HURWITZ MODULI VARIETIES PARAMETERIZING GALOIS COVERS OF AN ALGEBRAIC CURVE

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Abstract. Given a smooth, projective curve Y, a finite group G and a positive integer n we study smooth, proper families $X \to Y \times S \to S$ of Galois covers of Y with Galois group isomorphic to G branched in n points, parameterized by algebraic varieties S. When G is with trivial center we prove that the Hurwitz space $H^G_n(Y)$ is a fine moduli variety for this moduli problem and construct explicitly the universal family. For arbitrary G we prove that $H^G_n(Y)$ is a coarse moduli variety. For families of pointed Galois covers of (Y, y_0) we prove that the Hurwitz space $H^G_n(Y, y_0)$ is a fine moduli variety, and construct explicitly the universal family, for arbitrary group G. We use classical tools of algebraic topology and of complex algebraic geometry.

1. Introduction

Fulton constructed in [19], with an approach via fundamental groups, the Hurwitz spaces, complex manifolds $H^{d,n}$, whose points are in bijective correspondence with the equivalence classes of covers of degree d of $\mathbb{P}^1_{\mathbb{C}}$ simply branched in n points. These manifolds are connected by a classical result of Lüroth, Clebsch and Hurwitz (cf. [19, Proposition 1.5], [57, Lemma 10.15]). Given $d \geq 3$ and $n \geq 2d - 2$, Fulton studied in [19] families of simple covers of $\mathbb{P}^1_{\mathbb{Z}}$ of degree d branched in n points, parameterized by schemes over $\mathbb{Z}$. He constructed a universal family and proved that over $\mathbb{C}$ its parameter scheme, endowed with the canonical complex space structure, is biholomorphic to the Hurwitz space $H^{d,n}$.

The construction of the Hurwitz spaces may be extended as follows. Given a smooth, projective, irreducible curve Y, a transitive subgroup $G \subset S_d$, conjugacy classes $O_1, \ldots, O_k$ in G and positive integers $n_1, \ldots, n_k$, one constructs a complex manifold whose points are in bijective correspondence with the equivalence classes of covers of Y of degree d, whose monodromy group is G, with the following branching data: the number of branch points is $n = n_1 + \cdots + n_k$ and n_i of the branch points have local monodromies in O_i for every i. Similarly one may consider Hurwitz spaces which parameterize Galois covers of Y, with Galois group isomorphic to G and branching data as above, up to G-equivariant isomorphisms over Y. These types of Hurwitz spaces were first introduced by Fried in [17] for covers of $\mathbb{P}^1$ as a tool for the study of the arithmetic of the field extensions of $\mathbb{Q}[t]$, in particular in connection with the Inverse Galois Problem.

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A lot of work by various authors was devoted to determining the connected components of the Hurwitz spaces. To the author's knowledge the strongest result for $G = S_d$, $Y = \mathbb{P}^1$ and branching data with arbitrary set of conjugacy classes $O_1, \ldots, O_k$, $O_i \subset S_d$ was obtained by Kulikov, who proved that the Hurwitz spaces are connected, provided the branching data contains at least $3d-3$ transpositions [36, Theorem 3.3]. This result was extended in [55] to covers of a fixed curve Y of genus ≥ 1. The papers [37, 38, 6] are devoted to determining the number of connected components of the Hurwitz spaces when every n_i of the branching data is large enough. The paper [31] extends the connectivity result of Clebsch and Hurwitz to Hurwitz spaces of Galois covers of $\mathbb{P}^1$ with Galois group isomorphic to a Weyl group and branching data consisting of reflections. The Hurwitz spaces of Galois covers of $\mathbb{P}^1$ with Galois group isomorphic to the dihedral group D_n were studied in [8] and their connectedness was proved when a certain numerical type, related to the branching data is fixed.

Given a projective, nonsingular, irreducible curve Y, a finite group G and a positive integer n, we study smooth, proper families of Galois covers of Y, branched in n points, with Galois group isomorphic to G (G-covers), parameterized by algebraic varieties. We are concerned with the problem of whether the Hurwitz spaces are moduli varieties for appropriate categories of families of G-covers of Y, which means constructing universal families, parameterized by the Hurwitz spaces, or, when such families do not exist, proving that the Hurwitz spaces are coarse moduli varieties (cf. [43, Definition 5.6]). We consider two types of families of covers.

Let $y_0 \in Y$ be a marked point. A smooth, proper family of pointed G-covers of (Y, y_0) branched in n points, parameterized by an algebraic variety S, is a pair of morphisms $(p : X \to Y \times S, \eta : S \to X)$, where $\pi_2 \circ p : X \to S$ is proper, smooth with connected fibers, G acts by automorphisms on X, such that every fiber $p_s : X_s \to Y \times \{s\}$ is a G-cover branched in n points contained in $Y \setminus \{y_0\}$ and $\eta(s) \in p_s^{-1}(y_0)$ for $\forall s \in S$. We prove that the Hurwitz space $H_n^G(Y, y_0)$ which parameterizes the G-equivalence classes of the pointed G-covers of (Y, y_0) branched in n points is a fine moduli variety. Namely, we construct explicitly a family

$$(p : \mathcal{C}(y_0) \to Y \times H_n^G(Y, y_0), \zeta : H_n^G(Y, y_0) \to \mathcal{C}(y_0))$$

and prove that it is universal in the category of families of pointed G-covers of (Y, y_0) branched in n points, parameterized by algebraic varieties.

We denote by $H_n^G(Y)$ the Hurwitz space which parameterizes the G-equivalence classes of the G-covers of Y branched in n points. If the center of G is trivial we prove that $H_n^G(Y)$ is a fine moduli variety. Namely, we construct explicitly a family of G-covers $\pi : \mathcal{C} \to Y \times H_n^G(Y)$ and prove that it is universal in the category of smooth, proper families of G-covers of Y branched in n points, parameterized by algebraic varieties. If G is an arbitrary group we prove that $H_n^G(Y)$ is a coarse moduli variety for this category.

Fixing the branching data, by choosing conjugacy classes $O_1, \ldots, O_k$ in G and positive integers $n_1, \ldots, n_k$ as above, one obtains Hurwitz spaces, which are unions of connected components of $H_n^G(Y, y_0)$ or $H_n^G(Y)$. These Hurwitz spaces are fine moduli varieties for the categories of families of pointed G-covers of (Y, y_0), resp. families of G-covers of Y, provided $Z(G) = 1$, with the prescribed branching data, and they are coarse moduli varieties for these families for arbitrary G.

The problem of constructing the Hurwitz moduli spaces was already studied in [58] and [1] in a more general set-up, over arbitrary algebraically closed fields and families parameterized by schemes. The existence of universal families, or coarse moduli schemes was proved by very complicated constructions in the framework of the theory of stacks. Fulton wrote in [19, p. 547] that over $\mathbb{C}$ it is not difficult to construct analytically the universal families of simple covers of $\mathbb{P}^1$ parameterized by $H^{d,n}$. We give, over $\mathbb{C}$, a simple construction of the universal families of G-covers of Y with an approach via fundamental groups, as explicit as the classical construction of branched covering maps $X \to Y$ of a given Riemann surface [16]. We use classical tools of algebraic topology and of complex algebraic geometry, in particular the GAGA theory [50, 47]. The closely related topic of smooth, proper families of covers of degree d of a fixed curve Y with monodromy group a fixed transitive subgroup G of S_d will be treated in a paper of the author in preparation. The smooth, proper families of pointed covers of (Y, y_0) of degree d with a fixed monodromy group $G \subset S_d$ are studied in [34]

We think that our approach, via fundamental groups, to the Hurwitz spaces, as moduli varieties of appropriate categories of families of covers, will be accessible to a wider range of mathematicians who are interested in the Hurwitz spaces and the familes of covers of a fixed curve.

The covers $X \to Y$ with restricted monodromy group and the related Galois covers $C \to Y$ yield polarized abelian varieties isogenous to abelian subvarieties of the Jacobian variety $J(X)$ [10, 28, 7]. The smooth, proper families of such covers give morphisms of their parameter varieties to certain moduli spaces of polarized abelian varieties by means of the variations of the associated polarized Hodge structures of weight one. This indicates a perspective in the study of the abelian varieties of low dimension and of their moduli by means of the rich geometry of curves. The unirationality of the moduli spaces of three-dimensional abelian varieties $\mathcal{A}_3(1,1,d)$ and $\mathcal{A}_3(1,d,d)$ with $d \leq 4$ was proved in [29, 30] by means of families of simply ramified covers of elliptic curves of degree d branched in 6 points. In [2] it was proved that every sufficiently general principally polarized abelian variety of dimension 6 is isomorphic to a Prym-Tyurin variety of a cover of $\mathbb{P}^1$ of degree 27, branched in 24 points, with monodromy group $W(E_6) \subset S_{27}$.

The Hurwitz spaces of G-covers of $\mathbb{P}^1$ were intensively studied in connection with the Inverse Galois Problem. We refer to [9, 12, 48] for surveys on this subject. The problem of constructing families of covers of $\mathbb{P}^1$ parameterized by the Hurwitz spaces, such that every fiber is a cover of the corresponding equivalence class, was addressed in [18, Section 4] and [11] (see also [57, Chapter 10]). The constructed families, however, are not proper families of curves over the Hurwitz spaces, but families of étale covers of open subsets of $\mathbb{P}^1$.

The monograph [4] is devoted to the Hurwitz schemes (or stacks) and of their natural compactifications. The authors work with equivalence of covers different from the one considered so far and so different are the sets of equivalence classes of covers. Namely two G-covers $\pi : C \to D$ and $\pi' : C' \to D'$ are considered equivalent if there is a G-equivariant isomorphism $f : C \to C'$ and an isomorphism $h : D \to D'$ such that $\pi' \circ f = h \circ \pi$. In comparison, in our set-up $D = D' = Y$ is a fixed curve and $h = id_Y$.

In Section 2 we prove some properties of smooth, proper families of covers of Y, $X \to Y \times S \to S$ related to the branch locus $B \subset Y \times S$.

In Section 3 we give an explicit construction of a smooth, proper family of pointed G-covers of (Y, y_0) branched in n points $(p : \mathcal{C}(y_0) \to Y \times H^G_n(Y, y_0), \zeta)$ such that the fiber over every element of $H^G_n(Y, y_0)$ is a pointed G-cover of the G-equivalence class represented by the element (Theorem 3.20). In Proposition 3.18 we give the explicit form of p locally at the ramification points in analytic coordinates and later in Proposition 7.3 this is done for every smooth, proper family of G-covers of Y.

In Section 4 we give some generalizations of a result of Serre [51, Proposition 20] related to lifting of morphisms.

In Section 5 we prove that the family $(p : \mathcal{C}(y_0) \to Y \times H^G_n(Y, y_0), \zeta)$ constructed in Section 3 is universal, thus proving that $H^G_n(Y, y_0)$ is a fine moduli variety (Theorem 5.5). We mention that a key ingredient in the proof is the use of the criterion for extending morphisms from [33].

Section 6 is devoted to the G-covers of Y branched in n points. We give a structure of an algebraic variety of the Hurwitz space $H^G_n(Y)$ by patching affine charts $U(y)$, $y \in Y$, which are quotients of $H^G_n(Y, y)$ with respect to a natural action of $G/Z(G)$ (Proposition 6.6). If the center $Z(G)$ of G is trivial we construct a smooth, proper family of G-covers of Y branched in n points, $\pi : \mathcal{C} \to Y \times H^G_n(Y)$, such that the fiber over every point of $H^G_n(Y)$ is a G-cover of the G-equivalence class represented by the point (Theorem 6.14).

In Section 7 we prove that the family $\pi : \mathcal{C} \to Y \times H^G_n(Y)$ is universal, provided G has trivial center, thus proving that $H^G_n(Y)$ is a fine moduli variety (Theorem 7.4). If G is arbitrary, we prove in Theorem 7.6, verifying the conditions of [43, Definition 5.6], that $H^G_n(Y)$ is a coarse moduli variety for the category of smooth, proper families of G-covers of Y branched in n points, parameterized by algebraic varieties. The construction of $\pi : \mathcal{C} \to Y \times H^G_n(Y)$ as well as the proofs of Theorem 7.4 and Theorem 7.6 are reduced to the universal family $(p : \mathcal{C}(y_0) \to Y \times H^G_n(Y, y_0), \zeta)$ of pointed G-covers by means of a second action of G on $\mathcal{C}(y_0)$, constructed in Section 6, which lifts a natural action of G on $Y \times H^G_n(Y, y_0)$ and commutes with the action of G relative to the Galois cover $p : \mathcal{C}(y_0) \to Y \times H^G_n(Y, y_0)$. Finally in Theorem 5.8, Theorem 7.8 and Theorem 7.9 we give variants of the main theorems in which the families of G-covers of Y have local monodromies at the branch points in fixed conjugacy classes of G.

Notation and conventions. We assume the base field is $\mathbb{C}$. Algebraic varieties are reduced, separated, possibly reducible schemes of finite type, *points* are closed points. Fiber products and pullbacks are those defined in the category of schemes over $\mathbb{C}$. A cover $f : X \to Y$ of algebraic varieties is a finite, surjective morphism. If G is a finite group which acts faithfully by automorphisms on X, i.e. $G \to Aut(X)$ is injective, f is G-invariant, and $\overline{f} : X/G \to Y$ is an isomorphism, then f is a Galois cover with Galois group isomorphic to G. Given an algebraic variety $(X, \mathcal{O}_X)$ the canonically associated complex space is denoted by $(X^{an}, \mathcal{O}_{X^{an}})$ [47]. Its topological space is denoted by $|X^{an}|$. Given a topological space M and two paths $\alpha : I \to M$ and $\alpha' : I \to M$, $I = [0, 1]$, we write $\alpha \sim \alpha'$ if α' has the same end points as α and is homotopic to α (with homotopy leaving the endpoints fixed) [40, Chapter 2, § 2]. The set of paths homotopic to α is denoted by $[\alpha]$. The product of the paths α and β is denoted by $\alpha \cdot \beta$ and equals the path $\gamma : I \to M$, where $\gamma(t) = \alpha(2t)$ if $t \in [0, \frac{1}{2}]$, $\gamma(t) = \beta(2t - 1)$ if $t \in [\frac{1}{2}, 1]$. Given a covering space $p : M \to N$ of the topological space N, the map p is called topological covering map. Lifting a path α of N from initial point $z \in M$ the end point is denoted by $z\alpha$.

2. Smooth families of covers of a curve

Throughout the paper, with the exception of Section 4, Y is a smooth, projective, irreducible curve of genus $g \geq 0$, n is a positive integer and G is a finite group.

2.1. We recall some facts about the Hilbert scheme $Y^{[n]}$ which parameterizes the 0-dimensional subschemes of length n of Y [14]. There is a bijective correspondence between the effective divisors of Y of degree n and the 0-dimensional subschemes of Y of length n: every divisor $D = \sum_{i=1}^{r} n_i y_i$ corresponds to the closed subscheme of Y whose closed subset is $\mathrm{Supp}(D) = \{y_1, \ldots, y_r\}$ and the structure sheaf is the skyscraper sheaf $\oplus_{i=1}^{r} \mathcal{O}_{Y,y_i}/\mathfrak{m}_{y_i}^{n_i}$. Abusing the notation we will denote it again by D. We write $\deg D = n = \ell(D)$.

2.2. Let $Y^{(n)}$ be the symmetric product $Y^{(n)} = Y^n/S_n$ (cf. [52, Ch. III § 14]). This is a projective variety [25, Lecture 10] and it parameterizes the effective divisors of Y of degree n. We denote by $Y^{(n)}_*$ the open subset, which corresponds to the divisors without multiple points. It is the complement of the quotient Δ/S_n, where $\Delta \subset Y^n$ is the big diagonal. For every partition $\nu = (n_1, \ldots, n_r)$, of length $\ell(\nu) = r$, $n_1 \geq \cdots \geq n_r$, $n_1 + \cdots + n_r = n$, let us denote by $Y^{(n)}_\nu$ the set

$$Y^{(n)}_\nu = \{n_1 y_1 + \cdots + n_r y_r | y_i \neq y_j \text{ for } i \neq j\}.$$

Let us denote by $Y^{(n)}_r$ the set $\{D \in Y^{(n)} || \mathrm{Supp}(D)| = r\}$ and by $Y^{(n)}_{\leq r}$ the set $\{D \in Y^{(n)} || \mathrm{Supp}(D)| \leq r\}$. Consider the composition of morphisms $Y^r \to Y^n \to Y^{(n)}$, where the first one is

$$(y_1, \ldots, y_r) \mapsto (y_1, \underset{n_1}{\ldots}, y_1, \ldots, y_r, \underset{n_r}{\ldots}, y_r).$$

Its image is a closed, irreducible subset of $Y^{(n)}$ equal to the closure $\overline{Y^{(n)}_\nu}$. One has $Y^{(n)}_{\leq r} = \bigsqcup_{\ell(\nu)\leq r} \overline{Y^{(n)}_\nu}$ and $Y^{(n)}_\nu = \overline{Y^{(n)}_\nu} \setminus Y^{(n)}_{\leq r-1}$. Therefore $Y^{(n)}_\nu$ is an irreducible locally closed subset of $Y^{(n)}$ of dimension r. Represent ν as $(1^{r_1}, 2^{r_2}, \ldots, s^{r_s})$, where r_i is the number of times i occurs in $(n_1, \ldots, n_r)$. Every $D \in Y^{(n)}_\nu$ may be written in a unique way as $D = D_1 + 2D_2 + \cdots + sD_s$ where the divisor $D = D_1 + D_2 + \cdots + D_s$ has no multiple points. Let us denote the set of such s-tuples by $\left(Y^{(r_1)} \times \cdots \times Y^{(r_s)}\right)_*$. One obtains a bijective map

$$\left(Y^{(r_1)} \times \cdots \times Y^{(r_s)}\right)_* \longrightarrow Y^{(n)}_\nu \tag{1}$$

Let us denote by A the universal divisor $A = \{(y, D) | y \in \mathrm{Supp}\, D\}$, $A \subset Y \times Y^{(n)}$. Let $\Delta_Y \subset Y \times Y$ be the diagonal. Then A is the image of $\Delta_Y \times Y^{n-1}$ with respect to the quotient morphism $Y \times Y^n \to Y \times Y^{(n)}$, so A is an irreducible, closed subvariety of $Y \times Y^{(n)}$ of codimension 1.

Proposition 2.3. *Let (n, r) be a pair of positive integers, such that $r \in [1, n]$. In the set-up of § 2.2 the following properties hold:*

(i) *$Y^{(n)}$ is isomorphic to the Hilbert scheme $Y^{[n]}$ which parameterizes the 0-dimensional subschemes of Y of length n and the closed subscheme $A \subset Y \times Y^{(n)}$ is the corresponding universal family.*

(ii) *The set $Y^{(n)}_r$ is locally closed and it is a disjoint union of the irreducible, locally closed subsets $Y^{(n)}_\nu$ with $\ell(\nu) = r$, which are moreover smooth of dimension r.*

(iii) *For every partition* ν *of* n *of length* r*, represented in the form* $(1^{r_1},\dots,s^{r_s})$*,* $r_1+\dots+r_s=r$ *the map* $\left(Y^{(r_1)}\times\dots\times Y^{(r_s)}\right)_* \longrightarrow Y^{(n)}_\nu$ *given by*

$$(D_1,\dots,D_s)\mapsto D_1+2D_2+\dots+sD_s \tag{2}$$

is an isomorphism.

Proof. (i) The variety $Y\times Y^{(n)}$ is smooth, so A is an effective Cartier divisor. The projection $p:A\to Y^{(n)}$ is proper with finite fibers, so by Zariski's main theorem it is finite. Furthermore it is surjective and flat. Indeed, let $a\in A$, $b=p(a)$. Let $\mathcal{O}_a$ and $\mathcal{O}_b$ be the fibers of the structure sheaves of $Y\times Y^{(n)}$ and $Y^{(n)}$ respectively. Let $J_{A,a}=(f)$. Applying [41, 20.E] to $u:\mathcal{O}_a\to\mathcal{O}_a$, where $u(x)=xf$, one concludes that $\mathcal{O}_{A,a}=\mathcal{O}_a/(f)$ is $\mathcal{O}_b$-flat. The variety $Y^{(n)}$ is irreducible, so $p_*\mathcal{O}_A$ is a locally free sheaf of rank n, hence every fiber of $p:A\to Y^{(n)}$ is of length n. By the universal property of Hilbert schemes there exists a unique classifying morphism $\varphi:Y^{(n)}\to Y^{[n]}$ such that $A\subset Y\times Y^{(n)}$ is the pullback of the universal family $\mathcal{W}\subset Y\times Y^{[n]}$. Now, $Y^{[n]}$ is a smooth scheme [49, Theorem 4.3.5] and φ induces a bijection of the closed points of $Y^{(n)}$ and $Y^{[n]}$ (cf. § 2.1). Therefore φ is an isomorphism.

(ii) One has $Y^{(n)}_r = Y^{(n)}_{\le r}\setminus Y^{(n)}_{\le r-1}$, so $Y^{(n)}_r$ is locally closed. Clearly $Y^{(n)}_r=\bigsqcup_{\ell(\nu)=r}Y^{(n)}_\nu$. The last claim follows from (iii).

(iii) Consider the map $\psi:Y^{(r_1)}\times\dots\times Y^{(r_s)}\to Y^{(n)}$ given by (2). It is the quotient by S_n and by $S_{r_1}\times\dots\times S_{r_s}$ of the product of the diagonal morphisms $Y^{r_1}\times\dots\times Y^{r_s}\to Y^n$, therefore ψ is a morphism. Its image is closed, irreducible and equals $\overline{Y^{(n)}_\nu}$. The map of (iii) is the restriction of ψ on the preimage of $\overline{Y^{(n)}_\nu}\setminus\overline{Y^{(n)}_{\le r-1}}$ and it is bijective. By [26, Corollary 14.10] it suffices to verify that the differential $d\psi$ is injective at every point of $\left(Y^{(r_1)}\times\dots\times Y^{(r_s)}\right)_*$. This holds since for every $m\in\mathbb{N}$ the morphism $Y\to Y^{(m)}$, $y\mapsto my$, has injective differential at every point. Indeed, let $p\in Y$ and let $U\ni y$ be an embedded open disk with $t:U\to\mathbb{C}$ a coordinate at p. Then U^m/S_m is a coordinate neighborhood of mp in $Y^{(m)}$ with local coordinates the m elementary symmetric polynomials of $t\circ p_i:U^m\to U\to\mathbb{C}$, $i=1,\dots,m$. The map $U\to U^m/S_m$ has the form $t\mapsto(mt,\dots,\binom{m}{i})t^i,\dots)$ with derivative $(m,0,\dots,0)$ at $t=0$. □

Definition 2.4. Let n be a positive integer. Let X and S be algebraic varieties. A morphism $f:X\to Y\times S$ is called a *smooth family of covers of* Y *branched in* n *points* if $\pi_2\circ f:X\to Y\times S$ is a proper, smooth morphism such that for every $s\in S$ the fiber X_s is an irreducible curve and $f_s:X_s\to Y$ is a cover branched in n points

Lemma 2.5. *Let* $p:M\to N$ *be a finite morphism of algebraic varieties. Let* G *be a finite group which acts by automorphisms on* M *so that* p *is* G*-invariant.*

(i) *The quotient set* M/G *and the quotient map* $M\to M/G$ *have a structure of an algebraic variety and a finite morphism and* p *equals the composition of the induced finite morphisms* $M\to M/G\to N$.
(ii) *Suppose that* $M/G\to N$ *is an isomorphism. Then* $|M^{an}|/G\to|N^{an}|$ *is a homeomorphism.*

Proof. (i) Let $x \in M$. Let U be an affine open set in N which contains $p(x)$. Then $p^{-1}(U)$ is an affine open set which contains the orbit Gx. Apply [52, Ch. III Prop. 19].

