This manuscript is the AAM version of the DOI 10.1016/j.jmaa.2022.126782, published in Journal of Mathematical Analysis and Applications Volume 518, Issue 2, 15 February 2023, art. 126782

Convergence Theorems for Varying Measures Under Convexity Conditions and Applications

Valeria Marraffa and Bianca Satco

Abstract. In this paper, convergence theorems involving convex inequalities of Copson's type (less restrictive than monotonicity assumptions) are given for varying measures, when imposing convexity conditions on the integrable functions or on the measures. Consequently, a continuous dependence result for a wide class of differential equations with many interesting applications, namely measure differential equations (including Stieltjes differential equations, generalized differential problems, impulsive differential equations with finitely or countably many impulses and also dynamic equations on time scales) is provided.

Mathematics Subject Classification. Primary 28A33; Secondary 28C15, 28A20, 28A25.

Keywords. Convergence of measures, continuous dependence, measure differential equations, convex inequality.

1. Introduction

Convergence results for varying measures have significant applications to various fields of pure and applied sciences including stochastic processes, statistics, control and game theories, transportation problems, neural networks, signal and image processing (see, for example, [2, 5–8, 15, 20, 24, 28, 34]).

E.T. Copson in [4], weakening the monotonicity of a sequence of real numbers by changing it in a convex inequality involving k consecutive terms of the sequence, gave a sufficient condition to guarantee the convergence of bounded sequences of real numbers.

This research has been accomplished within the UMI Group TAA "Approximation Theory and Applications". It was supported by F.F.R. 2021 dell'Università degli Studi di Palermo: Prof. Valeria Marraffa and by INDAM-GNAMPA,prot. U-UFMBAZ-2022-000032 12-01-2011, Prof. Bianca Satco.

Recently, following this idea, in [23] the classical Monotone Convergence Theorem has been generalized by changing the monotonicity with a convexity condition on the involved functions.

In the present paper, we go further and continue the investigation started in [11,14–17,21,22,25,32], providing conditions to ensure convergence results for a sequence of functions $(f_n)_n$ integrable with respect to the measures of a sequence $(m_n)_n$, when the functions satisfy a convexity condition. If the sequence of measures $(m_n)_n$ satisfies a convexity condition, convergence theorems for the integrals are obtained as well.

The paper is organized as follows.

In Sect. 2, after the Introduction, convergence theorems for varying measures are given when the sequence of functions $(f_n)_n$ satisfies inequalities of Copson's type (in both increasing or decreasing manners) and the sequence of measures is setwisely or weakly convergent.

In Sect. 3, the convergence of the sequence $\left(\int_\Omega f dm_n\right)_n$ under a convexity condition on the sequence of measures $(m_n)_n$ is obtained, when again $(m_n)_n$ converges in setwise sense.

Finally, Sect. 4 provides a continuous dependence result for measure differential equations under Copson's type assumptions on the measures driving the equations. Such outcomes are important in applications since they allow one to approximate the solutions of a differential problem driven by a general finite Borel measure by solutions of differential problems driven by measures with nicer behavior (e.g. [13], [33] or [9], [10], [31] for the more general, set-valued setting).

Measure differential equations (which can be equivalently written as Stieltjes differential equations, see [19], [27]) proved themselves very useful in studying real life processes with dead times or abrupt changes occurring in their dynamics, e.g. [1], [19] or [29].

We remark that the main theorem of this section, proved for measure differential equations, could be used to get new continuous dependence results for generalized differential problems ([33]), for impulsive differential equations with finitely or countably many impulses ([19], [33]) and also for dynamic equations on time scales ([13]).

2. Convergence Results Under Convexity Conditions on the Functions

Let $(\Omega, \mathcal{A})$ be a measurable space and we denote by $\mathcal{M}^+(\Omega)$ the family of *finite nonnegative measures* on $(\Omega, \mathcal{A})$. Let $m, m_n \in \mathcal{M}^+(\Omega)$ for $n \in \mathbb{N}$ and let $f, f_n : \Omega \to \mathbb{R}$, for $n \in \mathbb{N}$, be measurable functions. The symbol $L^1(m)$ stands for the family of Lebesgue integrable functions with respect to (briefly w.r.t.) the measure m while $\int_A f dm$ is the Lebesgue integral of f over a set $A \in \mathcal{A}$.

We recall the following result.

Lemma 2.1. ([4], [23, Lemmas 1, 2]) *Let $(x_n)_n$ be a sequence of real numbers which satisfies the inequalities*

$$x_{n+k} \geq \sum_{j=1}^{k} \alpha_j x_{n+k-j}, \quad \text{for all } n \geq 1 \tag{1}$$

($x_{n+k} \leq \sum_{j=1}^{k} \alpha_j x_{n+k-j}$, for all *$n \geq 1$, respectively), where k is a fixed positive integer, the coefficients α_j are strictly positive and $\sum_{j=1}^{k} \alpha_j = 1$.*

Then the sequence $y_n = \min\{x_{n-1}, \ldots, x_{n-k}\}$, $n \geq k+1$, is increasing ($y_n = \max\{x_{n-1}, \ldots, x_{n-k}\}$, $n \geq k+1$ is decreasing, respectively). Moreover, if the sequence $(x_n)_n$ satisfies (1) and if $\lambda = \lim_{n\to\infty} x_n$ then $x_n \leq k\lambda$ for all $n \geq 1$.

In the whole Section we will consider a sequence of functions $(f_n)_n$ satisfying

$$f_{n+k}(x) \geq \sum_{J=1}^{k} \alpha_j f_{n+k-j}(x), \quad \text{for all } n \geq 1, x \in \Omega, \tag{2}$$

or, respectively, the reverse inequalities

$$f_{n+k}(x) \leq \sum_{j=1}^{k} \alpha_j f_{n+k-j}(x), \quad \text{for all } n \geq 1, x \in \Omega, \tag{3}$$

where k is a fixed positive integer, the coefficients α_j are strictly positive and $\sum_{j=1}^{k} \alpha_j = 1$.

