Control Engineering Practice 147 (2024) 105930

Contents lists available at ScienceDirect

Control Engineering Practice

journal homepage: www.elsevier.com/locate/conengprac

Linear unknown input-state observer for nonlinear dynamic models☆

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A R T I C L E I N F O

Keywords:
State observer
Unknown input observer
Disturbance observer
Nonlinear systems
Measurement noise
Robotics

A B S T R A C T

This paper proposes an unknown input observer for nonlinear systems with input decoupling via system invertibility. Starting from a suitable reformulation of the model of a generic nonlinear system, obtained by merging all system uncertainties with respect to an appropriate nominal linear model into a disturbance vector, the proposed observer can asymptotically copy both the system state and unknown inputs, even in the presence of measurement noise. Formal proof of the estimate convergence is demonstrated analytically. A comparison of the proposed method with existing solutions is shown in simulation, and the method's effectiveness in real-world scenarios is demonstrated by experimental results on a soft articulated robot.

1. Introduction

Knowing a mathematical model robust enough to encapsulate the complexity of a dynamic system is often crucial when trying to solve a control task (Guo & Zhao, 2015). Even with a model with the correct structure, imprecise knowledge of parameter values, which can vary over time, or failure in identifying external signals, hinder in general achievable control performance (Li, Yang, Chen, & Chen, 2014). A prevalent strategy is to collect all sources of system uncertainty, arising from variations in model parameters, unmodeled dynamics, and exogenous disturbances, into an unknown input vector to be estimated in real time.

Several techniques have been devised for this purpose, spanning from Extended State Observers (ESO) (Gu, Wang, Peng, Wang, & Han, 2022; Li & Xia, 2020; Li, Zhang, Luo, & Li, 2023; Talole, Kolhe, & Phadke, 2009; Wang et al., 2017), Disturbance Observers (DO) (Castillo, Sanz, Garcia, Qiu, Wang, & Xu, 2019; Potluri & Singh, 2015; Yu, Wang, Wang, & Chen, 2016), to Unknown Input-State Observers (UIO) (Dabladji, Ichalal, Arioui, & Mammar, 2016; Sundaram & Hadjicostis, 2007; Valcher, 1999). ESOs estimate both system state and unknown input according to a Luenberger-like solution and have demonstrated to be able to handle unknown inputs with a finite number of nonzero time derivatives, i.e., signals that are described as polynomial functions of time with finite order. However, they not only require extending the system state with additional variables but also cannot cope with generic unknown inputs with an infinite number of nonzero time derivatives. Generalized Extended State Observer (GESO) (Miklosovic, Radke, & Gao, 2006) are indeed designed assuming that the unknown input is bounded and constant at steady state (She et al., 2023). Generalized Proportional Integral Observers (GPIO) (Ramírez-Neria, Sira-Ramírez, Rodríguez-Angeles, & Luviano-Juárez, 2012; Sira-Ramírez, 2018) reaches better estimation performance, but require high gains to achieve fast convergence, which makes them more sensitive to measurement noise (Chen, Yang, Guo, & Li, 2016; Qiao & Sun, 2023). DO-based solutions require a slow variation of the noise and involve building an additional state observer when the state is not fully accessible, which increases the overall computational load. Conversely, UIOs are able to overcome the above limitations by decoupling the state estimation from the evolution of the unknown inputs (Valcher, 1999) and then reconstructing such inputs once state estimation convergence is achieved.

The original approaches to designing a UIO required meeting existence conditions that are often too strict in practical contexts, thus limiting their scope of application (Chen, Patton, & Zhang, 1996). An idea working around this problem and partially relaxing the above conditions was first presented in Jin, Tahk, and Park (1997), Saberi, Stoorvogel, and Sannuti (2000), where buffers of consecutive output samples were used to achieve the desired decoupling of the unknown input, but no exact observer design procedure was provided. An elegant solution was later proposed in Sundaram and Hadjicostis (2007) in the form of a delayed UIO (DUIO), along with a strategy to reconstruct both the system state and unknown inputs. More recently, promising improvements have been proposed in Chakrabarty, Ayoub, Żak, and Sundaram (2017) and subsequent derivations have found application

☆ This work has been funded by the Italian MIMIT Project FREECO2 - "Following, Reducing, rEcycling and rEuse agrifood chain CO2 footprint", under grant number F/310265/02/X56.

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https://doi.org/10.1016/j.conengprac.2024.105930
Received 15 November 2023; Received in revised form 1 March 2024; Accepted 25 March 2024
Available online 10 April 2024

in several fields (Azid, Kumar, Cirrincione, & Fagiolini, 2021; Fagiolini, Trumić, & Jovanović, 2020; Pedone & Fagiolini, 2020, 2022; Pedone, Trumić, Jovanović, & Fagiolini, 2022). Yet, the obtained observers suffer from a delay in the reconstructed signal, which hinders their real-time use, still involve complex algebraic conditions, work under the assumption of linearity of the models, and, finally, do not allow a complete a-priori determination of the estimation convergence rate.

Motivated, by the superior results of UIO-based solutions and a renewed interest in their design (cf. the very recent results with data-driven settings (Shi, Lian, & Jones, 2022; Turan & Ferrari-Trecate, 2021) and even with switching linear systems (Conte, Perdon, & Zattoni, 2021; Zattoni, Perdon, & Conte, 2022)), this paper proposes an approach that aims at simplifying the UIO design procedure, resolves the delay and measurement noise issues, and is also valid for a class of nonlinear dynamic models, thus broadening, in our view, its application scope. Precisely, by leveraging on the general reformulation of a nonlinear model as the sum of a linear nominal component and an uncertain one, which groups all system uncertainty and nonlinear terms, our method requires simpler algebraic conditions to achieve the input decoupling and allows direct feedback on the state estimation error. The obtained Linear UIO (LUIO), which also includes a robust estimation of the output vector and its time-derivative with respect to the measurement noise, has faster and more accurate estimation, and provides current estimates immediately usable in a control loop. Another nice feature of the proposed UIO is that, by using the strong observability and invertibility properties of the system, it finds an exact number κ of time derivatives of the system output y that allows reconstructing the generic unknown input exactly, without any assumption on its derivatives.

Contribution: The contribution of this paper includes at least the following: (1) The formalization of a generic nonlinear system into a simpler one in matrix form composed of a nominal linear system excited by an appropriate disturbance vector that groups all system uncertainties with respect to the nominal system; in this setting, all system matrices are linear and constant by design; (2) The design of an observer that explicitly provides estimates of the output vector and its first derivatives, up to a suitable order, that are robust to measurement noise; (3) An unknown input decoupling approach that exploits only the information deriving from the resulting system matrices and that is based on simpler algebraic rules with larger applicability; (4) Formal proofs of applicability in nonlinear domains with no estimation delays; (5) Simulations that demonstrate the superiority of the proposed approach over existing solutions and its robustness with respect to the measurement noise; (6) Experimental validation on a Soft Articulated Robot (SAR) with two degrees of freedom, demonstrating its applicability in real-world scenarios.

2. Model formulation and problem statement

Consider the continuous-time nonlinear model

$$\begin{aligned} \dot{x} &= f(x, u, w, t)\,, \\ y &= C x + \pi\,, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is a state vector, $u \in \mathbb{R}^p$ and $w \in \mathbb{R}^m$ are known and unknown input vectors, respectively, $y \in \mathbb{R}^c$ is an output vector and $\pi \in \mathbb{R}^c$ is a measurement noise signal, $C \in \mathbb{R}^{c\times n}$ is a constant matrix and $f \in \mathbb{R}^n$ is a nonlinear vector function. Without loss of generality, one can extract from the dynamic map f its *linear* time-invariant components, not depending on the unknown input w, i.e.

$$f(x, u, w, t) = A\,x + B\,u + \tilde{f}(x, u, w, t)\,,$$

with $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$. Then, defining the unknown input vector $\delta = \tilde{f} \in \mathbb{R}^{\upsilon}$, (1) is rearranged as

$$\begin{aligned} \dot{x} &= Ax + Bu + W\delta\,, \\ y &= Cx + \pi\,, \end{aligned} \tag{2}$$

with $W \in \mathbb{R}^{n\times \upsilon}$ being a full-column rank constant matrix. Within this setting, we aim at solving the following:

Problem 1. Given the model in (2), design a dynamic system asymptotically estimating the system state x and unknown input δ, using only information on the known input and output signals, $u(t)$ and $y(t)$, i.e. computing estimates $\hat{x}$ and $\hat{\delta}$ such that $\hat{x}(t) \to x(t)$ and $\hat{\delta}(t) \to \delta(t)$ for $t \to \infty$.

