry to produce a nonnegative weak solution for problem (1). The idea is simple. If the C^1-functional $I : W \to \mathbb{R}$ satisfies both the (C_c)-condition and a mountain pass geometry, then I has a critical point in W.

Lemma 1. $I : W \to \mathbb{R}$ *satisfies the* (C_c)*-condition for each positive constant* c*.*

Proof. Consider a sequence $\{u_n\} \subset W$ satisfying the (C_c)-condition with respect to the C^1-functional I and $c > 0$. So, we have

$$c = I(u_n) + c_n, \qquad \langle I'(u_n), u_n\rangle \to 0 \quad \text{as } n \to +\infty, \tag{3}$$

where $c_n \to 0$ as $n \to +\infty$. From (3), there is a sequence $\{\varepsilon_n\}$ of positive real numbers with $\varepsilon_n \to 0$ as $n \to +\infty$, such that

$$|\langle I'(u_n), v\rangle| \leq \frac{\varepsilon_n \|v\|}{1 + \|u_n\|} \quad \text{for all } v \in W \text{ and } n \in \mathbb{N}.$$

Now, choosing $v = u_n^- := \min\{0, u_n\}$, we deduce that u_n^- converges to zero in W. In fact, we have that $g(x, u_n^-)u_n^- = 0$ for all $n \in \mathbb{N}$ (by (g_0)) and so

$$\begin{aligned}
\langle I'(u_n), u_n^-\rangle &= \int_{\mathbb{R}^N} \left[|\nabla u_n|^{p-2}\nabla u_n \nabla u_n^- + \xi(x)|u_n|^{p-2}u_n u_n^-\right] dx \\
&\quad + \int_{\mathbb{R}^N} \frac{|\nabla u_n|^{2p-2}\nabla u_n}{\sqrt{1 + |\nabla u_n|^{2p}}} \nabla u_n^- dx - \int_{\mathbb{R}^N} g(x, u_n)u_n^- dx \\
&= \int_{\mathbb{R}^N} \left[|\nabla u_n^-|^p + \xi(x)|u_n^-|^p\right] dx + \int_{\mathbb{R}^N} \frac{|\nabla u_n^-|^{2p}}{\sqrt{1 + |\nabla u_n^-|^{2p}}} dx \\
&\geq \|u_n^-\|^p \to 0 \text{ as } n \to +\infty, \\
\Rightarrow \quad & u_n^- \to 0 \text{ in } W.
\end{aligned}$$

Next we choose $v = u_n^+ := \max\{0, u_n\}$. We can find $C_1 > 0$ such that

$$\begin{aligned}
&- \|u_n^+\|^p - \int_{\mathbb{R}^N} \frac{|\nabla u_n^+|^{2p}}{\sqrt{1 + |\nabla u_n^+|^{2p}}} dx + \int_{\mathbb{R}^N} g(x, u_n^+)u_n^+ dx \\
&= -\langle I'(u_n), u_n^+\rangle \leq \frac{\varepsilon_n \|u_n^+\|}{1 + \|u_n\|} \leq \varepsilon_n \leq C_1.
\end{aligned}$$

Using (g_0) we have

$$\int_{\mathbb{R}^N} pG(x,u_n)dx = \int_{\mathbb{R}^N} pG(x,u_n^+)dx \quad \text{for all } n\in\mathbb{N},$$
$$(\text{since } \textstyle\int_{\mathbb{R}^N} pG(x,u_n^-)dx = 0 \text{ for all } n\in\mathbb{N}),$$
$$\Rightarrow \quad I(u_n) = I(u_n^+) + \frac{1}{p}\|u_n^-\|^p + \frac{1}{p}\int_{\mathbb{R}^N}\left[\sqrt{1+|\nabla u_n^-|^{2p}}-1\right]dx.$$

Also, we get

$$0 \le \int_{\mathbb{R}^N}\left[\sqrt{1+|\nabla u_n^-|^{2p}}-1\right]dx \le \int_{\mathbb{R}^N}|\nabla u_n^-|^p dx \le \|u_n^-\|^p \to 0$$

as $n\to+\infty$. Since the sequence $\{I(u_n)\}$ is bounded (recall $I(u_n) = c - c_n$ and $c_n\to 0$ as $n\to+\infty$), we can find $C_2>0$ such that

$$-pC_2 \le I(u_n^+) = \|u_n^+\|^p + \int_{\mathbb{R}^N}\left[\sqrt{1+|\nabla u_n^+|^{2p}}-1\right]dx - \int_{\mathbb{R}^N} pG(x,u_n^+)dx \le pC_2. \tag{4}$$

