SUPERLINEAR $(p(z), q(z))$-EQUATIONS

NIKOLAOS S. PAPAGEORGIOU, CALOGERO VETRO

ABSTRACT. We consider Dirichlet boundary value problems for equations involving the $(p(z), q(z))$-Laplacian operator in the principal part and prove the existence of one and three nontrivial weak solutions, respectively. Here, the nonlinearity in the reaction term is allowed to depend on the solution, but does not satisfy the Ambrosetti-Rabinowitz condition. The hypotheses on the reaction term ensure that the Euler-Lagrange functional, associated to the problem, satisfies both the (C_c)-condition and a mountain pass geometry.

1. INTRODUCTION

We study the following Dirichlet boundary value problem:

$$(P_g) \qquad \begin{cases} -\Delta_{p(z)}u(z) - \Delta_{q(z)}u(z) = g(z, u(z)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Delta_{k(z)}u := div(|\nabla u|^{k(z)-2}\nabla u)$ is the $k(z)$-Laplacian, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is the nonlinearity, $\Omega \subset \mathbb{R}^n$ is an open bounded domain with smooth boundary, $p, q \in C(\overline{\Omega})$ are such that $q(z) < p(z)$ for all $z \in \overline{\Omega}$ and

$$1 < q^- := \inf_{z\in\Omega} q(z) \leq q(z) \leq q^+ := \sup_{z\in\Omega} q(z) < +\infty,$$
$$1 < p^- := \inf_{z\in\Omega} p(z) \leq p(z) \leq p^+ := \sup_{z\in\Omega} p(z) < +\infty.$$

Here, $g(z, \xi)$ (reaction term) is a Carathéodory function (i.e., for each $\xi \in \mathbb{R}$, $z \to g(z, \xi)$ is measurable and for a.a. $z \in \Omega$, $\xi \to g(z, \xi)$ is continuous). We make the following assumptions:

(g_1) there exist $a_1, a_2 \in [0, +\infty[$ and $\alpha \in C(\overline{\Omega})$ with $p^+ < \alpha^- \leq \alpha^+ < p^*(z)$ for all $z \in \overline{\Omega}$, satisfying

$$|g(z, \xi)| \leq a_1 + a_2|\xi|^{\alpha(z)-1} \quad \text{for all } (z, \xi) \in \Omega \times \mathbb{R},$$

with $p^*(z) = \dfrac{np(z)}{n - p(z)}$ if $p(z) < n$ and $p^*(z) = +\infty$ if $p(z) \geq n$;

(g_2) if $G(z, t) = \int_0^t g(z, \xi)d\xi$, then we have

$$\lim_{|t|\to+\infty} \frac{G(z, t)}{|t|^{p^+}} = +\infty \quad \text{uniformly for a.a. } z \in \Omega;$$

(g_3) $\displaystyle\lim_{t\to 0} \frac{g(z, t)}{|t|^{p^+-1}} = 0$ uniformly for a.a. $z \in \Omega$;

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(g_4) if $e(z,t) = g(z,t)t - p^+ G(z,t)$, then there exists $d \in L^1(\Omega)$ satisfying

$$e(z,t) \leq e(z,s) + d(z) \quad \text{for a.a. } z \in \Omega, \text{ all } 0 < t < s \text{ or } s < t < 0.$$

Let $W^{1,p(z)}(\Omega)$ be the generalized Lebesgue−Sobolev space given in Section 2 and $W_0^{1,p(z)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p(z)}(\Omega)$. We recall that for a weak solution of the problem (P_g) we mean a function $u \in W_0^{1,p(z)}(\Omega)$ satisfying

$$\int_\Omega |\nabla u|^{p(z)-2} \nabla u \nabla v dz + \int_\Omega |\nabla u|^{q(z)-2} \nabla u \nabla v dz = \int_\Omega g(z,u) v dz, \text{ for each } v \in W_0^{1,p(z)}(\Omega).$$

Recently there has been considerable interest on the existence and multiplicity of solutions of equations driven by the sum of a p-Laplacian and of a q-Laplacian with $1 < p < q < +\infty$, known as (p,q)-elliptic equations. Such equations were studied exclusively in the framework of constant exponents for the differential operators. We mention the works of Barile-Figueiredo [1], Chaves-Ercole-Miyagaki [2], Cingolani-Degiovanni [3], Marano-Mosconi-Papageorgiou [16], Motreanu-Vetro-Vetro [18], Mugnai-Papageorgiou [19], Sun-Zhang-Su [22] and the references therein. To the best of our knowledge there have been no works on such equations with variable exponents. Also, problems with a superlinear reaction term not satisfying the AR-condition were studied by Iturriaga-Lorca-Ubilla [13], Li-Yang [14], Mugnai-Papageorgiou [19], Papageorgiou-Rădulescu [20], Sun [21] (constant exponent equations) and Gasiński-Papageorgiou [11], Tan-Fang [23], Zhou [24] (variable exponent equations). The last three papers use $p(z)$-Laplace-type differential operators and the conditions on the reaction term are more restrictive (see hypothesis (f_∞^3) in [23] and hypothesis (h_4) in [24]).

Our approach uses variational methods based on critical point theory together with Morse theory (critical groups). We prove an existence theorem and a multiplicity theorem producing three nontrivial smooth solutions.

2. Mathematical background

We fix the notation as follows. By X and X^* we mean a Banach space and its topological dual, respectively. In addition, by $L^{p(z)}(\Omega)$ and $W^{1,p(z)}(\Omega)$ we mean the variable exponent Lebesgue space and the generalized Lebesgue−Sobolev space, respectively. Precisely, we have the variable exponent Lebesgue space $L^{p(z)}(\Omega)$ given as

$$L^{p(z)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \ : \ \text{u is measurable and } \rho_p(u) := \int_\Omega |u(z)|^{p(z)} dz < +\infty \right\},$$

endowed with the norm

$$\|u\|_{L^{p(z)}(\Omega)} := \inf \left\{ \lambda > 0 : \int_\Omega \left| \frac{u(z)}{\lambda} \right|^{p(z)} dz \leq 1 \right\}.$$

On the other hand, we consider the generalized Lebesgue−Sobolev space $W^{1,p(z)}(\Omega)$ defined by

$$W^{1,p(z)}(\Omega) := \{ u \in L^{p(z)}(\Omega) : |\nabla u| \in L^{p(z)}(\Omega) \},$$

endowed with the norm

$$\|u\|_{W^{1,p(z)}(\Omega)} = \|u\|_{L^{p(z)}(\Omega)} + \| \, |\nabla u| \, \|_{L^{p(z)}(\Omega)}.$$

It is well-known that, for specific constant m,

$$\|u\|_{L^{p(z)}(\Omega)} \leq m \|\nabla u\|_{L^{p(z)}(\Omega)} \quad \text{for all } u \in W_0^{1,p(z)}(\Omega),$$

(see Theorem 8.2.18, p. 263, Diening-Harjulehto-Hästö-Růžička [5]). Then $\|u\|_{W^{1,p(z)}(\Omega)}$ and $\| |\nabla u| \|_{L^{p(z)}(\Omega)}$ are equivalent norms on $W_0^{1,p(z)}(\Omega)$. So, we will use $\| |\nabla u| \|_{L^{p(z)}(\Omega)}$ to replace $\|u\|_{W^{1,p(z)}(\Omega)}$ and put

$$\|u\| = \| |\nabla u| \|_{L^{p(z)}(\Omega)} \quad \text{in } W_0^{1,p(z)}(\Omega).$$

We note that both $L^{p(z)}(\Omega)$ and $W^{1,p(z)}(\Omega)$, endowed with the above norms, are separable, reflexive and uniformly convex Banach spaces (see Fan-Zhang [7]). Also, the classical Sobolev embedding theorem was generalized by Fan-Zhao [9] in the following way.