For our purposes, the following notions are helpfully recalled or introduced. A signal $s(t)$ is *convergent* if $\lim_{t\to\infty} s(t) = 0$. A matrix A is *Hurwitz* if its eigenvalues have negative real parts. I_n is the identity and $0_{n_1\times n_2}$ is the $n_1 \times n_2$ zero matrix. Given a full-column rank matrix N_1, $N_1^{\dagger} = (N_1^{\top} N_1)^{-1} N_1^{\top}$ is its left pseudoinverse matrix; given a full-row rank matrix N_2, $N_2^{\dagger\dagger} = N_2^{\top}(N_2 N_2^{\top})^{-1}$ is its right pseudoinverse. Given a scalar function $s(t)$, $\mathbb{S}^{\kappa}(t)$ is the Taylor series vector, whose components pile $s(t)$ and its first derivatives up to the order κ, i.e. $\mathbb{S}^{\kappa}(t) = (s(t), \dot{s}(t), \dots, d^{\kappa} s(t)/dt^{\kappa})^{\top}$. Given a dynamic matrix A, an input matrix B, and an output matrix C, the κth order *observability matrix* associated with the pair (A, C) is recursively defined as $\mathcal{O}^{\kappa} = (C^{\top}, (\mathcal{O}^{\kappa-1} A)^{\top})^{\top}$, the κth order *invertibility matrix* $\mathcal{I}^{\kappa}$ associated with the triplet (A, B, C) is

$$\mathcal{I}^{\kappa} = \begin{pmatrix} 0 & 0 \\ \mathcal{O}^{\kappa-1} B & \mathcal{I}^{\kappa-1} \end{pmatrix}.$$

The dynamic model in (2) is said *invertible* from output y to δ, if δ is uniquely determined by the Taylor series vector, $\mathbb{Y}^{\kappa}$, of its output signal, for some κ. In this regard, denoting with $\mathcal{V}^{\kappa}$ the kth invertibility matrix associated with the triplet (A, W, C), the following two results hold (Sundaram & Hadjicostis, 2007):

Proposition 1. *System (2) is invertible from y to δ if, and only if, for some integer $k \le n$, the first υ columns of $\mathcal{V}^{\kappa}$ are linearly independent of each others and of the remaining columns of $\mathcal{V}^{\kappa}$, i.e. the columns $(0^{\top}, (\mathcal{V}^{\kappa-1})^{\top})^{\top}$. This occurs when*

$$\operatorname{rank}(\mathcal{V}^{\kappa}) - \operatorname{rank}(\mathcal{V}^{\kappa-1}) = \upsilon\,. \tag{3}$$

Proposition 2. *System (2) is strongly observable if, and only if, for some integer $\kappa \le n$, it holds*

$$\operatorname{rank}([\mathcal{O}^{\kappa}, \mathcal{V}^{\kappa}]) - \operatorname{rank}(\mathcal{V}^{\kappa}) = n\,. \tag{4}$$

The latter proposition ensures that all columns of $\mathcal{O}^{\kappa}$ are linearly independent of each other and of the columns of $\mathcal{V}^{\kappa}$. Starting from $\kappa = 0$, the existence and the value of κ are found iteratively by increasing its value until the invertibility condition in (3) is met; the system is not invertible if the condition is not satisfied for some $\kappa \le n$.

Finally, two definitions are helpful for this work purpose: a matrix F is said (A, C) *conditioned compliant* if there exists a matrix L such that $F = A - LC$; a dynamic system solving Problem 1 is termed a *LUIO* and its determination is the object of this work.

3. Design of linear unknown input-state observers

This section describes a general procedure to obtain a LUIO for the nonlinear model in (2) and, hence, to solve Problem 1. Such procedure uses only information about the first $\kappa + 1$ functions (from the order 0 to the κth order) of the Taylor series of the known input $u(t)$ and estimated noise-free output $\hat{\bar{y}}(t)$, which are piled into the vectors $\mathbb{U}^{\kappa}(t) = (u(t), \dot{u}(t), \dots, d^{\kappa} u(t)/dt^{\kappa})^{\top}$ and $\hat{\bar{\mathbb{Y}}}^{\kappa}(t) = (\hat{\bar{y}}(t), \dot{\hat{\bar{y}}}(t), \dots, d^{\kappa} \hat{\bar{y}}(t)/dt^{\kappa})^{\top}$, respectively. The estimated noise-free output $\hat{\bar{y}}(t)$, after having appropriately dealt with the measurement noise π, through a procedure that will be described later, ensuring that $\hat{\bar{y}}(t)$ converges to $\bar{y}(t) = C\,x(t)$ and, consequently, $\hat{\bar{\mathbb{Y}}}^{\kappa}$ converges to $\bar{\mathbb{Y}}^{\kappa}$, i.e. the Taylor Series Vector of the noise-free output.

Precisely, considering the factorization of the dynamic model in (2), one seeks a dynamic observer of the form

$$\begin{aligned} \dot{\hat{x}} &= A\,\hat{x} + B\,u + \varphi(\mathbb{U}^{\kappa}, \hat{\bar{\mathbb{Y}}}^{\kappa})\,, \\ \hat{y} &= \psi(\hat{x}, \mathbb{U}^{\kappa}, \hat{\bar{\mathbb{Y}}}^{\kappa}),\ \ \hat{\delta} = \theta(\hat{x}, \mathbb{U}^{\kappa}, \hat{\bar{\mathbb{Y}}}^{\kappa})\,, \end{aligned} \tag{5}$$

where $\hat{x}$ is an estimate of the system state, $\hat{\delta}$ is an estimate of the unknown input δ, $\hat{y}$ is the observer's output vector, and φ, ψ, and θ are functions to be defined. In accordance with this choice, the state estimation error, $\tilde{x} = x - \hat{x}$, evolves according to the dynamics

$$\begin{aligned}\dot{\tilde{x}} &= \dot{x} - \dot{\hat{x}} = \\ &= A\,x + B\,u + W\,\delta - A\,\hat{x} - B\,u - \varphi(\mathbb{U}^\kappa, \hat{\mathbb{Y}}^\kappa) = \\ &= A\,\tilde{x} + W\,\delta - \varphi(\mathbb{U}^\kappa, \hat{\mathbb{Y}}^\kappa)\,,\end{aligned}$$

whose convergence needs to be ensured by a suitable choice of the above functions. To this goal, inspired by Azid et al. (2021), one can write the expressions of the noise-free output $\bar{y}$ and its first κ derivatives, i.e.

$$\begin{aligned}\bar{y} &= Cx\,,\\ \dot{\bar{y}} &= CAx + CBu + CW\,\delta\,,\\ \ddot{\bar{y}} &= CA^2x + CABu + CB\,\dot{u} + CW\dot{\delta}\,,\\ &\vdots\\ \bar{y}^{(\kappa)} &= CA^\kappa x + \textstyle\sum_{i=0}^{\kappa-1} CA^{\kappa-1-i}\left(Bu^{(i)} + W\delta^{(i)}\right)\,,\end{aligned}$$

and then compactly write them as

$$\bar{\mathbb{Y}}^\kappa = \mathcal{O}^\kappa\,x + \mathcal{H}^\kappa\,\mathbb{U}^\kappa + \mathcal{V}^\kappa\,\Delta^\kappa\,, \tag{6}$$

where Δ^κ is the κth order Taylor series vector of δ, $\mathcal{H}^\kappa$ is the κth order invertibility matrix associated with the triplet (A, B, C), while $\mathcal{O}^\kappa$ and $\mathcal{V}^\kappa$ are the κth order observability and invertibility matrices previously defined.

