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Navier–Stokes Equations in the Half Space with Non Compatible Data

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Abstract. This paper considers the Navier–Stokes equations in the half plane with Euler-type initial conditions, i.e., initial conditions with a non-zero tangential component at the boundary. Under analyticity assumptions for the data, we prove that the solution exists for a short time independent of the viscosity. We construct the Navier–Stokes solution through a composite asymptotic expansion involving solutions of the Euler and Prandtl equations plus an error term. The norm of the error goes to zero with the square root of the viscosity. The Prandtl solution contains a singular term, which influences the regularity of the error. The error term is the sum of a first-order Euler correction, a first-order Prandtl correction, and a further error term. The use of an analytic setting is mainly due to the Prandtl equation.

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1. Introduction

In this paper, we shall study the solutions of the 3D Navier–Stokes(NS) equations in the half-space when the initial and boundary data are incompatible. At the boundary, we shall impose the no-slip boundary condition and, therefore, the incompatibility means that we shall consider initial data having non-zero tangential component at the boundary; we shall call these data *Euler type* or *not well prepared* initial data. We shall establish the existence and uniqueness of the NS solutions for a short time but independent of the viscosity. The central hypothesis we shall impose is the analyticity of the initial datum.

The study of the NS equations in the half-space (or half-plane in 2D) is a central subject in the mathematical theory of fluid dynamics because it is a prototypical case where one can study the interaction between a fluid and a wall; explaining such interactions and the resulting phenomena (like boundary layer formation, vorticity generation, and transitions) motivates a significant part of the interest in fluids. The introduction by S. Ukai of an exact formula for solutions to the Stokes equations in the half-space [1] had a significant impact on the mathematical theory of the NS equations; since Ukai's result, several papers appeared concerning the existence and uniqueness of solutions for the Navier–Stokes equations in the half-space with general initial data, see for example [2,3] and references therein. However, these results, on one side, do not allow Euler-type initial data, i.e., data with a non-zero tangential component at the boundary; on the other hand, they rely on estimates that would degrade in the zero viscosity limit.

Euler-type initial data are important classical configurations; among them, we mention the impulsively started disk or plate [4], and the flow generated by the interaction between a wall and a point vortex or a core vortex [5]. Incompatible data are an interesting subject also from a numerical point of view; in [6–10], one can find in-depth studies of how the lack of compatibility between initial and boundary data can lead to loss of numerical accuracy; and on the appropriate compatibility conditions, one should impose to ensure the required accuracy. Incompressible fluid dynamics Euler-type data typically arise when, to the Navier–Stokes equations, one imposes initial data that are stationary solutions of the Euler equations and are, therefore, of great interest for the numerical and theoretical study of the boundary layer theory. These

data, for example, have been used to show, numerically, that Prandtl's equations develop singularity, see the seminal papers [11] and [12], or even to test the Prandtl equations' effectiveness in reproducing the separation phenomena occurring at the boundary, see [13–15].

The study of the zero viscosity limit of the incompressible Navier–Stokes solutions is one of the most important and challenging subjects in the mathematical theory of fluid dynamics. Formally, one should have the convergence of the NS solutions to the Euler solutions. However, NS equations are a singular perturbation of the Euler equations, and, from the physical point of view, the interaction of the flow with the boundary creates sharp gradients in the normal directions; This situation requires the introduction of a boundary layer corrector to correct the discrepancy between the no-slip boundary condition and the Euler solution: classically, such a corrector is the Prandtl equation's solution. The lack of a completely satisfactory theory of the Prandtl boundary layer equations has been a significant obstacle in the characterization of the zero viscosity solutions of the NS equations. The monotonic setting [16] (through the use of the Crocco transformation as in the original Oleinik's work or through the use of energy estimates, as in [17–19]), and the analytic setting [20–25], are two instances where one can control the ill-posedness and instability phenomena recently discovered [26–31]; so that it is possible to prove the well-posedness of Prandtl's equations. More recently, in [32,33], they have achieved similar results in the Gevrey class (rather than analytic), assuming some structural hypothesis on the initial data (like the nondegeneracy of the singular point) or without such assumptions in [34].

