# IN THE SHADOWS OF A HYPERGRAPH: LOOKING FOR ASSOCIATED PRIMES OF POWERS OF SQUAREFREE MONOMIAL IDEALS

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ABSTRACT. The aim of this paper is to study the associated primes of powers of squarefree monomial ideals. Hypergraphs and squarefree monomial ideals are strongly connected. The cover ideal J(H) of a hypergraph H is the intersection of the primes corresponding to the edges of H. We define the *shadow* of H, as a certain set of smaller hypergraphs related to H. We then describe how the shadows of H preserve information about the associated primes of the powers of J(H). Some implications to the persistence property are studied.

#### INTRODUCTION

The primary decomposition of ideals in Noetherian rings is a fundamental result in commutative algebra and algebraic geometry. From a minimal primary decomposition, one can define the set of the associated primes by taking the radical of each ideal in the decomposition. Our goal in this paper is to investigate the associated primes of powers of squarefree monomial ideals.

Squarefree monomial ideals and powers of ideals are central objects in combinatorial and commutative algebra and algebraic geometry for the several connections they encode between these areas, see for instance [12].

There are several ways to relate a squarefree monomial ideal to a hypergraph. To serve our intent, we will associate to a hypergraph the squarefree monomial ideal with minimal primes corresponding to the edges of the hypergraph, and vice versa. This ideal is usually called the cover ideal of the hypergraph.

The associated primes of a squarefree monomial ideal are easy to describe, whereas computing the associated primes of a power could be really tricky. Currently, the set of the associated primes of a power of any squarefree monomial ideal is far from being fully understood. Recently, in [13] Kaiser, Stehlík and Škrekovski produced an example of a squarefree monomial ideal, precisely the cover ideal of a graph, which fails the persistence property, i.e., the set of the associated primes could "lose" some elements from a power to the next.

For ideals associated to a combinatorial object, one hopes to explain their behavior in terms of the original object. With this in mind, we define the shadow of a hypergraph, Definition 2.1, as a certain set of smaller hypergraphs related to the original one. We

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show that, see Theorem 2.8, Theorem 3.5, the shadows preserve information about the associated primes of a power of the cover ideal of the hypergraph.

The paper is organized as follows. In Section 1, we introduce the terminology and the basic results. In Section 2, we define the shadows of a hypergraph that are the new tool introduced in this paper. Then we start an investigation of the associated primes of a squarefree monomial ideal in terms of the shadows of the associated hypergraph. In particular, in this section, we deal with the second power. In Section 3, under some restrictive conditions, we broaden our investigation to any power. Finally, in Section 4, we apply the results of Section 3 to the persistence property.

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### 1. NOTATION AND BASIC FACTS

Let  $V := \{x_1, \ldots, x_n\}$  and  $R = K[V] = K[x_1, \ldots, x_n]$  be the standard polynomial ring in *n* variables over a field *K*. A squarefree monomial ideal  $I \subseteq R$  always has a unique minimal primary decomposition,  $I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t$ , as an intersection of squarefree prime ideals  $\mathfrak{p}_i = (x_{i_1}, \ldots, x_{i_s})$ . For more details and a full description of the topic we refer to Section 1.3 in [8].

This property establishes a one-to-one correspondence between squarefree monomial ideals and finite simple hypergraphs. First, recall that a (simple) **hypergraph** H is a pair H = (V, E) where  $V := \{x_1, \ldots, x_n\}$  is called the set of vertices of H and E is a collection of subsets of V. (We will only consider finite simple hypergraphs, these are also called *clutters* in the literature). We will denote, for a set  $U = \{x_{i_1}, \ldots, x_{i_s}\} \subseteq V$ , by

$$\mathfrak{p}_U := (x_{i_1}, \dots, x_{i_s}) \subseteq R$$

the prime ideal generated by the variables in U and by

$$x_U := x_{i_1} \cdots x_{i_n} \in R$$

the monomial given by the product of the variables in U.

Then, a hypergraph H = (V, E) unequivocally corresponds to the squarefree monomial ideal  $J(H) := \bigcap_{e \in E} \mathfrak{p}_e$ , called the **cover ideal** of H, and vice versa.

Let H = (V, E) be a hypergraph. A subset T of V is a **vertex cover** of H if every edge  $e \in E$  contains at least one element of T. A vertex cover T is a **minimal vertex cover** if no proper subset of T is a vertex cover. Minimal vertex covers are related to the minimal generators of J(H). Indeed, T is a minimal vertex cover of H if and only if  $x_T \in \mathcal{G}(J(H))$ , the set of monomials which minimally generates J(H). See [5] and [7] for a further investigation on cover ideals of hypergraphs. In this paper we are interested in the study of the associated prime ideals of the (regular) powers of J(H). Recall the following, classical, definition.

**Definition 1.1.** Let R be a ring and I an ideal of R. A prime ideal  $\mathfrak{p} \subset R$  is called an **associated prime ideal** of I if there exists some element  $m \in R/I$  such that  $\mathfrak{p} = \operatorname{Ann}(m)$ , the annihilator of m. The set of all associated prime ideals of I is denoted by  $\operatorname{ass}(I)$ .

By definition, the hypergraph H = (V, E) easily provides a description of all elements in  $\operatorname{ass}(J(H))$ . Indeed,  $\mathfrak{p}_U \in \operatorname{ass}(J(H))$  if and only if  $U \in E$ .

In order to describe the associated primes of the powers of J(H) the next lemma is an essential tool. Recall that for an hypergraph H = (V, E) and  $U \subseteq V$  the **induced subhypergraph** of H on U is the hypergraph  $H_U = (U, E(U))$  where  $E(U) = \{e \in E \mid e \subseteq U\}$ .

Lemma 1.2 (Lemma 2.11 [2]). Let H = (V, E) be a hypergraph. Let  $U \subseteq V$ , then  $\mathfrak{p}_U \in \operatorname{Ass}(R/J(H)^s) \Leftrightarrow \mathfrak{p}_U \in \operatorname{Ass}(K[U]/J(H_U)^s).$ 

Lemma 1.2 moves the problem of looking for associated prime ideals to maximal ideals. Indeed,  $\mathfrak{p}_U$  is associated to  $J(H)^s$ , if and only if it is associated to  $J(H_U)^s \subseteq K[U]$ , and  $\mathfrak{p}_U$  is the maximal ideal in K[U]. An immediate consequence of Lemma 1.2 is a first, well known, step in the description of the elements in ass  $J(H)^s$ .

**Lemma 1.3.** Let H be a hypergraph. Then  $\mathfrak{p}_e \in \operatorname{ass} J(H)^s$  for each integer  $s \ge 1$  and for each edge e of H.

A corollary of Lemma 1.2 will be useful in Section 3.

**Corollary 1.4.** Let H = (V, E) be a hypergraph on V. Let  $F \subseteq U \subseteq V$ , then

$$\mathfrak{p}_F \in \operatorname{Ass}(R/J(H)^s) \Leftrightarrow \mathfrak{p}_F \in \operatorname{Ass}(K[U]/J(H_U)^s).$$

*Proof.* It is an immediate consequence of Lemma 1.2. Indeed, since  $F \subseteq U$  we get  $(H_U)_F = H_F$ .