(ii) $p^{an} : M^{an} \to N^{an}$ is a finite holomorphic map [47, Prop. 3.2(vi)]. The induced map $|M^{an}|/G \to |N^{an}|$ is bijective and continuous. It is a closed map, in fact, the image of every closed subset $Z \subset |M^{an}|/G$ equals the image by p^{an} of its preimage in $|M^{an}|$ which is closed since p^{an} is a finite map. Therefore $|M^{an}|/G \to |N^{an}|$ is a homeomorphism. □

Proposition 2.6. *Let n be a positive integer. Let $f : X \to Y \times S$ be a smooth family of covers of Y branched in n points. Then*

(i) *f is finite, surjective and flat.*
(ii) *The discriminant scheme D of $f : X \to Y \times S$ (cf.* [3, Ch. VI n.6]*) is an effective relative Cartier divisor with respect to $\pi_2 : Y \times S \to S$ (cf.* [44, Lecture 10]*).*
(iii) *Let $B \subset Y \times S$ be the support of D. Let $X' = f^{-1}(Y \times S \setminus B)$. Then $f|_{X'} : X' \to Y \times S \setminus B$ is a finite, étale, surjective morphism.*
(iv) *$|X^{an}| \setminus f^{-1}(B) \to |(Y \times S)^{an}| \setminus B$ is a topological covering map.*
(v) *Suppose S is connected. Then there exists an integer N such that $\ell(D_s) = N$ for $\forall s \in S$.*
(vi) *For every $s \in S$ let B_s be the branch locus of $f_s : X_s \to Y$. Then the map $\beta : S \to Y^{(n)}$ given by $\beta(s) = B_s$ is a morphism.*
(vii) *The projection $B \to S$ is finite, étale, surjective of degree n.*
(viii) *Let G be a finite group which acts by automorphisms on X, so that f is G-invariant and the morphism $X_s/G \to Y$ induced by f_s is an isomorphism for every $s \in S$. Then the morphism $X/G \to Y \times S$ induced by f is an isomorphism.*

Proof. (i) For every $s \in S$, $f_s : X_s \to Y \times \{s\}$ is a finite, surjective morphism, so $f : X \to Y \times S$ is surjective with finite fibers. It is proper since $\pi_2 \circ f : X \to S$ is proper by hypothesis (cf. [39, Ch. 3 Prop. 3.16]). By Zariski's main theorem $f : X \to Y \times S$ is finite. The morphisms $X \to S$ and $Y \times S \to S$ are flat and for every $s \in S$, $X_s \to Y \times \{s\}$ is flat, therefore $f : X \to Y \times S$ is flat (cf. [41, (20.G)]).

(ii) The statement is local, so we may assume that S is connected. By (i) $f_* \mathcal{O}_X$ is a locally free sheaf. The discriminant ideal sheaf J_D is the image of the invertible sheaf $(\wedge^{\max} f_* \mathcal{O}_X)^{\otimes 2} \to \mathcal{O}_{Y \times S}$ (cf. [3, p.124]). Let $z \in \operatorname{Supp} D$, $s = \pi_2(z)$ and let d_z be the generator of $(J_D)_z$. Consider the local homomorphism $\mathcal{O}_s = \mathcal{O}_{S,s} \to \mathcal{O}_{Y \times S, z} = \mathcal{O}_z$. The image of d_z in $\mathcal{O}_{Y \times S, z} \otimes \mathbb{C}(s) = \mathcal{O}_{Y \times \{s\}, z}$ generates the discriminant ideal of $X_s \to Y \times \{s\}$ at the point z, so it is a non-zero-divisor. Applying [41, (20.E)] to $u : \mathcal{O}_z \to \mathcal{O}_z$. where $u(a) = a d_z$ one concludes that d_z is a non-zero-divisor in $\mathcal{O}_z$ and $\mathcal{O}_{D,z} = \mathcal{O}_z/(d_z)$ is $\mathcal{O}_s$-flat. Therefore D is an effective Cartier divisor of $Y \times S$ and $\pi_2|_D : D \to S$ is flat. This proves (ii).

(iii) This follows from [3, Ch. 6 Proposition (6.6)].

(iv) $f^{an} : X^{an} \setminus f^{-1}(B) \to (Y \times S)^{an} \setminus B$ is unramified and flat by [47, Prop. 3.1(iii)], hence it is locally biholomorphic by [23, Théorème 3.1]. Furthermore it is proper by [47, Prop. 3.2(v)], hence it is a topological covering map by [16, Prop. 4.22].

(v) The projection $g = \pi_2|_D : D \to S$ is a finite, surjective, flat morphism of schemes. Since S is connected, there exists an integer N such that $g_*\mathcal{O}_D$ is a locally free sheaf of rank N. One has $\ell(D_s) := h^0(\mathcal{O}_{D_s}) = N$ for every $s \in S$.

(vi) We may assume, without loss of generality, that S is connected. For every $s \in S$ one has $B_s = \operatorname{Supp} D_s$, $\ell(D_s) = N$, $|B_s| = n$. By (ii) the closed subscheme D of $Y \times S$ is a flat family of 0-dimensional subschemes of Y of length N. Let us apply Proposition 2.3 for the pair (N, n). The classifying morphism $h : S \to Y^{(N)}$ has image contained in $Y^{(N)}_n = \bigsqcup_{\nu, \ell(\nu)=n} Y^{(N)}_\nu$, therefore this image is contained in the locally closed subset $Y^{(N)}_\nu$ for some partition ν of N. Write ν in the form $(1^{n_1}, 2^{n_2}, \dots, k^{n_k})$, where $n_1 + \cdots + n_k = n$ and $n_1 + 2n_2 + \cdots kn_k = N$. Then the map $\beta : S \to Y^{(n)}_* \subset Y^{(n)}$ is the composition of morphisms

$$S \xrightarrow{h} Y^{(N)}_\nu \longrightarrow (Y^{(n_1)} \times \cdots \times Y^{(n_k)})_* \longrightarrow Y^{(n)}_*$$

where the middle one is the inverse of the isomorphism of Proposition 2.3(iii) and the last one is obtained from $Y^{n_1} \times \cdots \times Y^{n_k} \xrightarrow{=} Y^n$ taking the quotient of Y^n by S_n and of $Y^{n_1} \times \cdots \times Y^{n_k}$ by $S_{n_1} \times \cdots \times S_{n_k}$.

(vii) The universal divisor $A \subset Y \times Y^{(n)}$ has the property that the projection $A \to Y^{(n)}$ is finite, surjective, flat and is unramified over $Y^{(n)}_*$. The morphism $\beta : S \to Y^{(n)}$ has image contained in $Y^{(n)}_*$, so the pullback $A_S = S \times_{Y^{(n)}} A$ is a closed subscheme of $Y \times S$ and the morphism $A_S \to S$ is finite, surjective, flat and unramified. The underlying reduced subscheme of A_S is B and A_S is reduced by [42, p.184], so A_S coincides with B.

(viii) By (i) and Lemma 2.5 the morphism $X/G \to Y \times S$ induced by f is finite. It is bijective, and fits in the commutative diagram

$$\begin{tikzcd} X/G \arrow[rr] \arrow[dr] & & Y \times S \arrow[dl] \\ & S & \end{tikzcd}$$

whose vertical morphisms are proper. The scheme-theoretical fibers of $X/G \to S$ (over the closed points of S) are isomorphic to X_s/G by [35, Prop. A.7.1.3]. The assumption that $X_s/G \to Y \times \{s\}$ is an isomorphism for every $s \in S(\mathbb{C})$ implies by [22, Prop. 4.6.7] that every $s \in S(\mathbb{C})$ has an open neighborhood U such that $(X/G)_U \to Y \times U$ is a closed embedding. This implies that $X/G \to Y \times S$ is an isomorphism since X/G and $Y \times S$ are reduced schemes. $\square$

3. Parameterization of pointed G-covers

In the rest of the paper the elements $D \in Y^{(n)}_*$ are considered as subsets of Y of cardinality n. We start with some definitions and recall some known facts (see e.g. [32, Section 1])

Definition 3.1. Let G be a finite group.

(i) A G-cover of Y is a cover $p : C \to Y$, where C is a smooth, irreducible, projective curve such that G acts faithfully on the left on C by automorphisms of C, p is G-invariant and $\overline{p} : C/G \to Y$ is an isomorphism.

(ii) Two G-covers of Y, $p : C \to Y$ and $p_1 : C_1 \to Y$ are called G-equivalent if there exists a G-equivariant isomorphism $f : C \to C_1$ such that $p = p_1 \circ f$.

If the center of G is trivial, $Z(G) = 1$, such an isomorphism is unique if it exists.

(iii) Let $y_0 \in Y$. A pointed G-cover of (Y, y_0) is a couple $(p : C \to Y, z_0)$, where $p : C \to Y$ is a G-cover unramified at y_0 and $z_0 \in p^{-1}(y_0)$.

(iv) Let $(p : C \to Y, z_0)$ and $(p_1 : C_1 \to Y, w_0)$ be two pointed G-covers of (Y, y_0). They are called G-equivalent if there is a G-equivariant isomorphism $f : C \to C_1$ such that $p = p_1 \circ f$ and $f(z_0) = w_0$. Such an isomorphism is unique if it exists.

3.2. Let $(p : C \to Y, z_0)$ be a pointed G-cover of (Y, y_0) branched in n points, $n \geq 1$. Let $D = \{b_1, \dots, b_n\}$ be its branch locus. Let $C' = p^{-1}(Y \setminus D)$, $p' = p|_{C'}$. Endowing C and Y with the canonical Euclidean topologies of $|C^{an}|$ and $|Y^{an}|$ respectively, $p' : C' \to Y \setminus D$ is a topological covering map.

Let $\alpha : I \to Y \setminus D$, $I = [0, 1]$, be a closed path with $\alpha(0) = \alpha(1) = y_0$. Let us denote by $z_0\alpha$ the end point of its lifting $\alpha'_{z_0} : I \to C'$ with initial point $\alpha'_{z_0}(0) = z_0$. Let $g \in G$ be the unique element such that $gz_0 = \alpha'_{z_0}(1) = z_0\alpha$. One associates in this way with every element $[\alpha] \in \pi_1(Y \setminus D, y_0)$ an element $g \in G$. We let $g = m_{z_0}([\alpha])$. The map $m_{z_0} : \pi_1(Y \setminus D, y_0) \to G$ is a surjective homomorphism.

Let $\overline{U}_1, \dots, \overline{U}_n$ be embedded closed disks in $Y \setminus y_0$ which are disjoint and such that $b_i \in U_i$ for $\forall i$, where U_i is the interior of $\overline{U}_i$. For every $i = 1, \dots, n$ let us choose a path $\eta_i : I \to Y \setminus \cup_{j=1}^{n} U_j$ such that $\eta_i(0) = y_0$, $\eta_i(1) \in \partial\overline{U}_i$ and let $\gamma_i : I \to Y \setminus D$ be the closed path which starts at y_0, travels along η_i, then makes a counterclockwise loop along $\partial\overline{U}_i$ and returns back to y_0 along η_i^-. The condition that the branch locus of $p : C \to Y$ equals D is equivalent to the condition that $p' : C \setminus p^{-1}(D) \to Y \setminus D$ is unramified and

$$m_{z_0}([\gamma_1]) \neq 1, \dots, m_{z_0}([\gamma_n]) \neq 1. \tag{3}$$

Let $i \in [1, n]$. Varying $\overline{U}_1, \dots, \overline{U}_n$ and $\eta_1, \dots, \eta_n$ the elements $m_{z_0}([\gamma_i])$ belong to the same conjugacy class of G.

Definition 3.3. Given a pointed G-cover $(C, z_0) \to (Y, y_0)$ branched in D, $D \subset Y \setminus y_0$ the homomorphism $m_{z_0} : \pi_1(Y \setminus D, y_0) \to G$ and the pair (D, m_{z_0}) are called respectively *the monodromy homomorphism* and *the monodromy invariant* associated with the pointed G-cover.

Remark 3.4. We will use the terminology pointed topological G-covering map and monodromy homomorphism also for topological Galois covering maps $p : M \to N$, $p(z_0) = y_0$, where M and N are connected, locally connected topological spaces and $\theta : G \to \mathrm{Deck}(M/N)$ is a fixed isomorphism with the group of covering transformations of $p : M \to N$.

3.5. Let $y_0 \in Y$. Associating with a pointed G-cover $(C, z_0) \to (Y, y_0)$ its monodromy invariant (D, m_{z_0}) Riemann's existence theorem establishes a one-to-one correspondence between the set of G-equivalence classes $[p : C \to Y, z_0]$ of pointed G-covers of (Y, y_0) branched in n points and the set of pairs (D, m), where $D \in (Y \setminus y_0)^{(n)}_*$ and $m : \pi_1(Y \setminus D, y_0) \to G$ is a surjective homomorphism which satisfies Condition (3). We briefly recall why this correspondence is bijective (cf. [40], [16], [57], [32, Prop. 1.3]).

Let $p : (C, z_0) \to (Y, y_0)$ and $p_1 : (C_1, z_1) \to (Y, y_0)$ be two pointed G-covers with the same monodromy invariant (D, m). Let $C' = p^{-1}(Y \setminus D)$, $C'_1 = p_1^{-1}(Y \setminus D)$. Then there is a G-equivariant covering homeomorphism $f' : |C'^{an}| \to$

$|C_1'^{an}|$ such that $f'(z_0) = z_1$. It is biholomorphic since both $p^{an}|_{C'^{an}}$ and $p_1^{an}|_{C_1'^{an}}$ are locally biholomorphic and it can be extended to a G-equivariant biholomorphic covering map of the Riemann surfaces $f : C^{an} \to C_1^{an}$, which yields a G-equivariant covering isomorphism of the algebraic curves C and C_1. Hence the correspondence is injective.

Given a pair (D, m) one constructs a pointed G-cover $(C, z_0) \to (Y, y_0)$ whose monodromy invariant is (D, m) as follows. Let $\Gamma = \mathrm{Ker}(m)$. Let

$$C' = \{\Gamma[\alpha] | \alpha : I \to Y \setminus D \text{ is a path with } \alpha(0) = y_0\}. \tag{4}$$

Here $[\alpha]$ is the homotopy class of α in $Y \setminus D$. The map $p' : C' \to Y \setminus D$ is defined by $p'(\Gamma[\alpha]) = \alpha(1)$. One lets $z_0 = \Gamma[c_{y_0}]$, where c_{y_0} is the constant path $c_{y_0}(t) = y_0$ for $\forall t \in I$. One defines an action of G on C as follows: if $z = \Gamma[\alpha]$ and $g = m([\sigma])$, one lets $gz = \Gamma[\sigma \cdot \alpha]$. The group G acts transitively without fixed points on the fibers of $p' : C' \to Y \setminus D$. One endows C' with a Hausdorff topology by the following basis of open sets: for every path $\alpha : I \to Y \setminus D$ with $\alpha(0) = y_0$ and every embedded open disk $U \subset Y \setminus D$ which contains $\alpha(1)$ one lets

$$N_\alpha(U) = \{\Gamma[\alpha \cdot \tau] | \tau : I \to U \text{ is a path such that } \tau(0) = \alpha(1)\}$$

Then (C', p') is a connected covering space of $Y \setminus D$, the group G acts by covering transformations and p' induces a homeomorphism $C'/G \xrightarrow{\sim} Y \setminus D$. Let us verify that the monodromy homomorphism of the pointed topological G-covering map $p' : (C', z_0) \to (Y \setminus D, y_0)$ coincides with $m : \pi_1(Y \setminus D, y_0) \to G$. Let $\sigma : I \to Y \setminus D$ be a path in $Y \setminus D$ with $\sigma(0) = y_0$. The map $\tilde{\sigma} : I \to C'$ defined by $\tilde{\sigma}(s) = \Gamma[\sigma_s]$, where $\sigma_s : I \to Y \setminus D$ is the path $\sigma_s(t) = \sigma(st)$, is a lifting of σ in C' with initial point $z_0 = \Gamma[c_{y_0}]$ and terminal point $\tilde{\sigma}(1) = \Gamma[\sigma]$. In particular, if σ is a closed path and $m([\sigma]) = g$, the terminal point of $\tilde{\sigma}$ is $\Gamma[\sigma] = gz_0$. This shows that the monodromy homomorphism of $p' : (C', z_0) \to (Y \setminus D, y_0)$ coincides with $m : \pi_1(Y \setminus D, y_0) \twoheadrightarrow G$.

One endows C' with the unique complex analytic structure such that $p' : C' \to Y \setminus D$ is a holomorphic, locally biholomorphic map. Compactifying one obtains a holomorphic map of compact Riemann surfaces $p : C \to Y$ branched in D and the action of G on C' is extended to an action of G on C by biholomorphic maps. Finally C has a structure of a projective, nonsingular, irreducible curve, whose associated structure of a complex analytic variety coincides with the one above, $p : C \to Y$ is a morphism, the action of G is by algebraic automorphisms and $p : C \to Y$ is a Galois cover with Galois group G.

Definition 3.6. Let $g = g(Y)$, let n be a positive integer. Let G be a finite group which can be generated by $2g + n - 1$ elements. Let $y_0 \in Y$. We denote by $H_n^G(Y, y_0)$ the set of pairs (D, m) where $D \in (Y \setminus y_0)_*^{(n)}$ and $m : \pi_1(Y \setminus D, y_0) \twoheadrightarrow G$ is a surjective homomorphism which satisfies Condition (3).

3.7. The set $H_n^G(Y, y_0)$ is nonempty and it is bijective to the set of G-equivalence classes $[p : C \to Y, z_0]$ of pointed G-covers of (Y, y_0) (cf. § 3.5). One endows $H_n^G(Y, y_0)$ with a Hausdorff topology as follows. Let $D = \{b_1, \ldots, b_n\}$, let $\overline{U}_1, \ldots, \overline{U}_n$ be embedded closed disks in $Y \setminus y_0$ which are disjoint and such that $b_i \in U_i$ for $\forall i$, where U_i is the interior of $\overline{U}_i$. Let $N_D(U_1, \ldots, U_n) \subset (Y \setminus y_0)_*^{(n)}$ be the open set consisting of $E = \{y_1, \ldots, y_n\}$ such that $y_i \in U_i$ for every i. The inclusion $Y \setminus \cup_{i=1}^n U_i \hookrightarrow Y \setminus D$ is a deformation retract, so for every homomorphism

$m : \pi_1(Y \setminus D, y_0) \to G$ and every $E \in N_D(U_1, \dots, U_n)$ there is a unique homomorphism $m(E) : \pi_1(Y \setminus E, y_0) \to G$ such that the following diagram commutes

$$\begin{tikzcd} \pi_1(Y \setminus D, y_0) \arrow[dr, "m"'] & \pi_1(Y \setminus \cup_{i=1}^n U_i, y_0) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d] & \pi_1(Y \setminus E, y_0) \arrow[dl, "m(E)"] \\ & G & \end{tikzcd} \tag{5}$$

Given a closed path $\gamma : I \to Y \setminus E$ based at y_0 we denote by $[\gamma]_E$ its homotopy class in $Y \setminus E$. The homomorphism $m(E)$ is uniquely determined by the following property. For every closed path $\gamma : I \to Y \setminus \cup_{i=1}^n \overline{U}_i$ based at y_0 the following equality holds:

$$m([\gamma]_D) = m(E)([\gamma]_E) \text{ for } \forall E \in N_D(U_1, \dots, U_n). \tag{6}$$

We recall some known facts about $H_n^G(Y, y_0)$ (see e.g. [32, Section 1]). Let

$$N_{(D,m)}(U_1, \dots, U_n) = \{(E, m(E)) | E \in N_D(U_1, \dots, U_n)\}. \tag{7}$$

One defines Hausdorff topology on $H_n^G(Y, y_0)$ by choosing as a basis the family of all sets $N_{(D,m)}(U_1, \dots, U_n)$. Let $\delta : H_n^G(Y, y_0) \to (Y \setminus y_0)_*^{(n)}$ be the map defined by $\delta((D, m)) = D$. Then δ is a topological covering map. The topological covering space $H_n^G(Y, y_0)$ inherits the structure of a complex manifold from $(Y \setminus y_0)_*^{(n)}$ and δ is a holomorphic map. Furthermore by Théorème 5.1, Proposition 3.1 and Proposition 3.2 of [47] $H_n^G(Y, y_0)$ has a structure of an algebraic variety, $\delta : H_n^G(Y, y_0) \to (Y \setminus y_0)_*^{(n)}$ is a finite, étale, surjective morphism, and the associated complex analytic space and holomorphic map coincide with the ones defined above. Furthermore the algebraic variety $H_n^G(Y, y_0)$ is nonsingular and affine since this is true for $(Y \setminus y_0)_*^{(n)}$.

Consider the subset $B \subset Y \times H_n^G(Y, y_0)$ defined by

$$B = \{(y, (D, m)) | y \in D\}. \tag{8}$$

The map $id_Y \times \delta : Y \times H_n^G(Y, y_0) \to Y \times (Y \setminus y_0)_*^{(n)}$ is a finite, étale, surjective morphism and B is the preimage of the universal divisor $Y \times (Y \setminus y_0)_*^{(n)} \cap A$.

Definition 3.8. Let G be a finite group. Let n be a positive integer. Let $\mathcal{C}$ and S be algebraic varieties. A morphism $p : \mathcal{C} \to Y \times S$ is called a *smooth family of G-covers of Y branched in n points* if:

(i) p satisfies the conditions of Definition 2.4;
(ii) G acts on $\mathcal{C}$ on the left by automorphisms of $\mathcal{C}$, $p : \mathcal{C} \to Y \times S$ is G-invariant and $p_s : \mathcal{C}_s \to Y \times \{s\}$ is a G-cover for $\forall s \in S$ (cf. Definition 3.1).
(iii) Two such families $p : \mathcal{C} \to Y \times S$ and $p_1 : \mathcal{C}_1 \to Y \times S$ are called G-equivalent if there exists a G-equivariant isomorphism $f : \mathcal{C} \to \mathcal{C}_1$ such that $p = p_1 \circ f$.

It is clear that two families are equivalent if and only if there is an S-isomorphism $f : \mathcal{C} \to \mathcal{C}_1$ which is a covering G-isomorphism over Y for $\forall s \in S$, i.e. $p_s = (p_1)_s \circ f_s$ for $\forall s \in S$.

Definition 3.9. Let $y_0 \in Y$. A smooth family of pointed G-covers of (Y, y_0) branched in n points is a pair $(p : \mathcal{C} \to Y \times S, \zeta : S \to \mathcal{C})$, where p satisfies the conditions of Definition 3.8, $p_s : \mathcal{C}_s \to Y \times \{s\}$ is unramified at (y_0, s) for $\forall s \in S$ and $\zeta : S \to \mathcal{C}$ is a morphism such that $\zeta(s) \in p_s^{-1}(y_0, s)$ for $\forall s \in S$. Two such families $(p : \mathcal{C} \to Y \times S, \zeta)$ and $(p_1 : \mathcal{C}_1 \to Y \times S, \zeta_1)$ are called G-equivalent if there exists a G-equivariant isomorphism $f : \mathcal{C} \to \mathcal{C}_1$ such that $p = p_1 \circ f$ and $\zeta_1 = f \circ \zeta$

If two families are G-equivalent, then the G-equivariant isomorphism $f : \mathcal{C} \to \mathcal{C}_1$ is unique in the case of Definition 3.9 and, provided G has trivial center, it is unique in the case of Definition 3.8.