2.1. Setwisely Convergent Measures

We are now giving convergence theorems with convexity conditions on the functions and setwise converging measures.

We recall that a sequence $(m_n)_n$ *converges setwisely* to m ($m_n \xrightarrow{s} m$) if for every $A \in \mathcal{A}$

$$\lim_{n\to\infty} m_n(A) = m(A)$$

([21, Sect. 2.1], [17, Definition 2.3])

Definition 2.2. Let $(m_n)_n \subseteq \mathcal{M}(^+\Omega)$. We say that a sequence $(f_n)_n : \Omega \to \mathbb{R}$ is *uniformly* (m_n)-integrable on *Ω* if

$$\lim_{\alpha\to+\infty} \sup_{n\in\mathbb{N}} \int_{\{|f_n|>\alpha\}} |f_n| dm_n \ = 0. \tag{4}$$

If $f_n = f$ for all $n \in \mathbb{N}$, then we say that f is *uniformly* (m_n)-integrable on *Ω*.

In the proof of our convergence results, we will use the following proposition:

Proposition 2.3. ([11, Proposition 2.10 and Corollary 2.8]) *Let* $(m_n)_n \subseteq \mathcal{M}^+(\Omega)$ *converge setwisely to* $m \in \mathcal{M}^+(\Omega)$. *Moreover, let* $f : \Omega \to \mathbb{R}$ *be uniformly* (m_n)*-integrable on* Ω. *Then* $f \in L^1(m)$ *and for all* $A \in \mathcal{A}$

$$\lim_{n\to\infty} \int_A f\, dm_n = \int_A f\, dm. \tag{5}$$

We show that if $(f_n)_n$ satisfies a convexity condition of Copson's type the convergence holds for the sequence $(f_n)_n$, not only for the function f (see (6) below).

Theorem 2.4. *Let* $f_n : \Omega \to [0, +\infty]$, $n \in \mathbb{N}$, *be a sequence of measurable functions satisfying (2) and let* m *and* m_n, $n \in \mathbb{N}$, *belong to* $\mathcal{M}^+(\Omega)$. *Then there exists a measurable function* $f : \Omega \to [0, +\infty]$ *such that* $\lim_{n\to\infty} f_n(x) = f(x)$ *for all* $x \in \Omega$. *Suppose that*

(2.4.i) f *is uniformly* (m_n)*-integrable on* Ω;

(2.4.ii) $(m_n)_n$ *is setwisely convergent to* m.

Then, for all $A \in \mathcal{A}$,

$$\lim_{n\to\infty} \int_A f_n dm_n = \int_A f dm. \tag{6}$$

Proof. According to Copson's theorem applied to each $x \in \Omega$, we can find a function $f : \Omega \to [0, +\infty]$ such that

$$\lim_{n\to+\infty} f_n(x) = f(x) \ \text{ for all } x \in \Omega.$$

Since each function f_n is measurable, the function f is also measurable.

To prove the assertion, it is sufficient to prove the equality (6) for $A = \Omega$. Fix $x \in \Omega$ and define for each $n \geq k + 1$

$$g_n(x) = \min\{f_{n-1}(x), \ldots, f_{n-k}(x)\}.$$

Now by (2) and Lemma 2.1 it follows that

$$g_n(x) \leq g_{n+1}(x) \leq f_n(x)$$

and $\lim_{n\to+\infty} g_n(x) = f(x)$.

Therefore, applying the monotone convergence theorem for setwise converging measures ([17, Corollary 6.2]) to the increasing sequence $(g_n)_n$ we get

$$\lim_{n\to+\infty} \int_\Omega g_n dm_n = \int_\Omega f dm.$$

Observe that $\int_\Omega f dm \in [0, +\infty]$. If $\int_\Omega f dm = +\infty$, then since for all $n \in \mathbb{N}$ $g_n(x) \leq f_n(x)$, passing to the limit we obtain

$$\int_\Omega f dm = +\infty = \lim_{n\to+\infty} \int_\Omega g_n dm_n = \lim_{n\to+\infty} \int_\Omega f_n dm_n.$$

Assume that $\int_\Omega f dm < +\infty$ and consider $h_n(x) = \min\{l_{n-1}(x), \ldots, l_{n-k}(x)\}$ for $n > k$, where $l_n(x) = \sum_{j=1}^k f_{n-j}(x)$. The sequence $(l_n)_n$ satisfies the inequality

$$l_{n+k}(x) \geq \sum_{j=1}^{k} \alpha_j l_{n+k-j}(x), \quad \text{for all } n \geq 1, x \in \Omega;$$

therefore, by Lemma 2.1 it follows that $(h_n)_n$ is an increasing sequence with

$$\lim_{n\to+\infty} h_n(x) = k \cdot f(x)$$

for all $x \in \Omega$. Moreover, for every $n > k$

$$f_{n-k}(x) \leq h_n(x) \leq k \cdot f(x);$$

therefore, applying the Dominated Convergence Theorem for varying measures [30, Proposition 18, p.232] we conclude that

$$\lim_{n\to\infty} \int_\Omega f_n dm_n = \int_\Omega f dm.$$

□

If we consider now the decreasing case (3), we have the following

Theorem 2.5. *Let $f_n : \Omega \to [0, +\infty]$ be a sequence of measurable functions such that (3) holds and let m and m_n, $n \in \mathbb{N}$, belong to $\mathcal{M}^+(\Omega)$. Then there exists a measurable function $f : \Omega \to [0, +\infty]$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \Omega$.*

Suppose that

(2.5.i) $f_1, f_2, \ldots, f_k$ are uniformly (m_n)-integrable on Ω and $f_1, f_2, \ldots, f_k \in L^1(m)$;

(2.5.ii) $(m_n)_n$ is setwisely convergent to m.