Having said this, it is possible to prove the following first main result, describing when the sought LUIO exists and how φ, ψ, and θ must be chosen:

Theorem 1 (*Linear Unknown Input-state Observer (LUIO)*). *Given an integer $\kappa \le n$, the continuous-time dynamic system described by the equations*

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x} + Bu + \Phi(\hat{\mathbb{Y}}^\kappa - \mathcal{H}^\kappa\mathbb{U}^\kappa) + \Xi(\hat{\bar{y}} - \hat{y})\,,\\ \hat{y} &= C\hat{x}\,,\ \ \hat{\delta} = W^\dagger(\dot{\hat{x}} - A\hat{x} - Bu)\,,\end{aligned} \tag{7}$$

where $\hat{\bar{y}}$ and $\hat{\mathbb{Y}}^\kappa$ are estimates of the noise-free output and its corresponding κth order Taylor series vector is a LUIO for the model in (2), providing the estimates $\hat{x}$, $\hat{y}$, and $\hat{\delta}$ of the actual system state, output, and unknown input if, and only if, the following conditions hold:

- *(A1) (A, C) is observable (system observability),*
- *(A2) Ξ is such that $A - \Xi\,C = \Theta$, where Θ is a free Hurwitz matrix that is (A, C) conditioned compliant (free solution convergence),*
- *(A3) $\Phi\,\mathcal{V}^\kappa = (W, 0_{n\times c\kappa})$ (input decoupling),*
- *(A4) $\Phi\,\mathcal{O}^\kappa = 0_{n\times n}$ (residue cancellation),*
- *(A5) κ satisfies (3)–(4) (system invertibility and strong observability).*

Also, the dynamics of the state estimation error reads

$$\dot{\tilde{x}} = \Theta\,\tilde{x}\,. \tag{8}$$

Furthermore, under the hypothesis of high-frequency low-amplitude measurement noise and having defined vector

$$\rho = \begin{pmatrix}\int_0^t y(\tau)\,d\tau\\ \bar{\mathbb{Y}}^\kappa\end{pmatrix}$$

that includes also the integral of the system output y over the time, the dynamic system

$$\dot{\hat{\rho}} = P\hat{\rho} + EC_\rho(\rho - \hat{\rho})\,, \tag{9}$$

with $P = \begin{pmatrix}0_{\gamma\times c} & \mathbb{I}_\gamma\\ 0_{c\times c} & 0_{c\times\gamma}\end{pmatrix}$, $C_\rho = \begin{pmatrix}\mathbb{I}_c & 0_{c\times\gamma}\end{pmatrix}$, where $\gamma = c(\kappa+1)$, and E any matrix rendering $P - EC_\rho$ Hurwitz, provides robust estimates $\hat{\rho}$ of the output vector y and its first κ time derivatives, i.e. such that $\hat{\rho}$ converges to ρ. □

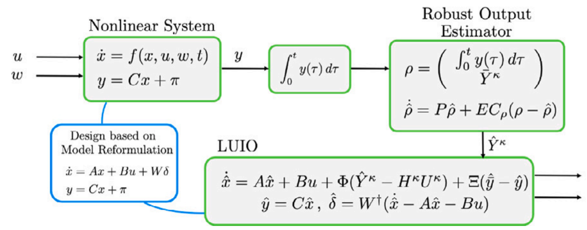

Fig. 1. Depiction of the proposed LUIO.

A depiction of the proposed LUIO is illustrated in Fig. 1. Before delving into the proof, let us clarify the meaning of the above conditions in the following:

Remark 1. Condition (A1) ensures that the system is observable and then all the eigenvalues of the observer can be placed at will through a feedback matrix Ξ. Condition (A2) specifies that Ξ has a structure allowing to obtain the exact desired dynamics described by matrix Θ, for the state estimation error. Conditions (A3) and (A4) decouple the state estimation error dynamics from the term depending on the κth observability matrix, i.e. vector $\mathcal{O}^\kappa\,x$ in (6), and the unknown input, respectively. Finally, (A5) implies that all the columns of the observability matrix $\mathcal{O}^\kappa$ are linearly independent of each other and of all columns of the invertibility matrix $\mathcal{V}^\kappa$, as shown later; it guarantees that the linear equations appearing in (A2) and (A3) have a solution, which in turn involves the existence of the solving matrix Φ.

Proof of Theorem 1. The result is proven in four steps. First, a method to deal with the noise signal $\pi(t)$ is designed; Secondly, a procedure allowing the asymptotic estimation of the κth order Taylor series vector $\bar{\mathbb{Y}}^\kappa$ is found; Third, a dynamic estimator ensuring the state estimation signal $\hat{x}(t)$ converge to $x(t)$; Finally, an input recovery rule reconstructing the unknown input vector $\delta(t)$ is derived.

Step 1 *- Estimation of the κth Taylor Series Vector of Noise-free System Output*. As commonly done in signal theory, the measurement noise π is written as the sum of Z sinusoidal signals, i.e.

$$\pi(t) = \textstyle\sum_{i=1}^Z a_i\,\sin(\omega_i\,t + \phi_i) \tag{10}$$

where a_i, ω_i and ϕ_i are the amplitude, frequency, and phase of the ith signal. Then, the system output in (2) reads

$$\begin{aligned}y(t) &= \bar{y}(t) + \pi(t) = \\ &= Cx(t) + \textstyle\sum_{i=1}^Z a_i\,\sin(\omega_i\,t + \phi_i)\,.\end{aligned} \tag{11}$$

It should be noted here that the integral over the time of the system output y practically coincides with that of the noise free output $\bar{y}$. To show this, consider the integral $\Pi_i(t)$ of the ith term composing the noise signal, i.e. the function

$$\Pi_i(t) = \tfrac{a_i}{\omega_i}\int\sin(\omega_i t + \phi_i) = -\tfrac{a_i}{\omega_i}\cos(\omega_i t + \phi_i)\,.$$

Given an interval $[0, t)$, one has

$$\begin{aligned}\int_0^t \pi(\tau)\,d\tau &= -\textstyle\sum_{i=1}^Z \tfrac{a_i}{\omega_i}\cos(\omega_i\,\tau + \phi_i)\Big|_0^t = \\ &= \textstyle\sum_{i=1}^Z \tfrac{a_i}{\omega_i}(\cos(\phi_i) - \cos(\omega_i\,t + \phi_i)) = \\ &= 2\textstyle\sum_{i=i}^Z \tfrac{a_i}{\omega_i}\sin(\phi_i + \omega_i t/2)\sin(\omega_i t)\,,\end{aligned}$$

where $\cos(\phi_i) - \cos(\omega_i t + \phi_i)$ has been compacted via a prosthaphaeresis formula. Then, the quantity $\int_0^t y(\tau)\,d\tau$ reads

$$\begin{aligned}&\int_0^t \bar{y}(\tau)\,d\tau + \int_0^t \pi(\tau)\,d\tau = \\ &= \int_0^t \bar{y}(\tau)\,d\tau + 2\textstyle\sum_{i=i}^Z \tfrac{a_i}{\omega_i}\sin(\phi_i + \omega_i t/2)\sin(\omega_i t)\,,\end{aligned}$$

Evaluating the Euclidean norm of the above formula yields

$$\begin{aligned}\left\|\int_0^t y(\tau)\,d\tau\right\|_2 &= \left\|\int_0^t \bar{y}(\tau)\,d\tau\right\|_2 + \left\|\int_0^t \pi_i(\tau)\,d\tau\right\|_2 \le \\ &\le \left\|\int_0^t \bar{y}(\tau)\,d\tau\right\|_2 + 2\textstyle\sum_{i=1}^Z |a_i|/|\omega_i|\,.\end{aligned}$$

As typical measurement noise signals have *low-amplitudes and high-frequencies* (cf. e.g. Khalil & Praly, 2014), it holds $a_i \ll \omega_i$ or, equivalently, $a_i/\omega_i \approx 0$, by which

$$\left\| \int_0^t y(\tau)\, d\tau \right\|_2 \approx \left\| \int_0^t \bar{y}(\tau)\, d\tau \right\|_2 .$$

Thereby, the effect of the noise components on the integral of the output signal is negligible, since it holds $\int_0^t y(\tau)\, d\tau \approx \int_0^t \bar{y}(\tau)\, d\tau$.