In the literature there are only few other results explicitly describing the elements in  $\operatorname{ass}(J(H)^s)$ . Most of them deal with the case that H is a graph, i.e., the edges all have cardinality 2. If H is a graph we will often denote it by the letter G. For instance, see proposition below, the authors of [3] describe the set  $\operatorname{Ass}(R/J(G)^2)$ . They prove that the new primes match the (minimal) odd cycles of G.

Recall that in a graph G = (V, E) a set of distinct vertices  $C = \{x_{i_1}, x_{i_2}, \ldots, x_{i_n}\} \subseteq V$ is called an *n*-cycle (or cycle of length *n*) if  $\{x_{i_j}, x_{i_{j+1}}\} \in E$  for each  $j \in \{1, \ldots, n\}$  and  $x_{i_{n+1}} := x_{i_1}$ . *C* is called an odd (even) cycle if *n* is odd (even). The vertices  $x_{i_j}, x_{i_{j+1}}$ connected by an edge  $\{x_{i_j}, x_{i_{j+1}}\}$  are called *adjacent* vertices in *C*. A chord of *C* is an edge of *G* joining two nonadjacent vertices. If *C* has no chord, we shall call it chordless.

**Proposition 1.5** (Corollary 3.4, [3]). Let G be a finite graph. A prime ideal  $\mathfrak{p} = (x_{i_1}, \ldots, x_{i_s})$  is in  $\operatorname{ass}(J(G)^2)$  if and only if:

- (a) s = 2 and  $\mathfrak{p} \in \operatorname{ass}(J(G))$ ; or
- (b) s is odd, and after re-indexing,  $\{x_{i_1}, x_{i_2}, \ldots, x_{i_s}\}$  is a chordless cycle of G.

## ERIN BELA, GIUSEPPE FAVACCHIO, AND NGHIA TRAN

# 2. Introducing the shadows

The authors of [2] give a description of the set  $\operatorname{ass}(J(H)^s)$  in terms of the coloring properties of the hypergraph H. However, their method is not very efficient to list all the elements in  $\operatorname{ass}(J(H)^s)$  for any given hypergraph H. In this section, we introduce a tool that can be useful for this aim: we define the shadows of a hypergraph. The motivating idea is to take information from some other hypergraphs, smaller than H, and to bring it to H.

The following is the definition of a shadow of H.

**Definition 2.1.** Let H = (V, E) be a hypergraph. We say that a hypergraph H' = (V', E') is a *shadow* of H if

- (a)  $V' \subseteq V$ ; and
- (b) |E| = |E'| (same cardinalities) and  $e \cap V' \in E'$  for each  $e \in E$ .

We denote by  $\mathcal{S}(H)$  the set of all the shadows of H. Note that two different elements in  $\mathcal{S}(H)$  have different vertex sets. Thus  $H' = (V', E') \in \mathcal{S}(H)$  will be also called the shadow of H on V'. By definition, H is always a shadow of itself on the vertex set V; we refer to this as the *trivial* shadow. However, not every subset of V produces a shadow of H, as we show in the following example.

**Example 2.2.** Consider the hypergraph H on the vertex set  $V = \{x_1, x_2, x_3, x_4, x_5\}$  with the edge set  $E = \{\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_1, x_4, x_5\}\}$ . The set  $\mathcal{S}(H)$  contains non-trivial elements, namely, shadows on the vertex sets  $V_1 := \{x_1, x_2, x_4\}, V_2 := \{x_1, x_3, x_4\}, V_3 := \{x_1, x_2, x_3, x_4\}, V_4 := \{x_1, x_3, x_4, x_5\}$  and  $V_5 := \{x_1, x_2, x_4, x_5\}$ . Indeed, we have

$$(V_1, \{\{x_1, x_2\}, \{x_2, x_4\}, \{x_1, x_4\}\}) \in \mathcal{S}(H), \text{ and}$$
  
 $(V_2, \{\{x_1, x_3\}, \{x_3, x_4\}, \{x_1, x_4\}\}) \in \mathcal{S}(H).$ 

Both of these shadows are graphs, more precisely they are 3-cycles. Additionally,

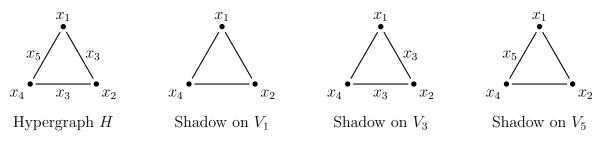
$$(V_3, \{\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_1, x_4\}\}) \in \mathcal{S}(H), (V_4, \{\{x_1, x_3\}, \{x_3, x_4\}, \{x_1, x_4, x_5\}\}) \in \mathcal{S}(H) \text{ and} (V_5, \{\{x_1, x_2\}, \{x_2, x_4\}, \{x_1, x_4, x_5\}\}) \in \mathcal{S}(H).$$

Furthermore, for instance, H has no shadow on the set  $V_6 := \{x_1, x_2, x_3\}$  since we get

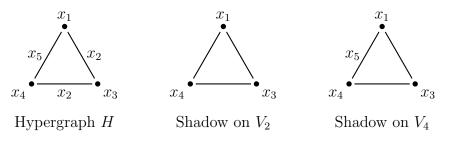
$$(V_6, \{\{x_1, x_2, x_3\}, \{x_2, x_3\}, \{x_1\}\}),$$

and this fails to be a simple hypergraph.

The hypergraph H and its shadows are showed in the following figures



where the edge  $\{a, b, v_1, \ldots, v_m\}$  is depicted as the segment  $a \bullet \frac{v_1 \ldots v_m}{\bullet} \bullet b$ . Similarly, we can re-picture the hypergraph H in the following form to better see the shadows on  $V_2$  and  $V_4$ :



In the following example, we show a hypergraph which only has trivial shadow.

**Example 2.3.** Let H be the hypergraph on the vertex set  $V = \{x_1, x_2, x_3, x_4, x_5\}$  and the edge set  $E = \{\{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_3, x_4, x_5\}, \{x_4, x_5, x_1\}, \{x_5, x_1, x_2\}\}$ . In this case, the set  $\mathcal{S}(H)$  has only one element, namely H. Indeed, notice that each edge of Hcontains vertices with "consecutive" indexes. Since any subset of V with two elements is contained in some edge, then H has no shadow on any set  $V' \subsetneq V$ . For instance, H has no shadow on the subset V' obtained from V by removing  $x_1$  since  $\{x_4, x_5\} \subset \{x_3, x_4, x_5\}$ .

J(H') is an ideal of K[V'] and there is a natural inclusion from K[V'] into K[V]. The ideal generated by the image of J(H') under this map, i.e., the ideal generated by  $\mathcal{G}(J(H')) \subseteq K[V]$ , is called cone ideal of J(H') in K[V].

The next lemma provides a connection between the monomial generators of J(H) and J(H') for a shadow H' of H.

**Lemma 2.4.** Let H = (V, E) be a hypergraph and  $H' = (V', E') \in \mathcal{S}(H)$  a shadow of H. Then  $\mathcal{G}(J(H')) \subseteq \mathcal{G}(J(H))$ .

*Proof.* The ideal J(H') is generated by monomials  $x_U$  where U is a minimal vertex cover of H'. By the definition of shadow, U is also a minimal vertex cover of H, and U does not involve the variables in  $V \setminus V'$ .

**Remark 2.5.** From Lemma 2.4, we have  $J(H') = K[V'] \cap J(H)$ . Thus, each element m in J(H') also belongs to J(H).

As a consequence of Lemma 2.4, we get the following result.