Proposition 1. *Assume that $p \in C(\overline{\Omega})$ with $p(z) > 1$ for each $z \in \overline{\Omega}$. If $\alpha \in C(\overline{\Omega})$ and $1 < \alpha(z) < p^*(z)$ for all $z \in \Omega$, then there exists a continuous and compact embedding $W^{1,p(z)}(\Omega) \hookrightarrow L^{\alpha(z)}(\Omega)$.*

In addition, from Theorem 1.11 of [9], we deduce that the embedding $L^{p(z)}(\Omega) \hookrightarrow L^{q(z)}(\Omega)$ is continuous, whenever $q, p \in C(\overline{\Omega})$ and $1 < q(z) < p(z)$ for all $z \in \Omega$. We recall another theorem from Fan-Zhao [9] (say, Theorem 1.3), which links $\| \cdot \|_{L^{p(z)}(\Omega)}$ to $\rho_p(\cdot)$.

Theorem 1. *Let $u \in L^{p(z)}(\Omega)$. Then, the following relations hold:*

(i) $\|u\|_{L^{p(z)}(\Omega)} < 1 \ (= 1,\ > 1) \Leftrightarrow \rho_p(u) < 1 \ (= 1,\ > 1)$;

(ii) *if* $\|u\|_{L^{p(z)}(\Omega)} > 1$, *then* $\|u\|^{p^-}_{L^{p(z)}(\Omega)} \leq \rho_p(u) \leq \|u\|^{p^+}_{L^{p(z)}(\Omega)}$;

(iii) *if* $\|u\|_{L^{p(z)}(\Omega)} < 1$, *then* $\|u\|^{p^+}_{L^{p(z)}(\Omega)} \leq \rho_p(u) \leq \|u\|^{p^-}_{L^{p(z)}(\Omega)}$.

Now, we consider the function $G : \Omega \times \mathbb{R} \to \mathbb{R}$ given as

$$G(z,t) = \int_0^t g(z,\xi) d\xi \quad \text{for all } t \in \mathbb{R},\ z \in \Omega,$$

and the functional $B : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ given as

$$B(u) = \int_\Omega G(z, u(z))\, dz, \qquad \text{for all } u \in W_0^{1,p(z)}(\Omega).$$

By the assumption (g_1), we deduce that $B \in C^1(W_0^{1,p(z)}(\Omega), \mathbb{R})$. From Proposition 1 we have that B admits the following compact derivative

$$\langle B'(u), v\rangle = \int_\Omega g(z, u(z)) v(z)\, dz, \qquad \text{for all } u, v \in W_0^{1,p(z)}(\Omega).$$

Define the functionals $A_1, A_2 : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ by

$$A_1(u) = \int_\Omega \frac{1}{p(z)} |\nabla u(z)|^{p(z)} dz \quad \text{and} \quad A_2(u) = \int_\Omega \frac{1}{q(z)} |\nabla u(z)|^{q(z)} dz \quad \text{for all } u \in W_0^{1,p(z)}(\Omega).$$

Clearly, $A_1, A_2 \in C^1(W_0^{1,p(z)}(\Omega), \mathbb{R})$ and

$$\langle A_1'(u), v\rangle = \int_\Omega |\nabla u|^{p(z)-2} \nabla u \nabla v dz \quad \text{and} \quad \langle A_2'(u), v\rangle = \int_\Omega |\nabla u|^{q(z)-2} \nabla u \nabla v dz$$

for all $u, v \in W_0^{1,p(z)}(\Omega)$.

Remark 1. $A_1' : W_0^{1,p(z)}(\Omega) \to W_0^{1,p(z)}(\Omega)^*$ is a mapping of type (S_+), that is, if $u_n \xrightarrow{w} u$ in $W_0^{1,p(z)}(\Omega)$ and $\limsup_{n\to+\infty}\langle A_1'(u), u_n - u\rangle \le 0$, then $u_n \to u$ in $W_0^{1,p(z)}(\Omega)$. The same holds for A_2'. Consequently, $A_1' + A_2'$ is a mapping of type (S_+).

We consider the functional $I : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ given as

$$I(u) = A_1(u) + A_2(u) - B(u) \quad \text{for all } u \in W_0^{1,p(z)}(\Omega),$$

and point out that $I(0) = 0$.

3. Main results

In this section, we prove that the problem (P_g) has at least one nontrivial weak solution. We need the following notion of (C_c)-condition.

Definition 1. Let X be a real Banach space and $I \in C^1(X, \mathbb{R})$. We say that I satisfies the (C_c)-condition if any sequence $\{u_n\} \subset X$ such that $I(u_n) \to c \in \mathbb{R}$ and $(1 + \|u_n\|)I'(u_n) \to 0$ in X^* as $n \to +\infty$ has a convergent subsequence.

Now, we prove the following lemma.

Lemma 1. *Let the assumptions $(g_1), (g_2), (g_4)$ be satisfied. Then the functional I satisfies the (C_c)-condition for each $c > 0$.*

Proof. Let $\{u_n\} \subset W_0^{1,p(z)}(\Omega)$ be a sequence satisfying the (C_c)-condition with respect to the functional I. So, we have

$$c = I(u_n) + c_n, \qquad \langle I'(u_n), u_n\rangle \to 0 \quad \text{as } n \to +\infty, \tag{1}$$

where $c_n \to 0$ as $n \to +\infty$.

We show that $\{u_n\}$ is a bounded sequence in $W_0^{1,p(z)}(\Omega)$. We argue by contradiction. So, suppose that $\{u_n\}$ is unbounded. We may assume that $\|u_n\| \to +\infty$ as $n \to +\infty$, by considering a subsequence if necessary. Also, we put

$$v_n = \frac{u_n}{\|u_n\|} \quad \text{for all } n \in \mathbb{N}.$$

Clearly, $\|v_n\| = 1$ for all $n \in \mathbb{N}$. Thus, we suppose that there exists $v \in W_0^{1,p(z)}(\Omega)$ such that

$$\begin{aligned} &v_n \xrightarrow{w} v \quad \text{in } W_0^{1,p(z)}(\Omega);\\ &v_n \to v \quad \text{in } L^{p^+}(\Omega) \text{ and } L^{\alpha(z)}(\Omega). \end{aligned}$$

Let $\Omega_0 := \{z \in \Omega : v(z) \neq 0\}$. We claim that $|\Omega_0| = 0$ ($|\Omega_0|$ denotes the Lebesgue measure of Ω_0). We argue by contradiction again. So, suppose that $|\Omega_0| > 0$. We note that

$$|u_n(z)| \to +\infty \quad \text{for a.a. } z \in \Omega_0 \quad \text{as } n \to +\infty,$$

since $v_n \to v \neq 0$ in Ω_0. Now, using (g_2), that is

$$\lim_{|t|\to+\infty} \frac{G(z,t)}{|t|^{p^+}} = +\infty \quad \text{uniformly for a.a. } z \in \Omega,$$

we get

$$\lim_{n\to+\infty} \frac{G(z,u_n(z))}{\|u_n\|^{p^+}} = \lim_{n\to+\infty} \frac{G(z,u_n(z))}{|u_n(z)|^{p^+}}|v_n(z)|^{p^+} = +\infty \quad \text{for a.a. } z \in \Omega_0. \tag{2}$$