Then, for all $A \in \mathcal{A}$,

$$\lim_{n\to\infty} \int_A f_n dm_n = \int_A f dm. \tag{7}$$

Proof. The existence of $f : \Omega \to [0, +\infty]$ such that $\lim_{n\to+\infty} f_n(x) = f(x)$ for all $x \in \Omega$ can be proved as in Theorem 2.4. It is sufficient now to show the equality (7) for $A = \Omega$. Fix $x \in \Omega$ and define for $n \geq k+1$,

$$g_n(x) = \max\{f_{n-1}(x), \ldots, f_{n-k}(x)\}.$$

By Lemma 2.1, $(g_n(x))_n$ is a decreasing sequence satisfying

$$f_{n-1}(x) \leq g_n(x) \leq g_{n-1}(x) \text{ for all } n > k;$$

moreover, by the definition of $g_n(x)$ we get

$$\lim_{n\to+\infty} g_n(x) = \lim_{n\to+\infty} f_n(x) = f(x).$$

Now observe that if f_1 and f_2 are in $L^1(m)$ and are uniformly (m_n)-integrable on Ω, then the same is true for $\max\{f_1, f_2\} = \frac{|f_1+f_2|+|f_1-f_2|}{2}$. Therefore, the function $g_{k+1}(x) = \max\{f_k(x), \ldots, f_1(x)\} \in L^1(m)$ and it is uniformly (m_n)-integrable.

By Proposition 2.3 it follows that

$$\lim_{n\to\infty} \int_\Omega g_{k+1} dm_n = \int_\Omega g_{k+1} dm.$$

Besides, $f_n(x) \leq g_{n+1}(x) \leq g_{k+1}(x)$ for $n > k$, so applying the Lebesgue convergence theorem for setwise convergent measures ([30, Proposition 18, p.232]) we get

$$\lim_{n\to\infty} \int_\Omega f_n dm_n = \int_\Omega f dm.$$

□

2.2. Weakly Convergent Measures

We are now considering convergence theorems with convexity conditions on the functions and weakly converging measures.

For the following two results we suppose that Ω is a locally compact Hausdorff space and $\mathcal{A}$ will be its Borel σ-algebra. We denote by $C_b(\Omega)$ the family of all bounded continuous functions on Ω.

We recall that a sequence $(m_n)_n$ *converges weakly* to m ($m_n \xrightarrow{w} m$, [21, Sect. 2.1]) if

$$\int_\Omega g dm_n \to \int_\Omega g dm, \quad \text{for all } g \in C_b(\Omega).$$

We have the following

Theorem 2.6. *Let* $f_n : \Omega \to [0, +\infty]$, $n \in \mathbb{N}$ *be a sequence of lower semi-continuous functions satisfying (2) and let* m *and* m_n, $n \in \mathbb{N}$, *belong to* $\mathcal{M}^+(\Omega)$. *Then there exists a measurable function* $f : \Omega \to [0, +\infty]$ *such that* $\lim_{n\to\infty} f_n(x) = f(x)$ *for all* $x \in \Omega$. *Suppose that*

(2.6.i) f *is uniformly* (m_n)*-integrable on* Ω*;*

(2.6.ii) f *is continuous;*

(2.6.iii) $(m_n)_n$ *is weakly convergent to* m.

Then, for all $A \in \mathcal{A}$,

$$\lim_{n\to\infty} \int_A f_n dm_n = \int_A f dm. \tag{8}$$

Proof. The proof follows as in Theorem 2.4, but in this case we have to apply the monotone convergence theorem for weakly convergent measures ([17, Theorem 6.1]) when $\int_\Omega f dm = \infty$.

If $\int_\Omega f dm < +\infty$ then, as in the previous result, we have that

$$f_{n-k}(x) \leq k \cdot f(x) \quad \text{for all } n > k \text{ and } x \in \Omega,$$

and since by (2.6.i) the function f is uniformly (m_n)-integrable it follows that the sequence $(f_n)_n$ is uniformly (m_n)-integrable on Ω. Consequently, by the Lebesgue convergence theorem for weakly convergent measures ([17, Corollary 5.1]), we get

$$\lim_{n\to\infty} \int_\Omega f_n dm_n = \int_\Omega f dm.$$

□

If condition (3) holds instead, then the following can be proved.

Theorem 2.7. *Let $f_n : \Omega \to [0, +\infty]$, $n \in \mathbb{N}$, be a sequence of lower semicontinuous functions such that (3) holds and let m and m_n, $n \in \mathbb{N}$, belong to $\mathcal{M}^+(\Omega)$. Then there exists a measurable function $f : \Omega \to [0, +\infty]$ such that $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in \Omega$.*

Suppose that

(2.7.i) $f_1, f_2, \ldots, f_k$ are uniformly (m_n)-integrable on Ω and $f_1, f_2, \ldots, f_k \in L^1(m)$;

(2.7.ii) f is continuous;

(2.7.iii) $(m_n)_n$ is weakly convergent to m.

Then, for all $A \in \mathcal{A}$,

$$\lim_{n\to\infty} \int_A f_n dm_n = \int_A f dm. \tag{9}$$

Proof. The proof follows as that of Theorem 2.5. Since

$$f_n(x) \leq g_n(x) \leq g_{k+1}(x) \quad \text{for all } n > k$$

and since by (2.7i) the function g_{k+1} is uniformly (m_n)-integrable on Ω, it follows that the sequence $(f_n)_n$ is uniformly (m_n)-integrable as well. Therefore, applying the Lebesgue convergence theorem for weakly converging measures ([17, Corollary 5.1]) we deduce that

$$\lim_{n\to\infty} \int_\Omega f_n dm_n = \int_\Omega f dm.$$

□

3. Convergence Results for Measures Satisfying Convexity Conditions

In this section we will consider limit theorems of the following type:

$$\int_\Omega f \, dm_n \to \int_\Omega f \, dm$$

where the sequence of measures $(m_n)_n$ satisfies a convexity condition.