**Lemma 2.6.** If  $(J(H')^s : m) = \mathfrak{p} \neq (1)$  for some prime ideal  $\mathfrak{p}$ , then  $m \notin J(H)^s$ .

Proof. Suppose  $m \in J(H)^s$ , then  $m = m_1 \cdots m_s M$  where the  $m_i$ 's are monomial minimal generators of J(H). Since m only contains the variables in V', each  $m_i$  will also have this property. That means,  $m_i \in J(H')$  for all  $i \in \{1, 2, \ldots, s\}$ . Hence  $m \in J(H')^s$ , which contradicts  $(J(H')^s : m) \neq (1)$ .

The next results show the first evidences that our construction really serves our purpose. We strongly use the classification in Proposition 1.5 and assume the existence of a graph  $G \in \mathcal{S}(H)$ . Then, we show that  $J(H)^2$  only has associated primes inherited from  $J(G)^2$ . The following lemma can be deduced from Corollary 3.4 in [3]. We also include a proof for the convenience of the reader.

# **Lemma 2.7.** Let $C_{2n+1} = (V, E)$ be an (2n+1)-cycle. Then $(J(C_{2n+1})^2 : x_V) = \mathfrak{p}_V$ .

Proof. A minimal cover of  $C_{2n+1}$  involves exactly n+1 vertices. Then  $J(C_{2n+1})^2$  is generated in degree 2n+2 and  $x_V \notin J(C_{2n+1})^2$ . Moreover,  $x_1x_V = x_{\{1,2,4,\ldots,2n\}} \cdot x_{\{1,3,5,\ldots,2n+1\}} \in J(C_{2n+1})^2$ . Analogously, we get  $x_ix_V \in J(C_{2n+1})^2$  for each  $x_i \in V$ .

**Theorem 2.8.** Let H = (V, E) be a hypergraph. If  $G \in \mathcal{S}(H)$  is an odd cycle (i.e.,  $G = C_{2n+1}$  for some positive integer n), then  $\mathfrak{p}_V \in \operatorname{ass} J(H)^2$ .

*Proof.* Let  $E = (e_1, \ldots, e_k)$ . Since  $G = (V', E') \in \mathcal{S}(H)$ , the edges of G are given by  $\{e'_1, \ldots, e'_k\}$  where  $e'_i = e_i \cap V'$ . By hypothesis, G is an odd cycle, so k = 2n + 1 for some positive integer n. Without loss of generality, we relabel the vertices of G so that

$$e'_{i} = \begin{cases} \{x_{i}, x_{i+1}\}, & \text{if } 1 \leq i \leq 2n, \\ \{x_{2n+1}, x_{1}\}, & \text{if } i = 2n+1. \end{cases}$$

From Proposition 1.5, we know that  $\mathfrak{p}_{V'} \in \operatorname{ass} J(G)^2$ , and Lemma 2.7 we have  $(J(G)^2 : x_{V'}) = \mathfrak{p}'_V$ , where  $x_{V'} = \prod_{i=1}^{2n+1} x_i$ . Then, we claim that  $(J(H)^2 : x_{V'}) = \mathfrak{p}_V$ . If  $x_j \in V'$ ,  $x_j x_{V'} \in J(G)^2 \subseteq J(H)^2$ . So  $x_j \in (J(H)^2 : x_{V'})$ . Moreover, if  $y_j \in V \setminus V'$ , then there exists an edge  $e_i \in E$  such that  $y_j \in e_i$ . Without loss of generality, one can assume that i = 1. Thus we have that

$$y_j x_{V'} = y_j x_1 x_2 \cdots x_{2n+1} = (y_j x_3 x_5 \cdots x_{2n+1}) (x_1 x_2 x_4 \cdots x_{2n}).$$

The right hand side of the above equality is in  $J(H)^2$  since it is the product of two vertex covers of H. Thus,  $y_j \in (J(H)^2 : x_{V'})$ . Finally,  $x_{V'} \notin J(H)^2$  since  $x_{V'} \notin J(H')^2$ .  $\Box$ 

**Example 2.9.** Let *H* be the hypergraph in Example 2.2. Since, for instance, the shadow of *H* on  $\{x_1, x_2, x_4\}$  is an odd cycle, we can state that

$$\mathfrak{p}_V = (x_1, x_2, x_3, x_4, x_5) \in \operatorname{ass}(J(H)^2).$$

Now we show that Theorem 2.8 works in a more general setting. We need some further notation. Let H = (V, E) be a hypergraph and let  $G = (V', E') \in \mathcal{S}(H)$  be a graph. For a subset  $U \subset V'$ , we denote by

$$\widehat{U} := \bigcup_{e \in E, e' \subseteq U} e \subseteq V.$$

**Corollary 2.10.** Let H be a hypergraph and H' a shadow of H. If  $C_{2n+1}$  is an odd cycle that is a subhypergraph of H', then  $\mathfrak{p}_{\widehat{C}_{2n+1}} \in \operatorname{ass}(J(H)^2)$ .

Proof. Say H' = (V', E'). We take the subhypergraph  $\tilde{H} := H_{\hat{C}_{2n+1}}$  of H on the vertex set  $\hat{C}_{2n+1}$ . Notice that  $\tilde{H}$  has a shadow on  $C_{2n+1}$ . That is the odd cycle  $C_{2n+1}$ . Thus, from Proposition 1.5 and Theorem 2.8,  $\mathfrak{p}_{\hat{C}_{2n+1}} \in \operatorname{ass}(J(\tilde{H})^2)$ . Moreover, from Lemma 1.2, we have  $\mathfrak{p}_{\hat{C}_{2n+1}} \in \operatorname{ass}(J(H)^2)$ .

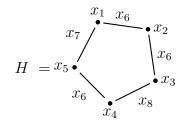
**Corollary 2.11.** Let H be a hypergraph and  $\tilde{H}$  a subhypergraph of H. If an odd cycle  $C_{2n+1} \in \mathcal{S}(\tilde{H})$ , then  $\mathfrak{p}_{\widehat{C}_{2n+1}} \in \operatorname{ass}((J(H)^2)$ .

**Example 2.12.** Let H = (V, E) be the hypergraph with the vertex set

$$V = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$$

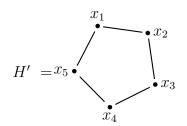
and the edge set

$$E = \{\{x_1, x_2, x_6\}, \{x_2, x_3, x_6\}, \{x_3, x_4, x_8\}, \{x_4, x_5, x_6\}, \{x_1, x_5, x_7\}\}$$



The shadow of H on the vertex set  $V' = \{x_1, x_2, x_3, x_4, x_5\}$  is

$$H' = (V', \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_1, x_5\}\}) \in \mathcal{S}(H).$$



We see that H' is a graph, precisely it is an odd cycle of length 5. By Theorem 2.8, we have that

$$\mathfrak{p}_V = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in \operatorname{ass}(J(H)^2).$$

Now, we take the shadow of H on the vertex set  $V'' = \{x_1, x_3, x_5, x_6, x_8\}$ . The shadow of H on V'' is

Note that H'' has a subhypergraph that is a cycle of length 3,  $C_3 = \{\{x_1, x_6\}, \{x_5, x_6\}, \{x_1, x_5\}\}$ . By Corollary 2.10, this cycle produces an element in  $\operatorname{ass}(J(H)^2)$ . So, we get

$$\mathfrak{p}_{\widehat{C}_3} = (x_1, x_2, x_4, x_5, x_6, x_7) \in \operatorname{ass}(J(H)^2).$$

In the following example we show that condition (b) in Definition 2.1 is necessary for Theorem 2.8.