Motivated by this, it is helpful for our purpose to define a new vector ρ, whose components pile the time integrals of the c components of the system output $y(t)$ and the κth Taylor series vector of the noise-free output $\bar{y}(t)$, i.e.

$$\rho = \begin{pmatrix} \rho_c \\ \bar{\mathbb{Y}}^\kappa \end{pmatrix} \in \mathbb{R}^\mu , \quad \text{with } \rho_c = \int_0^t y(\tau)\, d\tau \in \mathbb{R}^c , \tag{12}$$

with $\mu = c + \gamma$ and $\gamma = c(\kappa+1)$. In deriving the dynamics of ρ, it should be observed that it is perturbed by two unknown yet bounded signals, $\eta \in \mathbb{R}^c$ and π. Signal η is the effect of the change rates of the $(\kappa+1)$th derivatives of $\bar{y}(t)$, which is, in general, not negligible. The dynamics of ρ can then be written as

$$\begin{aligned} \dot{\rho} &= P\rho + P_b \eta + P_n \pi \\ y_\rho &= C_\rho \rho , \end{aligned} \tag{13}$$

where

$$P = \begin{pmatrix} 0_{\gamma\times c} & \mathbb{I}_\gamma \\ 0_{c\times c} & 0_{c\times\gamma} \end{pmatrix} , \; P_b = \begin{pmatrix} 0_{\gamma\times c} \\ \mathbb{I}_c \end{pmatrix} , \; P_n = \begin{pmatrix} \mathbb{I}_c \\ 0_{\gamma\times c} \end{pmatrix} ,$$
$$C_\rho = \begin{pmatrix} \mathbb{I}_c & 0_{c\times\gamma} \end{pmatrix} .$$

Noticing that ρ is observable by construction, one can design a Luenberger-like state observer of the form

$$\begin{aligned} \dot{\hat{\rho}} &= P\hat{\rho} + E\,(y_\rho - \hat{y}_\rho) , \\ \hat{y}_\rho &= C_\rho \hat{\rho} , \end{aligned} \tag{14}$$

where $\hat{\rho} = (\hat{\rho}_c^\top, \hat{\mathbb{Y}}^{\kappa\top})^\top$ is the estimated vector and E is a free design matrix to be chosen so that the estimation error $\tilde{\rho} = \rho - \hat{\rho}$, whose components are $\tilde{\rho}_c$ and $\tilde{\mathbb{Y}}^\kappa$, is convergent. Direct computation of the dynamics yields of $\tilde{\rho}$ gives

$$\begin{aligned} \dot{\tilde{\rho}} &= \dot{\rho} - \dot{\hat{\rho}} = \\ &= P\rho - P\hat{\rho} - EC\rho + EC\hat{\rho} - P_b\eta - P_\pi \pi = \\ &= \tilde{P}\tilde{\rho} - P_b\,\eta - P_n \pi , \end{aligned} \tag{15}$$

in which $\tilde{P} = P - EC$ is a matrix that has to be made Hurwitz through a suitable choice of E. This would ensure that $\tilde{\rho}(t)$ is asymptotically bounded signal. To prove this, one has to show that the effect of the unknown signals η and π are negligible. Assume that a suitable matrix E, whose existence is ensured by the observability property of ρ, has been chosen so that $\tilde{P}$ is Hurwitz and its eigenvalues are distinct, i.e. they are located at μ different positions. By algebra theory, there exist μ linearly independent eigenvectors, associated with the chosen eigenvalues, with which one can construct the so-called modal matrix T. Such matrix can be used to diagonalize $\tilde{P}$ by the change of coordinate $\epsilon = T\,\tilde{\rho}$, so that the linear dynamics in (15) reads

$$\begin{aligned} \dot{\epsilon} &= T\tilde{P}T^{-1}\epsilon - T^{-1}P_b\,\eta - T^{-1}P_n\,\pi = \\ &= \Lambda\,\epsilon - T^{-1}P_b\,\eta - T^{-1}P_n\,\pi , \end{aligned} \tag{16}$$

where Λ is diagonal and its nonzero entries are the chosen eigenvalues. Now, from a boundedness point of view, the worst case happens when the two unknown signals constantly take on their absolute maximum values, i.e. $\eta(t) = \eta_m$ and $\pi(t) = \pi_m$. In this case, the new state ϵ converges to the equilibrium point

$$\epsilon_q = -\Lambda^{-1}(\Gamma\,\eta_m + \Delta\,\pi_m) , \tag{17}$$

with $\Gamma = T^{-1}P_b$ and $\Delta = T^{-1}P_n$. The Euclidean norm of (17) is upper-bounded as follows:

$$\begin{aligned} \|\epsilon_q\|_2 &= |\lambda_m| \left(\|\Gamma\|_2 |\eta_m| + \|\Delta\|_2 |\pi_m| \right) \le \\ &\le |\lambda_m| \left(\|\Gamma\|_2 |\eta_m| + \|\Delta\|_2 \textstyle\sum_{i=1}^Z |a_i| \right) \end{aligned} \tag{18}$$

where λ_m is the largest eigenvalues of Λ^{-1} and having considered from (10) that

$$\|\pi\|_2 = \textstyle\sum_{i=1}^Z \|a_i \sin(\omega_i t + \phi_i)\|_2 \le \sum_{i=1}^Z |a_i| .$$

Now, being Λ diagonal, $\lambda_m = \frac{1}{\lambda_M}$, where λ_M is the largest eigenvalue of Λ. Then, the maximum value of the unknown input η_m corresponds to known maximum values of the first $(\kappa+1)$th time derivatives of the output signal $y(t)$; also, considering that typical measurement noise signals are characterized by low and bounded amplitudes and high frequencies (cf. e.g. Khalil & Praly, 2014), (18) suggests that, choosing a suitable low $|\lambda_m|$, which corresponds to a large $|\lambda_M|$, the effects of η_m and π_m can be made negligible. This choice implies that $\|\epsilon_q\|_2 \approx 0$, that, in turn, implies that the norm of the state trajectories satisfies $\|\epsilon\|_2 \le \|\epsilon_q\|_2 \approx 0$ and, hence, $\tilde{\rho} \approx 0$.