It is known (see [12], p 30) that if $(m_n)_n$ is a sequence of measures converging setwise to a set function m, then m is a measure if one of the following holds:

1) $(m_n)_n$ is an increasing sequence;
2) (Vitali–Hahn–Saks) m is finite-valued.

We want first to prove that if we substitute the monotonicity condition by one of the following inequalities of Copson type,

$$m_{n+k}(A) \geq \sum_{j=1}^{k} \alpha_j m_{n+k-j}(A), \quad \text{for all } n \geq 1, A \in \mathcal{A}, \tag{10}$$

or

$$m_{n+k}(A) \leq \sum_{j=1}^{k} \alpha_j m_{n+k-j}(A), \quad \text{for all } n \geq 1, A \in \mathcal{A}, \tag{11}$$

where k is a fixed positive integer, the coefficients α_j are strictly positive and $\sum_{j=1}^{k} \alpha_j = 1$, we still obtain that m is a measure.

Proposition 3.1. *Let $(m_n)_n$ be a sequence in $\mathcal{M}^+(\Omega)$ converging setwisely to a set function $m : \mathcal{A} \to \mathbb{R}$. If (10) holds then m is a measure.*

Proof. Fix $A \in \mathcal{A}$ and for $n \geq k+1$ let

$$\nu_n(A) = \min\{m_{n-1}(A), \ldots, m_{n-k}(A)\}.$$

Then $\nu_n(A) \leq m_n(A)$ for each $n > k$; indeed,

$$m_n(A) \geq \sum_{j=1}^{k} \alpha_j m_{n-j}(A) \geq \sum_{j=1}^{k} \alpha_j \nu_n(A) = \nu_n(A).$$

Moreover, $\nu_n(A) \leq \nu_{n+1}(A)$ for all $n \in \mathbb{N}$, since

$$\begin{aligned} \nu_{n+1}(A) &= \min\{m_n(A), \ldots, m_{n-k+1}(A)\} \\ &\geq \min\{m_{n-k}(A), \min\{m_n(A), \ldots, m_{n-k+1}(A)\}\} \\ &= \min\{m_n(A), \nu_n(A)\} = \nu_n(A). \end{aligned}$$

Therefore, $(\nu_n)_n$ is an increasing sequence of measures, thus it converges to a measure ν. We want to prove that the sequence $(m_n)_n$ converges to ν as well. Fix $A \in \mathcal{A}$. If $\nu(A) \to +\infty$ then as

$$\nu_n(A) \leq m_n(A) \quad \text{for all } n > k,$$

also $m_n(A) \to +\infty$. Assume that $\lim_{n\to+\infty} \nu_n(A) = \nu(A) < +\infty$. Then for every $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that whenever $n > n_\varepsilon$

$$\nu(A) - \varepsilon < \nu_n(A) < \nu(A).$$

If $1 \leq s \leq k$,

$$\begin{aligned} m_{n+s}(A) &\geq \alpha_s m_n(A) + \sum_{t\neq s} \alpha_t m_{n+s-t}(A) \\ &\geq \alpha_s m_n(A) + \sum_{t\neq s} \alpha_t \nu_{n+s}(A) \\ &= \alpha_s m_n(A) + (1-\alpha_s)\nu_{n+s}(A) \\ &\geq \alpha_s m_n(A) + (1-\alpha_s)(\nu(A) - \varepsilon). \end{aligned}$$

For each $n > n_\varepsilon$ there is $1 \leq \bar{s} \leq k$ for which $m_{n+\bar{s}}(A) = \nu_{n+k+1}(A)$. Then

$$\begin{aligned} \nu(A) \geq \nu_{n+k+1}(A) = m_{n+\bar{s}}(A) &\geq (1-\alpha_{\bar{s}})(\nu(A)-\varepsilon) + \alpha_{\bar{s}} m_n(A) \\ &= m_n(A) + (1-\alpha_{\bar{s}})(\nu(A) - \varepsilon - m_n(A)). \end{aligned}$$

Also $\nu(A) - \varepsilon < \nu_n(A) \leq m_n(A)$, so if α is the least of the coefficients $\alpha_{\bar{s}}$ satisfying

$$\nu(A) \geq m_n(A) + (1-\alpha)(\nu(A) - \varepsilon - m_n(A)) = \alpha m_n(A) + (1-\alpha)(\nu(A) - \varepsilon)$$

we get

$$\alpha m_n(A) \leq \nu(A) - (1-\alpha)(\nu(A) - \varepsilon).$$

Therefore

$$m_n(A) \leq \nu(A) + \frac{1-\alpha}{\alpha}\varepsilon$$

whence

$$\nu(A) - \varepsilon < \nu_n(A) \leq m_n(A) \leq \nu(A) + \frac{1-\alpha}{\alpha}\varepsilon.$$

This implies that the sequence $(m_n)_n$ setwise converges to ν which is a measure, and since by hypothesis $(m_n)_n$ converges to m, it follows that $m = \nu$ is a measure. □

An analogous result holds in the case of the reverse inequality in the convex combination, assuming that the measures $m_1, m_2, \ldots, m_k$ are finite-valued.

Proposition 3.2. *Let $(m_n)_n$ be a sequence in $\mathcal{M}^+(\Omega)$ converging setwise to a set function $m : \mathcal{A} \to \mathbb{R}$. If (11) holds and $m_1, m_2, \ldots, m_k$ are finite-valued, m is a measure.*

Proof. The proof is similar to that of Proposition 3.1. In this case considering for $n > k$ $\nu_n(A) = \max\{m_{n-1}(A), \ldots, m_{n-k}(A)\}$,

$$\nu_n(A) \geq m_n(A) \quad \text{and} \quad \nu_n(A) \geq \nu_{n+1}(A).$$

Since $\nu_{k+1}(A) = \max\{m_k(A), \ldots, m_1(A)\}$ is finite-valued, reasoning as before we obtain that there is a coefficient $0 < \alpha < 1$ such that

$$\nu(A) - \frac{1-\alpha}{\alpha}\varepsilon \leq m_n(A) \leq \nu(A) + \varepsilon$$

and the thesis follows. □

Besides, a sequence $(m_n)_n$ for which (10) holds can be shown to satisfy a domination condition.