The construction, see [25], of the analytic Prandtl solution allowed showing the convergence of the NS solution in the half-plane as well as in the half-space. In [35], the authors constructed the NS solution through an asymptotic matching between the Prandtl and the Euler solutions, plus a correction which was shown to be $O(\sqrt{\nu})$. The convergence to Euler and Prandtl solutions in their respective domain of validity immediately followed. See also [36] for a different geometry (the outside of a disk) and, more recently, the paper [37], where the authors obtained the above results by the use of energy methods and [38], where the inviscid limit was derived without using a boundary-layer corrector.

Recently, in the 2D half-plane, all the above has been improved, showing that analyticity is necessary only close to the boundary; this important result has been achieved first by Maekawa in [39], where zero vorticity close to the boundary was assumed; this result has been generalized to the half-space in [40]. Kukavica, Vicol, and Wang [41,42], in the half-plane, proved that the analyticity of the data only near the boundary and Sobolev regularity elsewhere is enough to obtain the validity of the inviscid limit. These results have been obtained through energy methods applied to the vorticity formulation of the Navier–Stokes equations without using the matched asymptotic expansion of the solution. This result has been extended to a 3D setting in [43].

In 1994, Kato initiated a different line of research. In [44], he showed that the convergence of the NS solutions to the Euler solution is equivalent to the vanishing of the dissipation in a sublayer of thickness $O(\nu)$, therefore smaller than the Prandtl layer. Temam and Wang, in [45], see also [46,47], obtained a condition leading to convergence based on the pressure gradient at the boundary, which is interesting because the appearance of high stress at the boundary is the first indicator of the deviation of the Navier–Stokes solutions from the Prandtl solution and the precursor of vortices formation and subsequent separation, see [13–15]. During the last 15 years, the Kato criterion has been improved, interpreted in terms of different quantities of physical interest, e.g., vorticity [48–50] or tangential velocity and velocity gradient at the boundary [51], combined with Oleinik's monotonicity setting [52]. Recently, in [53–55], the authors have derived interesting criteria based on the vorticity in the interior of the domain; it is impossible to report all the implications of this line of research here, and we refer the interested reader to [49] or to the recent review paper [56].

Further cases where it has been possible to establish the inviscid limit are flows with symmetries [57–61], flows with anisotropic viscosity [62] and, recently, in the Gevrey class [63,64].

As mentioned, we want to construct the Navier–Stokes equation solution starting from an initial datum with a non-zero tangential slip. Moreover, we want the solution's time of existence not to shrink to zero when the viscosity goes to zero; this is not trivial because, in 3D, the time of existence of the NS solutions is short, and in the presence of boundaries, the existence time, in general, would go to zero

with the viscosity. Therefore, we shall have to handle both the initial layer, due to the presence of the initial discontinuity, and the boundary layer, due to the mismatch between the no-slip boundary condition and the Euler solution. Following the paper [35], we shall decompose the solution to the Navier–Stokes equations as the sum of the solution to the Euler equations, the solution to the Prandtl equations with non-compatible data, and a remainder; due to the presence of the initial layer, we shall see that further decomposition of the remainder is necessary. In our procedure, we shall rely on the well-posedness result for Prandtl's equations with non-compatible data [21].

We also mention the recent work [65,66] that has tackled similar problems, although considering a linearized version of the NS equations.

1.1. Asymptotic Decomposition of the NS Solution

Consider the Navier–Stokes equations in the half-plane or in the half-space $\Pi^+ = \{(x, y) : y \geq 0\}$ where, in 3D, $x = (x_1, x_2)$:

$$\begin{aligned} \partial_t \mathbf{u} - \nu \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p &= 0, \\ \nabla \cdot \mathbf{u} &= 0, \\ \mathbf{u}\big|_{t=0} &= \mathbf{u}_0, \\ \gamma \mathbf{u} &= 0. \end{aligned} \tag{1.1}$$