Step 2 - *Estimation of System State and Unknown Inputs*. We can now move on to find the dynamic model that allows a convergent estimation of the state signal $x(t)$. To this end, consider first the Lyapunov candidate

$$V(\tilde{\rho}, \tilde{x}) = \tilde{\rho}^\top S\,\tilde{\rho} + \tilde{x}^\top N\,\tilde{x} , \tag{19}$$

where S and N are suitable positive definite matrices, and try to show that the convergence to zero of the estimation errors, $\tilde{\rho}$ and $\tilde{x}$, is decoupled. Towards this goal one can evaluate the time derivative of (19) that reads:

$$\begin{aligned} \dot{V} &= \frac{\partial V}{\partial \tilde{\rho}}\dot{\tilde{\rho}} + \frac{\partial V}{\partial \tilde{x}}\dot{\tilde{x}} = \\ &= 2\,\tilde{\rho}^\top S\,\dot{\tilde{\rho}} + 2\,\tilde{x}^\top N\,\dot{\tilde{x}} = \\ &= 2\tilde{\rho}^\top S\,\tilde{P}\,\tilde{\rho} + 2\,\tilde{x}^\top N\,\dot{\tilde{x}} = \\ &= \tilde{\rho}^\top (S\,\tilde{P} + \tilde{P}^\top S)\,\tilde{\rho} + 2\,\tilde{x}^\top N\,\dot{\tilde{x}} , \end{aligned} \tag{20}$$

which needs to be negative definite to establish our result. The first addend of (20) can be made negative definite with respect to ρ if $\tilde{P}$ solves the Lyapunov equation

$$S\tilde{P} + \tilde{P}^\top S = -Q_{\tilde{\rho}} \tag{21}$$

for a positive definite matrix $Q_{\tilde{\rho}}$. Also, it is known that such matrix is Hurwitz, unique, and given by:

$$S = \int_0^\infty e^{\tilde{P}^\top t} Q_{\tilde{\rho}}\, e^{\tilde{P}t}\, dt . \tag{22}$$

Using (21) gives

$$\frac{\partial V}{\partial \tilde{\rho}}\dot{\tilde{\rho}} = -\tilde{\rho}^\top Q_{\tilde{\rho}}\,\tilde{\rho} . \tag{23}$$

Step 3 - *Dynamic State Estimator*. The proof continues by referring to the observer's form in (5) and trying to find an affine form for the function $\varphi(\mathbb{U}^\kappa, \hat{\mathbb{Y}}^\kappa)$, which decouples the dynamics of the state estimation error from the unknown signal $\delta(t)$. Namely, one can choose

$$\begin{aligned} \varphi(\mathbb{U}^\kappa, \hat{\mathbb{Y}}^\kappa) &= \Phi_1\,\hat{\mathbb{Y}}^\kappa + \Phi_2\,\mathbb{U}^\kappa + \nu = \\ &= \Phi_1\left(\bar{\mathbb{Y}}^\kappa - \tilde{\mathbb{Y}}^\kappa\right) + \Phi_2\,\mathbb{U}^\kappa + \nu \end{aligned}$$

where $\hat{\mathbb{Y}}^\kappa = \bar{\mathbb{Y}}^\kappa - \tilde{\mathbb{Y}}^\kappa$ has been substituted, Φ_i, for $i = 1, 2$, are free matrices, and ν is a free signal. It can be already stated, by virtue of (23), that the error $\tilde{\rho}$ goes to zero, which in turn implies that $\tilde{\mathbb{Y}}^\kappa \to 0$ and consequently $\hat{\mathbb{Y}}^\kappa = \bar{\mathbb{Y}}^\kappa$, leading for the state estimation dynamics to the following:

$$\begin{aligned} \dot{\tilde{x}} &= A\,\tilde{x} + W\,\delta - \Phi_1\,\bar{\mathbb{Y}}^\kappa - \Phi_2\,\mathbb{U}^\kappa - \nu = \\ &= A\,\tilde{x} - \Phi_1(\mathcal{O}^\kappa x + \mathcal{H}^\kappa\,\mathbb{U}^\kappa + \mathcal{V}^\kappa\,\Delta^\kappa) - \Phi_2\,\mathbb{U}^\kappa + \\ &\quad + W\delta - \nu , \end{aligned}$$

where (6) has been used. This suggests choosing $\Phi_2 = -\Phi_1\,\mathcal{H}^\kappa$, so that the following term collection can be done in the state estimation dynamics:

$$\dot{\tilde{x}} = A\,\tilde{x} - \Phi_1(\mathcal{O}^\kappa x + \mathcal{V}^\kappa\Delta^\kappa) + W\delta - \nu . \tag{24}$$

Now, it can be seen that the behavior of the solution $\tilde{x}(t)$ of the differential equation above *undesirably* depends on the unknown input vector δ, its κth vector of the Taylor series Δ^κ, and the initial state estimation error $\tilde{x}(0)$. In contrast, a desired behavior for the state estimation is that of (8), with Θ a free Hurwitz matrix that is (A, C) conditioned compliant. By comparing (24) with (8), one can see that the above property is ensured if

$$A\,\tilde{x} - \Phi_1(\mathcal{O}^\kappa x + \mathcal{V}^\kappa \Delta^\kappa) + W\delta - v = \Theta\,\tilde{x}\,.$$

A possible way is to choose v as in the standard Luenberger's approach, i.e. $v = \Xi(\hat{y} - \bar{y})$, where Ξ is a free output injection matrix. Then, computing the estimated output vector $\hat{y}$ through the function $\psi(\hat{x}, \mathbb{U}^\kappa, \hat{\mathbb{Y}}^\kappa) = C\,\hat{x}$ (refer again to the observer's form in (5)) makes the left-hand side of the above expression equal to

$$A\,\tilde{x} - \Phi_1(\mathcal{O}^\kappa x + \mathcal{V}^\kappa \Delta^\kappa) + W\delta - \Xi\,\left(\bar{y} - \tilde{y}_c\right) + \Xi\,C\,\hat{x}$$

in which the substitution $\hat{\bar{y}} = \bar{y} - \tilde{y}_c$ has been done considering that $\tilde{y}_c$ is the vector composed of the first c components of $\tilde{\rho}$. By virtue of the fact that $\tilde{\rho} \to 0$, which implies that $\tilde{y}_c \to 0$, and $\bar{y} = C\,x$, we come to

$$(A - \Xi\,C)\,\tilde{x} - \Phi_1(\mathcal{O}^\kappa x + \mathcal{V}^\kappa \Delta^\kappa) + W\delta. \tag{25}$$

By comparing now (25) with the right-hand side of (8), the following two conditions are found:

$$\begin{aligned} A - \Xi\,C &= \Theta\,, \\ -\Phi_1(\mathcal{O}^\kappa x + \mathcal{V}^\kappa \Delta^\kappa) + W\delta &= 0\,. \end{aligned}$$

The existence of a matrix Ξ satisfying the equation in the first row, for some Hurwitz matrix Θ that is conditioned compliant, is ensured by (A, C) being observable (Condition A1). As for the equation in the second row, observing that δ is the first elements of vector Δ^κ, the second-row equation becomes

$$\Phi_1\,\mathcal{O}^\kappa x + \left(\Phi_1\,\mathcal{V}^\kappa - \left(W, 0_{n\times c\kappa}\right)\right)\Delta^\kappa = 0\,.$$

Now, as the above equation must be satisfied for all x and all Δ^κ, it follows that it must be

$$\Phi_1\,\mathcal{O}^\kappa = 0_{n\times n}\,,\ \text{and}\ \Phi_1\,\mathcal{V}^\kappa = \left(W, 0_{n\times c\kappa}\right)\,.$$

Now, putting all together one obtains $\varphi(\hat{\mathbb{Y}}^\kappa, \mathbb{U}^\kappa) = \Phi_1(\hat{\mathbb{Y}}^\kappa - \mathcal{H}^\kappa \mathbb{U}^\kappa) - \Xi(\hat{\bar{y}} - \hat{y})$, or equivalently $\varphi(\hat{\mathbb{Y}}^\kappa, \mathbb{U}^\kappa) = \Phi_1(\hat{\mathbb{Y}}^\kappa - \mathcal{H}^\kappa \mathbb{U}^\kappa) - \Xi(\bar{y} - \hat{y})$, and finally, renaming Φ_1 as Φ, the relations in the above formula becomes the ones expressed in Condition A2 and A3. Note now that the two conditions can be stacked vertically in the equation system

$$\Psi^\top \Phi^\top = Y^\top\,, \tag{26}$$

where $\Psi^\top = \begin{pmatrix} \mathcal{V}^{\kappa\top} \\ \mathcal{O}^{\kappa\top} \end{pmatrix}$ and $Y^\top = \begin{pmatrix} W^\top \\ 0_{c\kappa\times n} \\ 0_{n\times n} \end{pmatrix}$. The above system admits solution in terms of $\Phi^\top$ if, and only if,

$$\operatorname{rank}\left(\Psi^\top\right) = \operatorname{rank}\left(\Psi^\top | Y^\top\right) \tag{27}$$