Proposition 3.3. *Let $(m_n)_n$ be a sequence in $\mathcal{M}^+(\Omega)$ converging setwise to a set function $m : \mathcal{A} \to \mathbb{R}$ and satisfying (10). Then for all $A \in \mathcal{A}$ and $n \geq 1$, $m_n(A) \leq k m(A)$.*

Proof. Fix $A \in \mathcal{A}$. For $n \geq k+1$ consider the sequence of measures $(\nu_n)_n$ defined by

$$\nu_n(A) = m_{n-1}(A) + \cdots + m_{n-k}(A).$$

Then

$$\begin{aligned}
\nu_n(A) &\geq \sum_{j=1}^{k} \alpha_j m_{n-1-j}(A) + \cdots + \sum_{j=1}^{k} \alpha_j m_{n-k-j}(A) \\
&= \alpha_1(m_{n-2}(A) + \ldots m_{n-k-1}(A)) + \alpha_2(m_{n-3}(A) + \ldots m_{n-k-2}(A)) \\
&+ \cdots + \alpha_k(m_{n-1-k}(A) + \cdots + m_{n-2k}(A)) \\
&= \sum_{j=1}^{k} \alpha_j \nu_{n-j}(A)
\end{aligned}$$

so also the sequence $(\nu_n(A))_n$ verifies the Copson's inequality and if for $n \geq k + 1$

$$\eta_n(A) = \min\{\nu_{n-1}(A), \ldots, \nu_{n-k}(A)\}$$

then $\eta_n(A) \leq \eta_{n+1}(A)$ for all $n > k$, and it follows by Lemma 2.1 that the sequence $\eta_n(A)$ is divergent or it converges to some $\lambda(A)$. If it is divergent there is nothing to prove, otherwise assume that it is convergent to $\lambda(A)$ and so by the previous Proposition 3.1 it follows that λ is a measure and the sequence $(\nu_n)_n$ is also setwise convergent to λ.

On the other hand by Proposition 3.1 the sequence $(m_n(A))_n$ converges to $m(A)$, and from the definition of the sequence of measures $(\nu_n)_n$ it follows that $\lambda(A) = km(A)$.

To prove that $m_n(A) \leq km(A)$, we observe that

$$\begin{aligned}\eta_n(A) &= \min\{\nu_{n-1}(A), \ldots \nu_{n-k}(A)\} \\ &= \min\{(m_{n-2}(A) + \cdots + m_{n-k-1}(A)), \ldots, (m_{n-k-1}(A) \\ &\quad + \cdots + m_{n-2k}(A))\}\end{aligned}$$

and since the term $m_{n-k-1}(A)$ is an addend in every term we get

$$m_{n-k-1}(A) \leq \eta_n(A) \leq km(A).$$

Therefore

$$m_n(A) \leq \eta_{n+k+1}(A) \leq km(A) \quad \text{for all } n \in \mathbb{N} \text{ and } A \in \mathcal{A}$$

and the thesis follows. □

Now we are able to prove the convergence results of this section.

Theorem 3.4. *Let $f : \Omega \to \mathbb{R}$ be a nonnegative measurable function and let $(m_n)_n$ be a sequence in $\mathcal{M}^+(\Omega)$ convergent setwisely to a set function $m : \mathcal{A} \to \mathbb{R}$ and satisfying (10). Then for all $E \in \mathcal{A}$*

$$\lim_{n\to+\infty} \int_E f dm_n = \int_E f dm. \tag{12}$$

Proof. It is sufficient to prove the equality (12) for $E = \Omega$. For every $A \in \mathcal{A}$ and for all $n \geq k + 1$ let

$$\nu_n(A) = \min\{m_{n-1}(A), \ldots, m_{n-k}(A)\}.$$

Then as in Proposition 3.1 we get that the sequence $(\nu_n(A))_n$ is increasing and $\nu_n(A) \leq m_n(A)$ for all $n > k$. Moreover, the sequence $(\nu_n)_n$ converges to m. So it follows by the convergence theorem for monotone measures ([21, Theorem 2.1 (c)]) that

$$\lim_{n\to+\infty} \int_\Omega f d\nu_n = \int_\Omega f dm.$$

If $\int_\Omega f dm = +\infty$, then one can see that

$$+\infty = \int_\Omega f dm = \lim_{n\to+\infty} \int_\Omega f d\nu_n \leq \lim_{n\to+\infty} \int_\Omega f dm_n$$

and the assertion is proved.

Assume now that $\int_\Omega f dm < +\infty$, i.e. $f \in L^1(m)$. It follows by Proposition 3.3 that $m_n(A) \leq km(A)$ for all $n \in \mathbb{N}$ and $A \in \mathcal{A}$, also

$$\int_\Omega f d(km) = k \int_\Omega f dm < +\infty;$$

therefore, the statement follows from [21, Theorem 2.1 (b)] □

For the opposite inequality we have the next result.

Theorem 3.5. *Let $f : \Omega \to \mathbb{R}$ be a nonnegative measurable function and let $(m_n)_n$ be a sequence in $\mathcal{M}^+(\Omega)$ converging setwise to a set function $m : \mathcal{A} \to \mathbb{R}$. Assume that (11) holds and $\int_\Omega f dm_j < \infty$, for $j = 1, \dots, k$. Then for all $E \in \mathcal{A}$*

$$\lim_{n\to+\infty} \int_E f dm_n = \int_E f dm. \tag{13}$$

Proof. It is sufficient to prove the equality (13) for $E = \Omega$. For every $A \in \mathcal{A}$ and for all $n \geq k+1$ let

$$\nu_n(A) = \max\{m_{n-1}(A), \dots, m_{n-k}(A)\}.$$

Then as in Proposition 3.2 we get that the sequence $(\nu_n(A))_n$ is decreasing and for all $n > k$ $\nu_n(A) \geq m_n(A)$. Moreover, the sequence $(\nu_n)_n$ converges to the measure m. Also for all $n > k$

$$\int_\Omega f d\nu_{k+1} \geq \int_\Omega f d\nu_n \geq \int_\Omega f dm_n,$$

so it follows by [21, Theorem 2.1 (b)] that

$$\lim_{n\to+\infty} \int_\Omega f dm_n = \int_\Omega f dm.$$

□

4. Application to Measure Differential Equations

We apply in this last section a previously obtained convergence result in order to get a continuous dependence feature of a measure differential equation

$$dx(t) = f(t, x(t))dm, \quad x(0) = x_0, \tag{14}$$

where $\Omega = [0,1]$ and $\mathcal{A}$ is its Borel σ-algebra, $m \in \mathcal{M}^+([0,1])$ and $f : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$.