Using (g_1) and (g_2), we deduce that there exists a constant K such that

$$G(z,t)+K\geq 0 \quad \text{for a.a. } z\in\Omega, \text{ all } t\in\mathbb{R}.$$

From (2) (assumptions (g_1) and (g_2) permit us to use Fatou's lemma), we obtain

$$\lim_{n\to+\infty}\int_{\Omega_0}\frac{G(z,u_n(z))}{\|u_n\|^{p^+}}dz=+\infty.$$

Consequently, we have

$$\begin{aligned}\lim_{n\to+\infty}\int_{\Omega}\frac{G(z,u_n(z))}{\|u_n\|^{p^+}}dz &= \lim_{n\to+\infty}\int_{\Omega}\frac{G(z,u_n(z))+K}{\|u_n\|^{p^+}}dz\\ &\geq \lim_{n\to+\infty}\int_{\Omega_0}\frac{G(z,u_n(z))+K}{\|u_n\|^{p^+}}dz\\ &= \lim_{n\to+\infty}\int_{\Omega_0}\frac{G(z,u_n(z))}{\|u_n\|^{p^+}}dz=+\infty. \end{aligned}\tag{3}$$

From (1), we get

$$\begin{aligned} c&=I(u_n)+c_n\\ &=\int_{\Omega}\frac{1}{p(z)}|\nabla u_n|^{p(z)}dz+\int_{\Omega}\frac{1}{q(z)}|\nabla u_n|^{q(z)}dz-\int_{\Omega}G(z,u_n(z))dz+c_n\\ &\geq\frac{1}{p^+}\|u_n\|^{p^-}-\int_{\Omega}G(z,u_n(z))dz+c_n, \end{aligned}$$

for all $n\in\mathbb{N}$ such that $\|u_n\|\geq 1$. Thus, we have

$$\int_{\Omega}G(z,u_n(z))dz\geq\frac{1}{p^+}\|u_n\|^{p^-}-c+c_n\to+\infty \quad \text{as } n\to+\infty.\tag{4}$$

Again from (1), we get

$$\begin{aligned} c&=I(u_n)+c_n\\ &=\int_{\Omega}\frac{1}{p(z)}|\nabla u_n|^{p(z)}dz+\int_{\Omega}\frac{1}{q(z)}|\nabla u_n|^{q(z)}dz-\int_{\Omega}G(z,u_n(z))dz+c_n\\ &\leq\frac{1}{p^-}\|u_n\|^{p^+}+\frac{1}{q^-}\max\left\{\|\nabla u_n\|^{q^+}_{L^{q(z)}(\Omega)},\|\nabla u_n\|^{q^-}_{L^{q(z)}(\Omega)}\right\}-\int_{\Omega}G(z,u_n(z))dz+c_n\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(by Theorem 1)}\\ &\leq K_0\|u_n\|^{p^+}-\int_{\Omega}G(z,u_n(z))dz+c_n \quad \text{for all } n\in\mathbb{N} \text{ such that } \|u_n\|\geq 1, \end{aligned}$$

where $K_0=\frac{1}{p^-}+\frac{1}{q^-}\max\{K_q^{q^-},K_q^{q^+}\}$ with K_q to denote the constant of the continuous embedding $L^{p(z)}(\Omega)\hookrightarrow L^{q(z)}(\Omega)$. Thus, by (4), there exists $n_0\in\mathbb{N}$ such that

$$\|u_n\|^{p^+}\geq\frac{c}{K_0}+\frac{1}{K_0}\int_{\Omega}G(z,u_n(z))dz-\frac{c_n}{K_0}>0 \quad \text{for all } n\geq n_0.$$

Therefore

$$\lim_{n\to+\infty}\int_{\Omega}\frac{G(z,u_n(z))}{\|u_n\|^{p^+}}dz\leq\lim_{n\to+\infty}\frac{\int_{\Omega}G(z,u_n(z))dz}{\frac{c}{K_0}+\frac{1}{K_0}\int_{\Omega}G(z,u_n(z))dz-\frac{c_n}{K_0}}=K_0,\tag{5}$$

which leads to contradiction with (3) and hence $|\Omega_0| = 0$. Then we have $v(z) = 0$ for a.a. $z \in \Omega$. Since $I(tu_n)$ is a continuous function on $[0,1]$ with respect to the variable t, for each $n \in \mathbb{N}$ there exists $t_n \in [0,1]$ such that

$$I(t_n u_n) = \max_{t \in [0,1]} I(tu_n).$$

For $k > 1$, we set

$$r_n = k^{\frac{1}{p^-}} v_n \quad \text{ for all } n \in \mathbb{N}.$$

Since $v_n \to 0$ in $L^{\alpha(z)}(\Omega)$ and $v_n(z) \to 0$ for a.a. $z \in \Omega$ as $n \to +\infty$, using (g_1) and Krasnoselskii's theorem (see, for example, Gasiński-Papageorgiou [11], p. 407), we deduce that

$$\lim_{n \to +\infty} \int_\Omega G(z, r_n(z)) dz = 0.$$

So, there exists $n_1 \in \mathbb{N}$ such that $\int_\Omega G(z, r_n(z)) dz < \frac{k}{2p^+}$ for all $n \geq n_1$. Since $\|u_n\| \to +\infty$, we can find $n_2 \in \mathbb{N}$ (with $n_2 \geq n_1$) such that $0 < k^{\frac{1}{p^+}} \frac{1}{\|u_n\|} \leq 1$ for all $n \geq n_2$. Then

$$\begin{aligned} I(t_n u_n) &\geq I(r_n) \\ &= \int_\Omega \frac{1}{p(z)} |\nabla r_n|^{p(z)} dz + \int_\Omega \frac{1}{q(z)} |\nabla r_n|^{q(z)} dz - \int_\Omega G(z, r_n(z)) dz \\ &\geq \frac{1}{p^+} \|r_n\|^{p^-} - \int_\Omega G(z, r_n(z)) dz \quad (\|r_n\| = k^{\frac{1}{p^-}} > 1) \\ &\geq \frac{1}{p^+} k - \frac{1}{2p^+} k = \frac{1}{2p^+} k \quad \text{for all } n \geq n_2. \end{aligned}$$

Now, $k > 1$ is arbitrary and hence we infer that

$$I(t_n u_n) \to +\infty \quad \text{as} \quad n \to +\infty. \tag{6}$$

From

$$I(0) = 0 \quad \text{and} \quad c = I(u_n) + c_n,$$

we deduce that there exists $n_3 \in \mathbb{N}$ such that $t_n \in]0,1[$ for $n \geq n_3$. It follows that

$$\frac{d}{dt} I(tu_n)\big|_{t=t_n} = 0 \quad \Rightarrow \quad \langle I'(t_n u_n), t_n u_n \rangle = 0 \quad \text{for all } n \geq n_3.$$