In the above equation γ is the trace operator, i.e., for regular functions,

$$\gamma \mathbf{v} = \mathbf{v}(y = 0). \tag{1.2}$$

When the viscosity ν is small, its effects mainly concentrate in a layer near the boundary, whose thickness is proportional to $\varepsilon = \sqrt{\nu}$. For these kinds of singularly perturbed problems, one looks for solutions that, at the formal level, can be written as [67]:

$$\mathbf{u} = \mathbf{u}_{(0)}^{out} + \mathbf{u}_{(0)}^{inn} + \varepsilon \left(\mathbf{u}_{(1)}^{out} + \mathbf{u}_{(1)}^{inn} \right) + \cdots + \varepsilon^N \left(\mathbf{u}_{(N)}^{out} + \mathbf{u}_{(N)}^{in} \right) + O(\varepsilon^{N+1}), \tag{1.3}$$

having carried out the expansion up to order N. In the above expansion $\mathbf{u}_{(i)}^{out}(x, y, t)$ and $\mathbf{u}_{(i)}^{in}(x, Y, t)$ for $i = 0, \ldots, N$ describe the outer flow (away from the boundary) and the inner flow (close to the boundary) respectively, being $Y = y/\varepsilon$ the rescaled variable.

The rigorous proof that an expansion like (1.3) holds in the compatible case, was given in [35], where the authors proved that the solution of the NS equations can be written as:

$$\mathbf{u} = \mathbf{u}^E + \bar{\mathbf{u}}^P + \varepsilon \left(\mathbf{u}_1^E + \bar{\mathbf{u}}_1^P \right) + \varepsilon \mathbf{e}.$$

In the above expansion, the outer leading order term, $\mathbf{u}^E$, is the solution of the Euler equations:

$$\begin{aligned} \partial_t \mathbf{u}^E + \mathbf{u}^E \cdot \nabla \mathbf{u}^E + \nabla p^E &= 0, \\ \nabla \cdot \mathbf{u}^E &= 0, \\ \mathbf{u}^E\big|_{t=0} &= \mathbf{u}_0, \\ \gamma_n \mathbf{u}^E &= 0, \end{aligned} \tag{1.4}$$

where $\gamma_n \mathbf{v}$ is the normal component of the trace of $\mathbf{v}$.

The leading order inner solution, $\bar{\mathbf{u}}^P$, is linked to the solution of the Prandtl equations as follows: let u^P the solution of Prandtl's equations

$$\begin{aligned} (\partial_t - \partial_{YY}) u^P + u^P \partial_x u^P + v^P \partial_Y u^P - (\partial_t + u^E\big|_{y=0} \partial_x) u^E\big|_{y=0} &= 0, \\ u^P\Big|_{Y=0} = 0, \quad u^P\Big|_{Y \to +\infty} &= u^E(x, y = 0, t), \\ u^P\Big|_{t=0} &= u_0(x, y = 0), \end{aligned} \tag{1.5}$$

where

$$v^P = -\int_0^Y \partial_x u^P dY'. \tag{1.6}$$

We define the inner solution as:

$$\bar{\mathbf{u}}^P = (\tilde{u}^P, \varepsilon \bar{v}^P), \tag{1.7}$$

where

$$\tilde{u}^P = u^P - u^E\big|_{y=0}, \tag{1.8}$$

$$\bar{v}^P = \int_Y^{+\infty} \partial_x \tilde{u}^P dY'. \tag{1.9}$$

The $\bar{\mathbf{u}}^P$ defined above decays away from the boundary so that it does not interfere with the outer solution $\mathbf{u}^E$. Moreover $\bar{\mathbf{u}}^P$ cancels the tangential flow generated by $\mathbf{u}^E$. However, $\bar{\mathbf{u}}^P$ generates a normal inflow at the boundary $Y = 0$, see (1.9). This inflow is canceled by the first order outer solution $\varepsilon \mathbf{u}_1^E$. However, his outer solution generates tangential flow, which is canceled by the first-order inner solution $\varepsilon \bar{\mathbf{u}}_1^P$.

The overall remainder $\varepsilon \mathbf{e}$ closes the procedure. Notice, however, that the remainder, which at the formal level should be $O(\varepsilon^2)$, due to the nonlinear interactions between the boundary layer terms and the outer Euler terms, can be proven to be $O(\varepsilon)$ only.