A function $x : [0,1] \to \mathbb{R}^d$ is a solution of this problem if

$$x(t) = x_0 + \int_{[0,t)} f(s, x(s))dm(s) \quad \text{for every} \quad t \in [0,1],$$

where the integral is understood in Lebesgue sense.

We recall that every finite Borel measure on the real line coincides with the Lebesgue–Stieltjes measure induced by some non-decreasing left-continuous function (see [3, Theorem 3.21]), consequently looking for solutions in the described sense for such an equation is equivalent to looking for solutions of a Stieltjes differential equation (we refer to [19] or [27]).

A global existence and uniqueness result for measure differential equations under Lipschitz assumptions on the right-hand side, stated for the equivalent formulation with Stieltjes derivative, was given in [19, Theorem 7.3] (see also [13, Theorem 5.3], [19, Theorem 7.4] for local results). We can also refer to [33, Theorem 5.4].

Theorem 4.1. *Let* $f : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$ *satisfy:*

i) *for every* $x \in \mathbb{R}^d$, $f(\cdot, x)$ *is measurable;*
ii) $f(\cdot, x_0)$ *is Lebesgue-integrable w.r.t.* m*;*
iii) *there exists a map* $L : [0,1] \to [0,+\infty)$ *Lebesgue-integrable with respect to* m *such that*

$$\|f(t,x) - f(t,y)\| \le L(t)\|x-y\|, \quad \text{for } m-a.e.\ t \in [0,1],\ x, y \in \mathbb{R}^d.$$

Then (14) has a unique solution on $[0,1]$.

We remind the reader that a function $h : [0,1] \to \mathbb{R}$ is called regulated (we refer to [18] for a detailed discussion on this notion) if there exist

$$h(t+) = \lim_{t' \to t+} h(t'), \text{ for all } t \in [0,1), \quad g(s-) = \lim_{s' \to s-} h(s'), \text{ for all } s \in (0,1].$$

The following related concept ([18]) is very useful for getting compactness for regulated functions; a set $\mathcal{F}$ of $\mathbb{R}^d$-valued regulated functions on $[0,1]$ is said to be equi-regulated if for every $\varepsilon > 0$ and every $t_0 \in [0,1]$ there exists $\delta > 0$ such that, for all $f \in \mathcal{F}$,

i) $\|f(t) - f(t_0-)\| < \varepsilon$ whenever $t_0 - \delta < t < t_0$;
ii) $\|f(s) - f(t_0+)\| < \varepsilon$ whenever $t_0 < s < t_0 + \delta$.

We also recall, for completeness, a recent Gronwall inequality for measure differential equations.

Theorem 4.2. ([26, Corollary 4.5]) *Let* $u, K, L : [0,1) \to [0,+\infty)$ *be such that* $L, K \cdot L, u \cdot L$ *are Lebesgue-integrable w.r.t. the measure* $m \in \mathcal{M}^+([0,1])$*. If*

$$u(t) \le K(t) + \int_{[0,t)} L(s)u(s)dm(s), \text{ for every } t \in [0,1),$$

then

$$u(t) \le K(t) + \int_{[0,t)} K(s)L(s)e^{\int_{[s,t)} L(\tau)dm(\tau)}dm(s), \text{ for every } t \in [0,1).$$

We present now the main result of this section on the behavior of the solution of (14) when the measure m is varying.

Theorem 4.3. *Let* $f : [0,1] \times \mathbb{R}^d \to \mathbb{R}^d$ *satisfy:*

i) *for every* $x \in \mathbb{R}^d$, $f(\cdot, x)$ *is measurable;*
ii) $f(\cdot, x_0) \in L^1(m)$*;*
iii) *there exists a map* $L : [0,1] \to [0,+\infty)$ *Lebesgue-integrable with respect to* m *such that*

$$\|f(t,x) - f(t,y)\| \le L(t)\|x-y\|, \quad \text{for all } t \in [0,1],\ x, y \in \mathbb{R}^d;$$

iv) *there exists a map* $M : [0,1] \to [0,+\infty)$ *Lebesgue-integrable with respect to* m *such that*

$$\|f(t,x)\| \leq M(t), \quad \text{for all } t \in [0,1],\ x \in \mathbb{R}^d.$$

Let $(m_n)_n$ *be a sequence in* $\mathcal{M}^+([0,1])$ *setwise convergent to* $m \in \mathcal{M}^+([0,1])$ *and satisfying (10).*

Then the sequence $(x_n)_n$ *of solutions of the measure differential problems*

$$dx(t) = f(t,x(t))dm_n, \quad x(0) = x_0 \tag{15}$$

converges uniformly on $[0,1]$ *to the solution* x *of (14).*

Proof. Let us first note that by Proposition 3.3, $f(\cdot,x_0)$, L and M are also Lebesgue-integrable w.r.t. every m_n since $m_n \leq km$ for all $n \in \mathbb{N}$.

Besides, the assumptions on f ensure that for any function $y : [0,1] \to \mathbb{R}^d$, the map $f(\cdot,y(\cdot))$ is measurable (therefore, by hypothesis *iv*) and the previous observation, Lebesgue-integrable w.r.t. m and also w.r.t. m_n, for every $n \in \mathbb{N}$).