3.10. Our goal in this section is: given a positive integer n and a point $y_0 \in Y$ to construct a smooth family of pointed G-covers of (Y, y_0) branched in n points

$$\big(p : \mathcal{C}(y_0) \to Y \times H^G_n(Y, y_0), \zeta : H^G_n(Y, y_0) \to \mathcal{C}(y_0)\big)$$

with the property that every fiber $\big(\mathcal{C}(y_0)_{(D,m)} \to Y, \zeta(D, m)\big)$ is a pointed G-cover of (Y, y_0) with monodromy invariant (D, m). This is obtained by the following steps:

(i) One constructs explicitly a set $\mathcal{C}(y_0)'$, a surjective map $p' : \mathcal{C}(y_0)' \to Y \times H^G_n(Y, y_0) \setminus B$ and an action of G on $\mathcal{C}(y_0)'$.
(ii) One endows $\mathcal{C}(y_0)'$ with a Hausdorff topology such that $p' : \mathcal{C}(y_0)' \to Y \times H^G_n(Y, y_0) \setminus B$ becomes a topological covering map, one verifies that G acts by covering transformations and that p' is a topological G-covering map.
(iii) One endows $\mathcal{C}(y_0)'$ with a structure of a complex analytic manifold inherited from $Y \times H^G_n(Y, y_0) \setminus B$. The map p' becomes an étale Galois holomorphic covering map.
(iv) Using [47, Théorème 5.1] one endows $\mathcal{C}(y_0)'$ with a structure of an algebraic variety and p' becomes an étale Galois cover.
(v) One constructs $\mathcal{C}(y_0)$ and $p : \mathcal{C}(y_0) \to Y \times H^G_n(Y, y_0)$ by the normal closures of the irreducible components of $Y \times H^G_n(Y, y_0)$ in the fields of rational functions of the irreducible components of $\mathcal{C}(y_0)'$.

Remark 3.11. The construction of $\mathcal{C}(y_0)'$ and its topology in (i) and (ii) is similar to the construction of universal covering spaces (see e.g. [40, Ch. V Theorem 10.2] or [16, Ch. 1 § 5]). For $Y = \mathbb{P}_1$ Parts (i) – (iv) are equivalent to Emsalem's construction of the family of pointed étale G-morphisms of $\mathbb{P}_1$ parameterized by the Hurwitz space $H^G_n(\mathbb{P}_1, y_0)$ (cf. [11, § 6 and § 7.1]).

3.12. If α, α' are paths in $Y \setminus D$ we denote by $\alpha \sim_D \alpha'$ the homotopy of α and α' in $Y \setminus D$ and by $[\alpha]_D$ the homotopy class of α in $Y \setminus D$. In the set-up of § 3.7 let $E \in N_D(U_1, \ldots, U_n)$ and let α, α' be paths in $Y \setminus \cup_{i=1}^n U_i$. Then $\alpha \sim_D \alpha'$ if and only if $\alpha \sim_E \alpha'$. Let $(D, m) \in H^G_n(Y, y_0)$. We denote by Γ_m the kernel of $m : \pi_1(Y \setminus D, y_0) \twoheadrightarrow G$. Let α, α' be paths in $Y \setminus \cup_{i=1}^n U_i$ such that $\alpha(0) = \alpha'(0) = y_0$ and $\alpha(1) = \alpha'(1)$. Then by (5) $[\alpha' \cdot \alpha^-]_D \in \Gamma_m$ if and only if $[\alpha' \cdot \alpha^-]_E \in \Gamma_{m(E)}$, hence $\Gamma_m[\alpha]_D = \Gamma_m[\alpha']_D$ if and only if $\Gamma_{m(E)}[\alpha]_E = \Gamma_{m(E)}[\alpha']_E$.

3.13. We denote by $\mathcal{C}(y_0)'$ the set

$$\mathcal{C}(y_0)' = \{(\Gamma_m[\alpha]_D, D, m) \,|(D, m) \in H^G_n(Y, y_0),\ \alpha : I \to Y \setminus D,\ \alpha(0) = y_0\}.$$

Let $p' : \mathcal{C}(y_0)' \to Y \times H^G_n(Y, y_0) \setminus B$ be the map

$$p'\left(\Gamma_m[\alpha]_D, D, m)\right) = (\alpha(1), D, m).$$

We define a left action of G on $\mathcal{C}(y_0)'$ as follows. For every $g \in G$, if $g = m([\sigma]_D)$, one lets

$$g(\Gamma_m[\alpha]_D, D, m) = (\Gamma_m[\sigma \cdot \alpha]_D, D, m).$$

The map p' is G-invariant, the isotropy subgroup of every $z \in \mathcal{C}(y_0)'$ is trivial and G acts transitively on every fiber of p', hence p' induces a bijection $\mathcal{C}(y_0)'/G \xrightarrow{\sim} Y \times H_n^G(Y, y_0) \setminus B$.

Let $z = (\Gamma_m[\alpha]_D, D, m) \in \mathcal{C}(y_0)'$. Let $p'(z) = (y, (D, m))$, where $D = \{b_1, \ldots, b_n\}$, $y \in Y \setminus D$. Let $\overline{U}, \overline{U}_1, \ldots, \overline{U}_n$ be disjoint, embedded closed disks in Y with interiors $U, U_1, \ldots, U_n$ respectively, with the property that $\overline{U}_i \subset Y \setminus y_0$ for $\forall i$, $y \in U$, $b_i \in U_i$ for $\forall i$. Let $\alpha : I \to Y \setminus \cup_{i=1}^n \overline{U}_i$ be a path such that $\alpha(0) = y_0, \alpha(1) = y$. Consider the following subset of $\mathcal{C}(y_0)'$:

$$
\begin{aligned}
N_{(\alpha,D,m)}(U, U_1, \ldots, U_n) = \{&(\Gamma_{m(E)}[\alpha \cdot \tau]_E, E, m(E))| \\
&E \in N_D(U_1, \ldots, U_n), \tau : I \to U, \tau(0) = y\}
\end{aligned}
\tag{9}
$$

Proposition 3.14. *Let $p' : \mathcal{C}(y_0)' \to Y \times H_n^G(Y, y_0) \setminus B$ be the G-invariant map defined in § 3.13.*

(i) *The family of sets defined in* (9) *is a basis of a topology of $\mathcal{C}(y_0)'$.*
(ii) *The map p' is a topological covering map.*
(iii) *The topology defined in (i) is Hausdorff.*
(iv) *The group G acts on $\mathcal{C}(y_0)'$ freely by Deck transformations and p' induces a homeomorphism $\mathcal{C}(y_0)'/G \xrightarrow{\sim} Y \times H_n^G(Y, y_0) \setminus B$.*
(v) *The map $\zeta : H_n^G(Y, y_0) \to \mathcal{C}(y_0)'$ defined by*
$$\zeta(D, m) = (\Gamma_m[c_{y_0}]_D, D, m)\,,$$
where c_{y_0} is the constant loop, is a continuous section of $\pi_2 \circ p' : \mathcal{C}(y_0)' \to H_n^G(Y, y_0)$, such that $p' \circ \zeta(D, m) = (y_0, (D, m))$ for every $(D, m) \in H_n^G(Y, y_0)$.
(vi) *For every $(D, m) \in H_n^G(Y, y_0)$ the couple*
$$(p'_{(D,m)} : \mathcal{C}(y_0)'_{(D,m)} \to Y \setminus D, \zeta(D, m))$$
is a pointed topological G-covering map of $(Y \setminus D, y_0)$ with monodromy homomorphism equal to $m : \pi_1(Y \setminus D, y_0) \twoheadrightarrow G$.

Proof. (i) It is obvious that $\mathcal{C}(y_0)'$ is a union of the sets (9). Let
$$W' = N_{(\alpha',D',m')}(U', U'_1, \ldots, U'_n), \quad W'' = N_{(\alpha'',D'',m'')}(U'', U''_1, \ldots, U''_n).$$
Let $z \in W' \cap W''$ and let $z = (\Gamma_m[\beta]_D, D, m)$. Let $p'(z) = (y, (D, m))$, $D = \{b_1, \ldots, b_n\}$. Then $y \in U' \cap U''$, $b_i \in U'_i \cap U''_i$ for $i = 1, \ldots, n$ and furthermore $y \in Y \setminus \left(\cup_{i=1}^n \overline{U}'_i \bigcup \cup_{i=1}^n \overline{U}''_i\right)$. One has $m = m'(D) = m''(D)$,
$$\Gamma_m[\beta]_D = \Gamma_{m'(D)}[\alpha' \cdot \tau']_D = \Gamma_{m''(D)}[\alpha'' \cdot \tau'']_D.$$
Let us choose disjoint closed disks $\overline{U}, \overline{U}_1, \ldots, \overline{U}_n$ with interiors $U, U_1, \ldots, U_n$ respectively, such that $U \ni y$, $U_i \ni b_i, i = 1, \ldots, n$, $\overline{U} \subset U' \cap U''$, $\overline{U}_i \subset U'_i \cap U''_i$ for $\forall i$ and $\beta(I) \subset Y \setminus \cup_{i=1}^n \overline{U}_i$. Let $W = N_{(\beta,D,m)}(U, U_1, \ldots, U_n)$. We claim that $W \subset W' \cap W''$. Let $x \in W$, $x = (\Gamma_{m(E)}[\beta \cdot \tau]_E, E, m(E))$. It is clear from Diagram (5) that $m(E) = m'(E) = m''(E)$. One has $D \in N_{D'}(U'_1, \ldots, U'_n)$, so $\beta \sim_D \beta'$, where β' is a path in $Y \setminus \cup_{i=1}^n U'_i$. The equality $\Gamma_m[\beta']_D = \Gamma_m[\beta]_D = \Gamma_m[\alpha' \cdot \tau']_D$ implies $\Gamma_{m(E)}[\beta']_E = \Gamma_{m(E)}[\alpha' \cdot \tau']_E$ since $E \in N_{D'}(U'_1, \ldots, U'_n)$ (cf. § 3.12). Therefore
$$x = (\Gamma_{m(E)}[\beta \cdot \tau]_E, E, m(E)) = (\Gamma_{m'(E)}[\alpha' \cdot \tau' \cdot \tau]_E, E, m'(E))$$
belongs to W', since $\tau' \cdot \tau$ is an arc in U'. Similarly $x \in W''$. This shows that $W \subset W' \cap W''$.

(ii) Let $(y,(D,m)) \in Y \times H^G_n(Y,y_0) \setminus B$, let $D = \{b_1,\cdots,b_n\}$. Choose paths $\alpha_1,\ldots,\alpha_{|G|}$ in $Y \setminus D$ with $\alpha_i(0) = y_0, \alpha_i(1) = y$ such that $[\alpha_i \cdot \alpha_j^-]_D \notin \Gamma_m$ if $i \neq j$. Then for every path $\alpha : I \to Y \setminus D$ with $\alpha(0) = y_0$, $\alpha(1) = y$ there is a unique α_i such that $\Gamma_m[\alpha]_D = \Gamma_m[\alpha_i]_D$. Let $\overline{U}_1,\ldots,\overline{U}_n$ be disjoint embedded closed disks in Y with interiors $U_1,\ldots,U_n$ respectively, such that $b_i \in U_i$ for $i = 1,\ldots,n$ and $\alpha_j(I) \subset Y \setminus \cup_{i=1}^n \overline{U}_i$ for every $j = 1,\ldots,|G|$. Let $\overline{U} \subset Y$ be an embedded closed disk such that y belongs to its interior U and $\overline{U} \subset Y \setminus \cup_{i=1}^n \overline{U}_i$. Then

$$p'^{-1}\left(U \times N_{(D,m)}(U_1,\ldots,U_n)\right) = \bigcup_{j=1}^{|G|} N_{(\alpha_j,D,m)}(U,U_1,\ldots,U_n) \tag{10}$$

and moreover $N_{(\alpha_i,D,m)}(U,U_1,\ldots,U_n) \cap N_{(\alpha_j,D,m)}(U,U_1,\ldots,U_n) = \emptyset$ if $i \neq j$. In fact, it is clear that the left-hand set of (10) contains the right-hand set. Let $z = (\Gamma_\mu[\beta]_E, E, \mu)$ be a point of the left-hand set. Then $E \in N_D(U_1,\ldots,U_n)$, $\mu = m(E)$. Let $\beta \sim_E \beta'$, where β' is a path in $Y \setminus \cup_{i=1}^n \overline{U}_i$. Let $\Gamma_m[\beta']_D = \Gamma_m[\alpha_i \cdot \tau]_D$ for some $\tau : I \to U$. Then $\Gamma_{m(E)}[\beta']_E = \Gamma_{m(E)}[\alpha_i \cdot \tau]_E$ (cf. § 3.12). Hence $z = (\Gamma_{m(E)}[\alpha_i \cdot \tau]_E, E, m(E)) \in N_{(\alpha_i,D,m)}(U,U_1,\ldots,U_n)$. This proves Equality (10) and in particular shows that p' is a continuous map. Suppose that $(\Gamma_{m(E)}[\alpha_i \cdot \tau]_E, E, m(E)) = (\Gamma_{m(E)}[\alpha_j \cdot \tau']_E, E, m(E))$ for some $i \neq j$ and some paths τ, τ' in U. One has $\tau \sim_E \tau'$ since U is simply connected, therefore $[\alpha_j \cdot \alpha_i^-]_E \in \Gamma_{m(E)}$. This implies that $[\alpha_j \cdot \alpha_i^-]_D \in \Gamma_m$, which contradicts the choice of $\{\alpha_i\}_i$. This shows that the right-hand side of (10) is a disjoint union.

Every open set $N_{(\alpha,D,m)}(U,U_1,\ldots,U_n)$ as in (9) is mapped by p' bijectively onto the open subset $U \times N_{(D,m)}(U_1,\ldots,U_n)$ of $Y \times H^G_n(Y,y_0) \setminus B$. Since every open subset of $N_{(\alpha,D,m)}(U,U_1,\ldots,U_n)$ is a union of sets $N_{(\beta,A,\mu)}(V,V_1,\ldots,V_n)$ as in (9), this bijection, being a continuous map, is open, hence it is a homeomorphism. This proves (ii).

(iii) This follows from (ii) since $Y \times H^G_n(Y,y_0) \setminus B$ is a Hausdorff topological space.

(iv) By § 3.13 it suffices to prove that for every $g \in G$ the map $\varphi_g : \mathcal{C}(y_0)' \to \mathcal{C}(y_0)'$ defined by $\varphi_g(z) = gz$ is continuous. Every open subset W of $\mathcal{C}(y_0)'$ is a union of subsets of the topology basis (9) and $\varphi_g^{-1}(W) = g^{-1}W$. So it suffices to prove that φ_g transforms every $V = N_{(\alpha,D,m)}(U,U_1,\ldots,U_n)$ into an open set. Let $g = m([\sigma]_D)$, where σ is a closed path in $Y \setminus \cup_{i=1}^n \overline{U}_i$. Let $x = (\Gamma_{m(E)}[\alpha \cdot \tau]_E, E, m(E)) \in V$. One has by Diagram (5) that $g = m(E)([\sigma]_E)$, hence $gx = (\Gamma_{m(E)}[\sigma \cdot \alpha \cdot \tau]_E, E, m(E))$. One obtains that $gV \subset N_{(\sigma\cdot\alpha,D,m)}(U,U_1,\ldots,U_n)$. Replacing the pair g,α by $g^{-1}, \sigma \cdot \alpha$ one obtains the opposite inclusion, therefore

$$gN_{(\alpha,D,m)}(U,U_1,\ldots,U_n) = N_{(\sigma\cdot\alpha,D,m)}(U,U_1,\ldots,U_n). \tag{11}$$

(v) Suppose that $\zeta(D,m) \in N_{(\alpha,A,\mu)}(V,V_1,\ldots,V_n)$. Let $\underline{V} = (V_1,\ldots,V_n)$. Then $D \in N_A(V_1,\ldots,V_n)$, $m = \mu(D)$, therefore $\zeta^{-1}N_{(\alpha,A,\mu)}(V,\underline{V}) \subset N_{(A,\mu)}(\underline{V})$. Moreover one has $(\Gamma_m[c_{y_0}]_D, D, m) = (\Gamma_{\mu(D)}[\alpha \cdot \tau]_D, D, \mu(D))$ for some path $\tau : I \to V$ such that $\tau(0) = \alpha(1)$, $\tau(1) = y_0$, so $[\alpha \cdot \tau]_D \in \Gamma_{\mu(D)}$. This implies that $[\alpha \cdot \tau]_E \in \Gamma_{\mu(E)}$ for every $E \in N(\underline{V})$ (cf. § 3.12). Hence for every $(E,\mu(E)) \in N_{(A,\mu)}(\underline{V})$ one has

$$\zeta(E,\mu(E)) = (\Gamma_{\mu(E)}[c_{y_0}]_E, E, \mu(E)) = (\Gamma_{\mu(E)}[\alpha \cdot \tau]_E, E, \mu(E)) \in N_{(\alpha,A,\mu)}(V,\underline{V}),$$

so $\zeta^{-1}N_{(\alpha,A,\mu)}(V,\underline{V}) = N_{(A,\mu)(\underline{V})}$. This proves that ζ is a continuous map. The other statements of (v) are obvious.

(vi) Let $(D,m) \in H^G_n(Y,y_0)$. Identifying $(Y \setminus D) \times \{(D,m)\}$ with $Y \setminus D$, the set $\mathcal{C}(y_0)'_{(D,m)}$, the action of G on it, its induced topology, the map $p'_{(D,m)}$ and the point $\zeta(D,m)$ are the same as those defined in § 3.5, therefore the monodromy homomorphism of the pointed topological G-covering map $(p'_{(D,m)}, \zeta(D,m))$ of $(Y \setminus D, y_0)$ equals $m : \pi_1(Y \setminus D, y_0) \twoheadrightarrow G$. □

Proposition 3.15. *Let $p' : \mathcal{C}(y_0)' \to Y \times H^G_n(Y,y_0) \setminus B$ be the topological G-covering map of Proposition 3.14.*

(i) *Consider $Y \times H^G_n(Y,y_0)$ with the structure of a complex analytic manifold of dimension $n+1$ (cf. § 3.7). Then $\mathcal{C}(y_0)'$ has a unique structure of a complex analytic manifold of dimension $n+1$ such that p' is a holomorphic map. Furthermore p' is a finite, étale holomorphic covering map (cf.* [47, § 5.0]*).*

(ii) *Consider $Y \times H^G_n(Y,y_0)$ with the structure of an algebraic variety as in § 3.7. Then $\mathcal{C}(y_0)'$ and p' have a unique structure of an algebraic variety and a finite, étale surjective morphism to $Y \times H^G_n(Y,y_0) \setminus B$ whose associated complex analytic space and holomorphic map are those of (i). The algebraic variety $\mathcal{C}(y_0)'$ is nonsingular, equidimensional of dimension $n+1$.*

(iii) *The group G acts freely by covering automorphisms on the algebraic variety $\mathcal{C}(y_0)'$ and p' induces an isomorphism $\mathcal{C}(y_0)'/G \cong Y \times H^G_n(Y,y_0) \setminus B$.*

(iv) *The map $\zeta : H^G_n(Y,y_0) \to \mathcal{C}(y_0)'$ is a morphism of algebraic varieties. It is a closed embedding, a section of the morphism $\pi_2 \circ p' : \mathcal{C}(y_0)' \to H^G_n(Y,y_0) \setminus B$ and $p' \circ \zeta(D,m) = (y_0,(D,m))$ for $\forall (D,m) \in H^G_n(Y,y_0)$.*

Proof. (i) One defines a complex analytic structure on the topological covering space $\mathcal{C}(y_0)'$ as in [53, Ch. IX § 1]. Let $M = \mathcal{C}(y_0)'$ be the obtained complex manifold. Suppose that M' is another complex analytic manifold whose underlying topological space is $\mathcal{C}(y_0)'$, such that p' is a holomorphic map. We claim that $id : M \to M'$ is biholomorphic. Let $z \in \mathcal{C}(y_0)'$ and let $x = p'(z)$. Let (W,ψ) be a complex analytic chart of x such that W is evenly covered and let U be a neighborhood of z in M such that $p'|_U : U \to W$ is biholomorphic. Let (V,ϕ) be a chart of M' such that $z \in V \subset U$. Then $\psi \circ p' \circ \phi^{-1} : \phi(V) \to (\psi \circ p')(V)$ is a holomorphic bijective map of domains in $\mathbb{C}^{n+1}$. Hence by Clements' theorem it is biholomorphic [45, Ch. 5 Theorem 5]. Therefore the restriction of $id : M \to M'$ on V is biholomorphic. This shows that $id : M \to M'$ is biholomorphic. The topological covering map p' is proper with finite fibers, so the holomorphic map p' is finite. It is locally biholomorphic, so it is étale. It remains to verify that every irreducible component of $\mathcal{C}(y_0)'$ dominates an irreducible component of $Y \times H^G_n(Y,y_0) \setminus B$. Since these are complex manifolds, the irreducible components coincide with the connected components (cf. [21, Ch. 9 § 2 n.1]), so this property is obvious.

(ii) By [47, Théorème 5.1] the pair $(\mathcal{C}(y_0)', p')$ has a unique structure of a scheme of finite type over $\mathbb{C}$ and a finite étale morphism of schemes, such that the associated complex analytic space and holomorphic map are those of (i). The scheme $\mathcal{C}(y_0)'$ is separated and smooth by Proposition 3.1(viii) and Proposition 2.1(iv) of [47].

(iii) The group G acts by covering transformations of the topological covering map p' (cf. Proposition 3.14(iv)), so G acts by biholomorphic covering maps of the finite, étale, holomorphic covering map p'. By [47, Théorème 5.1 (1)] the group G acts by covering automorphisms of the finite, étale morphism of algebraic varieties

$p' : \mathcal{C}(y_0)' \to Y \times H_n^G(Y, y_0) \setminus B$. By Lemma 2.5 p' induces a finite morphism $\overline{p}' : \mathcal{C}(y_0)'/G \to Y \times H_n^G(Y, y_0) \setminus B$. It is bijective (cf. § 3.13) and $Y \times H_n^G(Y, y_0) \setminus B$ is smooth, therefore $\overline{p}'$ is an isomorphism.