**Example 2.13.** Let H = (V, E) be the hypergraph with the vertex set

$$V = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9\}$$

and the edge set

$$E = \{\{x_1, x_2, x_6, x_8\}, \{x_2, x_3, x_8, x_6\}, \{x_3, x_4, x_7, x_9\}, \\ \{x_4, x_5, x_6, x_8\}, \{x_1, x_5, x_7, x_9\}, \{x_8, x_9\}, \{x_6, x_7\}\}.$$

A Macaulay2 computation [10] shows that

$$\operatorname{ass}(J(H)^2) = \{ \mathfrak{p}_e \mid e \in E \}$$

We claim that ignoring the rule |E| = |E'| in the condition (b) of Definition 2.1 Theorem 2.8 does not hold. Indeed, we get on the vertex set  $V' = \{x_1, x_2, x_3, x_4, x_5\}$  the hypergraph

 $H' = (V', \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_1, x_5\}\}) \in \mathcal{S}(H).$ 

That is an odd cycle and, see Lemma 2.7, we have

$$\mathfrak{p}_{V'} = (x_1, x_2, x_3, x_4, x_5) \in \operatorname{ass}(J(H')^2).$$

In the last part of this section we prove that, under some suitable hypothesis, all the associated primes of  $J(H)^2$  come from some non-trivial shadow (we will see in Proposition 2.15). We need an auxiliary lemma.

**Lemma 2.14.** Let H = (V, E) be a hypergraph, and suppose  $(J(H)^s : m) = \mathfrak{p}_V$  for some monomial m. Let  $V' \subsetneq V$  be a proper subset such that  $e_i \cap e_j \subseteq V'$  for each  $e_i, e_j \in E$ ,  $i \neq j$ . Then  $y^{s-1}$  does not divide m for each  $y \in V \setminus V'$ .

Proof. Let y be an element in  $V \setminus V'$ . We write  $m = y^a m'$ , where, unless to rename,  $y \in e_1$ and y does not divide m'. If  $a \geq s$  then, since  $ym \in J(H)^s$ , we get  $ym = m_1 \cdots m_s M$ , where  $m_j$  corresponds to a minimal vertex cover of H for  $j \in \{1, \ldots, s\}$ . Thus y divides M and  $m = m_1 \cdots m_s(M/y)$ . This contradicts  $m \notin J(H)^s$ . Therefore, we can assume a = s - 1. We work by induction on  $r = |e_1 \setminus V'|$ . If r = 1, i.e.,  $e_1 = (e_1 \cap V') \cup \{y\}$ , then from  $ym \in J(H)^s$ , we get  $ym = (ym_1) \cdots (ym_s)M$ , where  $ym_j$  are minimal vertex covers of H. Note that, for each  $x_j \in e_1 \cap V'$ , we can see that  $x_j$  does not divide  $m_1, \ldots, m_s$ (these are minimal vertex covers) and  $x_j$  does not divide M (otherwise we can just delete y and get  $m \in J(H)^s$ ). This implies that  $m \notin (\mathfrak{p}_{e_1})^s$ . To get a contradiction, we just take some  $z \notin e_1$  and remember that by hypothesis  $zm \in J(H)^s$  but  $zm \notin (\mathfrak{p}_{e_1})^s$ .

If r > 1, i.e.,  $e_1 = (e_1 \cap V') \cup \{y_1, \ldots, y_r\}$ , then just note that  $V'' = V' \cup \{y_1, \ldots, \widehat{y_i}, \ldots, y_r\}$  satisfies the hypothesis of the theorem and  $e_1 = (e_1 \cap V'') \cup \{y_i\}$ .

**Proposition 2.15.** Let H = (V, E) be a hypergraph and  $H' = (V', E') \in \mathcal{S}(H)$  a shadow of H. Assume that  $e_i \cap e_j \subseteq V'$  for each  $e_i, e_j \in E$ , where  $i \neq j$ . If  $\mathfrak{p}_V \in \operatorname{ass} J(H)^2$ , then  $\mathfrak{p}_{V'} \in \operatorname{ass} J(H')^2$ .

*Proof.* By the definition of associated primes, there exists a monomial  $m \in K[V]$  such that  $(J(H)^2 : m) = \mathfrak{p}_V$ . Say  $V' = \{x_1, \ldots, x_a\}$  and  $V \setminus V' = \{y_1, \ldots, y_b\}$ . By Lemma 2.14  $y_j$  does not divide m for  $j = 1, \ldots, b$ . Then  $m \in K[V']$  and therefore  $(J(H')^2 : m) = \mathfrak{p}_{V'}$ .  $\Box$ 

### 3. A FIRST CASE

In this section we investigate the relations between a hypergraph and its shadows in a particular case of study. Precisely, we consider shadows that only differ from the starting hypergraph by one edge and one vertex.

In this section, we shall use the following notation.

Notation 3.1. Let H = (V, E) be a hypergraph and H' = (X, E') a shadow of H such that

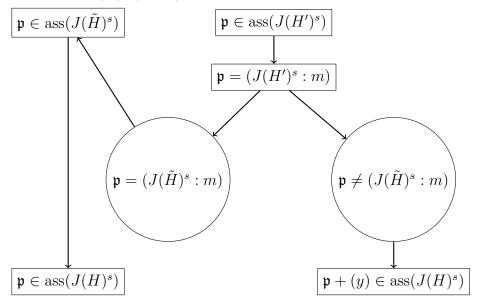
- (a)  $X = \{x_1, ..., x_n\}$  and  $V = X \cup \{y\}$ ; and
- (b) y only belongs to one edge, say  $e_y \in E$ .

After renaming, say  $e_y = \{x_1, \ldots, x_t, y\}$ . We set  $e := e'_y = \{x_1, \ldots, x_t\}$ , then we have  $H' = \{X, (E \setminus \{e_y\}) \cup \{e\}\}$ . Moreover, to shorten the notation,  $\tilde{H}$  will denote the subhypergraph of H on X. We denote by  $\mathfrak{p}_e$  and  $\mathfrak{p}_{e_y}$  the prime ideals generated by the variables in e and  $e_y$  respectively.

We remark that, in this setting, the hypergraphs H and H' share the vertex set X. Moreover, they share the same edges except for e.

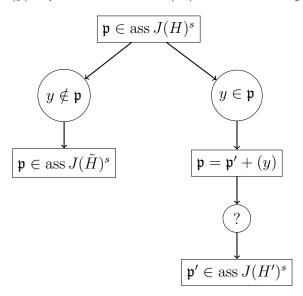
We will abuse notation: given a subset  $F \subseteq X \subseteq V$ , we will write  $\mathfrak{p}_F$  to denote both the ideals in K[X] and in K[V].