1.2. Statement of the Main Result

We give an informal statement of the main result of the present paper.

Theorem 1.1. (Informal statement) *Suppose that $\boldsymbol{u}_0$ has analytic regularity, with $\gamma_n \boldsymbol{u}_0 = 0$. Then, an analytic solution to the Navier–Stokes equations exists for a time independent of the viscosity; this solution can be written as*

$$\boldsymbol{u} = \boldsymbol{u}_{(0)}^{out} + \boldsymbol{u}_{(0)}^{inn} + \varepsilon \left(\boldsymbol{u}_{(1)}^{out} + \boldsymbol{u}_{(1)}^{inn} + \boldsymbol{e} \right), \tag{1.10}$$

where the outer solutions, $\boldsymbol{u}_{(0)}^{out}$ and $\boldsymbol{u}_{(1)}^{out}$, depend on (x, y, t), while the inner solutions, $\boldsymbol{u}_{(0)}^{inn}$ and $\boldsymbol{u}_{(1)}^{inn}$, depend on (x, Y, t) and are exponentially decaying outside the boundary layer.

Moreover $\boldsymbol{u}_{(0)}^{inn}$, $\boldsymbol{u}_{(1)}^{out}$ and $\boldsymbol{u}_{(1)}^{inn}$ have singular time derivatives at the initial time.

The rest of this subsection is devoted to illustrating the physical meaning of the terms appearing in the asymptotic expansion of the NS solution, Eq. (1.10).

1.2.1. The Leading-Order Outer Solution. The term $\mathbf{u}_{(0)}^{out}(x, y, t)$ is the solution $\mathbf{u}^E$ of the Euler equations Eq. (1.4). The existence of an analytic solution for the Euler equation is a well-known result in the literature, see among others [25,68,69]. Therefore, we shall refer to these results when constructing this part of the NS solution.

1.2.2. The Leading-Order Inner Solution. The term $\mathbf{u}_{(0)}^{inn} = \bar{\mathbf{u}}^P(x, Y, t)$ correlates to the solution of the Prandtl equations (1.5) through subtraction of a constant (in Y) function that makes it decay outside the BL. The incompatibility, in equations (1.5), between the initial data and the boundary data leads to the occurrence of a singular part $\mathbf{u}^S$ so that we must decompose $\bar{\mathbf{u}}^P$ in a regular and a singular part:

$$\bar{\mathbf{u}}^P = \bar{\mathbf{u}}^R + \mathbf{u}^S. \tag{1.11}$$

The singular part $\mathbf{u}^S = (u^S, \varepsilon v^S)$ is in the form of an initial layer; u^S solves the following system:

$$\begin{aligned} (\partial_t - \partial_{YY})u^S &= 0, \\ u^S\big|_{Y=0} &= -u_{00}(x), \\ u^S|_{t=0} &= 0, \end{aligned} \tag{1.12}$$

where u_{00} is the trace at the boundary of the tangential part of the initial NS datum, i.e.,

$$u_{00} = \gamma u_0; \tag{1.13}$$

clearly, if the initial and boundary data are compatible, $u_{00} = 0$ and, consequently, $u^S = 0$. One can write the explicit expression of u^S,

$$u^S = -\gamma u_0 \int_Y^{+\infty} \frac{e^{-\frac{\sigma^2}{4t}}}{\sqrt{\pi t}} d\sigma = -u_{00} \int_Y^{+\infty} \frac{e^{-\frac{\sigma^2}{4t}}}{\sqrt{\pi t}} d\sigma, \tag{1.14}$$

while v^S derives from the incompressibility condition,

$$v^S = \int_Y^{+\infty} \partial_x u^S dY'. \tag{1.15}$$

Notice how the tangential part u^S has Y- and t- derivatives that, at the initial time, are singular at the boundary. For the rest of the paper, we shall say that a function has a "Gaussian" singularity if it is bounded away from the origin $(Y, t) = (0, 0)$ and behaves near the origin like

$$t^{-\beta} p(Y/\sqrt{t}) e^{-Y^2/ct} \tag{1.16}$$

for some $\beta > 0$, $c > 0$ and some polynomial p. This kind of singularity is concentrated at the boundary at the initial time and can be tamed by multiplying the function by suitable powers of Y and t. The regular part $\bar{\mathbf{u}}^R$ can be proven to be more regular. In [21], one can find the proof that the Prandtl equation with incompatible data admits the solution given in (1.11).