Then let us see that $(\|x_n - x\|)_n$ is bounded on $[0,1]$. Indeed, for any $n \in \mathbb{N}$ and $t \in [0,1]$,

$$\begin{aligned}\|x_n(t) - x(t)\| &= \left\| \int_{[0,t)} f(s,x_n(s))dm_n(s) - \int_{[0,t)} f(s,x(s))dm(s) \right\| \\ &\leq \int_{[0,t)} \|f(s,x_n(s))\|\, dm_n(s) + \int_{[0,t)} \|f(s,x(s))\|\, dm(s)\end{aligned}$$

and using Proposition 3.3 brings us to

$$\begin{aligned}\|x_n(t) - x(t)\| &\leq \int_{[0,t)} \|f(s,x_n(s))\|\, d(km)(s) + \int_{[0,t)} \|f(s,x(s))\|\, dm(s) \\ &\leq (k+1) \int_{[0,t)} M(s)dm(s) \leq (k+1) \int_{[0,1)} M(s)dm(s).\end{aligned}$$

We can write, for each $t \in [0,1]$,

$$\begin{aligned}\|x_n(t) - x(t)\| &= \left\| \int_{[0,t)} f(s,x_n(s))dm_n(s) - \int_{[0,t)} f(s,x(s))dm(s) \right\| \\ &\leq \int_{[0,t)} \|f(s,x_n(s)) - f(s,x(s))\|\, dm_n(s) \\ &+ \left\| \int_{[0,t)} f(s,x(s))dm_n(s) - \int_{[0,t)} f(s,x(s))dm(s) \right\|.\end{aligned}$$

Applying Theorem 3.4,

$$\int_{[0,\cdot)} f(s,x(s))dm_n(s) \to \int_{[0,\cdot)} f(s,x(s))dm(s) \text{ pointwisely.}$$

But the sequence $\left(\int_{[0,\cdot)} f(s,x(s))dm_n(s)\right)_n$ is equi-regulated by [18, Theorem 3.10], as there is a nondecreasing function given on $[0,1]$ by

$$h(t) = \int_{[0,t)} \|f(s,x(s))\| d(km)(s)$$

satisfying, for every $0 \le t < t' \le 1$ and every $n \in \mathbb{N}$,

$$\begin{aligned} &\left\| \int_{[0,t)} f(s,x(s))dm_n(s) - \int_{[0,t')} f(s,x(s))dm_n(s) \right\| \\ &\quad = \left\| \int_{[t,t')} f(s,x(s))dm_n(s) \right\| \\ &\quad \le \int_{[t,t')} \|f(s,x(s))\| \, dm_n(s) \\ &\quad \le \int_{[t,t')} \|f(s,x(s))\| \, d(km)(s) \\ &\quad = h(t') - h(t). \end{aligned}$$

As it is well-known ([18, Theorem 3.3]), any equi-regulated, pointwisely convergent sequence of regulated functions converges uniformly, therefore

$$\int_{[0,\cdot)} f(s,x(s))dm_n(s) \to \int_{[0,\cdot)} f(s,x(s))dm(s) \text{ uniformly,}$$

i.e. for every $\varepsilon > 0$ one can find $n_\varepsilon \in \mathbb{N}$ such that

$$\left\| \int_{[0,t)} f(s,x(s))dm_n(s) - \int_{[0,t)} f(s,x(s))dm(s) \right\| < \varepsilon, \text{ for all } n \ge n_\varepsilon, \ t \in [0,1].$$

Using now the Lipschitz assumption on f, for every such n we get

$$\|x_n(t) - x(t)\| \le \int_{[0,t)} L(s)\|x_n(s) - x(s)\| dm_n(s) + \varepsilon, \text{ for every } t \in [0,1].$$

As $(\|x_n - x\|)_n$ is bounded on $[0,1]$, we can apply, for each $n \ge n_\varepsilon$, the Gronwall type result ([26, Corollary 4.5]), in order to deduce that

$$\|x_n(t) - x(t)\| \le \int_{[0,t)} L(s)\varepsilon e^{\int_{[s,t)} L(\tau)dm_n(\tau)} dm_n(s) + \varepsilon, \text{ for every } t \in [0,1).$$

Again by Theorem 3.4, the sequence $(\int_{[0,1)} L(\tau)dm_n(\tau))_n$ is convergent, therefore bounded, say by $M_1 > 0$, whence

$$\int_{[s,t)} L(\tau)dm_n(\tau) \le M_1, \quad \text{for all } s < t \in [0,1], \ n \in \mathbb{N}.$$

Consequently, for all $t \in [0,1]$,

$$\int_{[0,t)} L(s)e^{\int_{[s,t)} L(\tau)dm_n(\tau)} dm_n(s) \le \int_{[0,t)} L(s)e^{M_1} dm_n(s) \le M_1 e^{M_1}, \ \forall\, n \in \mathbb{N}$$

and so

$$\|x_n(t) - x(t)\| \le \varepsilon \left(1 + M_1 e^{M_1}\right), \text{ for every } t \in [0,1) \text{ and } n \ge n_\varepsilon.$$

Due to the left-continuity of x_n and x we get

$$\|x_n(1) - x(1)\| \leq \varepsilon \left(1 + M_1 e^{M_1}\right), \text{ for every } n \geq n_\varepsilon$$

and the proof is over. □

References

[1] Area, I., Fernández, F.J., Nieto, J.J., Tojo, F.A.F.: Concept and solution of digital twin based on a Stieltjes differential equation, Math. Methods Appl. Sci., (2022)

[2] Avendaño-Garrido, M.L., Gabriel-Argüelles, J.R., Torres Quintana, L., González-Hernández, J.: An approximation scheme for the Kantorovich-Rubinstein problem on compact spaces. J. Numer. Math. **26**(2), 63–75 (2018)