(iv) $p'^{-1}(\{y_0\} \times H_n^G(Y, y_0))$ is a closed algebraic subset of $\mathcal{C}(y_0)'$ and the restriction of p' on it is a finite étale morphism onto $\{y_0\} \times H_n^G(Y, y_0)$. Identifying $H_n^G(Y, y_0)$ with $\{y_0\} \times H_n^G(Y, y_0)$ the map ζ is its continuous section in the Euclidean topology by Proposition 3.14(v), hence $\zeta(H_n^G(Y, y_0))$ is a union of connected components of $p'^{-1}(\{y_0\} \times H_n^G(Y, y_0))$. These connected components in the Euclidean topology are connected components in the Zariski topology [47, Corollaire 2.6], so $\zeta(H_n^G(Y, y_0))$ is a closed algebraic subset of $\mathcal{C}(y_0)'$. The restriction of p' on it is a finite, étale, bijective morphism onto the smooth variety $\{y_0\} \times H_n^G(Y, y_0)$, hence it is an isomorphism. This shows that $\zeta : H_n^G(Y, y_0) \to \mathcal{C}(y_0)'$ is a morphism, it is a closed embedding which is a section of the morphism p'. The last statement is from Proposition 3.14(v). $\square$

3.16. Let H be a connected component of $H_n^G(Y, y_0)$. It is irreducible, since $H_n^G(Y, y_0)$ is smooth. Let $\mathcal{C}(y_0)'_H = p'^{-1}(H)$. Its closed subspace $\zeta(H)$ is pathwise connected and every fiber of $\mathcal{C}(y_0)'_H \to H$ is pathwise connected by Proposition 3.14(vi). Therefore $\mathcal{C}(y_0)'_H$ is pathwise connected. It is a Zariski open subset of the smooth algebraic variety $\mathcal{C}(y_0)'$, hence $\mathcal{C}(y_0)'_H$ is irreducible. Let us denote by $\mathcal{C}(y_0)_H$ the normalization of $Y \times H$ in the field of rational functions of $\mathcal{C}(y_0)'_H$ and let $p_H : \mathcal{C}(y_0)_H \to Y \times H$ be the corresponding finite, surjective morphism. The action of G on $\mathcal{C}(y_0)'$ can be uniquely extended to a faithful action of G on $\mathcal{C}(y_0)_H$ by algebraic automorphisms (cf. Proposition 3.15(iii)). The uniqueness of normalizations implies that there is a G-equivariant open embedding $j_H : \mathcal{C}(y_0)'_H \to \mathcal{C}(y_0)_H$ with image $p^{-1}(Y \times H \setminus B)$ such that $p \circ j_H = p'$. The G-invariant morphism $p_H : \mathcal{C}(y_0)_H \to Y \times H$ is a Galois cover. In fact the morphism $\mathcal{C}(y_0)_H/G \to Y \times H$ is finite, birational by Proposition 3.15(iii), and $Y \times H$ is smooth, so it is an isomorphism.

3.17. In the next proposition we study p_H at the ramification points. Let $(D, m) \in H$ and let $(b, (D, m)) \in (Y \times H) \cap B$, $D = \{b_1, \ldots, b_k, \ldots, b_n\}$, $b = b_k$. Let us choose local analytic coordinates s_i at b_i, such that $s_i(b_i) = 0$, $i = 1, \ldots, n$. Let $\epsilon \in \mathbb{R}^+, \epsilon \ll 1$ be such that the open sets $U_i = \{y | s_i(y) < \epsilon\}$ have disjoint closures $\overline{U}_i$, $i = 1, \ldots, n$ and $y_0 \in Y \setminus \cup_{i=1}^n \overline{U}_i$. Let $\gamma_1, \ldots, \gamma_n$ be closed paths based at y_0 as in § 3.2. Let $U = U_k$ and let $V = U \times N_{(D,m)}(U_1, \ldots, U_n)$. For every $v = (y, (E, m(E)) \in V$, where $y \in U$, $E = \{y_1, \ldots, y_n\}, y_i \in U_i$ let $t_i(v) = s_i(y_i)$ and let $t(v) = s_k(y)$.

Proposition 3.18. *The algebraic variety $\mathcal{C}(y_0)_H$ is smooth. Let*

$$x = (b, (D, m)) \in (Y \times H) \cap B, \quad D = \{b_1, \ldots, b_k, \ldots, b_n\}, \quad b = b_k$$

and let $p_H^{-1}(x) = \{w_1, \ldots, w_r\}$. Let $G(w_i)$ be the isotropy group of w_i. Then

(i) *For every $i = 1, \ldots, r$ $G(w_i)$ is a cyclic group generated by an element conjugated with $g_k = m([\gamma_k])$, $G(w_i)$ is of order $e = |g_k|$, where $er = |G|$.*

(ii) *Let $V = U \times N_{(D,m)}(U_1, \ldots, U_n)$ be as in § 3.17. Then $p_H^{-1}(V) = \bigsqcup_{i=1}^r W_i$, where W_i is an open, connected neighborhood of w_i, it is $G(w_i)$-invariant, and $p_H(W_i) = V$ for every $i = 1, \ldots, r$.*

(iii) *Let $i \in [1,r]$ and let $(W,w) = (W_i, w_i)$. Let $E \subset \mathbb{C} \times V$ be the analytic subset defined by the equation $z^e = t - t_k$ and let $p_1 : E \to V$ be the projection map.*
   (a) *There exists a biholomorphic map $\varphi : W \to E$ such that $p_1 \circ \varphi = p_H|_W$.*
   (b) *The composition $\psi = (z, t_1, \ldots, t_n) \circ \varphi : W \to \mathbb{C}^{n+1}$ maps W biholomorphically onto an open subset of $\mathbb{C}^{n+1}$.*
   (c) *There exists a primitive character $\chi : G(w) \to \mathbb{C}^*$ such that φ and ψ are $G(w)$-equivariant with respect to the actions of $G(w)$ on E and $\mathbb{C}^{n+1}$ defined respectively by $g(z,v) = (\chi(g)z, v)$ and $g(z, z_1, \ldots, z_n) = (\chi(g)z, z_1, \ldots, z_n)$.*

(iv) *There is a G-equivariant biholomorphic map $p_H^{-1}(V) \cong G \times^{G(w)} W$.*

Proof. We may assume, without loss of generality, that $k = 1$. Let us denote $p_H : \mathcal{C}(y_0)_H \to Y \times H$ by $p : M \to N$. The map $p^{an} : M^{an} \to N^{an}$ is a finite, surjective, holomorphic map and M^{an} is a reduced, normal complex space [47, Proposition 2.1]. Let $p^{-1}(V) = \bigsqcup_i W_i$ be the disjoint union of connected components of $p^{-1}(V)$. By [21, Ch. 9 § 2 n.1] every W_i is an open subset of M^{an}. The restriction $p|_{W_i} : W_i \to V$ is a finite holomorphic map. In fact, it has finite fibers and if A is a closed subset of W_i then A is closed in $p^{-1}(V)$, so $p(A)$ is closed in V since $p^{an} : M^{an} \to N^{an}$ is a closed map. Moreover $p|_{W_i} : W_i \to V$ is an open map by the Open Mapping Theorem (cf. [21, Ch. 5 § 4 n.3]). Therefore $p(W_i) = V$, since V is connected. We see that $W_i \cap p^{-1}(x) \neq \emptyset$ for every i, so $p^{-1}(V)$ has a finite number of connected components: $p^{-1}(V) = \bigsqcup_{i=1}^{\ell} W_i$.

Let $w \in p^{-1}(x)$ and let W be the connected component of $p^{-1}(V)$ which contains w. The set $p_H^{-1}(B \cap Y \times H)$ is a proper, closed algebraic subset of the algebraic variety $\mathcal{C}(y_0)_H = M$. Hence it has no interior points in the Euclidean topology of M (cf. [53, Ch. VII § 2 Lemma 1]), i.e. it is thin in M^{an}. The open set $W' = W \setminus p_H^{-1}(B \cap Y \times H)$ is connected [21, Ch. 7 § 4 n.2]. Let $V' = V \setminus B = V \setminus \{v | (t - t_1)(v) = 0\}$. Then $p^{-1}(V') \to V'$ is a topological covering map, since it is a restriction of $\mathcal{C}(y_0)' \to Y \times H_n^G(Y, y_0) \setminus B$, furthermore W' is a connected component of $p^{-1}(V')$, so $p|_{W'} : W' \to V'$ is a topological covering map as well. Let $b_0 \in U \setminus b_1$ and let $v_0 = (b_0, (D, m)) \in V'$. We claim that $\pi_1(V', v_0) \cong \mathbb{Z}$. Following the notation of [13] let $F_{0,2}(U) = \{(y_1, y_2) | y_i \in U, y_1 \neq y_2\}$. The topological space V' is homotopy equivalent to $F_{0,2}(U)$. The projection map $F_{0,2}(U) \to U$ given by $(y_1, y_2) \mapsto y_1$ is a locally trivial fiber bundle with fibers homeomorphic to $U \setminus b_1$ (cf. [13], or [5, Theorem 1.2]). By [27, Ch. 6 Sect. 6] one has an exact sequence of homotopy groups

$$\pi_2(U, b_1) \to \pi_1((U \setminus b_1) \times \{b_1\}, (b_0, b_1)) \to \pi_1(F_{0,2}(U), (b_0, b_1)) \to \pi_1(U, b_1) \to 1.$$

Therefore $\pi_1(V', v_0) \cong \pi_1(F_{0,2}(U), (b_0, b_1)) \cong \mathbb{Z}$. This implies that $p|_{W'} : W' \to V'$ is a topological Galois covering map whose group of Deck transformations $\mathrm{Deck}(W'/V')$ is isomorphic to the cyclic group $C_e \subset \mathbb{C}$ of order e for a certain integer $e \geq 1$.

Let $E \subset \mathbb{C} \times V$ be the analytic subset defined by $z^e = t - t_1$. This is a complex manifold of dimension $n + 1$, the holomorphic map $\phi : E \to \mathbb{C}^{n+1}$ defined by $\phi(z, v) = (z, t_1(v), \ldots, t_n(v))$ is injective and the Jacobian map $T_{(z,v)}\phi$ is an isomorphism for every $(z, v) \in E$. Therefore $\phi(E)$ is an open subset of $\mathbb{C}^{n+1}$ and $\phi : E \to \phi(E)$ is a biholomorphic map (cf. [15, Prop. 2.4]). The projection map

$p_1 : E \to V$ given by $(z, v) \mapsto v$ is a finite, surjective, holomorphic map [21, Ch. 2 § 3 n.5]. Let $E' = p_1^{-1}(V') = E \setminus \{z = 0\}$. Then the restriction of p_1, $p_1' : E' \to V'$ is locally biholomorphic. It is also proper [21, Ch. 9 § 2 n.4], hence $p_1' : E' \to V'$ is a topological covering map. The cyclic group $C_e = \{\omega^q | q \in \mathbb{Z}\}$, $\omega = exp\left(\frac{2\pi i}{e}\right)$, acts on E by $\omega^q(z, v) = (\omega^q z, v)$ and this action is by holomorphic automorphisms of E which preserve the fibers of $p_1 : E \to V$. Furthermore E' is connected, $p_1' : E' \to V'$ is a topological Galois covering map and $\mathrm{Deck}(E'/V') \cong C_e$. Indeed, C_e acts transitively on the fibers of $p_1'^{-1}(v_0)$, then, in order to prove that E' is connected, let us verify that $\pi_1(V', v_0)$ acts transitively on the right on $p_1'^{-1}(v_0)$. It suffices to prove the analogous statement replacing V by $\Delta \times \Delta$, where $\Delta = \{z \in \mathbb{C} | \, |z| < \epsilon\}$ and V' by $F_{0,2}(\Delta) = \{(z_1, z_2) | z_i \in \Delta, z_1 \neq z_2\}$. Let $(z, z_1, z_2) \in \mathbb{C} \times F_{0,2}(\Delta)$ satisfy $z^e = z_1 - z_2$. Let $\omega^q \in C_e$. Then the path

$$\alpha(\tau) = (\omega(\tau)^q z, \omega(\tau)^{qe} z_1, \omega(\tau)^{qe} z_2), \quad \omega(\tau) = \exp\left(\frac{2\pi i}{e}\tau\right) \tag{12}$$

is a lifting of a closed path in $F_{0,2}(\Delta)$ based at (z_1, z_2) which connects (z, z_1, z_2) with $(\omega^q z, z_1, z_2)$.

We conclude that there is a covering homeomorphism φ'

$$\begin{tikzcd} W' \arrow[rr, "\varphi'"] \arrow[dr, "p'|_{W'}"'] & & E' \arrow[dl, "p_1'"] \\ & V' & \end{tikzcd} \tag{13}$$

which is C_e-equivariant. It is biholomorphic since both $p'|_{W'}$ and p_1' are locally biholomorphic. According to [47, Proposition 5.3] there is a unique extension of φ' to a biholomorphic covering map φ

$$\begin{tikzcd} W \arrow[rr, "\varphi"] \arrow[dr, "p|_W"'] & & E \arrow[dl, "p_1"] \\ & V & \end{tikzcd} \tag{14}$$

The composition $\psi = \phi \circ \varphi : W \to \phi(E)$ is a biholomorphic map. We see that M^{an} is a nonsingular complex space, therefore $\mathcal{C}(y_0)_H$ is a smooth algebraic variety.

The restriction of (14) to $V \cap B$ yields three bijective maps, hence $p^{-1}(x) \cap W$ consists of one point. This shows that the number of connected components of $p^{-1}(V)$ equals $r = |p^{-1}(x)|$. We may enumerate them so that $w_i \in W_i, i = 1, \ldots, r$. Let $i \in [1, r]$ and let $(W, w) = (W_i, w_i)$. Let $G(w) = G(w_i) \subset G$ be the isotropy group of $w = w_i$. The group G acts on M^{an} by biholomorphic covering maps, hence it permutes transitively the connected components of $p^{-1}(V)$. Therefore $W = W_i$ is invariant under the action of $G(w)$, $p|_{W'} : W' \to V'$ is a topological Galois covering map and $G(w)$ is isomorphic to $Deck(W'/V')$. The composition $G(w) \xrightarrow{\sim} Deck(W'/V') \xrightarrow{\sim} C_e$ is a primitive character $\chi : G(w) \to C_e$ for which the statements of Part (iii) hold. Furtermore one has $e = |G(w)| = \frac{|G|}{r}$ as claimed in Part (i).

Next we prove that $G(w)$ is generated by an element conjugated with $m([\gamma_1])$. Let us consider the loop $\beta_1 : I \to U_1 \setminus b_1$ based at b_0 ($U = U_1$), defined by $s_1(\beta_1(\tau)) = e^{2\pi i\tau} s_1(b_0)$ (cf. § 3.17). Let $\beta : I \to V'$ be the loop $\beta(\tau) = (\beta_1(\tau), (D, m))$. Let $w_0 \in W', p'(w_0) = v_0$. Lifting β in W' with initial point w_0 the terminal point is $w_0\beta = gw_0$, where $g \in G(w)$ and $\varphi(gw_0) = exp(\frac{2\pi i}{e})\varphi(w_0)$ (apply (12) with $k = 1, z_1 = s_1(b_0), z_2 = s_1(b_1) = 0$). Therefore g generates $G(w)$. The closed path γ_1 is homotopic to $\eta_1 \cdot \beta_1 \cdot \eta_1^-$, where $\eta_1 : I \to Y \setminus D$ is a path such that $\eta_1(0) = y_0$, $\eta_1(1) = b_0$. Let $\eta : I \to Y \times H \setminus B$ be the path $\eta(\tau) = (\eta_1(\tau), (D, m))$. Let $z_0 = \zeta(D, m) \in \mathcal{C}(y_0)'_{(D,m)}$. Let $\tilde{\eta} : I \to \mathcal{C}(y_0)'_{(D,m)} \subset \mathcal{C}(y_0)'_H$ be the lifting of η with initial point $\tilde{\eta}(0) = z_0$. Then $\tilde{\eta}(1) \in p'^{-1}(v_0)$, so $\tilde{\eta}(1) = hw_0$ for some $h \in G$. Let $g_1 = m([\gamma_1])$. By Proposition 3.14(vi) lifting γ_1 in $\mathcal{C}(y_0)'$ with initial point z_0 the terminal point is g_1z_0. The lifted path is equal to the product $\tilde{\eta} \cdot \tilde{\beta} \cdot (\eta^-)^\sim$ of the liftings of η, β and η^- respectively. Now, since $\mathcal{C}(y_0)'_H \to Y \times H \setminus B$ is a topological Galois covering map and the terminal point of $(\eta^-)^\sim$ is g_1z_0, the initial point of $(\eta^-)^\sim$ is $g_1\tilde{\eta}(1) = g_1(hw_0)$. The initial point of $(\eta^-)^\sim$ is the terminal point of $\tilde{\eta} \cdot \tilde{\beta}$. This terminal point equals $z_0(\eta \cdot \beta) = (hw_0)\beta = h(w_0\beta) = h(gw_0)$, hence $g_1(hw_0) = h(gw_0)$, which implies $g = h^{-1}g_1h$. Parts (i), (ii) and (iii) are proved.

(iv) The map $G \times^{G(w)} W \to p^{-1}(V)$, given by $(g, z) \mapsto gz$, is biholomorphic since $p^{-1}(V) = \bigsqcup_{i=1}^r g_iW$, where $g_jg_i^{-1} \notin G(w)$ if $j \neq i$. □

3.19. Let $p : \mathcal{C}(y_0) \to Y \times H_n^G(Y, y_0)$ be the disjoint union of $p_H : \mathcal{C}(y_0)_H \to Y \times H$, where H runs over all connected components of $H_n^G(Y, y_0)$. There is a G-equivariant open embedding $\mathcal{C}(y_0)' \hookrightarrow \mathcal{C}(y_0)$ whose image is $p^{-1}(Y \times H_n^G(Y, y_0) \setminus B)$ and which we identify with $\mathcal{C}(y_0)'$. Let $\zeta : H_n^G(Y, y_0) \to \mathcal{C}(y_0)'$ be the morphism of Proposition 3.15(iv).

Theorem 3.20. *Let Y be a smooth, projective, irreducible curve of genus $g \geq 0$. Let n be a positive integer. Let $y_0 \in Y$. Let G be a finite group which can be generated by $2g + n - 1$ elements. The pair*

$$(p : \mathcal{C}(y_0) \to Y \times H_n^G(Y, y_0), \zeta : H_n^G(Y, y_0) \to \mathcal{C}(y_0)) \tag{15}$$

is a smooth family of pointed G-covers of (Y, y_0) branched in n points. For every $(D, m) \in H_n^G(Y, y_0)$ the pointed G-cover $(\mathcal{C}(y_0)_{(D,m)} \to Y, \zeta(D, m))$ of (Y, y_0) has monodromy invariant (D, m) (cf. § 3.3). Every pointed G-cover $(C \to Y, z_0)$ of (Y, y_0) branched in n points is G-equivalent to a unique pointed G-cover of (Y, y_0) of the family (15).

Proof. The composition $f : \mathcal{C}(y_0) \xrightarrow{p} Y \times H_n^G(Y, y_0) \xrightarrow{\pi_2} H_n^G(Y, y_0)$ is a proper morphism since p is finite and π_2 is proper. The map f is a morphism of smooth algebraic varieties. For every $z \in \mathcal{C}(y_0)$ the induced linear map on the tangent spaces T_zf is surjective with one-dimensional kernel. This is clear if $z \in \mathcal{C}(y_0)'$ and follows from Proposition 3.18(iii) if $p(z) \in B$. Therefore f is a smooth morphism [26, Ch. III Prop. 10.4]. For every $(D, m) \in H_n^G(Y, y_0)$ the fiber $\mathcal{C}(y_0)_{(D,m)}$ is a smooth, projective curve and the restriction $p_{(D,m)} : \mathcal{C}(y_0)_{(D,m)} \to Y \times \{(D, m)\}$ is a finite, surjective morphism. The Zariski open subset $\mathcal{C}(y_0)'_{(D,m)}$, the preimage of $(Y \setminus D) \times \{(D, m)\}$, is connected in the Euclidean topology, so it is irreducible. Hence $\mathcal{C}(y_0)_{(D,m)}$ is irreducible. Furthermore $\mathcal{C}(y_0)_{(D,m)}/G \cong Y \times \{(D, m)\}$ by Proposition 3.14(vi). The G-cover $\mathcal{C}(y_0)_{(D,m)} \to Y$ is ramified at every point of D by Proposition 3.18(i) and is unramified over $Y \setminus D$, where the restriction is G-equivalent to $\mathcal{C}(y_0)'_{(D,m)} \to Y \setminus D$. By Proposition 3.14(vi) the monodromy homomorphism of the pointed G-cover $(p_{(D,m)} : \mathcal{C}(y_0)_{(D,m)} \to Y, \zeta(D, m))$ of (Y, y_0) is $m : \pi_1(Y \setminus D, y_0) \twoheadrightarrow G$. The last statement is clear (cf. § 3.5). □

4. Lifting of morphisms

4.1. We consider separated schemes of finite type over the base field $k = \mathbb{C}$. Given such a scheme one denotes by X^{an} the associated complex space [47, § 1]. We recall that a morphism $(p, p^\sharp) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of schemes is called unramified at a point $x \in X(\mathbb{C})$ if $p_x^\sharp(\mathfrak{m}_{p(x)})\mathcal{O}_x = \mathfrak{m}_x$. This condition is equivalent to each of the following ones: a) $\Omega^1_{X/Y}(x) = 0$; b) $\Omega^1_{X^{an}/Y^{an}}(x) = 0$; c) $p^{an} : X^{an} \to Y^{an}$ is an immersion at x (cf.[3, Ch. VI Prop. 3.3], [47, Prop. 3.1(ii)], [24, Prop. 3.1], [46, Prop. 1.24]). If moreover p is flat at x, then p is étale at $x \in X(\mathbb{C})$. A morphism $p : X \to Y$ is étale at $x \in X(\mathbb{C})$ if and only if $p^{an} : X^{an} \to Y^{an}$ is locally biholomorphic at x (cf. [47, Prop. 3.1(iii)] and [23, Théorème 3.1]).

Proposition 4.2. *Let X, Y and Z be schemes of finite type over $\mathbb{C}$. Let $f : Z \to Y$ and $p : X \to Y$ be morphisms. Suppose that $p : X \to Y$ is unramified (at $\forall x \in X(\mathbb{C})$). Suppose that there exists a holomorphic map $h : Z^{an} \to X^{an}$ such that $f^{an} = p^{an} \circ h$*

$$\begin{tikzcd} & X^{an} \arrow[d, "p^{an}"] \\ Z^{an} \arrow[ur, "h"] \arrow[r, "f^{an}"'] & Y^{an} \end{tikzcd}$$

Then there exists a unique morphism $g : Z \to X$ such that $f = p \circ g$ and $g^{an} = h$.

Proof. We may assume, without loss of generality, that Z is connected. The diagonal morphism $\Delta : X \to X \times_Y X$ is a closed and open morphism since X and Y are separated schemes and $p : X \to Y$ is unramified. This implies that

$$\Delta^{an} : X^{an} \to (X \times_Y X)^{an} \cong X^{an} \times_{Y^{an}} X^{an}$$

is a closed and open immersion (cf. [15, Cor. 0.32] and [47, § 1.2]). One has the following Cartesian diagram (cf. [20, Prop. 9.3])

$$\begin{tikzcd} Z^{an} \arrow[r, "id \times h"] \arrow[d, "h"'] & Z^{an} \times_{Y^{an}} X^{an} \arrow[d, "h \times id"] \\ X^{an} \arrow[r, "\Delta^{an}"] & X^{an} \times_{Y^{an}} X^{an} \end{tikzcd}$$

It implies that $id \times h$ is a closed and open immersion. Let $Z \times_Y X = \bigsqcup_{i=1}^r W_i$ be the decomposition of the scheme $Z \times_Y X$ into connected components. According to [47, Cor. 2.6] Z^{an} is connected and

$$Z^{an} \times_{Y^{an}} X^{an} \cong (Z \times_Y X)^{an} = \bigsqcup_{i=1}^{r} W_i^{an}$$

is the decomposition into connected components. Therefore $\Gamma_h = (id \times h)(Z^{an})$, the graph of h, coincides with W_i^{an} for some i. Let $G = W_i$. Then the composition $G \hookrightarrow Z \times_Y X \xrightarrow{\pi_1} Z$ is an isomorphism since $G^{an} = \Gamma_h \hookrightarrow (Z \times_Y X)^{an} \to Z^{an}$ is a biholomorphic map [47, Prop. 3.1]. The composition $g = \pi_2|_G \circ (\pi_1|_G)^{-1} : Z \to G \to X$ is a morphism which satisfies $f = p \circ g$ and $h = g^{an}$. The uniqueness of g follows from the uniqueness of the closed subscheme $G \subset Z \times_Y X$ whose ideal sheaf satisfies $(\mathcal{J}_G)^{an} = \mathcal{J}_{\Gamma_h}$ (cf. [47, Prop. 1.3.1]). $\square$

Lemma 4.3. *Let X, Y and Z be complex spaces. Let $(f, \tilde{f}) : (Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$ and $(p, \tilde{p}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be holomorphic maps. Suppose that one of the following conditions holds:*

(i) *$(p, \tilde{p})$ is locally biholomorphic;*
(ii) *$(p, \tilde{p})$ is an immersion at every $x \in X$ and $(Z, \mathcal{O}_Z)$ is reduced.*

Suppose that there is a continuous lifting h of f: $f = p \circ h$

$$\begin{tikzcd} & {|X|} \arrow[d, "p"] \\ {|Z|} \arrow[ur, "h"] \arrow[r, "f"'] & {|Y|} \end{tikzcd}$$

Then h is the underlying continuous map of a holomorphic map $(h, \tilde{h}) : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$, which is a holomorphic lifting of $(f, \tilde{f})$: $(f, \tilde{f}) = (p, \tilde{p}) \circ (h, \tilde{h})$.