In the sequel, it will be helpful to introduce the following notation for the value at the boundary of the influx generated by the Prandtl solution:

$$\begin{aligned} g \equiv \gamma \bar{v}^P &= -\int_0^{+\infty} \partial_x \tilde{u}^P dY' = g^R + g^S, \\ g^R = -\int_0^{+\infty} \partial_x \tilde{u}^R dY' \qquad g^S &= -\int_0^{+\infty} \partial_x u^S dY' = -\sqrt{\frac{4}{\pi}} \sqrt{t} \partial_x u_0(x, y = 0). \end{aligned} \tag{1.17}$$

1.2.3. The First-Order Correction to the Outer Solution. The first order outer flow $\mathbf{u}^{out}_{(1)} = \mathbf{u}^E_{(1)}(x, y, t)$ is the solution of a linearized Euler's system:

$$\begin{aligned} \partial_t \mathbf{u}^E_{(1)} + \mathbf{u}^E_{(1)} \cdot \nabla \mathbf{u}^E + \mathbf{u}^E \cdot \nabla \mathbf{u}^E_{(1)} + \nabla p^E_{(1)} &= \mathbf{0}, \\ \nabla \cdot \mathbf{u}^E_{(1)} &= 0, \\ \gamma_n \mathbf{u}^E_{(1)} &= -\gamma \bar{v}^P, \\ \mathbf{u}^E_{(1)}\big|_{t=0} &= \mathbf{0}. \end{aligned} \tag{1.18}$$

The role of the correction to the Euler flow is to cancel the normal inflow generated at the boundary by the boundary layer corrector $\bar{\mathbf{u}}^P$. We shall decompose $\mathbf{u}^E_{(1)}$ in a regular and a singular part:

$$\mathbf{u}^E_{(1)} = \boldsymbol{w}^R + \boldsymbol{w}^S. \tag{1.19}$$

We shall see that one can write the singular part as $\sqrt{t}\boldsymbol{w}_b^S(x,y)$, where $\boldsymbol{w}_b^S$ does not depend on t. This time behavior means that the singularity, formed at $t=0$ in the inner flow $\mathbf{u}_{(0)}^{inn}$, does not remain confined in the boundary layer but instantaneously propagates in the whole space with an $O(\varepsilon)$ intensity. This is consistent with the elliptic-parabolic nature of the Navier–Stokes equations, leading to an infinite speed of propagating disturbances.

While the time derivative of $\mathbf{u}_{(1)}^E$ is everywhere singular at the initial time, its singularity, as t goes to zero, is less pronounced than the one of $\partial_t u^S$: the reason is that the singularity transfers to $O(\varepsilon)$ terms through the incompressibility condition that gives g^S in (1.17), and the integration in Y has a regularizing effect on the gaussian singularities. For the same reason, we shall see that the overall error $\mathbf{e}$ is regular, with a bounded time derivative.

1.2.4. The First-Order Correction to the Inner Solution. The first order inner flow $\mathbf{u}_{(1)}^{inn} = \bar{\mathbf{u}}_{(1)}^P(x,Y,t) = \left(\bar{u}_{(1)}^P, \varepsilon\bar{v}_{(1)}^P\right)$ is the solution of a linearized Prandtl's system,

$$\begin{aligned}(\partial_t - \partial_{YY})\bar{u}_{(1)}^P &= 0,\\ \gamma\bar{u}_{(1)}^P &= -\gamma u_{(1)}^E,\\ \bar{u}_{(1)}^P(t=0) &= 0,\end{aligned} \tag{1.20}$$

with the normal component given by the incompressibility condition,

$$\bar{v}_{(1)}^P = \int_Y^{+\infty} \partial_x \bar{u}_{(1)}^P dY'. \tag{1.21}$$

The role of $\bar{\mathbf{u}}_{(1)}^P$ is to correct the tangential component at the boundary generated by $\mathbf{u}_{(1)}^E$. The term $\bar{\mathbf{u}}_{(1)}^P$ admits a decomposition in a regular part and a singular part with a *gaussian* singularity less severe then the one in $\mathbf{u}_{(0)}^{inn}$.