Thus

$$\begin{aligned} I(t_n u_n) &= I(t_n u_n) - \frac{1}{p^+} \langle I'(t_n u_n), t_n u_n \rangle \\ &= \int_\Omega \frac{1}{p(z)} |\nabla t_n u_n|^{p(z)} dz + \int_\Omega \frac{1}{q(z)} |\nabla t_n u_n|^{q(z)} dz - \int_\Omega G(z, t_n u_n(z)) dz \\ &\quad - \int_\Omega \frac{1}{p^+} |\nabla t_n u_n|^{p(z)} dz - \int_\Omega \frac{1}{p^+} |\nabla t_n u_n|^{q(z)} dz + \frac{1}{p^+} \int_\Omega g(z, t_n u_n(z)) t_n u_n(z) dz \\ &= \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p^+} \right] t_n^{p(z)} |\nabla u_n|^{p(z)} dz + \int_\Omega \left[\frac{1}{q(z)} - \frac{1}{p^+} \right] t_n^{q(z)} |\nabla u_n|^{q(z)} dz \\ &\quad + \frac{1}{p^+} \int_\Omega [g(z, t_n u_n(z)) t_n u_n(z) - p^+ G(z, t_n u_n(z))] dz \\ &\leq \int_\Omega \left[\frac{1}{p(z)} - \frac{1}{p^+} \right] |\nabla u_n|^{p(z)} dz + \int_\Omega \left[\frac{1}{q(z)} - \frac{1}{p^+} \right] |\nabla u_n|^{q(z)} dz \end{aligned}$$

$$+\frac{1}{p^+}\int_\Omega([g(z,u_n(z))u_n(z)-p^+G(z,u_n(z))]+d(z))dz \quad (\text{by } (g_4))$$
$$=I(u_n)-\frac{1}{p^+}\langle I'(u_n),u_n\rangle+\frac{1}{p^+}\|d\|_{L^1(\Omega)}\to c+\frac{1}{p^+}\|d\|_{L^1(\Omega)} \text{ as } n\to+\infty,$$

a contradiction by (6). It follows that $\{u_n\}$ is a bounded sequence in $W_0^{1,p(z)}(\Omega)$. Note that $W_0^{1,p(z)}(\Omega)$ is a reflexive Banach space, and so, by considering a subsequence if necessary, there exists $u \in W_0^{1,p(z)}(\Omega)$ such that $u_n \xrightarrow{w} u$ in $W_0^{1,p(z)}(\Omega)$ and $u_n \to u$ in $L^{\alpha(z)}(\Omega)$. By using the Hölder inequality, we get

$$\begin{aligned}\int_\Omega g(z,u_n(z))(u_n(z)-u(z))dz &\le \int_\Omega |g(z,u_n(z))|\,|u_n(z)-u(z)|dz\\ &\le \int_\Omega (a_1+a_2|u_n(z)|^{\alpha(z)-1})|u_n(z)-u(z)|dz\\ &\le 2\|a_1+a_2|u_n|^{\alpha(z)-1}\|_{L^{\alpha'(z)}(\Omega)}\|u_n-u\|_{L^{\alpha(z)}(\Omega)}.\end{aligned}$$

So, we have

$$\lim_{n\to+\infty}\int_\Omega g(z,u_n(z))(u_n(z)-u(z))dz=0.$$

Now, by (1), we deduce that $\langle I'(u_n), u_n-u\rangle \to 0$ as $n\to+\infty$. It follows that

$$\begin{aligned}&\langle A_1'(u_n),u_n-u\rangle+\langle A_2'(u_n),u_n-u\rangle\\ &=\int_\Omega |\nabla u_n|^{p(z)-2}\nabla u_n\nabla(u_n-u)dz+\int_\Omega |\nabla u_n|^{q(z)-2}\nabla u_n\nabla(u_n-u)dz\\ &=\int_\Omega g(z,u_n)(u_n-u)dz+\langle I'(u_n),u_n-u\rangle\to 0 \text{ as } n\to+\infty.\end{aligned}$$

Since $A_1'+A_2'$ is a mapping of type (S_+), we obtain that $u_n\to u$ in $W_0^{1,p(z)}(\Omega)$. So, I satisfies the (C_c)-condition on $W_0^{1,p(z)}(\Omega)$. □

Our second result is the following lemma.

Lemma 2. *Let the assumptions $(g_1)-(g_3)$ be satisfied. Then the following assertions hold:*

(i) *there exist $\rho>0$ and $\delta>0$ such that $I(u)\ge\delta$ for each $u\in W_0^{1,p(z)}(\Omega)$ with $\|u\|=\rho$;*
(ii) *there exists $v\in W_0^{1,p(z)}(\Omega)$ such that $I(v)<0$ and $\|v\|>\rho$.*

Proof. (i): We say that the embeddings $W_0^{1,p(z)}(\Omega)\hookrightarrow L^{p^+}(\Omega)$ and $W_0^{1,p(z)}(\Omega)\hookrightarrow L^{\alpha(x)}(\Omega)$ are continuous and so there exist two constants C_{p^+} and C_α such that

$$\|u\|_{L^{p^+}(\Omega)}\le C_{p^+}\|u\| \quad\text{and}\quad \|u\|_{L^{\alpha(x)}(\Omega)}\le C_\alpha\|u\|. \tag{7}$$

Combining (g_1) and (g_3), we can verify that, for each $\varepsilon>0$, there exists a constant C_ε such that

$$G(z,t)\le\frac{\varepsilon}{p^+}|t|^{p^+}+C_\varepsilon|t|^{\alpha(z)} \quad \text{for a.a. } z\in\Omega, \text{ all } t\in\mathbb{R}. \tag{8}$$

If $u\in W_0^{1,p(z)}(\Omega)$ is such that $\|u\|<1$, using (7) and (8), we obtain

$$I(u)=\int_\Omega\frac{1}{p(z)}|\nabla u|^{p(z)}dz+\int_\Omega\frac{1}{q(z)}|\nabla u|^{q(z)}dz-\int_\Omega G(z,u)dz$$

$$\begin{aligned}
&\geq \frac{1}{p^+}\int_\Omega |\nabla u|^{p(z)}dz - \frac{1}{p^+}\varepsilon\int_\Omega |u|^{p^+}dz - C_\varepsilon\int_\Omega |u|^{\alpha(z)}dz \\
&\geq \frac{1}{p^+}\|u\|^{p^+} - \frac{1}{p^+}\varepsilon C_{p^+}^{p^+}\|u\|^{p^+} - C_\varepsilon C_\alpha^{\alpha^-}\|u\|^{\alpha^-} \\
&= \frac{1}{p^+}(1-\varepsilon C_{p^+}^{p^+})\|u\|^{p^+} - C_\varepsilon C_\alpha^{\alpha^-}\|u\|^{\alpha^-} \\
&= \left[\frac{1}{p^+}(1-\varepsilon C_{p^+}^{p^+}) - C_\varepsilon C_\alpha^{\alpha^-}\|u\|^{\alpha^- - p^+}\right]\|u\|^{p^+}.
\end{aligned}$$

Now, we choose $\varepsilon > 0$ and $\rho > 0$ such that

$$\sigma = \frac{1}{p^+}(1-\varepsilon C_{p^+}^{p^+}) - C_\varepsilon C_\alpha^{\alpha^-}\rho^{\alpha^- - p^+} > 0.$$

Then $I(u) \geq \sigma\rho^{p^+} = \delta > 0$ for every $u \in W_0^{1,p(z)}(\Omega)$ with $\|u\| = \rho$.