Here, we anticipate the results of this section. In the first part of the section, we investigate the relation linking associated primes of  $J(\tilde{H})^s$  and  $J(H')^s$  with the elements in  $\operatorname{ass}(J(H)^s)$ . We have seen in Corollary 1.4 that if  $\mathfrak{p} \in \operatorname{ass}(J(\tilde{H})^s)$  then  $\mathfrak{p} \in \operatorname{ass}(J(H)^s)$ . What about the associated prime of  $J(H')^s$ ? We will show that if  $\mathfrak{p} \in \operatorname{ass}(J(H')^s)$ , then either  $\mathfrak{p} + (y) \in \operatorname{ass}(J(H)^s)$  or  $\mathfrak{p} \in \operatorname{ass}(J(H)^s)$ . This depends on a further condition of a monomial m such that  $(J(H')^s : m) = \mathfrak{p}$ . The following diagram summarizes these results.



In the second part of the section, we will reverse the investigation. Starting from a prime associated to  $J(H)^s$ , we will look for which conditions allow us to find a relation with an

element in  $J(\tilde{H})^s$  or  $J(H')^s$ . Precisely, if  $\mathfrak{p} \in \operatorname{ass}(J(H)^s)$  and  $y \notin \mathfrak{p}$  then  $\mathfrak{p} \in \operatorname{ass}(J(\tilde{H})^s)$ . Moreover, if  $\mathfrak{p} = (y) + \mathfrak{p}'$ , it seems natural to ask if  $\mathfrak{p}' \in \operatorname{ass}(J(H')^s)$ , which we positively answer under an extra (restrictive) condition. In the next section, see 4.3, we will show that not all the primes  $(y) + \mathfrak{p}'$  associated to  $J(H)^s$  come from a prime  $\mathfrak{p}'$  in the shadow.



We start with an auxiliary result.

**Lemma 3.2.** Let  $m \in \mathcal{G}(J(H))$  be a monomial minimal generator of J(H). If y|m, then  $x_i \not\mid m$  for all  $x_i \in e$ .

*Proof.* In our setting, y only belongs to the edge  $e_y = \{x_1, \ldots, x_t, y\}$ . Since m is a minimal vertex cover of H, if  $x_i \in e = \{x_1, \ldots, x_t\}$  divides m, then  $\frac{m}{y}$  is also a vertex cover. This contradicts the minimality of m.

In order to relate the associated primes of  $J(H')^s$  to the associated primes of  $J(H)^s$ , the following proposition will be crucial.

**Proposition 3.3.** Let  $(J(H')^s : m) = \mathfrak{p}_F$  be a prime ideal, for some  $F \subseteq X$ . Then,

$$(J(H)^s:m) = \mathfrak{p}_F + \mathfrak{q}$$

where  $q \subseteq (y)$ . In other words, no monomial only involving the variables in  $X \setminus F$  belongs to  $(J(H)^s : m)$ .

Proof. Say  $F := \{x_{i_1}, \ldots, x_{i_k}\}$  and  $\{x_{\ell_1}, \ldots, x_{\ell_r}\} = X \setminus F$ . Recall that  $e = \{x_1, \ldots, x_t\}$ . First we show that  $\mathfrak{p}_F \subseteq (J(H)^s : m) \subsetneq (1)$ . From Lemma 2.6 we have  $m \notin J(H)^s$  and then  $(J(H)^s : m) \neq (1)$ . By hypothesis, for each  $x_j \in F$  we have  $x_j m \in J(H')^s$  i.e.  $m = m_1 \cdots m_s M$  for some monomials  $m_i \in J(H') \subseteq K[X]$ . But these monomials, see Remark 2.5 also belongs to J(H). Hence,  $x_j m \in J(H)^s$  and  $\mathfrak{p}_F \subseteq (J(H)^s : m) \subseteq K[V]$ .

In order to conclude the proof, take any monomial  $x_{\ell_1}^{a_1} \cdots x_{\ell_t}^{a_t}$  in variables in  $X \setminus F$ . Suppose that  $x_{\ell_1}^{a_1} \cdots x_{\ell_t}^{a_t} m = m_1 \cdots m_s M \in J(H)^s$ , where the  $m_j$ 's are minimal generators of J(H) in the variables in X. The monomials  $m_j \in J(H')$  and then  $x_{\ell_1}^{a_1} \cdots x_{\ell_t}^{a_t} \in (J(H')^s : m) = \mathfrak{p}_F$ , which is a contradiction.

**Lemma 3.4.** Let  $(J(H)^s:m) = \mathfrak{p}$  and  $y \notin \mathfrak{p}$ . Then  $(J(\tilde{H})^s:m) = \mathfrak{p}$ .

Proof. Say  $\mathfrak{p} = \mathfrak{p}_F$  for some  $F \subseteq X$ . First note that  $m \notin J(\tilde{H})^s$ . Indeed, if  $m = m_1 \cdots m_s \cdot M \in J(\tilde{H})^s$  with  $m_1, \ldots, m_s$  minimal vertex covers of  $\tilde{H}$ , then  $y^s m \in J(H)^s$ . This contradicts  $(J(H)^s : m) = \mathfrak{p}$ . We claim that  $(J(\tilde{H})^s : m) \supseteq \mathfrak{p}$ . Indeed, if  $x_j \in F$ , then  $x_j m \in J(H)^s \subseteq J(\tilde{H})^s$ . In order to obtain the assertion, we take a monomial  $T \notin \mathfrak{p}_F$  and assume that  $Tm \in J(\tilde{H})^s$ . Again from  $Tm = m_1 \cdots m_s \cdot M \in J(\tilde{H})^s$  with  $m_1, \ldots, m_s$  minimal vertex covers of  $\tilde{H}$ , we get  $Ty^s \in (J(H)^s : m)$  which contradicts the hypothesis.

**Theorem 3.5.** Let  $(J(H')^s : m) = \mathfrak{p}$ . Then, we have

- (a)  $(J(H)^s:m) = \mathfrak{p}$  if and only if  $(J(\tilde{H})^s:m) = \mathfrak{p}$ ;
- (b)  $(J(H)^s : m \cdot m_0) = \mathfrak{p} + (y)$ , for some monomial  $m_0 \notin \mathfrak{p}$ , if and only if  $(J(\tilde{H})^s : m) \neq \mathfrak{p}$ .

*Proof.* Note that  $y \notin \mathfrak{p}$ , so one implication in (a) follows from Lemma 3.4. Set  $\mathfrak{p}_F := \mathfrak{p} = (J(\tilde{H})^s : m)$  and say  $X \setminus F = \{x_{\ell_1}, \ldots, x_{\ell_r}\}.$ 

By Proposition 3.3, we have  $(J(H)^s:m) = \mathfrak{p} + \mathfrak{q}$  where either  $\mathfrak{q} = (0)$  or  $\mathfrak{q}$  is minimally generated by monomials  $y^a \cdot x_{\ell_1}^{a_1} x_{\ell_2}^{a_2} \cdots x_{\ell_r}^{a_r}$  for some a > 0 and  $a_1, \ldots, a_r \ge 0$ . We claim that  $\mathfrak{q} = (0)$ . Indeed, if  $T := y^a \cdot x_{\ell_1}^{a_1} x_{\ell_2}^{a_2} \cdots x_{\ell_r}^{a_r} \in \mathfrak{q}$ , we get  $\frac{T}{y^a} \in (J(\tilde{H})^s:m) = \mathfrak{p}_F$  which contradicts the hypothesis.