Proof. Case (i). In this case the stalk map $\tilde{p}_x : \mathcal{O}_{Y,p(x)} \to \mathcal{O}_{X,x}$ is an isomorphism for every $x \in X$. Let $z \in Z$, $x = h(z)$, $y = p(x)$. One defines a local homomorphism $\tilde{h}_z : \mathcal{O}_{X,x} \to \mathcal{O}_{Z,z}$ by $\tilde{h}_z(s_x) = \tilde{f}_z(\tilde{p}_x^{-1})(s_x)$. Let V be an open subset of X, let $U = h^{-1}(V)$. Let $s \in \Gamma(V, \mathcal{O}_X)$. For every $z \in U$ let $t_z = \tilde{h}_z(s_{h(z)})$. Let $t : U \to \bigsqcup_{z \in U} \mathcal{O}_{Z,z}$ be the map defined by $z \mapsto t_z$. We claim that $t \in \Gamma(U, \mathcal{O}_Z)$. For every $z \in U$ let V_x be an open neighborhood of $x = h(z)$, $V_x \subset V$, which is mapped by p biholomorphically onto $W_y \subset Y$. Let $U_z = h^{-1}(V_x)$. Let $r \in \Gamma(W_y, \mathcal{O}_Y)$ satisfy $(\widetilde{p|_{V_x}})(r) = s|_{V_x}$. Then $t|_{U_z} = \tilde{f}(r)$ belongs to $\Gamma(U_z, \mathcal{O}_Z)$. This shows that $t \in \Gamma(U, \mathcal{O}_Z)$. One defines in this way a morphism of sheaves $\tilde{h} : \mathcal{O}_X \to h_*\mathcal{O}_Z$ and $(h, \tilde{h}) : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ is a holomorphic lifting of $(f, \tilde{f}) : (Z, \mathcal{O}_Z) \to (Y, \mathcal{O}_Y)$.

Case (ii). Here one defines $\tilde{h}_z : \mathcal{O}_{X,x} \to \mathcal{O}_{Z,z}$ as follows. By hypothesis $\tilde{p}_x : \mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is surjective. Let $s_x = \tilde{p}_x(r_y)$. Let $t_z = \tilde{f}_z(r_y)$. We claim that t_z does not depend on the choice of r_y. In fact, let $s_x = \tilde{p}_x(r_y) = \tilde{p}_x(r'_y)$. There are neighborhoods V_x of x and W_y of y such that $p|_{V_x} : V_x \to W_y$ is a closed embedding, r_y and r'_y are germs of $r, r' \in \Gamma(W_y, \mathcal{O}_Y)$ and $\widetilde{p|_{V_x}}(r - r') = 0$. In particular $(r - r')(w) = 0$ for $\forall w \in p(V_x)$. Let $U_z = h^{-1}(V_x)$. Then $f(U_z) \subset p(V_x)$, therefore $\tilde{f}(r - r')(u) = (r - r')(f(u)) = 0$ for $\forall u \in U_z$. This implies that $\widetilde{f|_{U_z}}(r - r') = 0$ by [21, Ch. 4 § 3 n.3] since $(Z, \mathcal{O}_Z)$ is reduced. This shows that $\tilde{f}_z(r_y) = \tilde{f}_z(r'_y)$ as claimed, so the local homomorphism $\tilde{h}_z : \mathcal{O}_{X,x} \to \mathcal{O}_{Z,z}$ is well-defined. One concludes the proof as in Case (i). □

Combining Proposition 4.2 and Lemma 4.3 one obtains the following proposition

Proposition 4.4. *Let X, Y and Z be separated schemes of finite type over $\mathbb{C}$. Let $f : Z \to Y$ and $p : X \to Y$ be morphisms. Suppose one of the following conditions holds:*

(i) *p is étale;*
(ii) *p is unramified and Z is reduced.*

Suppose there is a continuous lifting c of f^{an}: $f^{an} = p^{an} \circ c$

$$\begin{tikzcd} & {|X^{an}|} \arrow[d, "p^{an}"] \\ {|Z^{an}|} \arrow[ur, "c"] \arrow[r, "f^{an}"'] & {|Y^{an}|} \end{tikzcd}$$

Then there is a morphism $(g, g^\sharp) : (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ such that $f = p \circ g$ and such that $g(z) = c(z)$ for $\forall z \in Z(\mathbb{C})$.

Recall that if $p : X \to Y$ is an étale cover of algebraic varieties, then $p^{an} : |X^{an}| \to |Y^{an}|$ is a topological covering map (see e.g. the proof of Proposition 2.6(iv)).

Corollary 4.5. *Let X, Y and Z be algebraic varieties. Let*

$$\begin{tikzcd} & X \arrow[d, "p"] \\ Z \arrow[ur, "g"] \arrow[r, "f"'] & Y \end{tikzcd}$$

be a commutative diagram of maps, where f is a morphism, p is an unramified morphism and $g : |Z^{an}| \to |X^{an}|$ is continuous. Then g is a morphism. In particular, if $p : X \to Y$ is an étale cover, then $\mathrm{Deck}(|X^{an}|/|Y^{an}|) \cong \mathrm{Aut}(X/Y)$.

Corollary 4.6. *Let $p : X \to Y$ be a map of sets. Suppose that Y has a structure of an algebraic variety $(Y, \mathcal{O}_Y)$ and X has two structures of algebraic varieties $(X, \mathcal{O}_X)$ and $(X, \mathcal{O}'_X)$ such that $p : X \to Y$ is unramified morphism for both structures of X. Suppose that the Euclidean topologies on X associated with $(X, \mathcal{O}_X)$ and $(X, \mathcal{O}'_X)$ coincide. Then the two Zariski's topologies on X coincide and $\mathcal{O}_X = \mathcal{O}'_X$.*

Proof. The map $id_X : X \to X$ is a continuous lifting of the two unramified morphisms, so by Proposition 4.4(ii) it is an isomorphism of the algebraic varieties $(X, \mathcal{O}_X)$ and $(X, \mathcal{O}'_X)$. □

5. Hurwitz moduli varieties parameterizing pointed G-covers

Let $\mathrm{Var}_{\mathbb{C}}$ be the category of algebraic varieties over $\mathbb{C}$.

Definition 5.1. For every $S \in \mathrm{Var}_{\mathbb{C}}$ we denote by $\mathcal{H}^G_{Y,n}(S)$ the set of all smooth families of G-covers of Y branched in n points $p : \mathcal{C} \to Y \times S$ modulo G-equivalence (cf. Definition 3.8). We denote by $\mathcal{H}^G_{(Y,y_0),n}(S)$ the set of all smooth families of pointed G-covers of (Y, y_0) branched in n points $(p : \mathcal{C} \to Y \times S, \zeta)$ modulo G-equivalence (cf. Definition 3.9).

5.2. Let $u : T \to S$ be a morphism of algebraic varieties. Given a smooth, proper morphism $\mathcal{C} \to S$ of reduced, separated schemes of finite type over $\mathbb{C}$, the pullback morphism $\mathcal{C}_T := \mathcal{C} \times_S T \to T$ is smooth and proper, in particular $\mathcal{C} \times_S T$ is a reduced scheme, since T is reduced (cf. [42, p.184]). Hence the scheme $\mathcal{C} \times_S T$ is isomorphic to the closed algebraic subvariety of $\mathcal{C} \times T$ whose set of points is the set-theoretical fiber product $\mathcal{C}(\mathbb{C}) \times_{S(\mathbb{C})} T(\mathbb{C})$.

Given a family of G-covers $p : \mathcal{C} \to Y \times S$ as in Definition 3.8 let $p_T : \mathcal{C}_T \to Y \times T$ be the morphism obtained from $\mathcal{C} \times_S T \to T$ and $\mathcal{C} \times_S T \to \mathcal{C} \to Y$.

This is the pullback family of covers. One defines an action of G on $\mathcal{C}_T$ induced by the action of G on the first factor of $\mathcal{C} \times_S T$. Then p_T is G-invariant and for every $t \in T$ the G-cover $(\mathcal{C}_T)_t \to Y$ is G-equivalent to $\mathcal{C}_{u(t)} \to Y$. Therefore $p_T : \mathcal{C}_T \to Y \times T$ satisfies the conditions of Definition 3.8. Clearly the pullbacks of G-equivalent families of G-covers are G-equivalent. This defines a moduli functor $\mathcal{H}^G_{Y,n} : \mathrm{Var}_{\mathbb{C}} \to (\mathrm{Sets})$

Given a family $(p : \mathcal{C} \to Y \times S, \zeta)$ of pointed G-covers of (Y, y_0) branched in n points as in Definition 3.9 and a morphism $u : T \to S$ the G-covers of the pullback family $p_T : \mathcal{C}_T \to Y \times T$ are unramified over y_0. Let $\zeta_T : T \to \mathcal{C}_T$ be the morphism $\zeta_T(t) = (\zeta(u(t)), t)$. One has $p_T(\zeta_T(t)) = (y_0, t)$. Therefore $(p_T : \mathcal{C}_T \to Y \times T, \zeta_T)$ satisfies the conditions of Definition 3.9. This defines a moduli functor $\mathcal{H}^G_{(Y,y_0),n} : \mathrm{Var}_{\mathbb{C}} \to (\mathrm{Sets})$

Proposition 5.3. *Let $(p : \mathcal{C} \to Y \times S, \zeta : S \to \mathcal{C})$ be a smooth family of pointed G-covers of (Y, y_0) branched in n points. Let $B \subset Y \times S$ be the branch locus of p. For every $s \in S$ let*

$$u(s) = (B_s, m_{\zeta(s)}) \in H^G_n(Y, y_0), \quad m_{\zeta(s)} : \pi_1(Y \setminus B_s, y_0) \twoheadrightarrow G \tag{16}$$

be the monodromy invariant of $(p_s : \mathcal{C}_s \to Y, \zeta(s))$. Then $u : S \to H^G_n(Y, y_0)$ is a morphism.

Proof. The map $\beta : S \to (Y \setminus y_0)^{(n)}_* \subset Y^{(n)}$ given by $\beta(s) = B_s$ is a morphism by Proposition 2.6(vi). The map u fits in the following commutative diagram

$$\begin{tikzcd} & H^G_n(Y, y_0) \arrow[d, "\delta"] \\ S \arrow[ur, "u"] \arrow[r, "\beta"'] & (Y \setminus y_0)^{(n)}_* \end{tikzcd}$$

Here δ, defined by $\delta(D, m) = D$, is a finite, surjective, étale morphism (cf. [32, Prop. 1.8] and § 3.7). By Corollary 4.5 it suffices to prove that $u : S \to H^G_n(Y, y_0)$ is a continuous map with respect to the Euclidean topologies of S^{an} and that of $H^G_n(Y, y_0)$ defined in § 3.7, i.e. we have to show that for every $s \in S$ and every neighborhood W of $u(s)$ the point s is internal of $u^{-1}(W)$. Let s_0 be an arbitrary point of S. Let $u(s_0) = (D, m)$, where $D = \{b_1, \ldots, b_n\}$, $z_0 = \zeta(s_0)$, $m = m_{z_0} : \pi_1(Y \setminus D, y_0) \twoheadrightarrow G$. Let $N_{(D,m)}(U_1, \ldots, U_n)$ be any of the open sets of the neighborhood basis of (D, m) in $H^G_n(Y, y_0)$ (cf. (7)). One has to prove that there exists a neighborhood $V \subset |S^{an}|$ of s_0 such that $u(V) \subset N_{(D,m)}(U_1, \ldots, U_n)$. There is a neighborhood $V_1 \subset |S^{an}|$ of s_0 such that $\beta(V_1) \subset N_D(U_1, \ldots, U_n)$ since β^{an} is a holomorphic map. The complex space S^{an} is locally connected (cf. [21, Ch. 9 § 3 n.1]) and $\mathcal{C}^{an} \setminus p^{-1}(B) \to (Y \times S)^{an} \setminus B$ is a topological covering map by Proposition 2.6(iv). Therefore there is an embedded open disk $U \subset Y \setminus \cup_{i=1}^n \overline{U}_i$, $y_0 \in U$ and a connected neighborhood V of s_0, such that $V \subset V_1$, $U \times V \subset Y \times S \setminus B$ and $p^{-1}(U \times V)$ is a disjoint union of connected open sets homeomorphic to $U \times V$. Let W be the connected component of $p^{-1}(U \times V)$ which contains $\zeta(s_0)$. One has $\zeta(V) \subset p^{-1}(\{y_0\} \times V) \subset p^{-1}(U \times V)$, so $\zeta(V) \subset W$ since V is connected. We claim that $u(V) \subset N_{(D,m)}(U_1, \ldots, U_n)$. For every $s \in V$ one has $u(s) = (\beta(s), m_{\zeta(s)})$, so one has to prove that $m_{\zeta(s)} = m(\beta(s))$ for $\forall s \in V$. It suffices to verify the equality $m_{\zeta(s)}([\gamma]_{\beta(s)}) = m(\beta(s))([\gamma]_{\beta(s)})$ for every closed path $\gamma : I \to Y \setminus \cup_{i=1}^n \overline{U}_i$,

$\gamma(0) = \gamma(1) = y_0$. Let $g = m([\gamma]_D)$. Then $m(\beta(s))([\gamma]_{\beta(s)}) = g$ according to (6). Hence we have to verity that for every $s \in V$ the lifting of $\gamma \times \{s\}$ in $\mathcal{C}_s \setminus p_s^{-1}(B_s)$ with initial point $\zeta(s)$ has terminal point $g\zeta(s)$. This follows from the covering homotopy property. Indeed, let F be the continuous map

$$F : [0,1] \times V \to Y \times S \setminus B, \quad F(t,s) = (\gamma(t), s).$$

The map $\tilde{F}_0 : \{0\} \times V \to \mathcal{C} \setminus p^{-1}(B)$, $\tilde{F}_0(0,s) = \zeta(s)$ is a continuous lifting of $F|_{\{0\}\times V}$. Let $\tilde{F}$

$$\begin{tikzcd} & \mathcal{C} \setminus p^{-1}(B) \arrow[d, "p"] \\ {[0,1] \times V} \arrow[ur, "\tilde{F}"] \arrow[r, "F"] & Y \times S \setminus B \end{tikzcd}$$

be the unique continuous lifting of F which extends $\tilde{F}_0$ (cf. [54, Ch. 2 § 2 Th. 3]). For every $s \in V$ the path $t \mapsto \tilde{F}(t,s)$ is the unique lifting in $\mathcal{C}_s \setminus p_s^{-1}(B_s)$ of $\gamma \times \{s\}$ with initial point $\zeta(s)$, so

$$\tilde{F}(1,s) = m_{\zeta(s)}([\gamma]_{\beta(s)})\zeta(s).$$

One has that $\tilde{F}(\{1\} \times V) \subset p^{-1}(\{y_0\} \times V) \subset p^{-1}(U \times V)$ and $\tilde{F}(1,s_0) = g\zeta(s_0) \in gW$, therefore $\tilde{F}(\{1\} \times V) \subset gW$, since V is connected. This shows that $\tilde{F}(1,s) = g\zeta(s)$, hence $m_{\zeta(s)}([\gamma]_{\beta(s)}) = g$ for $\forall s \in V$. □

We need the following result [33, Theorem 2] in the proof of Theorem 5.5 below. Here we state it in the form we will use it.

Proposition 5.4. *Let X, S and P be algebraic varieties. Let $h : X \to S$ be a smooth morphism whose nonempty fibers are irreducible curves. Let $P \to S$ be a proper morphism. Let U be an open subset of X such that $U \cap h^{-1}(s) \neq \emptyset$ for every $s \in h(X)$. Let $\varphi : U \to P$ be an S-morphism. Let $\Gamma = \{(x, \varphi(x)) | x \in U\} \subset U \times P$ be its graph and let $\overline{\Gamma} \subset X \times P$ be its closure. Suppose that the projection morphism $\overline{\Gamma} \to X$ has finite fibers. Then there is a unique extension of φ to X: an S-morphism $\tilde{\varphi} : X \to P$ such that $\tilde{\varphi}|_U = \varphi$.*

Theorem 5.5. *Let Y be a smooth, projective, irreducible curve of genus $g \geq 0$. Let n be a positive integer. Let $y_0 \in Y$. Let G be a finite group which can be generated by $2g + n - 1$ elements. The algebraic variety $H_n^G(Y, y_0)$ is a fine moduli variety for the moduli functor $\mathcal{H}^G_{(Y,y_0),n}$ of smooth families of pointed G-covers of (Y, y_0) branched in n points. The universal family is*

$$(p : \mathcal{C}(y_0) \to Y \times H_n^G(Y, y_0), \zeta : H_n^G(Y, y_0) \to \mathcal{C}(y_0)) \tag{17}$$

(cf. Theorem 3.20)

Proof. Let $[q : X \to Y \times S, \eta : S \to X] \in \mathcal{H}^G_{(Y,y_0),n}(S)$. Let $B \subset Y \times S$ be the branch locus and let $u : S \to H_n^G(Y, y_0)$, $u(s) = (B_s, m_{\eta(s)})$ be the morphism of Proposition 5.3. We want to prove that $(X \to Y \times S, \eta)$ is G-equivalent to the pullback by u of the family (17). This is the unique morphism with this property since the monodromy invariant classifies the pointed G-covers up to G-equivalence. For every $s \in S$ there exists a unique G-equivariant isomorphism $f_s : X_s \to \mathcal{C}(y_0)_{u(s)}$ such that $p_s \circ f_s = q_s$ and $f_s(\eta(s)) = \zeta(u(s))$. Let $f : X \to \mathcal{C}(y_0)$ be the G-equivariant map, which equals f_s on every X_s. One obtains the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & \mathcal{C}(y_0) \arrow[d, "p"] \\ Y \times S \arrow[r, "id\times u"] & Y \times H^G_n(Y, y_0) \end{tikzcd} \tag{18}$$

We want to prove that f is a morphism and that (18) is a Cartesian diagram. Let $\mathcal{B}$ be the branch locus of p. Then $B = (id \times u)^{-1}(\mathcal{B})$ is the branch locus of q. Let $X' = X \setminus q^{-1}(B)$, $f' = f|_{X'}$, $q' = q|_{X'}$. Restricting (18) on the complements of the branch loci one obtains the commutative diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "q'"'] & \mathcal{C}(y_0)' \arrow[d, "p'"] \\ Y \times S \setminus B \arrow[r, "(id\times u)'"] & Y \times H^G_n(Y, y_0) \setminus \mathcal{B} \end{tikzcd}$$

We claim that f' is continuous with respect to the Euclidean topologies of X'^{an} and $\mathcal{C}(y_0)'$ (cf. Proposition 3.14). Let $x \in X'_s$ and let $\lambda : I \to X'_s$ be a path such that $\lambda(0) = \eta(s)$, $\lambda(1) = x$. Then $q' \circ \lambda : I \to Y \setminus B_s$ is a path with initial point y_0. Lifting $q' \circ \lambda$ in $\mathcal{C}(y_0)'_{u(s)}$ with initial point $\zeta(u(s)) = (\Gamma_{m_{\eta(s)}}[c_{y_0}]_{B_s}, B_s, m_{\eta(s)})$ its terminal point is

$$f'(x) = f'_s(x) = (\Gamma_{m_{\eta(s)}}[q' \circ \lambda]_{B_s}, B_s, m_{\eta(s)}) \in \mathcal{C}(y_0)'_{u(s)} \tag{19}$$

(cf. § 3.5). For every $x_0 \in X'$ and every neighborhood N of $f'(x_0)$ in $\mathcal{C}(y_0)'$ we have to prove that x_0 is an internal point of $f'^{-1}(N)$. Let $q'(x_0) = (y, s_0)$, $D = \beta(s_0)$, $m = m_{\eta(s_0)} : \pi_1(Y \setminus D, y_0) \twoheadrightarrow G$, let $\lambda : I \to X'_{s_0}$ be a path such that $\lambda(0) = \eta(s_0)$, $\lambda(1) = x_0$ and let $\alpha = q'_{s_0} \circ \lambda : I \to Y \setminus D$. One has $\alpha(0) = y_0$, $\alpha(1) = y$ and $f'(x_0) = (\Gamma_m[\alpha]_D, D, m) \in \mathcal{C}(y_0)'$. Let $N_{(\alpha,D,m)}(U, U_1, \dots, U_n)$ be a neighborhood of $f'(x_0)$ as in § 3.13 contained in N. We want to show, that there exists a neighborhood W of x_0 such that $f'(W) \subset N_{(\alpha,D,m)}(U, U_1, \dots, U_n)$. The argument is similar to the one of Proposition 5.3. Shrinking U one can choose a connected neighborhood V of s_0 in $|S^{an}|$ such that $\beta(V) \subset N_D(U_1, \dots, U_n)$, $U \times V \subset Y \times S \setminus B$ and $q'^{-1}(U \times V)$ is a disjoint union of connected open sets homeomorphic to $U \times V$. Let W be the connected component of $q'^{-1}(U \times V)$ which contains x_0. We claim that $f'(W) \subset N_{(\alpha,D,m)}(U, U_1, \dots, U_n)$. Consider the homotopy

$$F : [0, 1] \times V \to Y \times S \setminus B, \quad F(t, s) = (\alpha(t), s)$$

By the covering homotopy property of topological covering maps (cf. [54, Ch. 2 § 2 Th. 3]) there exists a unique continuous lifting

$$\begin{tikzcd} & X' \arrow[d, "q'"] \\ {[0,1] \times V} \arrow[ur, "\tilde{F}"] \arrow[r, "F"] & Y \times S \setminus B \end{tikzcd}$$

such that $\tilde{F}(0, s) = \eta(s)$ for $\forall s \in V$. We have $q' \circ \tilde{F}(0, s) = q'(\eta(s)) = (y_0, s) = (\alpha(0), s)$. Let $s \in V$. The path $t \mapsto \tilde{F}(t, s)$ is a lifting of $\alpha \times \{s\}$ with initial point $\eta(s)$, so $q'(\tilde{F}(1, s)) = (\alpha(1), s) = (y, s) \in U \times V$. If $s = s_0$, then $\tilde{F}(1, s_0) = \lambda(1) =$

x_0. This implies that $\tilde{F}(\{1\}\times V)\subset W$ since V is connected. The map $q'|_W : W \to U\times V$ is a homeomorphism. Let $x\in W$ and let $q'(x)=(z,s)$. We construct a path in X'_s which connects $\eta(s)$ with x as follows. The path $\tilde{\alpha}_s(t)=\tilde{F}(t,s)$ has initial point $\eta(s)$ and terminal point $w\in W$ such that $q'(w)=(\alpha(1),s)=(y,s)$. Let $\tau : I\to U$ be a path such that $\tau(0)=y$, $\tau(1)=z$. Then $\mu=\tilde{\alpha}_s\cdot\big((q'|_W)^{-1}\circ(\tau\times\{s\})\big)$ is a path in X'_s which connects $\eta(s)$ with x and $q'_s\circ\mu=\alpha\cdot\tau$. Let $\beta(s)=E\in N_D(U_1,\dots,U_n)$. We showed in Proposition 5.3 that $m_{\eta(s)}=m(E)$. So, according to (19)

$$f'(x)=(\Gamma_{m(E)}[\alpha\cdot\tau]_E,E,m(E))$$

This shows that $f'(W)\subset N_{(\alpha,D,m)}(U,U_1,\dots,U_n)$. The claim that f' is continuous is proved.