1.2.5. The Error $\mathbf{e}(x,Y,t)$. The term $\mathbf{e}(x,Y,t)$ is an overall error that closes the asymptotic expansion. It satisfies a NS type equation with a source term and with a boundary condition that cancels the inflow generated by $\bar{\mathbf{u}}_{(1)}^P$ without generating tangential flow:

$$\begin{aligned}\partial_t\mathbf{e} + \left(\mathbf{u}_{(0)}^{NS} + \varepsilon\mathbf{u}_{(1)}^{NS}\right)\cdot\nabla\mathbf{e} + \mathbf{e}\cdot\nabla\left(\mathbf{u}_{(0)}^{NS} + \varepsilon\mathbf{u}_{(1)}^{NS}\right) + \varepsilon\mathbf{e}\cdot\nabla\mathbf{e} + \nabla p^e &= \boldsymbol{\Xi},\\ \nabla\cdot\mathbf{e} &= 0,\\ \gamma\mathbf{e} &= \left(0, -\gamma\bar{v}_{(1)}^P\right),\\ \mathbf{e}(t=0) &= 0,\end{aligned} \tag{1.22}$$

where we have defined the zero-th and first order approximation of the NS solution:

$$\mathbf{u}_{(0)}^{NS} = \mathbf{u}^E + \bar{\mathbf{u}}^P, \qquad \mathbf{u}_{(1)}^{NS} = \mathbf{u}_{(1)}^E + \bar{\mathbf{u}}_{(1)}^P.$$

The source term $\boldsymbol{\Xi}$ is generated by the discrepancies between the NS equation and the equations satisfied by the approximating terms in the asymptotic expansion. The explicit expression can be found in the Appendix A.

A key point in the proof of the regularity of $\mathbf{e}$ is that, in its equations, the singular terms multiply terms which go to zero with t or Y in a sufficiently fast way.

All the singular terms appearing in (1.10) have explicit expressions and finite L^2 norms in appropriate spaces of holomorphic functions. The expansion (1.10) implies in particular the validity of the inviscid limit in the energy norm with an $O(\sqrt{\varepsilon})$ rate, since $||\bar{\mathbf{u}}^P||_{L^2_{xy}} = \sqrt{\varepsilon}||\bar{\mathbf{u}}^P||_{L^2_{xY}}$.

1.2.6. Plan of the Paper. The organization of the paper is the following: in Sect. 2 we introduce the function spaces needed to prove the validity of the inviscid limit, and we analyze some of their properties. In Sect. 3 we introduce the abstract Cauchy–Kowalewski theorem; this theorem is essentially a fixed point result used to prove the existence of solutions to differential equations in a scale of Banach spaces. In Sect. 4 we present the main result of the paper. The other sections deal with the terms of order one and the remainder deriving from the asymptotic expansion of $\mathbf{u}^{NS}$.

2. Function Spaces

Define the strip $D(\rho)$, the angular sector $\Sigma(\theta)$ and the conoid $\Sigma(\theta, a)$ as:

$$D(\rho) = \{x \in \mathbb{C} : |Im(x)| < \rho\}, \tag{2.1}$$

$$\Sigma(\theta) = \{y \in \mathbb{C} : Re(y) > 0,\ |Im(y)| < Re(y)\tan(\theta)\}, \tag{2.2}$$

$$\begin{aligned}\Sigma(\theta, a) = &\{y \in \mathbb{C} : 0 < Re(y) \le a,\ |Im(y)| < Re(y)\tan(\theta)\}\\ &\cup\{y \in \mathbb{C} : Re(y) > a,\ |Im(y)| < a\tan(\theta)\}.\end{aligned} \tag{2.3}$$