(ii): Using (g_1) and (g_2), we deduce that, for all $M > 0$, there exists $C_M > 0$ such that

$$G(z,t) \geq M|t|^{p^+} - C_M \quad \text{for a.a. } z \in \Omega, \text{ all } t \in \mathbb{R}. \tag{9}$$

Let $\zeta \in W_0^{1,p(z)}(\Omega)$ such that $\zeta(z) > 0$ for all $z \in \Omega$, that is, $\zeta > 0$. From (9), for all $t > 1$, we get

$$\begin{aligned}
I(t\zeta) &= \int_\Omega \frac{t^{p(z)}}{p(z)}|\nabla\zeta|^{p(z)}dz + \int_\Omega \frac{t^{q(z)}}{q(z)}|\nabla\zeta|^{q(z)}dz - \int_\Omega G(z,t\zeta)dz \\
&\leq t^{p^+}\left[\int_\Omega \frac{1}{p(z)}|\nabla\zeta|^{p(z)}dz + \int_\Omega \frac{1}{q(z)}|\nabla\zeta|^{q(z)}dz - M\int_\Omega \zeta^{p^+}dz\right] + C_M|\Omega|.
\end{aligned}$$

If we choose $M > 0$ such that

$$\int_\Omega \frac{1}{p(z)}|\nabla\zeta|^{p(z)}dz + \int_\Omega \frac{1}{q(z)}|\nabla\zeta|^{q(z)}dz - M\int_\Omega \zeta^{p^+}dz < 0,$$

we obtain that $\lim_{n\to+\infty} I(t\zeta) = -\infty$. It follows that there exists $v = t_0\zeta \in W_0^{1,p(z)}(\Omega)$ such that $I(v) < 0$ and $\|v\| > \rho$. $\square$

Now, we recall the following version of the "Mountain Pass Theorem" (see Theorem 5.40, p. 118, Motreanu-Motreanu-Papageorgiou [17]).

Theorem 2. *If $I \in C^1(X,\mathbb{R})$ satisfies the (C_c)-condition, there exist $u_0, u_1 \in X$ and $\rho > 0$ such that*

$$\|u_1 - u_0\| > \rho, \quad \max\{I(u_0), I(u_1)\} < \inf\{I(u) : \|u - u_0\| = \rho\} = m_\rho,$$

$$\textit{and } c = \inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1} I(\gamma(t)) \textit{ with } \Gamma = \{\gamma \in C([0,1],X) : \gamma(0) = u_0, \gamma(1) = u_1\},$$

then $c \geq m_\rho$ and c is a critical value of I (i.e., there exists $\widehat{u} \in X$ such that $I'(\widehat{u}) = 0$ and $I(\widehat{u}) = c$).

Remark 2. A point $u \in W_0^{1,p(z)}(\Omega)$ is a local $W_0^{1,p(z)}(\Omega)$ minimizer of I, whenever we can find $\rho > 0$ such that $I(u) \leq I(u+h)$ for all $h \in W_0^{1,p(z)}(\Omega)$ with $\|h\| \leq \rho$. So, by the proof of Lemma 2(i), we get trivially that $u = 0$ is a local $W_0^{1,p(z)}(\Omega)$ minimizer of I.

Lemmas 1 and 2 ensure that I satisfies the hypotheses of Theorem 2 and hence I has a critical value $c \geq \delta$. Now we are ready for the existence theorem which produces one nontrivial weak solution for problem (P_g). The solution by Lemma 4.1 of Fukagai-Nurakawa [10] is in $C_0^1(\overline{\Omega})$.

Theorem 3. *Let the assumptions $(g_1) - (g_4)$ be satisfied. Then problem (P_g) has at least one nontrivial weak solution $u_0 \in C_0^1(\overline{\Omega})$.*

We recall some auxiliary notions and notation. The Banach space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone given by

$$C_+ = \{u \in C_0^1(\overline{\Omega}) \, : \, u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

Let X be a Banach space, $I \in C^1(X, \mathbb{R})$, $c \in \mathbb{R}$. We introduce the sets:

$$K_I = \{u \in X : I'(u) = 0\}, \quad K_I^c = \{u \in K_I : I(u) = c\}, \quad I^c = \{u \in X : I(u) \leq c\}.$$

Let (Y_1, Y_2) be a topological pair with $Y_2 \subset Y_1 \subset X$. For every $k \in \mathbb{N}_0$, let $H_k(Y_1, Y_2)$ be the k^{th}-relative singular homology group with integer coefficients (for $k \in -\mathbb{N}$, we have $H_k(Y_1, Y_2) = 0$). Let $u_0 \in K_I^c$ be isolated. The critical groups of I at u_0 are defined by

$$C_k(I, u_0) = H_k(I^c \cap U, I^c \cap U \setminus \{u_0\}) \quad \text{for all } k \in \mathbb{N}_0,$$

with U a neighborhood of u_0 such that $K_I \cap I^c \cap U = \{u_0\}$. The excision property of singular homology implies that this definition is independent of the isolating neighborhood U. Suppose $I \in C^1(X, \mathbb{R})$ satisfies the (C_c)-condition and $\inf I(K_I) > -\infty$. Let $c < \inf I(K_I)$. The critical groups of I at infinity are defined by

$$C_k(I, \infty) = H_k(X, I^c) \quad \text{for all } k \in \mathbb{N}_0.$$

This definition is independent of the choice of $c < \inf I(K_I)$ (see [17], Remark 6.60, p. 159). Next we compute the critical groups of the energy functional I at infinity.

Proposition 2. *Let the assumptions $(g_1) - (g_4)$ be satisfied. Then $C_k(I, \infty) = 0$ for all $k \in \mathbb{N}_0$.*

Proof. Let $\partial B_1 = \{u \in W_0^{1,p(z)}(\Omega) : \|u\| = 1\}$. By the assumptions (g_1) and (g_2), for all $M > 0$ there exists $C_M > 0$ such that

$$G(z,t) \geq M|t|^{p^+} - C_M \quad \text{for a.a. } z \in \Omega, \text{ all } t \in \mathbb{R}. \tag{10}$$

Then for $u \in \partial B_1$ and $t > 1$, we have

$$I(tu) \leq t^{p^+} \left[\int_\Omega \frac{1}{p(z)} |\nabla u|^{p(z)} dz + \int_\Omega \frac{1}{q(z)} |\nabla u|^{q(z)} dz - M \int_\Omega |u|^{p^+} dz \right] + C_M |\Omega|.$$

Recall that $M > 0$ is arbitrary. Hence we infer that

$$I(tu) \to -\infty \quad \text{as } \; t \to +\infty.$$

Now, for $u \in \partial B_1$ and $t > 1$, we have

$$\begin{aligned}
&\frac{d}{dt} I(tu) = \langle I'(tu), u \rangle = \frac{1}{t} \langle I'(tu), tu \rangle \\
=& \frac{1}{t} \left[\int_\Omega |\nabla tu|^{p(z)} dz + \int_\Omega |\nabla tu|^{q(z)} dz - \int_\Omega g(z, tu)\, tu \, dz \right] \\
\leq& \frac{1}{t} \left[p^+ \int_\Omega \frac{1}{p(z)} |\nabla tu|^{p(z)} dz + p^+ \int_\Omega \frac{1}{q(z)} |\nabla tu|^{q(z)} dz - p^+ \int_\Omega G(z, tu) dz \right.
\end{aligned}$$

$$
\begin{aligned}
&\quad - \int_\Omega e(z,u)dz + \|d\|_{L^1(\Omega)}\Big] \quad (\text{by } (g_4)) \\
&= \frac{1}{t}\left[p^+ I(tu) - \int_\Omega e(z,u)dz + \|d\|_{L^1(\Omega)}\right] \to -\infty \quad \text{as } t \to +\infty.
\end{aligned}
$$