We apply now Corollary 4.5 to the commutative diagram

$$\begin{tikzcd}
 & \mathcal{C}(y_0)' \arrow[d, "p'"] \\
X' \arrow[ur, "f'"] \arrow[r, "(id\times u)'\circ q'"'] & Y\times H_n^G(Y,y_0)\setminus\mathcal{B}
\end{tikzcd}$$

and conclude that $f' : X'\to\mathcal{C}(y_0)'$ is a morphism.

Let $\mathcal{C}(y_0)_S$ be the fiber product of $\mathcal{C}(y_0)\to H_n^G(Y,y_0)$ and $u : S\to H_n^G(Y,y_0)$. The composition $X'\xrightarrow{f'}\mathcal{C}(y_0)'\hookrightarrow\mathcal{C}(y_0)$ yields an S-morphism $\varphi : X'\to\mathcal{C}(y_0)_S$ which fits in the following commutative diagram of morphisms

$$\begin{tikzcd}
X \arrow[dr, "q"'] & X' \arrow[l, hook'] \arrow[r, "\varphi"] \arrow[d] & \mathcal{C}(y_0)_S \arrow[dl, "p_S"] \\
 & Y\times S \arrow[d, "\pi_2"] & \\
 & S &
\end{tikzcd}$$

The graph Γ of φ is contained in the set-theoretical fiber product $X\times_{Y\times S}\mathcal{C}(y_0)_S$ which is a Zariski closed subset of $X\times\mathcal{C}(y_0)_S$, so it contains the closure $\overline{\Gamma}$. Therefore the projection morphism $\overline{\Gamma}\to X$ has finite fibers. Applying Proposition 5.4 we conclude that φ can be extended to an S-morphism $\tilde{\varphi} : X\to\mathcal{C}(y_0)_S$. For every $s\in S$ the composition $X_s\xrightarrow{\tilde{\varphi}_s}(\mathcal{C}(y_0)_S)_s\xrightarrow{\sim}\mathcal{C}(y_0)_{u(s)}$ is a morphism which coincides with f'_s on X'_s. Hence this composition equals f_s. This implies on one hand that f is equal to the composition $X\xrightarrow{\tilde{\varphi}}\mathcal{C}(y_0)_S\longrightarrow\mathcal{C}(y_0)$, so f is a morphism, and on the other hand that $\tilde{\varphi}_s$ is a G-equivariant isomorphism for every $s\in S$. Applying [22, Proposition (4.6.7)] we conclude that $\tilde{\varphi} : X\to\mathcal{C}(y_0)_S$ is a G-equivariant isomorphism. It is clear that $p_S\circ\tilde{\varphi}=q$ and $\tilde{\varphi}(\eta(s))=\varphi(\eta(s))=\zeta_S(s)$ for every $s\in S$. Therefore (18) is a Cartesian diagram, so $(q : X\to Y\times S,\eta)$ is G-equivalent to the pullback of the family (17) by the morphism $u : S\to H_n^G(Y,y_0)$. □

Definition 5.6. Let $O_1,\dots,O_k$ be conjugacy classes of G, $O_i\neq O_j$ if $i\neq j$. Let $\underline{n}=n_1O_1+\cdots+n_kO_k$ be a formal sum, where $n_i\in\mathbb{N}$. Let $|\underline{n}|=n_1+\cdots+n_k=n$. We say that a pointed G-cover $(p : C\to Y,z_0)$ of (Y,y_0) branched in n points is of branching type $\underline{n}$ if, for every $i=1,\dots,k$, n_i of the branch points of p have local monodromies in O_i, i.e. if its monodromy invariant $(D,m) = (D,m_{z_0})$ satisfies the property (cf. § 3.2)

$$n_i \quad \text{of the elements} \quad m([\gamma_j]) \quad \text{belong to} \quad O_i \quad \text{for} \quad i = 1,\dots,k. \tag{20}$$

5.7. Let $H^G_{\underline{n}}(Y,y_0)$ be the subset of $H^G_n(Y,y_0)$ consisting of the elements (D,m) which satisfy Condition (20). One has

$$H^G_n(Y,y_0) = \bigsqcup_{|\underline{n}|=n} H^G_{\underline{n}}(Y,y_0)$$

and every $H^G_{\underline{n}}(Y,y_0)$ is an open subset in the Euclidean topology of $H^G_n(Y,y_0)$. Therefore every nonempty $H^G_{\underline{n}}(Y,y_0)$ is a union of connected components of $|H^G_n(Y,y_0)^{an}|$, so every nonempty $H^G_{\underline{n}}(Y,y_0)$ is a union of connected components in the Zariski topology of $H^G_n(Y,y_0)$ [47, Cor. 2.6] and inherits the structure of algebraic variety from $H^G_n(Y,y_0)$.

Suppose $H^G_{\underline{n}}(Y,y_0) \neq \emptyset$. Let us denote by $p_{\underline{n}} : \mathcal{C}_{\underline{n}}(y_0) \to Y \times H^G_{\underline{n}}(Y,y_0)$ the restriction of the family $p : \mathcal{C}(y_0) \to Y \times H^G_n(Y,y_0)$ and let $\zeta_{\underline{n}} : H^G_{\underline{n}}(Y,y_0) \to \mathcal{C}_{\underline{n}}(y_0)$ be the restriction of the morphism $\zeta : H^G_n(Y,y_0) \to \mathcal{C}(y_0)$. Let us denote by

$$\mathcal{H}^G_{(Y,y_0),\underline{n}} : \mathrm{Var}_{\mathbb{C}} \to (\mathrm{Sets})$$

the moduli functor which associates with every algebraic variety S the set $\{[\mathcal{C} \to Y \times S, \eta]\}$ of smooth families of pointed G-covers of (Y,y_0) of branching type $\underline{n}$ modulo G-equivalence and with every morphism $T \to S$ the pullback of families of G-covers. Theorem 5.5 implies the following one.

Theorem 5.8. *Let Y be a smooth, projective, irreducible curve. Let $y_0 \in Y$. Let G be a finite group. Let $O_1,\dots,O_k$ be congugacy classes in G, $O_i \neq O_j$ if $i \neq j$. Let $\underline{n} = n_1O_1 + \cdots + n_kO_k$ be a formal sum, where $n_i, i = 1,\dots,k$ are positive integers. Suppose that $H^G_{\underline{n}}(Y,y_0) \neq \emptyset$. The algebraic variety $H^G_{\underline{n}}(Y,y_0)$ is a fine moduli variety for the moduli functor $\mathcal{H}^G_{(Y,y_0),\underline{n}}$. The universal family is*

$$(p_{\underline{n}} : \mathcal{C}_{\underline{n}}(y_0) \to Y \times H^G_{\underline{n}}(Y,y_0), \zeta_{\underline{n}} : H^G_{\underline{n}}(Y,y_0) \to \mathcal{C}_{\underline{n}}(y_0)). \tag{21}$$

6. Parameterization of G-covers

6.1. Let $p : C \to Y$ be a G-cover branched in $D \subset Y$, $|D| = n \geq 1$. Endowing C and Y with the canonical Euclidean topologies of $|C^{an}|$ and $|Y^{an}|$ respectively, consider the topological covering map $p' : C' \to Y \setminus D$, where $C' = p^{-1}(Y \setminus D)$, $p' = p|_{C'}$. For every $y \in Y \setminus D$ and every $z \in p^{-1}(y)$ let $m_z : \pi_1(Y \setminus D, y) \to G$ be the monodromy epimorphism defined in § 3.2: $m_z[\alpha] = g$ if $gz = z\alpha$. Every m_z satisfies Condition ((3)) with closed paths based at y. Every two m_{z_1} and m_{z_2} are *pathwise connected* (cf. [56, § 1.3], [32, Section 1]): there is a path $\tau : I \to Y \setminus D$, $\tau(0) = y_1 = p(z_1)$, $\tau(1) = y_2 = p(z_2)$, image of a path in C' with initial point z_1 and terminal point z_2, such that

$$m_{z_2}([\alpha]) = m_{z_1}[\tau \cdot \alpha \cdot \tau^{-1}] := m^{\tau}_{z_1}([\alpha])$$

for every $[\alpha] \in \pi_1(Y \setminus D, y_2)$. The set $\underline{m}_p = \{m_z | z \in p^{-1}(Y \setminus D)\}$ forms an equivalence class with respect to pathwise connectedness in the set of epimorphisms $m : \pi_1(Y \setminus D, y) \twoheadrightarrow G$, where $y \in Y \setminus D$.

Definition 6.2. Given a G-cover $p : C \to Y$ branched in $D \subset Y$ the pair $(D, \underline{m}_p)$ is called the *monodromy invariant* of p.

Definition 6.3. Let $g = g(Y)$, let n be a positive integer. Let G be a finite group which can be generated by $2g + n - 1$ elements. We denote by $H_n^G(Y)$ the set of pairs $(D, \underline{m})$, where $D \in Y_*^{(n)}$ and $\underline{m}$ is an equivalence class of pathwise connected epimorphisms $m : \pi_1(Y \setminus D, y) \to G$, where $y \in Y \setminus D$, which satisfy Condition ((3)).

6.4. $H_n^G(Y) \neq \emptyset$ and the map $H_n^G(Y) \to Y_*^{(n)}$ given by $((D, \underline{m}) \to D)$ is surjective. Riemann's existence theorem yields that the mapping $[p : C \to Y] \mapsto (D, \underline{m}_p)$ stabilizes a bijective correspondence between the set of G-equivalence classes of G-covers branched in n points and the set $H_n^G(Y)$.

Let $y \in Y \setminus D$. Let $m : \pi_1(Y \setminus D, y) \to G$ be an epimorphism. Then $m_1 : \pi_1(Y \setminus D, y) \to G$ is pathwise connected with m if and only if $m_1 = gmg^{-1}$ for some $g \in G$. Let $U(y) \subset H_n^G(Y)$ be the subset $\{(D, \underline{m}) | y \notin D\}$. One has $H_n^G(Y) = \cup_{y \in Y} U(y)$. The map $H_n^G(Y, y) \to U(y)$, defined by $(D, m) \to (D, \underline{m})$ is invariant with respect to the action of G on the set $H_n^G(Y, y)$ defined by $h*(D, m) = (D, hmh^{-1})$. This action induces a free action of $\overline{G} = G/Z(G)$ on $H_n^G(Y, y)$. The set $U(y)$ is bijective to the quotient set ${}_G\backslash H_n^G(Y, y) =_{\overline{G}} \backslash H_n^G(Y, y)$.

Remark 6.5. The set $H_n^G(\mathbb{P}^1)$ is the one denoted by $\mathcal{H}_n^{in}(G)$ in [18, § 1.2].

Proposition 6.6. *For every $y \in Y$ the action of G on $H_n^G(Y, y)$ defined by $h * (D, m) = (D, hmh^{-1})$ is an action by covering automorphisms of the étale cover $\delta : H_n^G(Y, y) \to (Y \setminus y)_*^{(n)}$, where $\delta(D, m) = D$. The set $H_n^G(Y)$ can be endowed with a structure of an algebraic variety which has the following properties.*

(i) *The map $H_n^G(Y) \to Y_*^{(n)}$ defined by $(D, \underline{m}) \to D$ is a surjective, étale, finite morphism.*
(ii) *For every $y \in Y$ the set $U(y)$ is an affine open subset in $H_n^G(Y)$ and the map $\nu : H_n^G(Y, y) \to U(y)$ defined by $\nu(D, m) = (D, \underline{m})$ is an étale Galois cover with respect to the $*$-action with Galois group $\overline{G} = G/Z(G)$.*
(iii) *$H_n^G(Y)$ is a quasi-projective variety. If $Y \cong \mathbb{P}^1$ it is an affine variety.*

Proof. The action of $\overline{G} = G/Z(G)$ on $H_n^G(Y, y)$ is by covering homeomorphisms of the topological covering map $\delta : H_n^G(Y, y) \to (Y \setminus y)_*^{(n)}$ since

$$h * N_{(D,m)}(U_1, \dots, U_n) = N_{(D,hmh^{-1})}(U_1, \dots, U_n) \quad \text{for} \quad \forall h \in G.$$

The map δ is on the other hand an étale cover of affine varieties, so by Corollary 4.5 this action is by automorphisms of $H_n^G(Y, y)$. Endow $U(y)$ with a structure of an affine variety as the quotient ${}_{\overline{G}}\backslash H_n^G(Y, y)$. The associated Euclidean topology, i.e. that of $({}_{\overline{G}}\backslash H_n^G(Y, y))^{an}$, is the quotient topology of $H_n^G(Y, y)^{an}$ by Lemma 2.5. Let us verify the patching condition for the subsets $U(y) \subset H_n^G(Y)$, $y \in Y$. Let $y_1, y_2 \in Y$, $y_1 \neq y_2$. The set $U(y_1) \cap U(y_2)$ is Zariski open in $U(y_i)$ for $i = 1, 2$ since it is the preimage of $(Y \setminus \{y_1, y_2\})_*^{(n)}$ with respect to the morphism $U(y_i) \to (Y \setminus y_i)_*^{(n)}$. The Euclidean topology of $U(y_1) \cap U(y_2)$ inherited from $U(y_i)$ has a basis consisting of the open sets $\nu_i(N_{(D,m_i)}(U_1, \dots, U_n))$, where $D \subset Y \setminus \{y_1, y_2\}$, $m_i : \pi_1(Y \setminus D, y_i) \to G$, $\cup_{j=1}^n \overline{U}_j \subset Y \setminus \{y_1, y_2\}$ and $\nu_i : H_n^G(Y, y_i) \to U(y_i)$ is the quotient map. Now, given $D \in (Y \setminus \{y_1, y_2\})_*^{(n)}$ and $U_1, \dots, U_n$ as in § 3.7 such that $\cup_{j=1}^n \overline{U}_j \subset Y \setminus \{y_1, y_2\}$ let us choose a path $\tau : I \to Y \setminus \cup_{j=1}^n \overline{U}_j$ such that $\tau(0) = y_1$, $\tau(1) = y_2$. Then one has

$$\nu_1(N_{(D,m_1)}(U_1, \dots, U_n)) = \nu_2(N_{(D,m_1^\tau)}(U_1, \dots, U_n)).$$

This shows that the two Euclidean topologies on $U(y_1)\cap U(y_2)$ coincide. Applying Corollary 4.6 to the map

$$U(y_1)\cap U(y_2)\to (Y\setminus\{y_1,y_2\})^{(n)}_*,\quad (D,\underline{m})\mapsto D$$

we conclude that the two structures of algebraic varieties on $U(y_1)\cap U(y_2)$ inherited from $U(y_1)$ and $U(y_2)$ coincide. This shows that one can define on $H^G_n(Y)$ a structure of a reduced scheme over $\mathbb{C}$ such that every $U(y)$ is an affine open subset of $H^G_n(Y)$. The map $H^G_n(Y)\to Y^{(n)}_*$, defined by $(D,\underline{m})\mapsto D$ is a finite, étale, surjective morphism since these properties hold for $U(y)\to(Y\setminus y)^{(n)}_*$ for $\forall y\in Y$. This implies, in particular, that $H^G_n(Y)$ is a reduced, separated scheme of finite type over $\mathbb{C}$, i.e. an algebraic variety, since the open subset $Y^{(n)}_*\subset Y^{(n)}$ has these properties. Parts (i) and (ii) are proved. Part (iii) is proved in [32, Proposition 1.9]. □

Let $y_0\in Y$. Let us define a left action of G on $Y\times H^G_n(Y,y_0)$ by

$$h*(y,(D,m))=(y,(D,hmh^{-1})). \tag{22}$$

The open subset $Y\times H^G_n(Y,y_0)\setminus B$ (cf. (8)) is G-invariant and by Proposition 6.6 G acts on it by covering automorphisms of the étale cover

$$Y\times H^G_n(Y,y_0)\setminus B\to Y\times(Y\setminus y_0)^{(n)}_*\setminus A$$

(cf. § 2.2). For every $h\in G$ and every $z=(\Gamma_m[\alpha]_D,D,m)\in\mathcal{C}(y_0)'$ let us define $h*z\in\mathcal{C}(y_0)'$ as follows. Let $h=m([\eta]_D)$, where η is a loop based at y_0. Let

$$h*(\Gamma_m[\alpha]_D,D,m)=(\Gamma_{hmh^{-1}}[\eta^-\cdot\alpha]_D,D,hmh^{-1}). \tag{23}$$

Proposition 6.7. *The following properties hold.*

(i) $(h,z)\mapsto h*z$ *is a left action of* G *on the set* $\mathcal{C}(y_0)'$.

(ii) *The two actions of* G *on* $\mathcal{C}(y_0)'$ *defined by* $(g,z)\mapsto gz$ *and* $(h,z)\mapsto h*z$ *commute.*

(iii) $p'(h*z)=h*p'(z)$ *for* $\forall h\in G$ *and* $\forall z\in\mathcal{C}(y_0)'$.

(iv) *The* $*$*-action of* G *on* $\mathcal{C}(y_0)'$ *is an action by covering automorphisms of the composed étale cover*

$$\mathcal{C}(y_0)'\xrightarrow{p'}Y\times H^G_n(Y,y_0)\setminus B\longrightarrow Y\times(Y\setminus y_0)^{(n)}_*\setminus A. \tag{24}$$

(v) *The* $*$*-action of* G *on* $\mathcal{C}(y_0)'$ *can be uniquely extended to a left action of* G *on* $\mathcal{C}(y_0)$ *by covering automorphisms of the composed finite morphism*

$$\mathcal{C}(y_0)\xrightarrow{p}Y\times H^G_n(Y,y_0)\xrightarrow{id\times\delta}Y\times(Y\setminus y_0)^{(n)}_*.$$

This action commutes with the action of G *on* $\mathcal{C}(y_0)$ *relative to the Galois cover* $p:\mathcal{C}(y_0)\to Y\times H^G_n(Y,y_0)$ *and* p *is equivariant:* $p(h*z)=h*p(z)$ *for* $\forall h\in G$ *and* $\forall z\in\mathcal{C}(y_0)$.

(vi) *For every* $h\in G$ *the automorphism of* $\mathcal{C}(y_0)$ *defined by* $z\mapsto h*z$ *induces a* G*-equivalence between* $\mathcal{C}(y_0)_{(D,m)}\to Y$ *and* $\mathcal{C}(y_0)_{(D,hmh^{-1})}\to Y$ *for every* $(D,m)\in H^G_n(Y,y_0)$.

Proof. Let $z=(\Gamma_m[\alpha]_D,D,m)\in\mathcal{C}(y_0)'$.

(i) Let $h_1,h_2\in G$, $h_i=m([\eta_i])_D$.

$$\begin{aligned}(h_1h_2)*z&=(\Gamma_{h_1h_2m(h_1h_2)^{-1}}[\eta_2^-\cdot\eta_1^-\cdot\alpha]_D,D,h_1h_2m(h_1h_2)^{-1})\\ h_2*z&=(\Gamma_{h_2mh_2^{-1}}[\eta_2^-\cdot\alpha]_D,D,h_2mh_2^{-1}).\end{aligned}$$

$h_1 = m([\eta_1]_D) = h_2 m h_2^{-1}([\eta_2^- \cdot \eta_1 \cdot \eta_2]_D)$ and $(\eta_2^- \cdot \eta_1 \cdot \eta_2)^- = \eta_2^- \cdot \eta_1^- \cdot \eta_2$, hence

$$h_1 * (h_2 * z) = (\Gamma_{h_1(h_2 m h_2^{-1})h_1^{-1}}[(\eta_2^- \cdot \eta_1^- \cdot \eta_2) \cdot \eta_2^- \cdot \alpha]_D, D, h_1(h_2 m h_2^{-1})h_1^{-1}).$$

We see that $(h_1 h_2) * z = h_1 * (h_2 * z)$.

(ii) Let $g = m([\sigma]_D)$, $h = m([\eta]_D)$. Then

$$h * (gz) = h * (\Gamma_m[\sigma \cdot \alpha]_D, D, m) = (\Gamma_{hmh^{-1}}[\eta^- \cdot \sigma \cdot \alpha]_D, D, hmh^{-1})$$
$$h * z = (\Gamma_{hmh^{-1}}[\eta^- \cdot \alpha]_D, D, hmh^{-1}).$$

One has $g = m([\sigma]_D) = hmh^{-1}([\eta^- \cdot \sigma \cdot \eta]_D)$, so

$$g(h * z) = (\Gamma_{hmh^{-1}}[\eta^- \cdot \sigma \cdot \eta \cdot \eta^- \cdot \alpha]_D, D, hmh^{-1}).$$

We see that $h * (gz) = g(h * z)$.

(iii) Let $y = \alpha(1)$. Then $p'(z) = (y, (D, m))$ and according to (23)

$$p'(h * z) = ((\eta^- \cdot \alpha)(1), D, hmh^{-1}) = (y, D, hmh^{-1}) = h * p'(z).$$

(iv) Let $N_{(\alpha,D,m)}(U, U_1, \ldots, U_n)$ be as in (9). Let $h \in G$, $h = m([\eta]_D)$, where η is a closed path contained in $Y \setminus \cup_{i=1}^n \overline{U}_i$. Then

$$h * N_{(\alpha,D,m)}(U, U_1, \ldots, U_n) = N_{(\eta^- \cdot \alpha, D, hmh^{-1})}(U, U_1, \ldots, U_n).$$

This shows that the $*$-action of G on $\mathcal{C}(y_0)'$ is an action by covering homeomorphisms of the topological covering map $\mathcal{C}(y_0)' \to Y \times (Y \setminus y_0)^{(n)}_* \setminus A$. This map is a composition of étale covers (24), so by Corollary 4.5 the $*$-action is an action by covering automorphisms of the composed étale cover.

(v) It is clear from the definition of $\mathcal{C}(y_0)$ as the disjoint union of normalizations (cf. § 3.19) that the $*$-action of G on $\mathcal{C}(y_0)'$ can be uniquely extended to a left action of G on the algebraic variety $\mathcal{C}(y_0)$. The other statements follow from (iv), (ii) and (iii).

(vi) This follows from (v). □

Our next goal is, provided $Z(G) = 1$, to construct a smooth family of G-covers of Y branched in n points $\pi : \mathcal{C} \to Y \times H^G_n(Y)$ such that for every $(D, \underline{m}) \in H^G_n(Y)$ the G-cover $\mathcal{C}_{(D,\underline{m})} \to Y$ has monodromy invariant $(D, \underline{m})$.