In what follows, we shall deal with spaces of functions analytic in some of the above domains; in those spaces, the chosen paths of integration in the x variable are lines parallel to the real axis, while we shall adopt piecewise linear paths for the y and Y variables.

$$\Gamma(b) = \{x \in \mathbb{C} : Im(x) = b\}, \tag{2.4}$$

$$\begin{aligned}\Gamma(\theta', a) = &\{y \in \mathbb{C} : 0 < Re(y) \le a,\ Im(y) = Re(y)\tan(\theta')\} \cup\\ &\{y \in \mathbb{C} : Re(y) > a,\ Im(y) = a\tan(\theta')\}.\end{aligned} \tag{2.5}$$

Definition 2.1. The space $H^{l,\rho}$ is the set of all complex functions $f(x)$ such that

1. f is analytic in $D(\rho)$;
2. $|f|_{l,\rho} = \sum_{|\alpha|\le l} \sup_{|\lambda|<\rho} |\partial_x^\alpha f(\cdot + i\lambda)|_{L^2(\mathbb{R})} < \infty$.

When dealing with the 3D Navier Stokes system, x is a 2D vector, and α is a multi-index, while in the two-dimensional case, x is a scalar. The use of an L^2 norm instead of a generic L^p norm allows an important characterization for Hardy spaces, see [70] for a proof:

Theorem 2.1 (Paley–Wiener Theorem for the strip). *Let $f \in L^2(\mathbb{R})$, $\rho > 0$, denote with $\hat{f}$ the Fourier transform of f. The following are equivalent:*

1. *f is the restriction to the real line of a function holomorphic on the strip $D(\rho)$, with $\sup_{|\lambda|<\rho} |f(\cdot + i\lambda)|_{L^2(\mathbb{R})} < \infty$;*
2. *$e^{\rho|\xi|}\hat{f} \in L^2(\mathbb{R})$.*

We shall use extensively this result, and we shall often work in the Fourier variable ξ' corresponding to the physical variable x: to simplify the notation, we shall denote with the same symbol a function f and its Fourier transform with respect to the x variable, and similarly we shall use the same notation for a pseudodifferential operator and its symbol.

2.1. Function Spaces for the Outer Flow: Zero-th and First-Order Euler Equations

This section presents the appropriate function spaces for studying Euler equations and the first-order correction to the Euler equations. First, we introduce the space of functions, depending on x and y. The primary tool in constructing the solution of the Euler equations is the half-plane Leray projector, which

allows as many derivatives in x as in y. Therefore, we define the function space with (x, y) dependence as follows.

Definition 2.2. $H^{l,\rho,\theta}$ is the set of all functions $f(x, y)$ such that:

1. f is analytic in $D(\rho) \times \Sigma(\theta, a)$;
2. $|f|_{l,\rho,\theta} = \sum_{|\alpha_1|+\alpha_2 \le l} \sup_{|\theta'|<\theta} ||\partial_y^{\alpha_2} \partial_x^{\alpha_1} f|_{0,\rho}|_{L^2(\Gamma(\theta',a))} < \infty$.

The initial value for the Euler equations is given in $H^{l,\rho,\theta}$, with $l \ge 6 > 1 + d/2$ and $\theta < \pi/4$.

We now pass to introducing the function spaces with time dependence. In all the spaces defined below, the width of the analyticity domain diminishes linearly with the time t.

For a given Banach scale $\{X_\rho\}_{0<\rho\le\rho_0}$, with $X_{\rho''} \subset X_{\rho'}$ and $|\cdot|_{\rho'} \le |\cdot|_{\rho''}$ when $\rho' \le \rho'' \le \rho_0$, denote with $B^j_\beta([0, T], X_{\rho_0})$ the set of all functions f such that, for $k = 0, \ldots, j$, $\partial_t^k f$ is continuous from $[0, \tau]$ to $X_{\rho_0-\beta\tau}$ $\forall 0 < \tau \le T \le \rho_0/\beta$, with norm

$$|f|_{k,\rho,\beta} = \sum_{j=0}^{k} \sup_{t\in[0,T]} |\partial_t^j f(t)|_{\rho_0-\beta t}. \tag{2.6}$$

The following spaces are those where one can prove the existence of the outer solution.