From the hypothesis of strong observability of the system (4), the rows of matrix $\Psi^\top$ are linearly independent. Furthermore, the invertibility condition (3) ensures that the first v rows of matrix $\mathcal{V}^{\kappa\top}$ are linearly independent of all other rows of it and of each other. On the other hand, the unknown input matrix $W \in \mathbb{R}^{n\times v}$ is full-column rank for hypothesis and has a rank equal to v and, consequently, its transpose $W^\top \in \mathbb{R}^{v\times n}$ is a full-row rank matrix with rank also equal to v. Considering that the remainder of $Y^\top$ is composed by null values and recalling the shape of matrix $\mathcal{V}^{\kappa\top}$, this implies that condition (27) is satisfied if and only if the rows of $W^\top$ are in the span of the rows of sub-matrix $W^\top\,\mathbb{O}^{\kappa-1\top}$ and, the latter, if always satisfied by virtue of the invertibility condition (3) from which holds that $\operatorname{rank}\left(W^\top\,\mathbb{O}^{\kappa-1\top}\right) = \operatorname{rank}\left(W^\top\right)$. In this setting a solution in terms of $\Phi^\top$, and consequently Φ, exists and we can continue the proof. Having satisfied all the above conditions, the state estimation dynamics is rendered convergent by a suitable choice of Ξ.

Now, using the state estimation error dynamics in (8), the second addend of the time derivative in (20) becomes

$$\frac{\partial V}{\partial \tilde{x}}\dot{\tilde{x}} = 2\,\tilde{x}^\top N\,\dot{\tilde{x}} = 2\tilde{x}^\top N\,\Theta\,\tilde{x} = \tilde{x}^\top(N\,\Theta + \Theta^\top N)\,\tilde{x}\,, \tag{28}$$

which can be made negative definite with respect to $\tilde{x}$ by choosing

$$N = \int_0^\infty e^{\Theta^\top t} Q_x\, e^{\Theta t}\,dt \tag{29}$$

where Q_x is a positive definite matrix; such matrix N is the unique, positive definite solution of the Lyapunov equation $N\,\Theta + \Theta^\top N = -Q_x$, and it exists since Θ is Hurwitz by design. This implies that

$$\frac{\partial V}{\partial \tilde{x}}\dot{\tilde{x}} = -\tilde{x}^\top Q_x\,\tilde{x}^\top\,. \tag{30}$$

Substituting the matrices N and S obtained from (22)–(29) in the time derivative (20) gives

$$\dot{V} = -\tilde{\rho}^\top Q_{\tilde{\rho}}\,\tilde{\rho} - \tilde{x}^\top Q_x\,\tilde{x}\,, \tag{31}$$

which proves the overall convergence of $\tilde{\rho}$ and $\tilde{x}$.

Step 4 *- Unknown Input Reconstruction*. Now that the state estimate $\hat{x}(t)$ is guaranteed to asymptotically converge to $x(t)$, it can be used to retrieve the input estimate signal $\hat{\delta}(t)$. Precisely, from (2), one first gets $W\delta = \dot{x} - Ax - Bu$. Since W is full column rank by construction, one can multiply both members of the above equation by W's left-pseudoinverse, i.e.

$$(W^\top W)^{-1} W^\top W\,\delta = (W^\top W)^{-1} W^\top(\dot{x} - A\,x - B\,u)\,,$$

and obtain $\delta = (W^\top W)^{-1} W^\top(\dot{x} - A\,x - B\,u)$; then, since $x(t) \simeq \hat{x}(t)$, the state vector can be replaced with $\hat{x}(t)$ and its time derivative can be derived by assuming a similar form for the observer's dynamics, i.e. $\dot{\hat{x}} = A\,\hat{x} + B\,u + W\,\hat{\delta}$. Referring for the final time to the observer's form in (5), this leads to the choice of $\theta(\hat{x}, \mathbb{U}^\kappa, \hat{\mathbb{Y}}^\kappa) = W^\dagger(\dot{\hat{x}} - A\,\hat{x} - B\,u)$, which is the formula in the third row of (7). If so, the input estimation error becomes

$$\begin{aligned} \tilde{\delta} = \delta - \hat{\delta} &= W(\dot{x} - A\,x + B\,u - \dot{\hat{x}} + A\,\hat{x} - B\,u) = \\ &= W(\dot{\tilde{x}} - A\,\tilde{x})\,, \end{aligned}$$

whose convergence to zero is ensured by $\tilde{x}(t)$ being convergent. Finally, it can be noticed that the convergence speed of $\tilde{\delta}$ follows that of $\tilde{x}$, which concludes the proof. □

4. Application to soft articulated robots

The proposed method is applied to a SAR setup. After recalling the nonlinear model of the robot and exploiting the information deriving from the measured output and input vectors, the existence of the LUIO is tested and the observer is designed; then, simulations show the effectiveness of the method and its superiority to state-of-the-art solutions (cf. e.g. Sundaram & Hadjicostis, 2007); finally, the hardware of the experimental setup is described and the attained results are reported to validate the real applicability of the method. It should be emphasized that in both cases the design and structure of the controller will be seen as a black box.

4.1. Existence and design of the LUIO

A two-degree-of-freedom Soft Articulated Robot (SAR), with joint configuration vector $q = (q_1, q_2)^\top$ and input torque vector $\tau = (\tau_1, \tau_2)^\top$, is described by the nonlinear model

$$M(q)\,\ddot{q} + h(q, \dot{q}) = \tau\,, \tag{32}$$

where $M(q) = \{M_{ij}(q)\}$, for $i, j = 1, 2$, is the inertia matrix and $h(q, \dot{q}) = (h_1(q), h_2(q))^\top$ is a vector piling all Coriolis, centrifugal, and

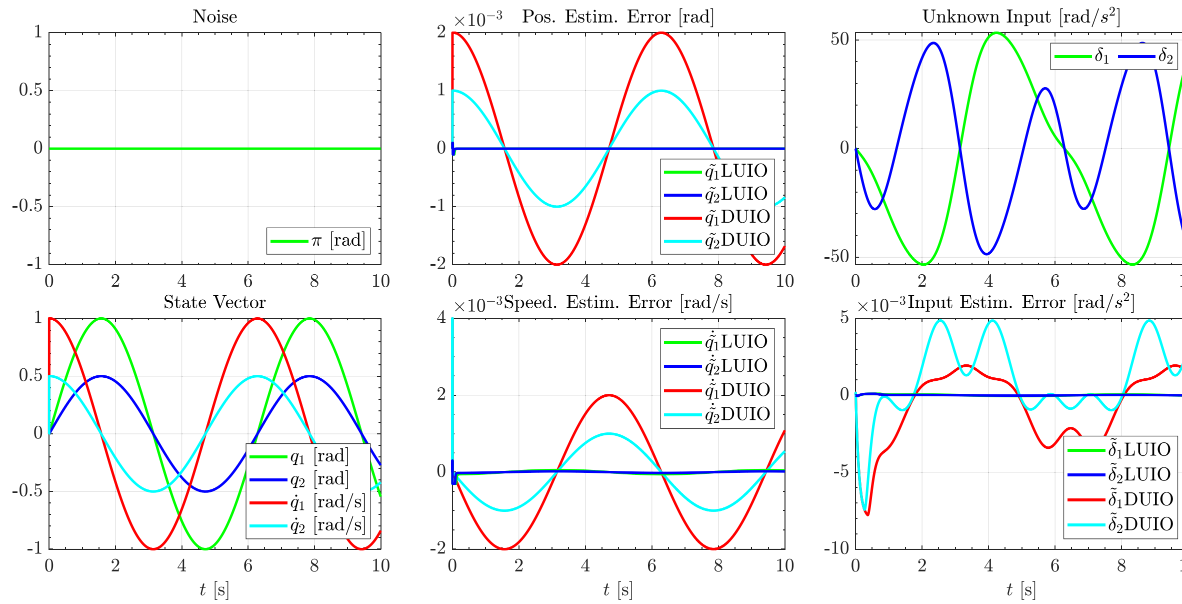

Fig. 2. Simulation results with a two-degree-of-freedom SAR. A LUIO and a DUIO estimate the full state and unknown inputs by using only link position information q and known inputs τ. The proposed linear observer reconstructs faster and more precisely both the system's state and unknown inputs.