Next we prove that $\{u_n^+\}$ is bounded in W. We assume that $\{u_n^+\}$ is an unbounded sequence. Let $\|u_n^+\|\to+\infty$ as $n\to+\infty$ (we can pass to a subsequence if necessary). Also, let $\|u_n^+\|>0$ for all $n\in\mathbb{N}$ and denote

$$v_n = \frac{u_n^+}{\|u_n^+\|} \quad \text{for all } n\in\mathbb{N}.$$

So, $\|v_n\|=1$ for all $n\in\mathbb{N}$. We can find $v\in W$ such that

$$v_n \xrightarrow{w} v \quad \text{in } W;$$
$$v_n \to v \quad \text{in } L^s(\mathbb{R}^N) \text{ and } L^\alpha(\mathbb{R}^N) \quad s,\alpha\in]p,p^*[.$$

Let $\Omega := \{x\in\mathbb{R}^N : v(x)\neq 0\}$. We claim that $|\Omega|=0$, where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^N$. By contradiction again, assume $|\Omega|>0$. We have

$$u_n^+(x)\to+\infty \quad \text{for } x\in\Omega, \quad \text{as } n\to+\infty,$$

since $v_n\to v\neq 0$ in Ω. So, we can assume that $u_n^+(x)>0$ for $x\in\Omega$. By hypothesis (g_2), we have

$$\lim_{n\to+\infty}\frac{G(x,u_n^+)}{\|u_n^+\|^p} = \lim_{n\to+\infty}\frac{G(x,u_n^+)}{(u_n^+)^p}(v_n)^p = +\infty \quad \text{uniformly for } x\in\Omega. \tag{5}$$

Also, using (g_0), we have

$$G(x,z)\ge 0 \quad \text{for } x\in\mathbb{R}^N, \text{ all } z\in\mathbb{R}.$$

From (5), by Fatou's lemma (hypotheses (g_1) and (g_2) permit its use), we deduce that

$$\lim_{n\to+\infty}\int_\Omega \frac{G(x,u_n^+)}{\|u_n^+\|^p}dx = +\infty.$$

So, we have

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} \frac{G(x,u_n^+)}{\|u_n^+\|^p}dx \geq \lim_{n\to+\infty}\int_\Omega \frac{G(x,u_n^+)}{\|u_n^+\|^p}dx = +\infty. \tag{6}$$

From (4), we get

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{G(x,u_n^+)}{\|u_n^+\|^p}dx &\leq \frac{1}{p} + \frac{1}{p\|u_n^+\|^p}\int_{\mathbb{R}^N}\left[\sqrt{1+|\nabla u_n^+|^{2p}}-1\right]dx + \frac{C_2}{\|u_n^+\|^p}\\ &\leq \frac{2}{p} + \frac{C_2}{\|u_n^+\|^p},\end{aligned}$$

which implies

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^N} \frac{G(x,u_n^+)}{\|u_n^+\|^p}dx \leq \frac{2}{p}.$$

This is a contradiction with (6), therefore $|\Omega| = 0$. We conclude that $v(x) = 0$ for a.a. $x \in \mathbb{R}^N$. Since $I(tu_n^+)$ is a continuous function on $[0,1]$ with respect to the variable t, for each $n \in \mathbb{N}$, we can find $t_n \in [0,1]$ such that

$$I(t_n u_n^+) = \max_{t\in[0,1]} I(tu_n^+).$$

For $k > 1$, we put

$$r_n = k^{\frac{1}{p}} v_n = \frac{k^{\frac{1}{p}}}{\|u_n^+\|}u_n^+ \in W \quad \text{for all } n \in \mathbb{N}.$$

Now, as $v_n \to 0$ in $L^s(\mathbb{R}^N)$ with $s \in [p,p^*[$ and $v_n(x) \to 0$ for a.a. $x \in \mathbb{R}^N$ as $n \to +\infty$, using hypothesis (g_1) and Krasnoselskii's theorem (see, for instance, Motreanu-Motreanu-Papageorgiou [6], p. 41), we have

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} G(x,r_n)dx = 0. \tag{7}$$

As $\|u_n^+\| \to +\infty$, there is $n_0 \in \mathbb{N}$ such that $0 < k^{\frac{1}{p}}\frac{1}{\|u_n^+\|} \leq 1$ for all $n \geq n_0$. Also, from (7), we can find $n_1 \geq n_0$ such that

$$\int_{\mathbb{R}^N} G(x,r_n)dx < \frac{k}{2p} \quad \text{for all } n \geq n_1.$$