[3] Bruckner, A.M., Bruckner, J.B., Thomson, B.S.: Real Analysis. Prentice-Hall, Upper Saddle River (1997)

[4] Copson, E.T.: On a generalisation of monotonic sequences. Proc. Edinb. Math. Soc. **17**, 159–164 (1970)

[5] Costarelli, D., Croitoru, A., Gavriluţ, A., Iosif, A., Sambucini, A.R.: The Riemann-Lebesgue integral of interval-valued multifunctions. Mathematics **8**(12), 1–17 (2020)

[6] Croitoru, A., Gavriluţ, A., Iosif, A., Sambucini, A.R.: Convergence Theorems in Interval-Valued Riemann-Lebesgue Integrability, Mathematics, **10** (3), (2022), art. 450; https://doi.org/10.3390/math10030450

[7] Croitoru, A., Gavrilut, A., Iosif, A., Sambucini, A.R.: A note on convergence results for varying interval valued multisubmeasures. Mathematical Foundation of Computing (2021). https://doi.org/10.3934/mfc.2021020

[8] Dębicki, K., Engelke, S., Hashorva, E.: Generalized Pickands constants and stationarymax-stable processes. Extremes **20**, 493–517 (2017)

[9] Di Piazza, L., Marraffa, V., Satco, B.: Closure properties for integral problems driven by regulated functions via convergence results. J. Math. Anal. Appl. **466**, 690–710 (2018)

[10] Di Piazza, L., Marraffa, V., Satco, B.: Approximating the solutions of differential inclusions driven by measures. Ann. Mat. Pura Appl. **198**, 2123–2140 (2019)

[11] Di Piazza, L., Marraffa, V., Musiał, K., Sambucini, A.: Convergence for varying measures, J. Math. Anal. Appl., https://doi.org/10.1016/j.jmaa.2022.126782; ArXiv:2209.00354 (2022)

[12] Doob, J.L.: Measure Theory, Springer-Verlag, 1994

[13] Federson, M., Mesquita, J.G., Slavík, A.: Measure functional differential equations and functional dynamic equations on time scales. J. Diff. Eqs. **252**, 3816–3847 (2012)

[14] Feinberg, E.A., Kasyanov, P.O., Zgurovsky, M.Z.: Uniform Fatou Lemma. J. Math. Anal. Appl. **144**, 550–567 (2016)

[15] Feinberg, E.A., Kasyanov, P.O., Zgurovsky, M.Z.: Partially observable total-cost Markov decision processes with weakly continuous transition probabilities. Math. Oper. Res. **41**, 656–681 (2016)

[16] Feinberg, E.A., Kasyanov, P.O., Liang, Y.: Fatou's Lemma for weakly converging measures under the uniform integrability condition. Theory Prob. Appl. **64**(4), 615–630 (2020)

[17] Feinberg, E.A., Kasyanov, P.O., Liang, Y.: Fatou's Lemma in its classic form and Lebesgue's Convergence Theorems for varying measures with applications to MDPs. Theory Prob. Appl. **65**(2), 270–291 (2020)

[18] Fraňková, D.: Regulated functions. Math. Bohem. **116**, 20–59 (1991)

[19] Frigon, M., López Pouso, R.: Theory and applications of first-order systems of Stieltjes differential equations. Adv. Nonlinear Anal. **6**, 13–36 (2017)

[20] García Trillos, N., Sanz-Alonso, D.: Continuum limits of posteriors in graph Bayesian inverse problems. SIAM J. Math. Anal. **50**, 4020–4040 (2018)

[21] Hernandez-Lerma, O., Lasserre, J.B.: Fatou's Lemma and Lebesgue's Convergence Theorem for measures. J. Appl. Math. Stoch. Anal. **13**(2), 137–146 (2000)

[22] Lasserre, J.B.: On the setwise convergence of sequences of measures. J. Appl. Math. and Stoch. Anal. **10**(2), 131–136 (1997)

[23] Linero-Bas, A., Nieves-Roldán, D.: A generalization of the monotone convergence theorem. Mediterr. J. Math. **8**, 158 (2021)

[24] Liu, S., Bousquet, O., Chaudhuri, K.: Approximation and convergence properties of generative adversarial learning, NIPS'17: Proceedings of the 31st International Conference on Neural Information Processing Systems, (2017), 5551-5559

[25] Ma, L.: Sequential convergence on the space of Borel measures, arXiv preprint arXiv:2102.05840 (2021)

[26] Marquez, I.A.: Notes on the linear equation with Stieltjes derivative, Electronic Journ. of qualitative Theory of Differential equations, **42**, (2021), 1-18

[27] Monteiro, G.A., Satco, B.: Distributional, Differential and Integral Problems: Equivalence and Existence Results. Electronic Jour. Qualit. Theory Differ, Equ (2017)

[28] Pap, E., Gavriluţ, A.: Integrability of an Interval-valued Multifunction with respect to an Interval-valued Set Multifunction. Iranian Journal of Fuzzy Systems **15**(3), 47–63 (2018)

[29] Pouso, R.L., Márquez, I.A.: General existence principles for Stieltjes differential equations with applications to mathematical biology. J. Differential Equations **264**, 5388–5407 (2018)

[30] Royden, H.L.: Real Analysis, Collier-Macmillan Canada, 1968

[31] Satco, B.: Continuous dependence results for set-valued measure differential problems. Electr. Jour. Qualit. Th. Diff. Equat. **79**, 1–15 (2015)

[32] Serfozo, R.: Convergence of Lebesgue integrals with varying measures, Indian J. Stat., Serie A, **44** (39), (1982), 380-402

[33] Slavík, A.: Well-posedness results for abstract generalized differential equations and measure functional differential equations. J. Differ. Eqs. **259**, 666–707 (2015)

[34] Thorpe, M., Theil, F., Johansen, A.M., Cade, N.: Convergence of the k-means minimization problem using Γ-convergence. SIAM J. Appl. Math. **75**, 2444–2474 (2015)