6.8. Let G be a finite group with trivial center. We use the following construction due to H. Völklein [57, § 10.1.3.1]. Let

$$\begin{aligned} \mathcal{C}' = \{(y, D, m) | D \in Y^{(n)}_*, y \in Y \setminus D, \\ m : \pi_1(Y \setminus D, y) \twoheadrightarrow G \quad \text{satisfies Condition ((3))}\} \end{aligned} \tag{25}$$

Let $\underline{B} = \{(y, (D, \underline{m})) | y \in D\} \subset Y \times H^G_n(Y)$. Consider the map

$$\pi' : \mathcal{C}' \to Y \times H^G_n(Y) \setminus \underline{B}, \quad (y, D, m) \stackrel{\pi'}{\mapsto} (y, (D, \underline{m})).$$

One defines a left action of G on $\mathcal{C}'$ by

$$g(y, D, m) = (y, D, gmg^{-1}). \tag{26}$$

This action is free since $Z(G) = 1$, π' is G-invariant and the quotient set $\mathcal{C}'/G$ is bijective to $Y \times H^G_n(Y) \setminus \underline{B}$.

Let $y_0 \in Y$. Let $\mathcal{C}[y_0]' = \pi'^{-1}(Y \times U(y_0) \setminus \underline{B})$. Let $\kappa' : \mathcal{C}(y_0)' \to \mathcal{C}[y_0]'$ be the map defined by

$$\kappa'((\Gamma_m[\alpha]_D, D, m)) = (\alpha(1), D, m^\alpha). \tag{27}$$

This map is G-equivariant. Indeed, let $g = m([\sigma]_D)$. Then $g(\Gamma_m[\alpha]_D, D, m) = (\Gamma_m[\sigma\cdot\alpha]_D, D, m)$ is transformed by κ' in $(\sigma\cdot\alpha(1), D, m^{\sigma\cdot\alpha}) = (\alpha(1), D, gm^{\alpha}g^{-1})$. The map κ' fits in the following commutative diagram

$$\begin{tikzcd} \mathcal{C}(y_0)' \arrow[rr, "\kappa'"] \arrow[d, "p'"'] & & \mathcal{C}[y_0]' \arrow[d, "\pi'"] \\ Y\times H_n^G(Y,y_0)\setminus B \arrow[rr, "(id\times\nu)'"] & & Y\times U(y_0)\setminus \underline{B} \end{tikzcd} \tag{28}$$

where $\nu : H_n^G(Y, y_0) \to U(y_0)$, given by $\nu(D, m) = (D, \underline{m})$, is an étale Galois cover with Galois group G (cf. Proposition 6.6(ii)) and $(id_Y\times\nu)'$ is the quotient morphism of the $*$-action of G (cf. (22). This action is moreover free since $Z(G) = 1$.

Lemma 6.9. *Suppose G has trivial center. The $*$-action of G on $\mathcal{C}(y_0)'$ (cf. Proposition 6.7) is free, $\kappa' : \mathcal{C}(y_0)' \to \mathcal{C}[y_0]'$ is invariant with respect to it and every fiber of κ' is a G-orbit. The set $\mathcal{C}[y_0]'$ has a structure of a quotient algebraic variety variety ${}_G\backslash\mathcal{C}(y_0)'$, $\kappa' : \mathcal{C}(y_0)' \to \mathcal{C}[y_0]'$ is an étale Galois cover and the map $\pi' : \mathcal{C}[y_0]' \to Y \times U(y_0)\setminus \underline{B}$ is an étale Galois cover whose Galois group is isomorphic to G with respect to the action (26) of G on $\mathcal{C}[y_0]'$. The morphism $\kappa' : \mathcal{C}(y_0)' \to \mathcal{C}[y_0]'$ is equivariant with respect to the actions of G as Galois groups of the covers p' and π'.*

Proof. The morphism p' is equivariant with respect to the $*$-action of G by Proposition 6.7(iii) and G acts without fixed points on $Y\times H_n^G(Y, y_0)\setminus B$, so the $*$-action of G on $\mathcal{C}(y_0)'$ is free. Let $h = m([\eta]_D)$, $z = (\Gamma_m[\alpha]_D, D, m) \in \mathcal{C}(y_0)'$. Then

$$k'(h * z) = (\eta^{-}\cdot\alpha(1), D, (hmh^{-1})^{\eta^{-}\cdot\alpha}) = (\alpha(1), D, m^{\alpha}) = \kappa'(z).$$

Let $\kappa'(z) = \kappa'(z_1)$, where $z_1 = (\Gamma_{m_1}[\beta]_D, D, m_1)$. Then $\alpha(1) = \beta(1)$ and $m^{\alpha} = m_1^{\beta}$. Let $\eta = \alpha\cdot\beta^{-}$, $h = m([\eta]_D)$. Then $[\beta]_D = [\eta^{-}\cdot\alpha]_D$, $m_1 = m^{\alpha\cdot\beta^{-}} = hmh^{-1}$. Therefore $z_1 = h * z$.

By Proposition 6.7 the group $G\times G$ acts on $\mathcal{C}(y_0)'$ by automorphisms as $(g, h)z = g(h*z)$. This action is free and $(id_Y\times\nu)'\circ p' : \mathcal{C}(y_0)' \to Y\times U(y_0)\setminus\underline{B}$ is the associated étale Galois cover with Galois group isomorphic to $G\times G$. Let us endow the set $\mathcal{C}[y_0]'$ with the structure of the quotient algebraic variety ${}_G\backslash\mathcal{C}(y_0)'$ with respect to the $*$-action (cf. Lemma 2.5). The map $\kappa' : \mathcal{C}(y_0)' \to \mathcal{C}[y_0]'$ becomes an étale Galois cover with Galois group isomorphic to G. The action $(g, z) \mapsto gz$ descends to $\mathcal{C}[y_0]'$ as the action (26), since κ' is G-equivariant. Hence $\pi' : \mathcal{C}[y_0]' \to Y \times U(y_0)\setminus\underline{B}$ is an étale Galois cover. □

6.10. The topological space $|\mathcal{C}[y_0]'^{an}|$ is the quotient by G of the topological space $|\mathcal{C}(y_0)'^{an}|$ (cf. Lemma 2.5). Let $(y, D, m) \in \mathcal{C}[y_0]'$. Let $m = m_0^{\alpha}$, where

$$m_0 : \pi_1(Y\setminus D, y_0) \twoheadrightarrow G, \quad \alpha : I \to Y\setminus D, \quad \alpha(0) = y_0,\ \alpha(1) = y.$$

Let $N_{(\alpha,D,m_0)}(U, U_1, \ldots, U_n)$ be as in (9). Then $\kappa'(N_{(\alpha,D,m_0)}(U, U_1, \ldots, U_n))$ is a neighborhood of (y, D, m) in the Euclidean topology of $\mathcal{C}[y_0]'$. Let us denote it by $N_{(y,D,m)}(U, U_1, \ldots, U_n)$. One has by (27) that

$$\begin{aligned} N_{(y,D,m)}(U, U_1, \ldots, U_n) = \{(z, E, m(E)^{\tau}) | z \in U, E \in N_D(U_1, \ldots, U_n), \\ \tau : I \to U, \tau(0) = y, \tau(1) = z\}. \end{aligned} \tag{29}$$

Varying the embedded open disks $U \ni y$, $U_i \ni b_i$, $i = 1, \ldots, n$, one obtains a neighborhood basis of (y, D, m) in the topological space $|\mathcal{C}[y_0]'^{an}|$.

Proposition 6.11. *Suppose G has trivial center. The set $\mathcal{C}'$ (cf.* (25)*) has a structure of an algebraic variety such that for every $y \in Y$ the map ${}_G\backslash\mathcal{C}(y)' \to \mathcal{C}[y]' \subset \mathcal{C}'$ is an open embedding. The map $\pi' : \mathcal{C}' \to Y \times H_n^G(Y) \setminus \underline{B}$ is an étale Galois cover with Galois group isomorphic to G, where G acts as in* (26)*.*

Proof. One has $\mathcal{C}' = \cup_{y\in Y}\mathcal{C}[y]'$ and each $\mathcal{C}[y]'$ has the structure of an algebraic variety defined in Lemma 6.9. Let us verify the patching conditions. Let $y_1, y_2 \in Y$, $y_1 \neq y_2$. Then $\mathcal{C}[y_1]' \cap \mathcal{C}[y_2]'$ is Zariski open in $\mathcal{C}[y_i]'$, $i = 1, 2$, since it is the preimage of $Y \times (Y \setminus \{y_1, y_2\})^{(n)}$ with respect to the composed morphism $\mathcal{C}[y_i]' \to Y \times U(y_i) \to Y \times (Y \setminus y_i)^{(n)}$. It is clear from § 6.10 that the two Euclidean topologies on $\mathcal{C}[y_1]' \cap \mathcal{C}[y_2]'$ induced by $\mathcal{C}[y_1]'$ and $\mathcal{C}[y_2]'$ coincide. Applying Corollary 4.6 to the map $\mathcal{C}[y_1]' \cap \mathcal{C}[y_2]' \to Y \times (Y \setminus \{y_1, y_2\})^{(n)}$ given by $(y, D, m) \mapsto (y, D)$ we conclude that the two structures of algebraic varieties on $\mathcal{C}[y_1]' \cap \mathcal{C}[y_2]'$ inherited from $\mathcal{C}[y_1]'$ and $\mathcal{C}[y_2]'$ coincide. This shows that one can endow $\mathcal{C}'$ with a structure of a reduced scheme over $\mathbb{C}$ such that every $\mathcal{C}[y]'$ is a Zariski open subset of $\mathcal{C}'$. The map $\pi' : \mathcal{C}' \to Y \times H_n^G(Y) \setminus \underline{B}$ given by $\pi'(y, D, m) = (y, (D, \underline{m}))$ is an étale Galois cover since this property holds for every $U(y)$ by Lemma 6.9. This implies, in particular, that $\mathcal{C}'$ is a separated scheme of finite type over $\mathbb{C}$ since these properties hold for $Y \times H_n^G(Y) \setminus \underline{B}$. This shows that $\mathcal{C}'$ is an algebraic variety. □

6.12. Let H be a connected component of $H_n^G(Y)$. Let $\mathcal{C}'_H = \pi'^{-1}(Y \times H \setminus \underline{B})$. We claim that this algebraic variety is irreducible. It suffices to prove the irreducibility of $\mathcal{C}'_{H\cap U(y_0)} = \pi'^{-1}(Y \times H \cap U(y_0) \setminus \underline{B})$ for every $y_0 \in Y$. Let $\tilde{H}$ be a connected component of $H_n^G(Y, y_0)$ which maps surjectively to $H \cap U(y_0)$. Then $\mathcal{C}(y_0)'_{\tilde{H}}$ is irreducible by § 3.16 and it maps surjectively onto $\mathcal{C}'_{H\cap U(y_0)}$ by the morphism $\kappa' : \mathcal{C}(y_0)' \to \mathcal{C}[y_0]'$ (cf. Lemma 6.9), hence $\mathcal{C}'_{H\cap U(y_0)}$ is irreducible. Let $\mathcal{C}_H$ be the normalization of $Y \times H$ in the field $\mathbb{C}(\mathcal{C}'_H)$ and let $\pi_H : \mathcal{C}_H \to Y \times H$ be the corresponding finite, surjective morphism. The action of G on $\mathcal{C}'_H$ (cf. Proposition 6.11) can be uniquely extended to an action of G on $\mathcal{C}_H$ by algebraic automorphisms. Let

$$\mathcal{C} = \bigsqcup_{H\subset H_n^G(Y)} \mathcal{C}_H.$$

Let $\pi : \mathcal{C} \to Y \times H_n^G(Y)$ be the finite morphism which restricts to π_H over every connected component $H \subset H_n^G(Y)$. The G-invariant morphism $\pi : \mathcal{C} \to Y \times H_n^G(Y)$ is a Galois cover with Galois group G.

For every $y_0 \in Y$ let $\mathcal{C}[y_0] = \pi^{-1}(Y \times U(y_0))$ and let $\kappa : \mathcal{C}(y_0) \to \mathcal{C}[y_0]$ be the morphism, extension of $\kappa' : \mathcal{C}(y_0)' \to \mathcal{C}[y_0]'$ relative to the normalizations. Every $\mathcal{C}[y]$ is a Zariski open, dense subset of $\mathcal{C}$ and

$$\mathcal{C} = \cup_{y\in Y}\mathcal{C}[y].$$

Lemma 6.13. *Let $Z(G) = 1$. Let $y_0 \in Y$. The $*$-actions of G on $Y \times H_n^G(Y, y_0)$ and $\mathcal{C}(y_0)$ defined in* (22) *and Proposition 6.7(v) are without fixed points. There is a commutative diagram of morphisms, extension of Diagram* (28)*,*

(30)
$$\begin{tikzcd} \mathcal{C}(y_0) \arrow[r, "\kappa"] \arrow[d, "p"] & \mathcal{C}[y_0] \arrow[d, "\pi"] \\ Y \times H_n^G(Y, y_0) \arrow[r, "id\times\nu"] & Y \times U(y_0) \end{tikzcd}$$

where $\nu(D,m)=(D,\underline{m})$, the two horizontal morphisms are invariant with respect to the free $*$-actions of G and are isomorphic to the respective quotient morphisms. The variety $\mathcal{C}[y_0]$ is smooth. The morphism κ is equivariant with respect to the actions of G as Galois groups of the covers p and π.

Proof. The morphism p is equivariant with respect to the $*$-actions of G by Proposition 6.7(v) and G acts without fixed points on $Y\times H_n^G(Y,y_0)$ since $Z(G)=1$. Hence the $*$-action of G on $\mathcal{C}(y_0)$ is without fixed points. The morphism $\kappa:\mathcal{C}(y_0)\to\mathcal{C}[y_0]$ is invariant with respect to the $*$-action of G since this property holds for $\kappa':\mathcal{C}(y_0)'\to\mathcal{C}[y_0]'$ by Lemma 6.9. The quotient algebraic variety ${}_G\backslash\mathcal{C}(y_0)$ is well-defined by Proposition 6.7(v) and Lemma 2.5. The smoothness of $\mathcal{C}(y_0)$ (cf. Proposition 3.18) implies the smoothness of ${}_G\backslash\mathcal{C}(y_0)$ since the $*$-action of G is free. The morphism $(id_Y\times\nu)\circ p:\mathcal{C}(y_0)\to Y\times U(y_0)$ is finite and surjective, so the same holds for the morphism ${}_G\backslash\mathcal{C}(y_0)\to Y\times U(y_0)$. Furthermore ${}_G\backslash\mathcal{C}(y_0)$ contains a Zariski open, dense subset isomorphic to ${}_G\backslash\mathcal{C}(y_0)'\cong\mathcal{C}[y_0]'$. Therefore $\kappa:\mathcal{C}(y_0)\to\mathcal{C}[y_0]$ induces an isomorphism ${}_G\backslash\mathcal{C}(y_0)\xrightarrow{\sim}\mathcal{C}[y_0]$ by the uniqueness of normalizations. The morphism κ is G-equivariant since this property holds for κ' (cf. Lemma 6.9). □

Theorem 6.14. *Let Y be a smooth, projective, irreducible curve of genus $g\geq 0$. Let n be a positive integer. Let G be a finite group which can be generated by $2g+n-1$ elements. Suppose G has trivial center. Then the morphism*

$$\pi:\mathcal{C}\to Y\times H_n^G(Y) \tag{31}$$

is a smooth family of G-covers of Y branched in n points. For every $(D,\underline{m})\in H_n^G(Y)$ the G-cover $\mathcal{C}_{(D,\underline{m})}\to Y$ has monodromy invariant $(D,\underline{m})$. Every G-cover $C\to Y$ branched in n points is G-equivalent to a unique G-cover of Y of the family (31).

Proof. The algebraic variety $\mathcal{C}$ is smooth since every one of its Zariski open subsets $\mathcal{C}[y]\cong{}_G\backslash\mathcal{C}(y)$ is smooth by Lemma 6.13. The composition $\mathcal{C}\xrightarrow{\pi}Y\times H_n^G(Y)\to H_n^G(Y)$ is proper since π is finite and Y is projective. This is a morphism of smooth varieties of relative dimension 1 and for every $z\in\mathcal{C}$ the induced linear map on the tangent spaces is surjective. Indeed, this property holds for $\pi_2\circ p:\mathcal{C}(y_0)\to H_n^G(Y,y_0)$ for $\forall y_0\in Y$ (cf. Theorem 3.20), the morphism $\pi_2\circ p$ is equivariant with respect to the free $*$-actions of G and one applies Lemma 6.13. By [26, Ch. III Prop. 10.4] we conclude that $\mathcal{C}\to H_n^G(Y)$ is a smooth morphism.

Let $(D,\underline{m})\in H_n^G(Y)$. Let $y_0\in Y\setminus D$ and let $m:\pi_1(Y\setminus D,y_0)\to G$ belong to $\underline{m}$. One has $\mathcal{C}_{(D,\underline{m})}=\pi^{-1}(Y\times\{(D,\underline{m})\})$ and by (30)

$$\kappa^{-1}(\mathcal{C}_{(D,\underline{m})})=\bigsqcup_{h\in G}\mathcal{C}(y_0)_{(D,hmh^{-1})}$$

where $k|_{\mathcal{C}(y_0)_{(D,m)}}:\mathcal{C}(y_0)_{(D,m)}\to\mathcal{C}_{(D,\underline{m})}$ is a G-equivariant isomorphism. Using Theorem 3.20 we conclude that $\mathcal{C}_{(D,\underline{m})}\to Y$ has monodromy invariant $(D,\underline{m})$. The last statement is clear from § 6.1 and § 3.5. □

6.15. The map $\pi:\mathcal{C}\to Y\times H_n^G(Y)$ has the following local analytic form at the ramification points. Let $\pi(z)=(b,(D,\underline{m}))$, where $D=\{b_1,\dots,b_k,\dots b_n\}$, $b=b_k$. The isotropy group $G(z)\subset G$ is cyclic of order $e\geq 2$. There are local analytic coordinates $(s, t_1, \dots, t_n)$ of $\mathcal{C}$ at z such that the map π and the action of $G(z)$ are given locally at z by

$$\pi : (s, t_1, \dots, t_n) \mapsto (s^e + t_k, t_1, \dots, t_k, \dots, t_n)$$
$$g(s, t_1, \dots, t_n) = (\chi(g)s, t_1, \dots, t_n)$$

where $\chi : G(z) \to \mathbb{C}^*$ is a primitive character of $G(z)$. This follows from Proposition 3.18 and Lemma 6.13, since the horizontal morphisms κ and $id_Y \times \nu$ in (30) are locally biholomorphic.

7. Hurwitz moduli varieties parameterizing G-covers

In this section we assume that Y is a smooth, projective, irreducible curve of genus $g \geq 0$, n is a positive integer and G is a finite group which can be generated by $2g + n - 1$ elements.

Proposition 7.1. *Let $q : X \to Y \times S$ be a smooth family of G-covers of Y branched in n points. Let $B \subset Y \times S$ be the branch locus of q. Let $v : S \to H_n^G(Y)$ be the map*

$$v(s) = (B_s, \underline{m}_s), \quad s \in S \tag{32}$$

where $(B_s, \underline{m}_s)$ is the monodromy invariant of $q_s : X_s \to Y$ (cf. Definition 6.2). Then v is a morphism.

Proof. Let $\beta : S \to Y^{(n)}$ be the morphism defined by $\beta(s) = B_s$ (cf. Proposition 2.6(vi)). One has $H_n^G(Y) = \cup_{y \in Y} U(y)$, every $U(y)$ is a Zariski open subset and $v^{-1}(U(y)) = \beta^{-1}((Y \setminus y)^{(n)})$ is a Zariski open subset of S for every $y \in Y$. Proving that v is a morphism is a local matter so we may assume, without loss of generality, that there is a point $y_0 \in Y$ such that every $q_s : X_s \to Y$ is unramified at y_0. Then $v(S) \subset U(y_0) \subset H_n^G(Y)$. One has $\{y_0\} \times S \subset Y \times S \setminus B$. Let $T = q^{-1}(\{y_0\} \times S)$. The composition $\mu : T \to \{y_0\} \times S \to S$ is finite, étale, G-invariant and $T/G \cong S$. In fact, $X/G \cong Y \times S$ by Proposition 2.6(viii), $T \to \{y_0\} \times S$ is a pullback of $q : X \to Y \times S$ by $\{y_0\} \times S \to Y \times S$ and one applies [35, Prop. A 7.1.3]. Let $q_T : X_T = X \times_S T \to Y \times T$ be the pullback family. The morphism $\theta : T \to X \times_S T$ defined by $\theta(t) = (t, t)$ satisfies $q_T \circ \theta(t) = (y_0, t)$ for $\forall t \in T$, so $(X_T \to X \times T, \theta)$ is a smooth family of pointed G-covers of (Y, y_0). For every $t \in T$, if $s = \mu(t)$, one has $(X_T)_t = X_s \times \{t\}$, the monodromy invariant of $(X_s, t) \to (Y, y_0)$ is $(B_s, m_t : \pi_1(Y \setminus B_s, y_0) \to G)$, m_t belongs to $\underline{m}_s$ and $m_{ht} = h m_t h^{-1}$ for $\forall h \in G$. We obtain the following commutative diagram of maps

$$\begin{tikzcd} T \arrow[r, "u"] \arrow[d, "\mu"'] & H_n^G(Y, y_0) \arrow[d, "\nu"] \\ S \arrow[r, "v"] & U(y_0) \end{tikzcd}$$

where $u(t) = (B_{\mu(t)}, m_t)$, $v(s) = (B_s, \underline{m}_s)$ and the vertical maps are quotient morphisms with respect to the actions of G defined respectively by $t \mapsto ht$ and $h * (D, m) = (D, hmh^{-1})$ for $\forall h \in G$ (cf. Proposition 6.6). By Proposition 5.3 the map u is a morphism, hence v is a morphism. □

The next proposition is a partial inverse of Proposition 7.1.

Proposition 7.2. *Let $v : S \to H^G_n(Y)$ be a morphism. For every $s \in S$ there exists a Zariski open neighborhood U of s, an étale Galois cover $\mu : \tilde{U} \to U$ with Galois group $\overline{G} = G/Z(G)$ and a smooth family of G-covers of Y branched in n points $q : X \to Y \times \tilde{U}$ such that $v|_U \circ \mu$ equals the morphism $\tilde{v} : \tilde{U} \to H^G_n(Y)$ associated with $q : X \to Y \times \tilde{U}$.*

Proof. Let $v(s) \in U(y_0)$ for some $y_0 \in Y$. Let $U = v^{-1}(U(y_0))$. Let $\nu : H^G_n(Y, y_0) \to U(y_0)$ be the étale morphism $\nu(D, m) = (D, \underline{m})$ (cf. Proposition 6.6(ii)). Let $\tilde{U} = U \times_{U(y_0)} H^G_n(Y, y_0)$. One has a Cartesian diagram

$$\begin{array}{ccc} \tilde{U} & \xrightarrow{u} & H^G_n(Y, y_0) \\ {\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\nu} \\ U & \xrightarrow{v|_U} & U(y_0) \end{array}$$

in which $\mu : \tilde{U} \to U$ is an étale Galois cover with Galois group $\overline{G}$ since this property holds for ν (cf.[35, Prop. A 7.1.3]). Let $(q : X \to Y \times \tilde{U}, \eta : \tilde{U} \to X)$ be the pullback by u of the universal family (17). Then the morphism $\tilde{v} : \tilde{U} \to H^G_n(Y)$ associated with $q : X \to Y \times \tilde{U}$ equals $\nu \circ u$. Hence $v|_U \circ \mu = \tilde{v}$. □

In the next proposition we give the local analytic form at the ramification points of an arbitrary smooth family of G-covers of Y branched in n points.