Now we prove item (b). With the notation as above, we have  $(J(H)^s : m) = \mathfrak{p} + \mathfrak{q}$ . First assume  $(J(\tilde{H})^s : m) \neq \mathfrak{p}$ . Then  $\mathfrak{q}$  is not the zero ideal. Consider the non-empty set

 $\{b \in \mathbb{N} \mid y^b \text{ divides } M \text{ for some } M \in \mathfrak{q}\},\$ 

and let *a* be its minimum element. Let  $T := y^a \cdot x_{\ell_1}^{a_1} x_{\ell_2}^{a_2} \cdots x_{\ell_r}^{a_r} \in \mathfrak{q}$  be a monomial minimal generator in  $\mathfrak{q}$ . We collect some relevant facts:

- a > 0, by Proposition 3.3;
- $m\frac{T}{u} \notin J(H)^s$ , by the minimality of T;
- $x_{\ell_1}^{a_1} x_{\ell_2}^{a_2} \cdots x_{\ell_r}^{a_r} \cdot m \frac{T}{y} \notin J(H)^s$ , by the minimality of a;

• 
$$y \cdot m \frac{I}{y} = mT \in J(H)^s$$
.

Then, we get  $\left(J(H)^s: m\frac{T}{y}\right) = \mathfrak{p} + (y)$ , and  $\mathfrak{p} + (y) \in \operatorname{ass}(J(H)^s)$ .

Vice versa, assume  $(J(H)^s : m \cdot m_0) = \mathfrak{p} + (y)$ , for some monomial  $m_0 \notin \mathfrak{p}$ . So, we have  $ymm_0 \in J(H)^s$  and say  $ymm_0 = ym_1 \cdot m_2 \cdots m_s \cdot M \in J(H)^s$  with  $ym_1, \ldots, m_s$  corresponding to minimal vertex covers of H. Then, we get  $mm_0 = m_1 \cdot m_2 \cdots m_s \cdot M \in J(\tilde{H})^s$ , i.e.,  $m_0 \in (J(\tilde{H})^s : m)$ . Since  $m_0$  does not involve the variables in  $\mathfrak{p}$ , we get a contradiction.

11

In particular, the next result shows that item (a) in Theorem 3.5 is always satisfied if  $\mathfrak{p}_e \not\subseteq \mathfrak{p}$ .

# **Proposition 3.6.** Let $(J(H')^s : m) = \mathfrak{p}$ . If $\mathfrak{p}_e \not\subseteq \mathfrak{p}$ , then $(J(H)^s : m) = \mathfrak{p}$ .

Proof. Say  $\mathfrak{p} = \mathfrak{p}_F$  with  $F := \{x_{i_1}, \ldots, x_{i_k}\}$  and  $\{x_{\ell_1}, \ldots, x_{\ell_r}\} = X \setminus F$ . By Proposition 3.3 we have  $(J(H): m) = \mathfrak{p} + \mathfrak{q}$  where  $\mathfrak{q}$  is an ideal minimally generated by monomials which are not only in variables  $\{x_{\ell_1}, \ldots, x_{\ell_r}\} = X \setminus F$ ; i.e., a minimal generator of  $\mathfrak{q}$  is a monomial  $y^b x_{\ell_1}^{a_1} \cdots x_{\ell_r}^{a_r}$  for some  $a_1, \ldots, a_r \ge 0$  and b > 0. Assume on the contrary that  $\mathfrak{q} \ne 0$ . Take any minimal generator in  $\mathfrak{q}$ , say  $T := y^b x_{\ell_1}^{a_1} \cdots x_{\ell_r}^{a_r}$ . Then  $m \cdot T = m_1 \cdots m_s \cdot M \in J(H)^s$  where the  $m_i$ 's are minimal vertex covers of H. Note that y does not divide M. Otherwise, we get  $x_{\ell_1}^{a_1} x_{\ell_2}^{a_2} \cdots x_{\ell_r}^{a_r} y^{b-1} \in (J(H)^s : m)$ , contradicting the minimality of T. Then we can write, after relabeling,  $m_i = ym'_i$  for  $i = 1, \ldots, b, m \cdot T = (ym'_1) \cdots (ym'_b) \cdot m_{b+1} \cdots m_s \cdot M \in J(H)^s$ . Say  $x_1 \in \mathfrak{p}_e$  and  $x_1 \notin \mathfrak{p}$ , then we get

$$m \cdot T\frac{x_1^b}{y^b} = (x_1m_1') \cdots (x_1m_b') \cdot m_{b+1} \cdots m_s \cdot M \in J(H)^s.$$

But  $m \cdot T \frac{x_1^b}{y^b}$  only contains variables of X. Then  $T \frac{x_1^b}{y^b} \in (J(H') : m) = \mathfrak{p}$ . By Proposition 3.3, this is a contradiction since  $T \frac{x_1^b}{y^b}$  only contains variables not in  $\mathfrak{p}$ .

Recall that by Corollary 1.4, a prime associated to  $J(H)^s$  either belongs to  $\operatorname{ass}(J(\tilde{H})^s)$  or it contains the variable y. This is summarized in the following statement.

Corollary 3.7. We have

$$\operatorname{ass}(J(H)^s) = \operatorname{ass}(J(\tilde{H})^s) \cup \mathcal{A},$$

where if  $\mathfrak{p} \in \mathcal{A}$ , then  $y \in \mathfrak{p}$ .

Question 3.8. Do the elements in  $\mathcal{A}$ , mentioned in Corollary 3.7, all come from the shadow? More precisely, if  $\mathfrak{p} = \mathfrak{p}' + (y) \in \operatorname{ass} J(H)^s$ , then is  $\mathfrak{p}' \in \operatorname{ass} J(H')^s$ ?

We will show in the next section, see Example 4.3, that such question has in general a negative answer. By the way, in the next theorem, we positively answer this question under a suitable condition.

**Theorem 3.9.** Let  $\mathfrak{p} = \mathfrak{p}' + (y) \in \operatorname{ass}(J(H)^s)$ . If  $\mathfrak{p} \notin \operatorname{ass}(J(H)^s : y)$ , then  $\mathfrak{p}' \in \operatorname{ass}(J(H')^s)$ .

*Proof.* Take the short exact sequence

$$0 \to \frac{K[V]}{J(H)^s : y} \to \frac{K[V]}{J(H)^s} \to \frac{K[V]}{J(H)^s + (y)} \to 0.$$

From theorem 6.3 in [11] we have that

$$\operatorname{Ass}(K[V]/J(H)^s) \subseteq \operatorname{Ass}(K[V]/J(H)^s : y) \bigcup \operatorname{Ass}(K[V]/J(H)^s + (y)).$$

Denoted by J' the cone ideal of  $J(H')^s$  in the ring K[V], note that  $K[V]/J(H)^s + (y) = K[V]/J' + (y)$ . Since, by hypothesis  $\mathfrak{p} \in \operatorname{Ass}(K[V]/J(H)^s + (y))$ , then  $\mathfrak{p} \in \operatorname{ass}(J' + (y))$ , i.e.  $\mathfrak{p}' \in \operatorname{ass}(J(H')^s)$ .

**Remark 3.10.** Question 3.8 has a positive answer if  $(J(H)^s : m) = \mathfrak{p} + (y)$  for some  $m \in K[X]$ .

In the next examples we show how to describe all the associated prime ideals of  $J(H)^s$  from  $\operatorname{ass}(J(\tilde{H})^s)$  and  $\operatorname{ass}(J(H')^s)$ .