It follows that $\frac{d}{dt}I(tu) < 0$ for all $t > 1$ big. The implicit function theorem implies that we can find $s \in C(\partial B_1)$ such that $s > 0$ and $I(s(u)u) = \rho_0$, where

$$p^+\rho_0 - \int_\Omega e(z,u)dz + \|d\|_{L^1(\Omega)} < 0.$$

We extend $s(\cdot)$ on $W_0^{1,p(z)}(\Omega) \setminus \{0\}$ by $s_0(u) = \frac{1}{\|u\|}s(\frac{u}{\|u\|})$ for all $u \in W_0^{1,p(z)}(\Omega) \setminus \{0\}$. We have $s_0 \in C(W_0^{1,p(z)}(\Omega) \setminus \{0\})$ and $I(s_0(u)u) = \rho_0$. Also, we have

$$I(u) = \rho_0 \Rightarrow s_0(u) = 1. \tag{11}$$

Therefore, if we define

$$\widehat{s}_0(u) = \begin{cases} 1 & \text{if } I(u) \le \rho_0, \\ s_0(u) & \text{if } \rho_0 < I(u), \end{cases} \tag{12}$$

then we have $\widehat{s}_0 \in C(W_0^{1,p(z)}(\Omega) \setminus \{0\})$ (see (11)). Next, we consider the deformation $h : [0,1] \times (W_0^{1,p(z)}(\Omega) \setminus \{0\}) \to W_0^{1,p(z)}(\Omega) \setminus \{0\}$ defined by

$$h(t,u) = (1-t)u + t\widehat{s}_0(u)u \quad \text{for all } t \in [0,1], \text{ all } u \in W_0^{1,p(z)}(\Omega).$$

We have:

- $h(0,u) = u$ for all $u \in W_0^{1,p(z)}(\Omega) \setminus \{0\}$,
- $h(1,u) = \widehat{s}_0(u)u + I^{\rho_0}$ (see (12)),
- $h(t,\cdot)\big|_{I^{\rho_0}} = id\big|_{I^{\rho_0}}$ (see (11), (12)).

From these facts we infer that

$$I^{\rho_0} \quad \text{is a strong deformation retract of } (W_0^{1,p(z)}(\Omega) \setminus \{0\}). \tag{13}$$

Consider the radial retraction $r_1 : (W_0^{1,p(z)}(\Omega) \setminus \{0\}) \to \partial B_1$ defined by

$$r_1(u) = \frac{u}{\|u\|} \quad \text{for all } u \in (W_0^{1,p(z)}(\Omega) \setminus \{0\}).$$

We introduce the deformation $\widehat{h} : [0,1] \times (W_0^{1,p(z)}(\Omega) \setminus \{0\}) \to (W_0^{1,p(z)}(\Omega) \setminus \{0\})$ defined by

$$\widehat{h}(t,u) = (1-t)u + t\,r_1(u) \quad \text{for all } t \in [0,1], \text{ all } u \in (W_0^{1,p(z)}(\Omega) \setminus \{0\}).$$

With this deformation we see that

$$(W_0^{1,p(z)}(\Omega) \setminus \{0\}) \text{ is deformable into } \partial B_1. \tag{14}$$

In addition using radial retraction $r_1(\cdot)$, we see that

$$\partial B_1 \text{ is a retract of } (W_0^{1,p(z)}(\Omega) \setminus \{0\}). \tag{15}$$

From (14), (15) and Theorem 6.5, p. 325 of Dugundji [6], we infer that

$$\partial B_1 \text{ is a deformation retract of } (W_0^{1,p(z)}(\Omega) \setminus \{0\}). \tag{16}$$

From (13) and (16), it follows that

$$\begin{aligned} & I^{\rho_0} \text{ and } \partial B_1 \text{ are homotopy equivalent,} \\ (17) \ \Rightarrow \quad & H_k(W_0^{1,p(z)}(\Omega), I^{\rho_0}) = H_k(W_0^{1,p(z)}(\Omega), \partial B_1) \quad \text{for all } k \in \mathbb{N}_0 \\ & \qquad \text{(see Motreanu-Motreanu-Papageorgiou [17], p. 143).} \end{aligned}$$

The Sobolev space $W_0^{1,p(z)}(\Omega)$ is infinite dimensional. Hence

$$\begin{aligned} & \partial B_1 \text{ is contractible (see Gasiński-Papageorgiou [12], Problems 4.154, 4.159]),} \\ \Rightarrow \quad & H_k(W_0^{1,p(z)}(\Omega), \partial B_1) = 0 \quad \text{for all } k \in \mathbb{N}_0 \\ & \qquad \text{(see Motreanu-Motreanu-Papageorgiou [17], p. 147),} \\ \Rightarrow \quad & H_k(W_0^{1,p(z)}(\Omega), I^{\rho_0}) = 0 \quad \text{for all } k \in \mathbb{N}_0 \quad \text{(see (17)).} \end{aligned}$$

As usual we assume that K_I is finite (or otherwise we already have a infinity of nontrivial solutions which are in $C_0^1(\overline{\Omega})$ by the nonlinear regularity theory, see [10]). So, choosing ρ_0 such that $p^+\rho_0 - \int_\Omega e(z,u)dz + \|d\|_{L^1(\Omega)} < 0$, we get

$$\begin{aligned} & C_k(I, \infty) = H_k(W_0^{1,p(z)}(\Omega), I^{\rho_0}) \quad \text{for all } k \in \mathbb{N}_0, \\ \Rightarrow \quad & C_k(I, \infty) = 0 \quad \text{for all } k \in \mathbb{N}_0. \end{aligned}$$

□

4. Three nontrivial weak solutions

In this section, we establish the existence of at least three nontrivial weak solutions, by using an additional assumption on the reaction term $g(z, \xi)$. Precisely, we have:

(g_5) $g(z, 0) = 0$ for all $z \in \Omega$ and $g(z, \xi) \geq 0$ for all $z \in \Omega$, all $\xi \in [0, +\infty[$.