gravity moments (Siciliano, Khatib, & Kröger, 2008); more precisely, it holds:

$$M_{11}(q) = J_1 + m_1 l_1^2/4 + J_2 + m_2(l_1^2 + l_2^2/4 + l_1 l_2 c_2)\,,$$
$$M_{12}(q) = J_2 + m_2(l_2^2/4 + l_1 l_2 c_2/2)\,,$$
$$M_{22}(q) = J_2 + m_2 l_2^2/2\,,$$

and

$$h_1(q) = -\tfrac{1}{2} m_2 l_1 s_2 \dot q_1 \dot q_2 - \tfrac{1}{2} m_2 l_1 s_2 (\dot q_1 + \dot q_2)\dot q_2 + \\ + g s_1(\tfrac{1}{2} m_1 l_1 + m_2 l_1) + \tfrac{1}{2} m_2 l_2\, g s_{12}\,,$$
$$h_2(q) = \tfrac{1}{2} m_2 l_1 s_2 \dot q_1^2 + \tfrac{1}{2} m_2 l_2\, g s_{12}\,,$$

in which J_i, m_i, l_i are the inertia, the mass and the length of the ith link, g is the gravity acceleration, and where the shorthand $c_i = \cos(q_i)$, $s_i = \sin(q_i)$, and $s_{ij} = \sin(q_i + q_j)$ have been used. Every joint configuration is assumed to be directly measured.

Following the strategy described in Section 2, one can choose the state vector $x = (q^\top, \dot q^\top)^\top$, the input vector $u = \tau$ and express the system model as in (1) with $f(x,u,w) = \big(f_1(x,u,w), f_2(x,u,w), f_3(x,u,w), f_4(x,u,w)\big)^\top$, where

$$f_1(x,u,w) = x_3, \quad f_2(x,u,w) = x_4,$$
$$\zeta(x,u,w) = \begin{pmatrix} f_3(x,u,w) \\ f_4(x,u,w) \end{pmatrix} = M(q)^{-1}\,(\tau - h(q,\dot q))$$

Then, in order to separate linear and nonlinear terms and reach the form of (2), one can consider the nominal expression of the inertia matrix, namely $\bar M(q)$, i.e. the typical inertia matrix of a two-degree soft articulated robot composed (Pedone et al., 2022) via the parameters reported in Table 1, to obtain

$$\tilde f(x,u,w) = \big(\tilde f_1(x,u,w), \tilde f_2(x,u,w), \tilde\zeta(x,u,w)^\top\big) = \\ = \begin{pmatrix} 0_{2\times 1} \\ -M(q)^{-1}\,h(q,\dot q) - \big(M(q)^{-1} - \Delta M(q)\big)\,\tau \end{pmatrix}$$

where $\Delta M(q) = M(q)^{-1} - \bar M(q)^{-1}$, and hence leads to the following system

$$\begin{aligned} \dot x &= Ax + Bu + \tilde f(x,u,w) \\ y &= Cx + \pi\,, \end{aligned} \tag{33}$$

where π is the unknown measurement noise vector and

$$A = \begin{pmatrix} 0_{2\times2} & \mathbb{I}_2 \\ 0_{2\times2} & 0_{2\times2} \end{pmatrix}, \; B = \begin{pmatrix} 0_{2\times2} \\ \bar M(q)^{-1} \end{pmatrix}, \; C = \begin{pmatrix} \mathbb{I}_2 & 0_{2\times2} \end{pmatrix},$$

Finally, collecting the unknown term $\tilde f(x,u,w)$ in a suitable unknown vector $\delta \in \mathbb{R}^2$, we obtain to the following final form

$$\dot x = Ax + Bu + W\,\delta\,, \tag{34}$$

where $W = \begin{pmatrix} 0_{2\times2} \\ \mathbb{I}_2 \end{pmatrix}$.

Moreover, to verify the existence of a LUIO, one has first to check the existence of an integer κ satisfying Condition A1 and A5. For this purpose, it is worth saying that the system model is observable with $\kappa = 2$ (indeed, $\mathcal{O}^1 = (C^\top, (CA)^\top)^\top = \mathbb{I}_4$) and thus Condition A1 is met. Furthermore, Condition A5 (or equivalently (3) and (4)) is met also for $\kappa = 2$ and with $\upsilon = 2$. Direct computation yields

$$\mathcal{V}^1 = \begin{pmatrix} 0 & 0 \\ CW & 0 \end{pmatrix} = 0_{4\times4}\,,$$
$$\mathcal{V}^2 = \begin{pmatrix} 0 & 0 & 0 \\ CW & 0 & 0 \\ CAW & CW & 0 \end{pmatrix} = \begin{pmatrix} 0_{4\times2} & 0_{4\times4} \\ \mathbb{I}_2 & 0_{2\times4} \end{pmatrix},$$

and

$$\mathbb{O}^2 = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix} = \begin{pmatrix} \mathbb{I}_4 \\ 0_{2\times4} \end{pmatrix}$$

from which $\operatorname{rank}\mathcal{V}^2 - \operatorname{rank}\mathcal{V}^1 = \upsilon = 2$ and $\operatorname{rank}(\mathcal{O}^2, \mathcal{V}^2) - \operatorname{rank}V^1 = n = 4$. Therefore, the minimum integer satisfying the LUIO existence is $\kappa = 2$, which also means that the first $\upsilon = 2$ columns of $\mathcal{V}^2$ are linearly independent. The LUIO design can now proceed by solving the matrix system $\Psi^\top \Phi^\top = \Upsilon^\top$ in (26), which, for (34), reads

$$\begin{pmatrix} \mathbb{V}^{2\top} \\ \mathbb{O}^{2\top} \end{pmatrix} \Phi^\top = \begin{pmatrix} W^\top \\ 0_{4\times4} \\ 0_{4\times4} \end{pmatrix},$$

and thus

$$\begin{pmatrix} 0_{2\times4} & \mathbb{I}_2 \\ 0_{4\times4} & 0_{4\times2} \\ \mathbb{I}_4 & 0_{4\times2} \end{pmatrix} \Phi^\top = \begin{pmatrix} 0_{2\times2} & \mathbb{I}_2 \\ 0_{8\times2} & 0_{8\times2} \end{pmatrix}. \tag{35}$$

Direct inspection of the involved matrices shows that the first matrix on the left side of (35) is full column rank. This allows finding

$$\Phi^\top = (\Psi^\top\Psi)^{-1}\Psi^\top\Upsilon^\top = \\ = \begin{pmatrix} 0_{4\times6} & \mathbb{I}_4 \\ \mathbb{I}_2 & 0_{2\times8} \end{pmatrix} \begin{pmatrix} 0_{2\times2} & \mathbb{I}_2 \\ 0_{8\times2} & 0_{8\times2} \end{pmatrix} = \begin{pmatrix} 0_{4\times2} & 0_{4\times2} \\ 0_{2\times2} & \mathbb{I}_2 \end{pmatrix},$$