**Example 3.11.** Let *H* be the hypergraph on the vertex set  $V = \{x_1, x_2, x_3, x_4, x_5, y\}$  and the edge set

$$E = \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_1, x_5\}, \{x_1, x_3, y\}\}.$$

Set  $X := \{x_1, x_2, x_3, x_4, x_5\}$ . Then the shadow of H on  $X := \{x_1, x_2, x_3, x_4, x_5\}$  is

$$H' = (X, \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_1, x_5\}, \{x_1, x_3\}\}).$$

Moreover, the subhypergraph of H on X is

$$\dot{H} = (X, \{\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_1, x_5\}\}).$$

A Macaulay2 computation shows that

ass 
$$J(H')^3 = \{(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_5), (x_1, x_5), (x_1, x_3)\} \cup \{(x_1, x_2, x_3)\}$$

and

ass  $J(\tilde{H})^3 = \{(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_5), (x_1, x_5)\} \cup \{(x_1, x_2, x_3, x_4, x_5)\}.$ 

From Theorem 3.5, we know that  $(x_1, x_2, x_3, y) \in \operatorname{ass}(J(H)^3)$ . Moreover, one can check that

$$\operatorname{ass}(J(H)^3) = \operatorname{ass}(J(\tilde{H})^3) \cup \{(x_1, x_3, y), (x_1, x_2, x_3, y)\}$$

**Example 3.12.** Let H be the hypergraph on the vertex set  $V = \{x_1, x_2, x_3, x_4, x_5, y\}$  given by

$$H = (V, \{\{x_1, x_2, x_3\}, \{x_1, x_4\}, \{x_2, x_4\}, \{x_2, x_5\}, \{x_3, x_5\}, \{x_4, x_5, y\}\}).$$

Set  $X := \{x_1, x_2, x_3, x_4, x_5\}$ ; then the shadow of *H* on *X* is

$$H' = (X, \{\{x_1, x_2, x_3\}, \{x_1, x_4\}, \{x_2, x_4\}, \{x_2, x_5\}, \{x_3, x_5\}, \{x_4, x_5\}\}).$$

Moreover, the subhypergraph of H on X is

$$\tilde{H} = \left(X, \{\{x_1, x_2, x_3\}, \{x_1, x_4\}, \{x_2, x_4\}, \{x_2, x_5\}, \{x_3, x_5\}\}\right).$$

Using Macaulay2, we compute that

ass 
$$J(H')^2 = \{(x_1, x_2, x_3), (x_1, x_4), (x_2, x_4), (x_2, x_5), (x_3, x_5), (x_4, x_5)\} \cup \cup \{(x_1, x_2, x_3, x_4), (x_1, x_2, x_3, x_5), (x_2, x_4, x_5)\}$$

and

ass 
$$J(H')^3 = \{(x_1, x_2, x_3), (x_1, x_4), (x_2, x_4), (x_2, x_5), (x_3, x_5), (x_4, x_5)\} \cup \cup \{(x_1, x_2, x_3, x_4), (x_1, x_2, x_3, x_5), (x_2, x_4, x_5)\} \cup$$

 $\cup \{(x_1, x_2, x_3, x_4, x_5)\}.$ 

We also know that  $\operatorname{ass}(J(\tilde{H})^3)$  and  $\operatorname{ass}(J(\tilde{H})^2)$  share the same elements, precisely

$$\{(x_1, x_2, x_3), (x_1, x_4), (x_2, x_4), (x_2, x_5), (x_3, x_5)\}$$

$$\cup \{ (x_1, x_2, x_3, x_4), (x_1, x_2, x_3, x_5), (x_1, x_2, x_3, x_4, x_5) \}.$$

Then, from Proposition 2.15 and Theorem 3.5, we have

$$\operatorname{ass}(J(H)^2) = \operatorname{ass}(J(\tilde{H})^2) \cup \{(x_4, x_5, y), (x_2, x_4, x_5, y)\}$$

What about  $\operatorname{ass}(J(H)^3)$ ? The element  $(x_1, x_2, x_3, x_4, x_5)$  appears both in  $\operatorname{ass}(J(\tilde{H})^3)$ and  $\operatorname{ass}(J(H')^3)$  and it contains  $(x_4, x_5)$ . One can check that

$$ass(J(H')^3:m) = (x_1, x_2, x_3, x_4, x_5)$$

and

$$\operatorname{ass}(J(H)^3:m) = (x_1, x_2, x_3, x_4, x_5)$$

where  $m := x_1 x_2^2 x_3 x_4^2 x_5^2$ .

Thus, by Theorem 3.5,  $(x_1, x_2, x_3, x_4, x_5, y) \notin \operatorname{ass}(J(H)^3)$  and one can check that

$$\operatorname{ass}(J(H)^3) = \operatorname{ass}(J(\tilde{H})^3) \cup \{(x_4, x_5, y), (x_2, x_4, x_5, y)\}$$

**Example 3.13.** Let H be the hypergraph on the vertex set  $V := \{x_1, x_2, x_3, x_4, x_5, y\}$ , given by

$$H = (V, \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_1, x_5, y\}, \{x_2, x_3, x_4, x_5\}\}).$$

Let H' be the shadow of H on the vertex set  $X := \{x_1, x_2, x_3, x_4, x_5\}$  and  $\tilde{H}$  the subhypergraph of H on X. Then

$$H' = (X, \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_1, x_5\}, \{x_2, x_3, x_4, x_5\}\})$$

and

$$\hat{H} = (X, \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_4\}, \{x_2, x_3, x_4, x_5\}\}).$$

A Macaulay2 computation shows that

ass 
$$J(H')^2 = \{(x_1, x_2), (x_1, x_3), (x_1, x_4), (x_1, x_5), (x_2, x_3, x_4, x_5)\} \cup \{(x_1, x_2, x_3, x_4, x_5)\}$$

and

ass 
$$J(\hat{H})^2 = \{(x_1, x_2), (x_1, x_3), (x_1, x_4), (x_2, x_3, x_4, x_5)\} \cup \{(x_1, x_2, x_3, x_4, x_5)\}.$$

The element  $(x_1, x_2, x_3, x_4, x_5)$  appears both in  $\operatorname{ass}(J(\tilde{H})^3)$  and  $\operatorname{ass}(J(H')^3)$  and it contains  $(x_1, x_5)$ . One can check that

$$(J(H')^2 : x_1 x_2 x_3 x_4 x_5) = (x_1, x_2, x_3, x_4, x_5)$$

but  $x_1x_2x_3x_4x_5 \in J(\tilde{H})^2$ . Thus

$$\operatorname{ass}(J(H)^2) = \operatorname{ass}(J(\tilde{H})^2) \cup \{(x_1, x_5, y), (x_1, x_2, x_3, x_4, x_5, y)\}$$

14

#### IN THE SHADOWS OF A HYPERGRAPH

# 4. An application to the persistence property

In this section, we apply the results of Section 3 to the persistence problem. A squarefree monomial ideal I is said to have the persistence property if  $\operatorname{ass}(I^s) \subseteq \operatorname{ass}(I^{s+1})$  for any integer s > 0. The authors of [13] describe an example of a cover ideal of a graph failing the persistence property. We show how to construct, starting from a hypergraph whose cover ideal fails the persistence property, a new hypergraph whose cover ideal fail such property. We use the notation introduced in Section 3.