So, we have

$$I(t_n u_n^+) \geq I(r_n)$$

$$
\begin{aligned}
&= \frac{1}{p}\int_{\mathbb{R}^N} [|\nabla r_n|^p + \xi(x)|r_n|^p]\, dx + \frac{1}{p}\int_{\mathbb{R}^N}\left[\sqrt{1+|\nabla r_n|^{2p}} - 1\right] dx \\
&\quad - \int_{\mathbb{R}^N} G(x, r_n)dx \\
&\geq \frac{1}{p}\|r_n\|^p - \int_{\mathbb{R}^N} G(x, r_n)dx \\
&\geq \frac{1}{p}k - \frac{1}{2p}k \quad \text{ for } n \text{ large enough (that is, for all } n \geq n_1) \\
&= \frac{1}{2p}k.
\end{aligned}
$$

We conclude that

$$
I(t_n u_n^+) \to +\infty \quad \text{as} \quad n \to +\infty \quad (k > 1 \text{ is arbitrary}). \tag{8}
$$

Since

$$
I(0) = 0 \quad \text{and} \quad \{I(u_n^+)\} \quad \text{bounded (see (4))},
$$

it follows that we can find $n_2 \geq n_1$ such that $t_n \in]0, 1[$ for all $n \geq n_2$. So, we have $0 < t_n u_n^+(x) < u_n^+(x)$ for $n \geq n_2$ whenever $x \in \mathbb{R}^N$ is such that $u_n^+(x) > 0$.

Now, by hypothesis (g_3) we get

$$
\begin{aligned}
\int_{\mathbb{R}^N} \sigma(x, t_n u_n^+)dx &\leq \int_{\mathbb{R}^N} \sigma(x, u_n^+)dx + \int_{\mathbb{R}^N} \beta(x)dx \\
&= \int_{\mathbb{R}^N} \sigma(x, u_n^+)dx + \|\beta\|_{L^1(\mathbb{R}^N)}
\end{aligned} \tag{9}
$$

for all $n \geq n_2$.

Let $h : [0, +\infty[\to \mathbb{R}$ be the function defined by

$$
h(t) = \sqrt{1 + t^{2p}} - 1 - \frac{1}{2}\frac{t^{2p}}{\sqrt{1 + t^{2p}}}.
$$

Since h is nondecreasing, using (2) and (9), we have

$$
\begin{aligned}
I(t_n u_n^+) &= I(t_n u_n^+) - \frac{1}{2p}\langle I'(t_n u_n^+), t_n u_n^+\rangle + o(1) \\
&= \frac{1}{2p}\|t_n u_n^+\|^p + \frac{1}{p}\int_{\mathbb{R}^N}\left[\sqrt{1+|\nabla t_n u_n^+|^{2p}} - 1 - \frac{1}{2}\frac{|\nabla t_n u_n^+|^{2p}}{\sqrt{1+|\nabla t_n u_n^+|^{2p}}}\right] dx \\
&\quad + \frac{1}{2p}\int_{\mathbb{R}^N} \sigma(x, t_n u_n^+)dx + o(1)
\end{aligned}
$$

$$\leq \frac{1}{2p}\|u_n^+\|^p + \frac{1}{p}\int_{\mathbb{R}^N}\left[\sqrt{1+|\nabla u_n^+|^{2p}} - 1 - \frac{1}{2}\frac{|\nabla u_n^+|^{2p}}{\sqrt{1+|\nabla u_n^+|^{2p}}}\right]dx$$
$$+ \frac{1}{2p}\int_{\mathbb{R}^N}\sigma(x,u_n^+)dx + \frac{1}{2p}\|\beta\|_{L^1(\mathbb{R}^N)} + o(1)$$
$$= I(u_n^+) - \frac{1}{2p}\langle I'(u_n^+), u_n^+\rangle + \frac{1}{2p}\|\beta\|_{L^1(\mathbb{R}^N)} + o(1)$$
$$= I(u_n^+) - \frac{1}{2p}\langle I'(u_n), u_n^+\rangle + \frac{1}{2p}\|\beta\|_{L^1(\mathbb{R}^N)} + o(1)$$
$$\leq C_3 \quad \text{for all } n \geq n_2 \text{ and some } C_3 > 0 \text{ (see (2))},$$

which contradicts (8). Therefore, $\{u_n^+\}$ is a bounded sequence in W. Since $u_n^- \to 0$ in W, we conclude that $\{u_n\}$ is bounded in W.