Now, we consider the function $G_+ : \Omega \times \mathbb{R} \to \mathbb{R}$ given as

$$G_+(z, t) = \int_0^t g(z, \xi^+) d\xi \quad \text{for all } t \in \mathbb{R},\ z \in \Omega,$$

and the functional $J : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ given as

$$J(u) = A_1(u) + A_2(u) - \int_\Omega G_+(z, u(z))\, dz, \quad \text{for all } u \in W_0^{1,p(z)}(\Omega).$$

By the assumptions $(g_1) - (g_5)$, we deduce that $J \in C^1(W_0^{1,p(z)}(\Omega), \mathbb{R})$ and

$$\langle J'(u), v\rangle = \langle A_1'(u), v\rangle + \langle A_2'(u), v\rangle - \int_\Omega g(z, u^+(z)) v(z)\, dz, \quad \text{for all } v \in W_0^{1,p(z)}(\Omega).$$

Assume that $\{u_n\} \subset W_0^{1,p(z)}(\Omega)$ is such that $(1 + \|u_n\|)J'(u_n) \to 0$ as $n \to +\infty$. Then, there exists a sequence $\{\varepsilon_n\}$ of nonnegative real numbers such that $\varepsilon_n \to 0$ as $n \to +\infty$ and

$$|\langle J'(u_n), v\rangle| \leq \frac{\varepsilon \|v\|}{1 + \|u_n\|} \quad \text{for all } n \in \mathbb{N}\text{ , all } v \in W_0^{1,p(z)}(\Omega).$$

If $v = v_n = \min\{0, u_n\}$, then we get

$$\int_\Omega |\nabla v_n|^{p(z)} dz \leq \int_\Omega |\nabla v_n|^{p(z)} dz + \int_\Omega |\nabla v_n|^{q(z)} dz \leq \frac{\varepsilon_n \|v_n\|}{1 + \|u_n\|} \quad \text{for all } n \in \mathbb{N},$$
$$\text{(since } g(z, u^+(z)) v_n(z) = 0 \text{ for all } z \in \Omega)$$

$$\Rightarrow \quad \min\left\{\|\nabla v_n\|^{p^+}_{L^{p(z)}(\Omega)}, \|\nabla v_n\|^{p^-}_{L^{p(z)}(\Omega)}\right\} \to 0 \quad \text{as } n \to +\infty,$$
$$\Rightarrow \quad \|v_n\| = \|\nabla v_n\|_{L^{p(z)}(\Omega)} \to 0 \quad \text{as } n \to +\infty.$$

It follows that the functional J satisfies the (C_c)-condition if and only if it satisfies the (C_c)-condition with respect to all the sequences $\{u_n\} \subset W_0^{1,p(z)}(\Omega)$ such that $u_n(z) \geq 0$ for all $z \in \Omega$, all $n \in \mathbb{N}$. Now, for all $u \in W_0^{1,p(z)}(\Omega)$ such that $u(z) \geq 0$ for all $z \in \Omega$, we have

$$\begin{aligned} J(u) &= A_1(u) + A_2(u) - \int_\Omega G_+(z, u(z))\,dz \\ &= A_1(u) + A_2(u) - \int_\Omega G(z, u(z))\,dz = I(u), \end{aligned}$$

since $G_+(z, u(z)) = \int_0^{u(z)} g(z, \xi^+)d\xi = \int_0^{u(z)} g(z, \xi)d\xi = G(z, u(z))$ for all $z \in \Omega$. By Lemma 1 the functional J satisfies the (C_c)-condition for all the sequences $\{u_n\} \subset W_0^{1,p(z)}(\Omega)$ such that $u_n(z) \geq 0$ for all $z \in \Omega$, all $n \in \mathbb{N}$. Clearly, Lemma 2 also holds for the functional J. The above facts (by Theorem 2) imply that there exists a function $u_0 \in W_0^{1,p(z)}(\Omega)$ such that

$$\langle A_1'(u_0), v\rangle + \langle A_2'(u_0), v\rangle = \int_\Omega g(z, u_0^+(z))v(z)\,dz, \qquad \text{for all } v \in W_0^{1,p(z)}(\Omega).$$

If we choose $v = \min\{0, u_0\}$, we deduce that $v = 0$, since $g(z, u_0^+(z))v(z) = 0$ for all $z \in \Omega$. It follows that $u_0(z) \geq 0$ for all $z \in \Omega$ and hence $g(z, u_0^+(z)) = g(z, u_0(z))$ for all $z \in \Omega$. So, the function u_0 satisfies

$$\langle A_1'(u_0), v\rangle + \langle A_2'(u_0), v\rangle = \int_\Omega g(z, u_0(z))v(z)\,dz, \qquad \text{for all } v \in W_0^{1,p(z)}(\Omega),$$

and this implies that u_0 is a nonnegative nontrivial weak solution of problem (P_g).

Finally we give the existence theorem which produces one nonnegative nontrivial weak solution for problem (P_g).

Theorem 4. *Let the assumptions $(g_1) - (g_5)$ be satisfied. Then problem (P_g) has at least one nonnegative nontrivial weak solution.*

Next, we can compute the critical groups of the energy functional J at the constant sign solution u_0. In the sequel, by $\delta_{k,m}$ we denote the Kronecker symbol defined by $\delta_{k,m} = 1$ if $k = m$ and $\delta_{k,m} = 0$ if $k \neq m$, where m is the Morse index of u_0.

Proposition 3. *Let the assumptions $(g_1) - (g_4)$ be satisfied. Then $C_k(J, u_0) = \delta_{k,1}\mathbb{Z}$ for all $k \in \mathbb{N}_0$.*

Proof. Clearly $K_J \subseteq C_+$, hence without loss of generality, we suppose that $K_J = \{0, u_0\}$. Since $u = 0$ is a local minimizer of J and u_0 a critical point of J of mountain pass type, there exists $\delta > 0$ (see proof of Lemma 2(i)) such that

$$0 = J(0) < \delta \leq J(u_0).$$

Let $\nu_- < 0 < \nu_+ < \delta$ and consider the inclusions $J^{\nu_-} \subseteq J^{\nu_+} \subseteq W_0^{1,p(z)}(\Omega)$. Next, we consider the following corresponding long exact sequence of singular homology groups (see [17], p. 129):

$$\cdots \to H_k(W_0^{1,p(z)}(\Omega), J^{\nu_-}) \xrightarrow{i_*} H_k(W_0^{1,p(z)}(\Omega), J^{\nu_+}) \xrightarrow{\partial_*} H_{k-1}(J^{\nu_+}, J^{\nu_-}) \to \cdots, \tag{18}$$

with i_* being the homomorphism induced by the inclusion $i : (W_0^{1,p(z)}(\Omega), J^{\nu_-}) \to (W_0^{1,p(z)}(\Omega), J^{\nu_+})$ and ∂_* is the boundary homomorphism. Since $K_J = \{0, u_0\}$ and $\nu_- < 0 = J(0)$, we have (by Proposition 2)

$$H_k(W_0^{1,p(z)}(\Omega), J^{\nu_-}) = C_k(J, \infty) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{19}$$

Also, we have $0 = J(0) < \nu_+ < J(u_0)$. So, we have

$$H_k(W_0^{1,p(z)}(\Omega), J^{\nu_+}) = C_k(J, u_0) \quad \text{for all } k \in \mathbb{N}_0. \tag{20}$$

Analogously, we get

$$H_{k-1}(J^{\nu_+}, J^{\nu_-}) = C_{k-1}(J, 0) = \delta_{k-1,0}\mathbb{Z} = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{21}$$

From (19)-(21) and the exactness of (18), we infer that only the tail of that chain (i.e., $k = 1$) is nontrivial. From the rank theorem, the exactness of (18), and using (19) and (21), we obtain

(22)

$$\text{rank } H_1(W_0^{1,p(z)}(\Omega), J^{\nu_+}) = \text{rank ker } \partial_* + \text{ rank im } \partial_* = \text{rank im } i_* + \text{ rank im } \partial_* \leq 1.$$