**Theorem 4.1.** Let H = (V, E) be a hypergraph where  $V = X \cup \{y\}$  such that

- 1. there exists only one edge  $e_y \in E$  containing y;
- 2. *H* has a shadow on *X*, say  $H' = (X, E') \in \mathcal{S}(H)$ .

Suppose J(H') fails the persistence property and let  $\mathfrak{p}' \in \operatorname{ass}(J(H')^s)$  and  $\mathfrak{p}' \notin \operatorname{ass}(J(H')^{s+1})$ for some s > 0. Set  $\tilde{H} := H_X$  the subhypergraph of H on X. If the following conditions hold,

3.  $\mathfrak{p}' \notin ass(J(\tilde{H})^s); and$ 4.  $\mathfrak{p}' + (y) \notin ass(J(H)^{s+1} : y),$ 

then J(H) fails the persistence property.

*Proof.* By hypothesis we have  $\mathfrak{p}' \in \operatorname{ass}(J(H')^s)$  and  $\mathfrak{p}' \notin \operatorname{ass}(J(H)^s)$ . So by Theorem 3.5, one gets that

$$\mathfrak{p}' + (y) \in \operatorname{ass}(J(H)^s).$$

Moreover, the hypothesis also ensures that  $\mathfrak{p}' \in \operatorname{ass}(J(H')^{s+1})$  and  $\mathfrak{p}'+(y) \notin \operatorname{ass}(J(H)^{s+1}: y)$ . Hence, from Theorem 3.9, we have  $\mathfrak{p}'+(y) \notin \operatorname{ass}(J(H)^{s+1})$ .

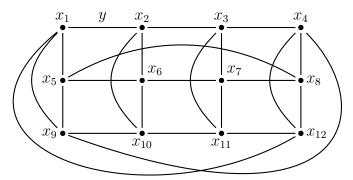
**Example 4.2.** In [13], Theorem 11 provides an example of a graph failing the persistence property. The graph, denoted by  $H_4$ , has the vertex set on  $X := \{x_1, \ldots, x_{12}\}$  and the edge set

$$E := \{\{x_1, x_2\}, \{x_1, x_5\}, \{x_1, x_9\}, \{x_1, x_{12}\}, \{x_2, x_3\}, \{x_2, x_6\}, \{x_2, x_{10}\}, \{x_3, x_4\}, \\ \{x_3, x_7\}, \{x_3, x_{11}\}, \{x_4, x_8\}, \{x_4, x_9\}, \{x_4, x_{12}\}, \{x_5, x_6\}, \{x_5, x_8\}, \{x_5, x_9\}, \\ \{x_6, x_7\}, \{x_6, x_{10}\}, \{x_7, x_8\}, \{x_7, x_{11}\}, \{x_8, x_{12}\}, \{x_9, x_{10}\}, \{x_{10}, x_{11}\}, \{x_{11}, x_{12}\}\}.$$

The persistence property fails since  $\operatorname{ass}(J(H_4)^3) \not\subseteq \operatorname{ass}(J(H_4)^4)$ . In particular,  $\mathfrak{p}_X \in \operatorname{ass}(J(H_4)^3) \setminus \operatorname{ass}(J(H_4)^4)$ .

We consider now the hypergraph H on vertex set  $V := X \cup \{y\}$ , constructed from  $H_4$  by adding the variable "y" only to the edge  $\{x_1, x_2\}$ :

$$H = (X \cup \{y\}, (E \setminus \{\{x_1, x_2\}\}) \cup \{\{x_1, x_2, y\}\}).$$



With this construction,  $H_4$  is the shadow of H on the set X. Moreover, the subhypergraph of H on X is

$$\ddot{H} = (X, E \setminus \{\{x_1, x_2\}\}).$$

By Theorem 4.1, H fails the persistence property and

$$\mathfrak{p}_V \in \operatorname{ass}(J(H)^3) \setminus \operatorname{ass}(J(H)^4)$$

One can check, for instance using Macaulay2, that actually  $\mathfrak{p}_X \notin ass(J(\tilde{H})^3)$  and  $\mathfrak{p}_V \notin ass(J(H)^4 : y)$ .

**Example 4.3.** Take the hypergraph H' on the vertex set  $X = \{x_1, \ldots, x_{12}, x_{13}\}$  and the edge set

$$E = \{\{x_1, x_2, x_{13}\}, \{x_1, x_5\}, \{x_1, x_9, x_{13}\}, \{x_1, x_{12}, x_{13}\}, \{x_2, x_3, x_{13}\}, \{x_2, x_6, x_{13}\}, \\ \{x_2, x_{10}, x_{13}\}, \{x_3, x_4, x_{13}\}, \{x_3, x_7, x_{13}\}, \{x_3, x_{11}, x_{13}\}, \{x_4, x_8, x_{13}\}, \{x_4, x_9, x_{13}\}, \\ \{x_4, x_{12}, x_{13}\}, \{x_5, x_6, x_{13}\}, \{x_5, x_8, x_{13}\}, \{x_5, x_9, x_{13}\}, \{x_6, x_7, x_{13}\}, \{x_6, x_{10}, x_{13}\}, \\ \{x_7, x_8, x_{13}\}, \{x_7, x_{11}, x_{13}\}, \{x_8, x_{12}, x_{13}\}, \{x_9, x_{10}, x_{13}\}, \{x_{10}, x_{11}, x_{13}\}, \{x_{11}, x_{12}, x_{13}\}\}$$

It was constructed from  $H_4$ , see example 4.2, by adding a new variable " $x_{13}$ " to all the edges but  $\{x_1, x_5\}$ . Consider now the hypergraph H on vertex set  $V := X \cup \{y\}$ , constructed from H' by adding the variable "y" only to the edge  $\{x_1, x_5\}$ :

$$H = (X \cup \{y\}, (E \setminus \{\{x_1, x_5\}\}) \cup \{\{x_1, x_5, y\}\})$$

With this construction, H' is the shadow of H on the set X. A computation with Macaulay2 shows that  $\mathfrak{p}_V \in J(H)^4$ , but  $\mathfrak{p}_X \notin J(H')^4$ . Indeed, we found two minimal monomials  $m_1, m_2$  such that  $\mathfrak{p}_V = (J(H)^4 : m_1) = (J(H)^4 : m_2)$  that are

$$m_1 := x_1^2 x_2^3 x_3^3 x_4^2 x_5^2 x_6^3 x_7^2 x_8^3 x_9^3 x_{10}^2 x_{11}^3 x_{12}^3 y, \quad m_2 := x_1^2 x_2^2 x_3^2 x_4^2 x_5^2 x_6^2 x_7^2 x_8^2 x_9^2 x_{10}^2 x_{11}^2 x_{12}^2 x_{13} y,$$

Both are divisible by y.

Then the conditions in the statement of Theorem 4.1 are not satisfied. By Theorem 3.9, we get  $\mathfrak{p}_V \in \operatorname{ass}(J(H)^4 : y)$ . Using Macaulay2, one can check that even if the hypergraph H' fails the persistence property, and in particular  $\mathfrak{p}_X \in \operatorname{ass}(J(H')^3) \setminus \operatorname{ass}(J(H')^4)$ , we have  $\operatorname{ass}(J(H)^3) \subseteq \operatorname{ass}(J(H)^4)$ .

#### IN THE SHADOWS OF A HYPERGRAPH

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