By the reflexivity of W, we can find $u \in W$ such that

$$u_n \xrightarrow{w} u \text{ in } W \text{ and } u_n \to u \text{ in } L^r(\mathbb{R}^N) \text{ for all } r \in [p, p^*[.$$

From (g_1), using also the Hölder inequality, we get

$$\int_{\mathbb{R}^N} g(x,u_n)(u_n-u)dx$$
$$\leq \int_{\mathbb{R}^N} |g(x,u_n)|\,|u_n-u|dx$$
$$\leq \int_{\mathbb{R}^N} (k(x)+c|u_n|^{\alpha-1})|u_n-u|dz$$
$$\leq \|k\|_{L^{s'}(\mathbb{R}^N)}\|u_n-u\|_{L^s(\mathbb{R}^N)} + c\|u_n\|^{\alpha-1}_{L^\alpha(\mathbb{R}^N)}\|u_n-u\|_{L^\alpha(\mathbb{R}^N)},$$

with $s, \alpha \in [p, p^*[$ and $s' = s(s-1)^{-1}$.

It follows that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} g(x,u_n)(u_n-u)dx = 0.$$

Also, we have $\langle I'(u_n), u_n - u\rangle \to 0$ as $n \to +\infty$ (by (3)). Therefore, we get

$$\langle J(u_n), u_n-u\rangle + \int_{\mathbb{R}^N}\frac{|\nabla u_n|^{2p-2}\nabla u_n}{\sqrt{1+|\nabla u_n|^{2p}}}\nabla(u_n-u)dx$$
$$= \int_{\mathbb{R}^N} g(z,u_n)(u_n-u)dz + \langle I'(u_n), u_n-u\rangle \to 0 \quad \text{as} \quad n \to +\infty.$$

We can assume that (we pass to a subsequence if necessary)

$$\limsup_{n\to+\infty}\langle J(u_n), u_n-u\rangle \leq 0 \tag{10}$$

or

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^N}\frac{|\nabla u_n|^{2p-2}\nabla u_n}{\sqrt{1+|\nabla u_n|^{2p}}}\nabla(u_n-u)dx\le 0. \tag{11}$$

If (10) holds, then the convexity of the function $\|\cdot\|$ ensures that

$$\limsup_{n\to+\infty}\|u_n\|\le\|u\|.$$

If (11) holds, then the convexity of the function $\sqrt{1+t^{2p}}-1$ ensures again that

$$\limsup_{n\to+\infty}\|u_n\|\le\|u\|.$$

The uniform convexity of W gives us $u_n\to u$ in W (see Motreanu-Motreanu-Papageorgiou [6], p. 30). So, I satisfies the (C_c)-condition on W. □

Lemma 2. *If hypotheses $(g_0)-(g_4)$ hold, then we have:*

(i) *there is $\zeta\in W$, $\zeta>0$ such that $I(t\zeta)\to-\infty$ as $t\to+\infty$;*
(ii) *there are $\rho>0$ and $\delta>0$ such that $I(u)\ge\delta$ for all $u\in W$ with $\|u\|=\rho$.*

Proof. (i): By hypotheses (g_0) and (g_2), we can find $C_\theta>0$ (for an appropriate constant $\theta>0$) such that

$$G(x,z)\ge\frac{\theta}{p}|z|^p-C_\theta\quad\text{ for all } z\ge 0.$$

Let $\mathcal{B}:=B(0,1)=\{x\in\mathbb{R}^N:|x|\le 1\}$ and consider a positive function $v\in C_0^\infty(\mathcal{B})$. Let ζ be the extension of v to zero out of $\mathcal{B}$. So, for all $t>1$ we have

$$\begin{aligned}I(t\zeta)=&\frac{t^p}{p}\int_{\mathbb{R}^N}[|\nabla\zeta|^p+\xi(x)|\zeta|^p]\,dx+\frac{1}{p}\int_{\mathbb{R}^N}(\sqrt{1+t^{2p}|\nabla\zeta|^{2p}}-1)dx\\&-\int_{\mathcal{B}}G(x,t\zeta)dx\le\frac{t^p}{p}\left[2\|\zeta\|^p-\theta\int_{\mathcal{B}}\zeta^p dx\right]+C_\theta|\mathcal{B}|.\end{aligned}$$

Now, we choose $\theta>0$ satisfying

$$2\|\zeta\|^p-\theta\int_{\mathcal{B}}\zeta^p dx<0,$$

so that $\lim\limits_{t\to+\infty}I(t\zeta)=-\infty$.