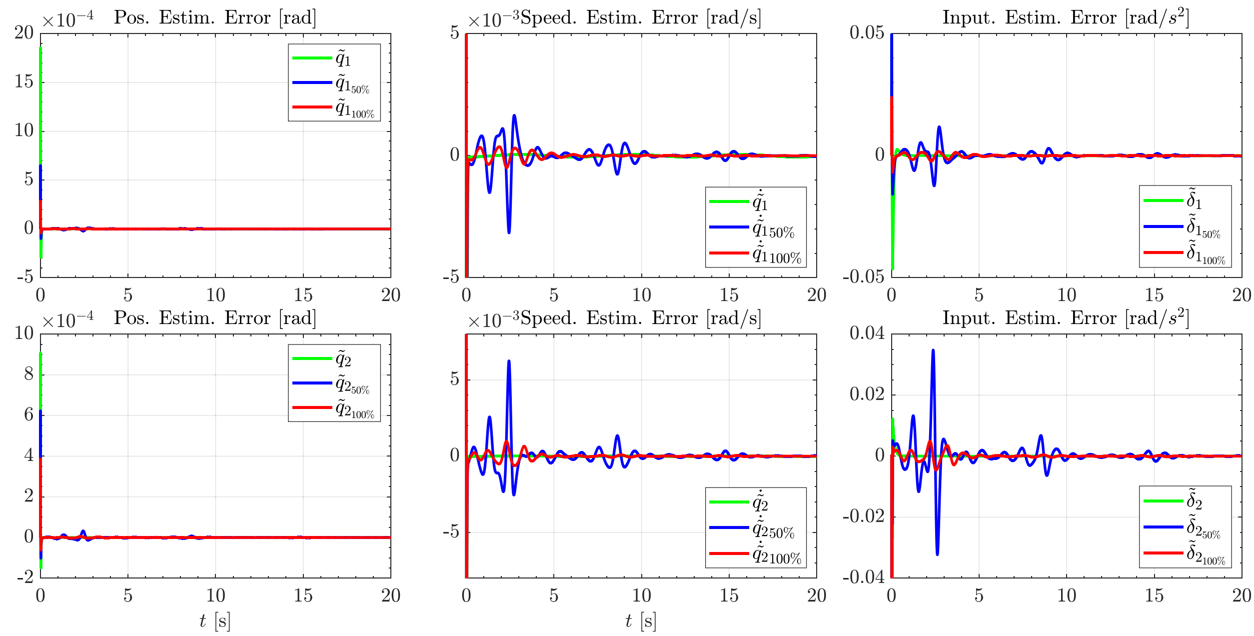

Fig. 3. Simulation results with a two-degree-of-freedom SAR with a parameter variations until 100% with respect to the nominal ones. The proposed linear observer reconstructs system state and unknown inputs with almost identical performance in all simulations conducted with estimation errors convergent to zero for $t \to \infty$.

Table 1
Nominal parameters of the soft articulated robot.

Link masses	$m_1 = m_2 = 0.45$ kg
Link inertia	$I_1 = I_2 = 0.0045$ kg m^2
Link lengths	$l_1 = l_2 = 0.1$ m
Agonistic	$k_a = 0.0026$ N m, $a_a = 8.9995$ rad^{-1}
Antagonistic	$k_b = 0.0011$ N m, $a_b = 8.9989$ rad^{-1}

and finally $\Phi = \begin{pmatrix} 0_{2\times4} & 0_2 \\ 0_{2\times4} & \mathbb{I}_2 \end{pmatrix}$.

Once the existence of a LUIO for system (33) has been demonstrated, the next step is to deal with the measurement noise signals. Following the procedure in Section 3, we define a new vector

$$\rho = \begin{pmatrix} \rho_c \\ \mathbb{Y}^\kappa \end{pmatrix} = \left(\int_0^t x_1 d\tau, \int_0^t x_2 d\tau, x_1, x_2, \dot{x}_1, \dot{x}_2, \ddot{x}_1, \ddot{x}_2 \right),$$

whose dynamics is in the form of (13), with

$$P = \begin{pmatrix} 0_{6\times2} & \mathbb{I}_6 \\ 0_{2\times2} & 0_{2\times6} \end{pmatrix}, \; P_b = \begin{pmatrix} 0_{6\times2} \\ \mathbb{I}_2 \end{pmatrix}, \; P_n = \begin{pmatrix} \mathbb{I}_2 \\ 0_{6\times2} \end{pmatrix},$$
$$C_\rho = \begin{pmatrix} \mathbb{I}_2 & 0_{2\times6} \end{pmatrix}.$$

As discussed in Section 3, the existence of a matrix E ensuring the estimation law in (14) is guaranteed by the above system being observable.

4.2. Simulation and comparison with existing solutions

The proposed solution is compared with a DUIO whose implementation follows the design procedure in Sundaram and Hadjicostis (2007) after replacing delays with derivatives. To make the results comparable, the eigenvalues of both estimators have the same position, namely in $p = (-100, -99.9, -99.8, -99.7)$. For the LUIO, this is obtained via the conditioned compliant matrix $\Theta = \begin{pmatrix} -199.5\,\mathbb{I}_2 & \mathbb{I}_2 \\ -999.5\,\mathbb{I}_2 & 0_{2\times2} \end{pmatrix}$. According to the procedure expressed in Section 3, to make the unknown inputs η and π negligible, we start to consider that from qbrobotics (2022) $\eta_m = 6.33\ \frac{rad}{s^3}$ and that the maximum allowable link positions, here denoted with q_{1_m} and q_{2_m}, respectively, are equal to $= 6.28$ rad. Furthermore, a possible strategy to deal with the measurement noise is to overestimate its amplitude considering it equal to the maximum link positions, i.e. $|\pi| = |q_{1_m}| = |q_{2_m}| = 6.28\,rad$. In this setting, the eigenvalues of $\tilde{P}$ are placed in $p_p = -\left(10^4, 50, 45, 40, 35, 30, 25, 20\right)$, from which $|\lambda_m| = \left|\frac{1}{10^4}\right|$, and consequently (18) reads

$$\begin{aligned} \|\epsilon_q\|_2 \; &= \; |\lambda_m| \left(\|\Gamma\|_2 |\eta_m| + \|\Delta\|_2 |\pi_m| \right) \leq \\ &\leq \; 10^{-4} \cdot 12.61 \leq 1.2 \cdot 10^{-3} \end{aligned} \tag{36}$$

Fig. 2 depicts the attained results of a noise-free simulation and shows a faster and more precise estimation of the proposed LUIO. Instead, Fig. 3 shows the estimation performance of the LUIO with system parameter variations up to 100%. More specifically, in the first scenario, all robot parameters take their respective nominal values as in Table 1, while, in the second and third scenarios, they are perturbed by an increment of 50% and 100%, respectively. For all three scenarios, the LUIO design is based on the same offline model in (34), i.e., the system matrices used to design the LUIO are the same for all conducted simulations. The deviations of the parameter of the nonlinear system in (32) do not substantially affect the estimation performance, which remain almost identical in all the three cases considered, with estimation errors convergent to zero for $t \to \infty$.

Another interesting feature of the proposed approach is its very low computational load in terms of CPU utilization. To demonstrate this, using Simulink Real-time Code Generation, the Matlab/Simulink scheme including the soft-articulation robot and the LUIO, with a system parameter deviation of 100% has been compiled and linked as a stand-alone application. Then, the obtained application has been run on a low-cost Raspberry PI 4 Model B system (Raspberry Pi, 2024) with a scheduling time of 10^{-3} seconds. The CPU utilization obtained on 20 runs of the application has an average of 5.12% and a standard deviation of 0.023%. This confirms a very low computational cost and guarantees practical implementability of the proposed solution even on low-cost platforms.

Fig. 4 shows the estimation performance of the LUIO and of the DUIO when the output vector is perturbed via a white measurement noise. More specifically, the noise signal is simulated through the white noise block in Simulink with a noise power $p_w = 3.5 \cdot 10^{-6}$ and with a sampling time equal to simulations one. The proposed method to deal with the measurement noise is able to ensure an asymptotic reconstruction of the free-noise output, with its relating κth derivatives,