Proposition 7.3. *Let $q : X \to Y \times S$ be a smooth family of G-covers of Y branched in n points. Let $B \subset Y \times S$ be the branch locus of q. Let $\beta : S \to Y^{(n)}_*$ be the morphism defined by $\beta(s) = B_s$ (cf. Proposition 2.6(vi)). Let $x \in X$ be a point such that $q(x) = (b, s_0) \in B$. Let $B_{s_0} = D = \{b_1, \dots, b_k, \dots, b_n\}$, $b = b_k$. Let $y_0 \in Y \setminus D$ and let $U_i \ni b_i$, $s_i : U_i \to \mathbb{C}$ be as in § 3.17. Denote the restriction of $(s_1, \dots, s_n) \circ \beta$ on $\beta^{-1}(N_D(U_1, \dots, U_n))$ by $(\beta_1, \dots, \beta_n)$. There exist open neighborhoods $V \subset |S^{an}|$ of s_0 and $W \subset |X^{an}|$ of x such that $\beta(V) \subset N_D(U_1, \dots, U_n)$, $q(W) = U \times V$ and the following properties hold.*

- (i) *The isotropy group $G(x)$ is cyclic of order $e \geq 2$. Let $F \subset \mathbb{C} \times U \times V$ be the analytic subset $F = \{(z, t, s) | z^e = t - \beta_k(s)\}$ and let $q_1 : F \to U \times V$ be the projection map. There exists a biholomorphic map $\phi : W \to F$ such that $q|_W = q_1 \circ \phi$.*
- (ii) *The composition $\psi = (z, id_V) \circ \phi : W \to \mathbb{C} \times V$ maps W biholomorphically onto an open neighborhood of $(0, s_0)$.*
- (iii) *W is $G(x)$-invariant and there exists a primitive character $\chi : G(x) \to \mathbb{C}^*$ such that ϕ and ψ are $G(x)$-equivariant with respect to the actions of $G(x)$ on F and $\mathbb{C} \times V$ defined respectively by $g(z, t, s) = (\chi(g)z, t, s)$ and $g(z, s) = (\chi(g)z, s)$.*
- (iv) *There is a G-equivariant biholomorphic map $q^{-1}(U \times V) \cong G \times^{G(x)} W$.*

Proof. Replacing S by an étale cover of a Zariski open neighborhood of s_0, as in the proof of Proposition 7.1 we may assume that there exists a morphism $\eta : S \to X$ such that $(q : X \to Y \times S, \eta)$ is a smooth family of pointed G-covers of (Y, y_0). Let $u : S \to H^G_n(Y, y_0)$ and $f : X \to \mathcal{C}(y_0)$ be the morphisms of Theorem 5.5 (cf. (18)). Let $u(s_0) = (D, m)$, $f(x) = w$. Denote $H^G_n(Y, y_0)$ by H, $N_{(D,m)}(U_1, \dots, U_n)$ of § 3.17 by N, the neighborhood of $w \in \mathcal{C}(y_0)$ of Proposition 3.18(iii) by $\mathcal{W}$. One has by Theorem 5.5 that $X \cong S \times_H \mathcal{C}(y_0)$. Therefore $X^{an} \cong S^{an} \times_{H^{an}} \mathcal{C}(y_0)^{an}$ (cf.

[47, § 1.2]). Let $W = (f^{an})^{-1}(\mathcal{W})$. Then $W \cong \mathcal{W} \times_{\mathcal{C}(y_0)^{an}} X^{an}$ (cf. [15, Prop. 0.27]. Hence

$$W \cong X^{an} \times_{\mathcal{C}(y_0)^{an}} \mathcal{W} \cong (S^{an} \times_{H^{an}} \mathcal{C}(y_0)^{an}) \times_{\mathcal{C}(y_0)^{an}} \mathcal{W} \cong S^{an} \times_{H^{an}} \mathcal{W}. \tag{33}$$

Let $V = u^{-1}(N)$. Then $\beta|_V = \delta|_N \circ u|_V$. We have $t_i = s_i \circ \delta|_N$, $i = 1, \ldots, n$ (cf. § 3.17), therefore $\beta_i|_V = t_i \circ u^{an}|_V$. Let E be the analytic subset of $\mathbb{C} \times U \times N$ defined by the equation $z^e = t - t_k$ (cf. Proposition 3.18(iii)). The inverse image $E_1 = (id_{\mathbb{C}} \times id_U \times u^{an})^{-1}(E)$ is the closed complex subspace of $\mathbb{C} \times U \times V$ whose ideal sheaf is generated by the holomorphic function $h(z,t,s) = z^e - (t - \beta_k(s))$ (cf. [15, Prop. 0.27]). Let $\varphi : \mathcal{W} \to E$ be the biholomorphic map of Proposition 3.18(iii). By base change $\varphi_1 : W_1 = \mathcal{W} \times_E E_1 \to E \times_E E_1 \cong E_1$ is a biholomorphic map of complex spaces and one has the following commutative diagram in which every square is Cartesian

$$\begin{tikzcd} W_1 \arrow[r, "\sim", "\varphi_1"'] \arrow[d] & E_1 \arrow[r, hook] \arrow[d] & \mathbb{C} \times U \times V \arrow[r] \arrow[d, "id_{\mathbb{C}} \times id_U \times u^{an}|_V"] & U \times V \arrow[r] \arrow[d] & V \arrow[r, hook] \arrow[d] & S^{an} \arrow[d, "u^{an}"] \\ \mathcal{W} \arrow[r, "\sim", "\varphi"'] & E \arrow[r, hook] & \mathbb{C} \times U \times N \arrow[r] & U \times N \arrow[r] & N \arrow[r, hook] & H^{an} \end{tikzcd} \tag{34}$$

Therefore the external rectangle is Cartesian (cf. [20, Prop. 4.16]). Comparing with (33) we conclude that $W \cong W_1$. The complex space E_1 is reduced since W is an open subspace of X^{an} which is reduced. Therefore $E_1 = F$. We may thus replace $\varphi_1 : W_1 \to E_1$ by a biholomorphic map $\phi : W \to F$ in (34). The composition of the bottom maps $\mathcal{W} \to U \times N$ in (34) equals $p|_{\mathcal{W}}$ (cf. Proposition 3.18(iii)), so by the Cartesian diagram (18) of Theorem 5.5 the composition of the top maps $W \to U \times V$ in (34) equals $q|_W$, therefore $q|_W = q_1 \circ \phi$. Part (i) is proved. The other parts follow similarly from Proposition 3.18, parts (iii) and (iv), by pullback replacing $U \times N$ by $\mathbb{C} \times N$ and $U \times V$ by $\mathbb{C} \times V$ in (34). □

Theorem 7.4. *Let Y be a smooth, projective, irreducible curve of genus $g \geq 0$. Let n be a positive integer. Let G be a finite group which can be generated by $2g+n-1$ elements. Suppose G has trivial center. The algebraic variety $H^G_n(Y)$ is a fine moduli variety for the moduli functor $\mathcal{H}^G_{Y,n}$ of smooth families of G-covers of Y branched in n points (cf. § 5.1). The universal family is (cf. Theorem 6.14)*

$$\pi : \mathcal{C} \to Y \times H^G_n(Y). \tag{35}$$

Proof. Let $[q : X \to Y \times S] \in \mathcal{H}^G_{Y,n}(S)$. Let $v : S \to H^G_n(Y)$, $v(s) = (\beta(s), \underline{m}_s)$ be the morphism of Proposition 7.1. We want to prove that $q : X \to Y \times S$ is G-equivariant to the pullback by v of the family (35). This is the unique morphism with this property since the monodromy invariant classifies the G-covers up to G-equivalence. Since $Z(G) = 1$, for every $s \in S$ there exists a unique G-equivariant isomorphism $\varphi_s : X_s \to \mathcal{C}_{v(s)}$ such that $\pi_{v(s)} \circ \varphi_s = (id_Y \times v) \circ q_s$. Let $\varphi : X \to \mathcal{C}$ be the G-equivariant map which equals φ_s on every X_s. One obtains the following commutative diagram of maps

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "q"'] & \mathcal{C} \arrow[d, "\pi"] \\ Y \times S \arrow[r, "id \times v"] & Y \times H^G_n(Y) \end{tikzcd} \tag{36}$$

We aim to prove that φ is a morphism and (36) is a Cartesian diagram. One has that $H_n^G(Y) = \cup_{y\in Y} U(y)$ is a covering of Zariski open sets (cf. Proposition 6.6) and $\varphi^{-1}(\pi^{-1}(Y\times U(y))) = q^{-1}((id_Y\times v)^{-1}(Y\times U(y)))$ is a Zariski open subset of X for $\forall y\in Y$. Proving that φ is a morphism is a local matter so we may assume, without loss of generality, that there exists a point $y_0\in Y$ such that $q_s : X_s\to Y$ is unramified at y_0 for every $s\in S$. Then $v(S)\subset U(y_0)$ and $\varphi(X)\subset \mathcal{C}_{U(y_0)} = \mathcal{C}[y_0]$. Let $T = q^{-1}(\{y_0\}\times S)$ and let $(q_T : X_T \to Y\times T, \theta)$ be the smooth family of pointed G-covers of (Y, y_0) defined in the proof of Proposition 7.1. Consider the following commutative diagram of morphisms

$$\begin{tikzcd} X_T \arrow[r, "f"] \arrow[d, "q_T"'] & \mathcal{C}(y_0) \arrow[r, "\kappa"] \arrow[d, "p"] & \mathcal{C}[y_0] \arrow[d, "\pi"] \\ Y\times T \arrow[r, "id\times u"] & Y\times H_n^G(Y,y_0) \arrow[r, "id\times\nu"] & Y\times U(y_0) \end{tikzcd}$$

where the left square is from (18) and the right one from (30). There are two actions of G on $X_T = X\times_S T$, namely $g(x,t) = (gx,t)$ and $h*(x,t) = (x,ht)$, and these actions commute. We claim that $f(h*z) = h*f(z)$ for $\forall h\in G, \forall z\in X_T$ (cf. (23)). It suffices to prove this for $\forall z\in q_T^{-1}(Y\times T\setminus B_T)$. Let $(x,t)\in X_s'\times\{t\}$, where $s=\mu(t)$. Let $\lambda : I\to X_s'$ be a path with $\lambda(0)=t, \lambda(1)=x$. Then the path $\lambda\times\{t\}$ connects $\theta(t)=(t,t)$ with (x,t) in $X_s'\times\{t\} = (X\times_S T)_t'$, so by (19)

$$f(x,t) = (\Gamma_{m_t}[q_s\circ\lambda]_{\beta(s)}, \beta(s), m_t).$$

Let $h = m_t([\eta]_{\beta(s)})$ and let $\tilde\eta$ be the lifting of η in X_s' such that $\tilde\eta(0)=t$, $\tilde\eta(1)=ht$. Then $\tilde\eta^-\cdot\lambda$ is a path in X_s' which connects ht with x, therefore $(\tilde\eta^-\cdot\lambda, ht)$ is a path in $X_s'\times\{ht\} = (X\times_S T)_{ht}'$ which connects $\theta(ht)=(ht,ht)$ with (x,ht). Therefore by (19)

$$\begin{aligned} f(h*(x,t)) = f(x,ht) &= (\Gamma_{m_{ht}}[q_s\circ(\tilde\eta^-\cdot\lambda)]_{\beta(s)}, \beta(s), m_{ht}) \\ &= (\Gamma_{hm_th^{-1}}[\eta^-\cdot(q_s\circ\lambda)]_{\beta(s)}, \beta(s), hm_th^{-1}) \\ &= h*f(x,t). \end{aligned}$$

By Lemma 6.13 this defines a commutative diagram of quotient morphisms

$$\begin{tikzcd} {}_G\backslash X_T \arrow[r] \arrow[d] & {}_G\backslash\mathcal{C}(y_0) \arrow[r, "\cong"] \arrow[d] & \mathcal{C}[y_0] \arrow[d] \\ {}_G\backslash Y\times T \arrow[r] & {}_G\backslash Y\times H_n^G(Y,y_0) \arrow[r, "\cong"] & Y\times U(y_0) \end{tikzcd} \tag{37}$$

By [35, Prop. A.7.1.3] one has ${}_G\backslash X_T\cong X$ since ${}_G\backslash T\cong S$. Furthermore the restriction of $\kappa\circ f$ on every fiber $(X_T)_t$ is the composition of the G-equivariant isomorphisms

$$X_s\times\{t\}\to\mathcal{C}(y_0)_{(\beta(s),m_t)}\to\mathcal{C}_{(\beta(s),\underline{m}_s)}.$$

Hence Diagram (37) is, up to the open embedding $\mathcal{C}[y_0]\hookrightarrow\mathcal{C}$, the same as Diagram (36). This proves that $\varphi : X\to\mathcal{C}$ is a G-equivariant morphism.

Consider the decomposition of φ

$$\begin{tikzcd} X \arrow[r, "\varphi_S"] \arrow[dr] & \mathcal{C}_S \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ & S \arrow[r, "v"] & H_n^G(Y) \end{tikzcd}$$

where the right square is Cartesian. The morphisms $X \to S$ and $\mathcal{C}_S \to S$ are proper and smooth and φ_S induces an isomorphism on every scheme-theoretical fiber $X_s \xrightarrow{\sim} (\mathcal{C}_S)_s \cong \mathcal{C}_{v(s)}$ for $\forall s \in S(\mathbb{C})$, therefore by [22, Prop. (4.6.7)] $\varphi_S : X \to \mathcal{C}_S$ is an isomorphism. This shows that the family of G-covers $q : X \to Y \times S$ is G-equivalent to the pullback by $v : S \to H_n^G(Y)$ of the family $\pi : \mathcal{C} \to Y \times H_n^G(Y)$. $\square$

7.5. Let M be an algebraic variety. For every algebraic variety S denote by $\operatorname{Hom}(S, M)$ the set of morphisms from S to M. Let us denote by $\operatorname{Spec}\mathbb{C}$ the algebraic variety with one point $0 \in \mathbb{C}$. Then $\operatorname{Hom}(\operatorname{Spec}\mathbb{C}, M) = M(\mathbb{C})$ is a set bijective to M. The mapping $h_M(S) = \operatorname{Hom}(S, M)$ defines a contravariant functor $h_M : \mathrm{Var}_{\mathbb{C}} \to (\mathrm{Sets})$ from the category of algebraic varieties to the category of sets.

In the next theorem we use the definition of coarse moduli variety of [43, Definition 5.6] adapted to the category $\mathrm{Var}_{\mathbb{C}}$ of algebraic varieties over $\mathbb{C}$.

Theorem 7.6. *Let Y be a smooth, projective, irreducible curve of genus $g \geq 0$. Let n be a positive integer. Let G be a finite group which can be generated by $2g+n-1$ elements. The mapping which with every $[\mathcal{C} \to Y \times S] \in \mathcal{H}_{Y,n}^G(S)$ associates the morphism $v(S) : S \to H_n^G(Y)$ of Proposition 7.1 is a well-defined natural transformation of contravariant functors $\phi : \mathcal{H}_{Y,n}^G \to h_{H_n^G(Y)}$. The couple $(H_n^G(Y), \phi)$ is a coarse moduli variety for the moduli functor $\mathcal{H}_{Y,n}^G$.*

Proof. If $\mathcal{C} \to Y \times S$ is G-equivalent to $\mathcal{C}_1 \to Y \times S$, then for every $s \in S$ the G-covers $\mathcal{C}_s \to Y$ and $(\mathcal{C}_1)_s \to Y$ have the same monodromy invariant, so both families define the same morphism $v(S) : S \to H_n^G(Y)$. If $u : T \to S$ is a morphism and $\mathcal{C}_T \to Y \times T$ is the pullback of $\mathcal{C} \to Y \times S$, then for every $t \in T$, $(\mathcal{C}_T)_t \to Y$ is G-equivalent to $\mathcal{C}_{u(t)} \to Y$, so $v(T) = v(S) \circ u$. This shows that the collection of mappings

$$\phi(S) : \mathcal{H}_{Y,n}^G(S) \to \operatorname{Hom}(S, H_n^G(Y))$$

is a well-defined natural transformation $\phi : \mathcal{H}_{Y,n}^G \to h_{H_n^G(Y)}$.

If $S = \operatorname{Spec}\mathbb{C}$, then $\mathcal{H}_{Y,n}^G(\operatorname{Spec}\mathbb{C})$ is the set of G-equivalence classes of G-covers of Y branched in n points and $\phi(\operatorname{Spec}\mathbb{C})$ transforms every $[p : C \to Y]$ in its monodromy invariant $(D, \underline{m})$. Hence

$$\phi(\operatorname{Spec}\mathbb{C}) : \mathcal{H}_{Y,n}^G(\operatorname{Spec}\mathbb{C}) \to H_n^G(Y)$$

is a bijection by Riemann's existence theorem. This is Condition (i) of [43, Definition 5.6]. Let us verify Condition (ii). Suppose there is an algebraic variety N and a natural transformation $\psi : \mathcal{H}_{Y,n}^G \to h_N$. By Yoneda's lemma we have to prove that there exists a unique morphism $f : H_n^G(Y) \to N$ such that for every $[\mathcal{C} \to Y \times S] \in \mathcal{H}_{Y,n}^G(S)$ if the morphisms $v(S) : S \to H_n^G(Y)$ and $w(S) : S \to N$ are the images of $[\mathcal{C} \to Y \times S]$ by $\phi(S)$ and $\psi(S)$ respectively, then the following diagram commutes

$$\begin{tikzcd} S \arrow[r, "v(S)"] \arrow[dr, "w(S)"'] & H^G_n(Y) \arrow[d, "f"] \\ & N \end{tikzcd} \tag{38}$$

The uniqueness of f, if it exists, is clear since $\phi(\operatorname{Spec}\mathbb{C})$ is bijective, so $\psi(\operatorname{Spec}\mathbb{C})$ determines in a unique way the map of sets $f : H^G_n(Y) \to N$. Let us prove the existence of such morphism. Define the map of sets $f : H^G_n(Y) \to N$ as follows. For every $(D, \underline{m}) \in H^G_n(Y)$ let $[\mathcal{C}_{(D,\underline{m})} \to Y] \in \mathcal{H}^G_{Y,n}(\operatorname{Spec}\mathbb{C})$ be the equivalence class of G-covers with monodromy invariant $(D, \underline{m})$. Let $f(D, \underline{m}) = \psi([\mathcal{C}_{(D,\underline{m})} \to Y])$. The equality of maps of sets $w(S) = f \circ v(S)$ holds since $v(S)$ and $w(S)$ are functorial with respect to base change so, by the definition of f, evaluating at every $s \in S$, one verifies that Diagram (38) commutes. The variety $H^G_n(Y)$ is a union of the Zariski open subsets $U(y)$: $H^G_n(Y) = \cup_{y\in Y} U(y)$. In order to prove that f is a morphism it suffices to prove that $f|_{U(y_0)} : U(y_0) \to N$ is a morphism for every $y_0 \in Y$. Let $S = H^G_n(Y, y_0)$ and let $(\mathcal{C}(y_0) \to Y \times H^G_n(Y, y_0), \zeta)$ be the universal family of pointed G-covers of (Y, y_0) (cf. Theorem 3.20). Let $w(S) : S \to N$ be the associated morphism. We saw in Proposition 6.7(vi) that for $\forall h \in G$ the map defined by $z \mapsto h * z$ is a G-equivalence between $\mathcal{C}(y_0)_{(D,m)} \to Y$ and $\mathcal{C}(y_0)_{(D,hmh^{-1})} \to Y$. Therefore $w(S) : S \to N$ is G-invariant with respect to the action of G on $H^G_n(Y, y_0)$ defined by $h * (D, m) = (D, hmh^{-1})$. By Proposition 6.6 $U(y_0) \cong {}_G\backslash H^G_n(Y, y_0)$ and clearly one has equality of maps $w(S) = f|_{U(y_0)} \circ \nu$, where $\nu : H^G_n(Y, y_0) \to U(y_0)$ is the quotient morphism. Therefore $f|_{U(y_0)}$ is a morphism by the universal property of quotient varieties. $\square$

7.7. Let $\underline{n} = n_1O_1 + \cdots + n_kO_k$, $|\underline{n}| = n$ be as in § 5.6. Let $D \in Y^{(n)}_*$, let $y_1, y_2 \in Y \setminus D$, let $m_1 : \pi_1(Y \setminus D, y_1) \twoheadrightarrow G$ and $m_2 : \pi_1(Y \setminus D, y_2) \twoheadrightarrow G$ be two path-connected epimorphisms (cf. § 6.1). Then m_1 satisfies Condition ((20)) if and only if m_2 satisfies it. We say that a G-cover $p : C \to Y$ branched in n points is of branching type $\underline{n}$ if its monodromy invariant $(D, \underline{m})$ has the property that every epimorphism of $\underline{m}$ satisfies Condition ((20)). We denote by $H^G_{\underline{n}}(Y)$ the set of $(D, \underline{m}) \in H^G_n(Y)$ of this type. One has

$$H^G_n(Y) = \bigsqcup_{|\underline{n}|=n} H^G_{\underline{n}}(Y),$$

every nonempty $H^G_{\underline{n}}(Y)$ is a union of connected components in the Zariski topology of $H^G_n(Y)$ and $H^G_{\underline{n}}(Y)$ inherits the structure of algebraic variety from $H^G_n(Y)$. Let us denote by

$$\mathcal{H}^G_{Y,\underline{n}} : \mathrm{Var}_{\mathbb{C}} \to (\mathrm{Sets})$$

the moduli functor, which associates with every algebraic variety S the set $\{[X \to Y \times S]\}$ of smooth families of G-covers of Y of branching type $\underline{n}$ modulo G-equivalence and with every morphism $T \to S$ the pullback of such families of G-covers. If the center of G is trivial let us denote by

$$\pi_{\underline{n}} : \mathcal{C}_{\underline{n}} \to Y \times H^G_{\underline{n}}(Y)$$

the restriction of the family $\pi : \mathcal{C} \to Y \times H^G_n(Y)$ (cf. Theorem 6.14). Theorem 7.4 and Theorem 7.6 imply the following ones.

Theorem 7.8. *Let Y be a smooth, projective, irreducible curve. Let G be a finite group with trivial center. Let $\underline{n} = n_1O_1 + \cdots + n_kO_k$, $|\underline{n}| = n$ be as in* § 5.6. *Suppose $H^G_{\underline{n}}(Y) \neq \emptyset$. The algebraic variety $H^G_{\underline{n}}(Y)$ is a fine moduli variety for the moduli functor $\mathcal{H}^G_{Y,\underline{n}}$ of smooth families of G-covers of Y of branching type $\underline{n}$. The universal family is*

$$\pi_{\underline{n}} : \mathcal{C}_{\underline{n}} \to Y \times H^G_{\underline{n}}(Y).$$

Theorem 7.9. *Let Y be a smooth, projective, irreducible curve. Let G be a finite group. Let $\underline{n} = n_1O_1 + \cdots + n_kO_k$, $|\underline{n}| = n$ be as in* § 5.6. *Suppose $H^G_{\underline{n}}(Y) \neq \emptyset$. The mapping which with every $[\mathcal{C} \to Y \times S] \in \mathcal{H}^G_{Y,\underline{n}}(S)$ associates the morphism $v(S) : S \to H^G_{\underline{n}}(Y)$ of Proposition 7.1 is a well-defined natural transformation of contravariant functors $\phi : \mathcal{H}^G_{Y,\underline{n}} \to h_{H^G_{\underline{n}}(Y)}$. The couple $(H^G_{\underline{n}}(Y), \phi)$ is a coarse moduli variety for the moduli functor $\mathcal{H}^G_{Y,\underline{n}}$ of smooth families of G-covers of Y of branching type $\underline{n}$.*

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