(ii): By hypotheses (g_1) and (g_4), we can find $C_\varepsilon>0$ (for each $\varepsilon>0$) such that

$$G(x,z)\le\frac{1}{p}(l(x)+\varepsilon)|z|^p+C_\varepsilon|z|^\alpha\quad\text{ for } x\in\mathbb{R}^N,\text{ all } z\in\mathbb{R}. \tag{12}$$

We stress that $W \hookrightarrow L^p(\mathbb{R}^N)$ and $W \hookrightarrow L^\alpha(\mathbb{R}^N)$ are continuous. Therefore, we can find constants C_p and C_α such that

$$\|u\|_{L^p(\mathbb{R}^N)} \le C_p\|u\| \quad \text{and} \quad \|u\|_{L^\alpha(\mathbb{R}^N)} \le C_\alpha\|u\| \ \text{ for all } u \in W. \tag{13}$$

For $u \in W$ with $\|u\| < 1$, by (13) and (12), we have

$$\begin{aligned} I(u) &= \frac{1}{p}\int_{\mathbb{R}^N} [|\nabla u|^p + \xi(x)|u|^p]\, dx + \int_{\mathbb{R}^N} (\sqrt{1+|\nabla u|^{2p}} - 1)dx \\ &\quad - \int_{\mathbb{R}^N} G(x,u)dx \\ &\ge \frac{1}{p}\|u\|^p - \frac{\|l\|_{L^\infty(\mathbb{R}^N)} + \varepsilon}{p}\int_{\mathbb{R}^N} |u|^p dx - C_\varepsilon \int_{\mathbb{R}^N} |u|^\alpha dx \\ &\ge \frac{1}{p}(1 - (\|l\|_{L^\infty(\mathbb{R}^N)} + \varepsilon)C_p^p)\|u\|^p - C_\varepsilon C_\alpha^\alpha \|u\|^\alpha \\ &= \left[\frac{1}{p}(1 - (\|l\|_{L^\infty(\mathbb{R}^N)} + \varepsilon)C_p^p) - C_\varepsilon C_\alpha^\alpha \|u\|^{\alpha - p}\right]\|u\|^p. \end{aligned}$$

Let $\varepsilon > 0$ and $\rho \in]0,1[$ be such that

$$\sigma = \frac{1}{p}(1 - (\|l\|_{L^\infty(\mathbb{R}^N)} + \varepsilon)C_p^p) - C_\varepsilon C_\alpha^\alpha \rho^{\alpha-p} > 0.$$

We conclude that $I(u) \ge \sigma\rho^p = \delta > 0$ for all $u \in W$ such that $\|u\| = \rho$. □

Remark 2. If t_0 is large enough, the function $w = t_0\zeta \in W$ satisfies $I(w) < 0$ and $\|w\| > \rho$.

Now, Lemmata 1 and 2 ensure that I satisfies the hypotheses of Theorem 1 and hence I has a critical value $c \ge m_\rho$. So, we can find $\widehat{u} \in X$ such that

$$\begin{aligned} \langle I'(\widehat{u}), v\rangle &= \int_{\mathbb{R}^N} \left[|\nabla \widehat{u}|^{p-2}\nabla\widehat{u}\nabla v + \xi(x)|\widehat{u}|^{p-2}\widehat{u}v\right] dx \\ &\quad + \int_{\mathbb{R}^N} \frac{|\nabla\widehat{u}|^{2p-2}\nabla\widehat{u}}{\sqrt{1+|\nabla\widehat{u}|^{2p}}}\nabla v dx - \int_{\mathbb{R}^N} g(x,\widehat{u})v dx = 0 \end{aligned}$$

for all $v \in W$.

If we choose $v = \widehat{u}^-$, we get

$$0 = \|\widehat{u}^-\|^p + \int_{\mathbb{R}^N} \frac{|\nabla\widehat{u}^-|^{2p}}{\sqrt{1+|\nabla\widehat{u}^-|^{2p}}}dx - \int_{\mathbb{R}^N} g(x,\widehat{u}^-)\widehat{u}^- dx \ge \|\widehat{u}^-\|^p.$$

It follows that $\widehat{u}^- = 0$ and so $\widehat{u} \ge 0$ and $\widehat{u} \ne 0$.

Therefore we can state the following existence theorem for problem (1).

Theorem 2. *If hypotheses $(g_0) - (g_4)$ hold, then problem* (1) *has at least one nonnegative nontrivial weak solution.*

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