Since u_0 is a critical point of J of mountain pass type. So,

$$C_1(J, u_0) \neq 0. \tag{23}$$

Form (20), (22), (23) and recalling that only for $k = 1$ the chain (18) is nontrivial, we deduce that $C_k(J, u_0) = \delta_{k,1}\mathbb{Z}$ for all $k \in \mathbb{N}_0$. □

Proposition 4. *Let the assumptions $(g_1)-(g_5)$ be satisfied. Then $C_k(I, u_0) = C_k(J, u_0)$ for all $k \in \mathbb{N}_0$.*

Proof. Consider the homotopy

$$h(t, u) = (1 - t)I(u) + tJ(u) \quad \text{for all } (t, u) \in [0, 1] \times W_0^{1,p(z)}(\Omega).$$

Assume there exist $\{t_n\} \subset [0, 1]$ and $\{u_n\} \subset W_0^{1,p(z)}(\Omega)$ such that

$$t_n \to t, \quad u_n \to u_0 \quad \text{in } W_0^{1,p(z)}(\Omega) \quad \text{and} \quad h'_u(t_n, u_n) = 0 \quad \text{for all } n \in \mathbb{N}. \tag{24}$$

By (24) we get

$$\begin{aligned} &(1 - t_n)\left[\langle A'_1(u_n), v\rangle + \langle A'_2(u_n), v\rangle - \textstyle\int_\Omega g(z, u_n(z))v(z)\, dz\right] \\ &+ t_n\left[\langle A'_1(u_n), v\rangle + \langle A'_2(u_n), v\rangle - \textstyle\int_\Omega g(z, u_n^+(z))v(z)\, dz\right] = 0, \end{aligned}$$

for all $v \in W_0^{1,p(z)}(\Omega)$, which leads to

$$\langle A'_1(u_n), v\rangle + \langle A'_2(u_n), v\rangle = \int_\Omega g(z, u_n^+(z))v(z)\, dz + (1 - t_n)\int_\Omega g(z, -u_n^-(z))v(z)\, dz$$

for all $v \in W_0^{1,p(z)}(\Omega)$. Therefore

$$\begin{cases} -\Delta_{p(z)}u_n(z) - \Delta_{q(z)}u_n(z) = g(z, u_n^+(z)) + (1 - t_n)g(z, -u_n^-(z)) & \text{for a.a. } z \in \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

We know that there exist $a \in]0, 1[$ and $M > 0$ such that

$$u_n \in C^{1,a}(\overline{\Omega}) \quad \text{and} \quad \|u_n\|_{C^{1,a}(\overline{\Omega})} \leq M \quad \text{for all } n \in \mathbb{N}.$$

By (24) and as $C^{1,a}(\overline{\Omega})$ is compactly embedded in $C^1(\overline{\Omega})$, it follows that $u_n \to u_0$ in $C^1(\overline{\Omega})$.

Now, since $u_0 \in D_+$, we can find $n_0 \in \mathbb{N}$ such that $u_n \in D_+$ for all $n \geq n_0$. Then $\{u_n \ : n \geq n_0\}$ are distinct positive solutions of (P_g), which leads to contradiction as K_J must be finite (by assumption). Consequently (24) can not happen and hence we obtain that $C_k(I, u_0) = C_k(J, u_0)$ for all $k \in \mathbb{N}_0$ (it is a direct consequence of the homotopy invariance of critical groups, see [4, Theorem 5.2]). $\square$

By reasoning in a similar way as above, but using the function $G_- : \Omega \times \mathbb{R} \to \mathbb{R}$ given as

$$G_-(z,t) = \int_0^t g(z, -\xi^-)d\xi \quad \text{for all } t \in \mathbb{R},\ z \in \Omega,$$

and the functional $\widehat{J} : W_0^{1,p(z)}(\Omega) \to \mathbb{R}$ given as

$$\widehat{J}(u) = A_1(u) + A_2(u) - \int_\Omega G_-(z, u(z))\, dz, \qquad \text{for all } u \in W_0^{1,p(z)}(\Omega),$$

one can derive the existence of nonpositive solutions for problem (P_g). Indeed, it is immediate to show that the functional $\widehat{J}$ satisfies the (C_c)-condition if and only if it satisfies the (C_c)-condition with respect to all the sequences $\{u_n\} \subset W_0^{1,p(z)}(\Omega)$ such that $u_n(z) \leq 0$ for all $z \in \Omega$, all $n \in \mathbb{N}$. Clearly, Lemma 2 also holds for the functional $\widehat{J}$. The above facts (by Theorem 2) imply that there exists a function $v_0 \in W_0^{1,p(z)}(\Omega)$ that is a nonpositive nontrivial weak solution of problem (P_g).

Theorem 5. *Let the assumptions $(g_1) - (g_5)$ be satisfied. Then problem (P_g) has at least one nonpositive nontrivial weak solution $v_0 \in -C_+$.*

Remark 3. The weak solution v_0 given by Theorem 5 is such that $v_0 \in -D_+$.

A similar line of reasoning as in Proposition 3 allows us to establish the following proposition.

Proposition 5. *Let the assumptions $(g_1) - (g_4)$ be satisfied. Then $C_k(I, v_0) = C_k(\widehat{J}, v_0)$ for all $k \in \mathbb{N}_0$.*

Finally we give the existence theorem which produces three nontrivial weak solutions for problem (P_g).

Theorem 6. *Let the assumptions $(g_1) - (g_5)$ be satisfied. Then problem (P_g) has at least three nontrivial weak solutions.*

Proof. From Theorems 4 and 5, by reasoning as in the proof of Proposition 3, we retrieve the two constant sign solutions $u_0 \in D_+$ and $v_0 \in -D_+$. If we assume $K_I = \{0, u_0, v_0\}$ which means that u_0 and v_0 are the only nontrivial solutions of P_g, then by Proposition 3 we have

$$C_k(I, u_0) = C_k(I, v_0) = \delta_{k,1}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{25}$$

In addition, we have

$$C_k(I, \infty) = 0 \quad \text{for all } k \in \mathbb{N}_0. \tag{26}$$

Finally, we recall that $u = 0$ is local minimizer of $I(\cdot)$. Hence

$$C_k(I, 0) = \delta_{k,0}\mathbb{Z} \quad \text{for all } k \in \mathbb{N}_0. \tag{27}$$

From (25)-(27) and the Morse relation

$$\sum_{u \in K_I} \sum_{k \in \mathbb{N}_0} \text{rank } C_k(I, u)t^k = \sum_{k \in \mathbb{N}_0} \text{rank } C_k(I, \infty)t^k + (1 + t) \sum_{k \in \mathbb{N}_0} \beta_k t^k \quad \text{for all } t \in \mathbb{R},$$

where $\beta_k \in \mathbb{N}$, we infer that there must be a third solution. Indeed for $t = -1$ we get $2(-1)^1 + (-1)^0 = 0$ which leads to the contradiction $(-1)^1 = 0$. Hence, we can find $w_0 \in K_I$ with $w_0 \notin \{0, u_0, v_0\}$. This completes the proof, since w_0 is the third nontrivial solution of Problem (P_g). □

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(N.S. Papageorgiou) Department of Mathematics, National Technical University, Zografou campus, 15780, Athens, Greece
Email address: `npapg@math.ntua.gr`

(C. Vetro) Department of Mathematics and Computer Science, University of Palermo, Via Archirafi 34, 90123, Palermo, Italy
Email address: `calogero.vetro@unipa.it`