Definition 2.3. $H^{l,\rho,\theta}_{\beta,T}$, with $T \le \min\{\rho/\beta, \theta/\beta\}$, is the space of all functions $f(x, y, t)$ such that $f \in \bigcap_{i=0}^{l} B^i_\beta([0, T], H^{l-i,\rho,\theta})$, with norm

$$|f|_{l,\rho,\theta,\beta,T} = \sum_{i=0}^{l} \sup_{t\in[0,T]} |\partial_t^i f(\cdot, \cdot, t)|_{l-i,\rho-\beta t,\theta-\beta t}. \tag{2.7}$$

The above space is the natural space for analytic solutions of the Euler equations; see Theorem 4.1 in [25].

Definition 2.4. $\tilde{H}^{l,\rho,\theta}_{\beta,T,1}$ is the set of all functions $f(t, x, y)$ such that

1. $f \in B^0_\beta([0, T], H^{l,\rho,\theta}) \cap B^1_\beta([0, T], H^{l-1,\rho,\theta})$;
2. $|f|_{l,\rho,\theta,\beta,T,1} = \sup_{t\in[0,T]} |f|_{l,\rho-\beta t,\theta-\beta t} + \sup_{t\in[0,T]} |\partial_t f|_{l-1,\rho-\beta t,\theta-\beta t} < \infty$.

The above space is the natural space where to prove the existence of the correction of Euler flow, $\mathbf{u}^E_{(1)}$: the singularity deriving from Prandtl equations implies that only one time-derivative is allowed; on the other hand, the regularity in y depends only on operators (like the Leray projector) with some symmetry between the behavior in x and y, so that we can use as many derivatives with respect to y as for x.

2.2. Function Spaces for the Inner Flow: Zero-th and First Order Boundary Layer Equations

We now introduce the space of functions analytic in x and Y, in the strip $D(\rho)$ and in the cone $\Sigma(\theta)$, respectively. Moreover, we impose exponential decay in the Y-variable so that for large Y (away from the boundary, in the outer flow), the boundary layer-type solutions do not influence the Euler solutions.

Definition 2.5. The space $K^{l,\rho,\theta,\mu}$ with $\mu > 0$ is the set of all functions $f(x, Y)$ such that

1. f is analytic in $D(\rho) \times \Sigma(\theta)$;
2. $\partial_Y^{\alpha_1} \partial_x^{\alpha_2} f \in C^0(\Sigma(\theta), H^{0,\rho})$ for any $\alpha_1 \le 2$ and $|\alpha_2| \le l - \alpha_1$;
3. $|f|_{l,\rho,\theta,\mu} = \sum_{\alpha_1\le 2} \sum_{|\alpha_2|\le l-\alpha_1} \sup_{Y\in\Sigma(\theta)} e^{\mu Re(Y)} |\partial_Y^{\alpha_1} \partial_x^{\alpha_2} f(\cdot, Y)|_{0,\rho} < \infty$.

The asymmetry on the regularity requirements in x and Y is due to the diffusion in Y that allows regularity only up to second-order Y-derivatives (unless one imposes stronger compatibility conditions).

We now introduce the functions depending on x and t. The presence of the diffusion allows regularity in t only up to first-order derivative.

Definition 2.6. $K^{l,\rho}_{\beta,T}$ is the set of all functions $f(x,t) \in B^0_\beta([0,T], H^{l,\rho}) \cap B^1_\beta([0,T], H^{l-1,\rho})$, with norm

$$|f|_{l,\rho,\beta,T} = \sum_{i=0}^{1} \sum_{|\alpha|\le l} \sup_{0\le t\le T} |\partial_t^i \partial_x^\alpha f(\cdot,t)|_{0,\rho-\beta t}. \tag{2.8}$$

The next space is where we shall prove the existence of the regular part of the solutions of the zeroth and first-order BL equations. The diffusion in Y allows regularity only up to second-order Y-derivatives and first-order t-derivatives.

Definition 2.7. The space $K^{l,\rho,\theta,\mu}_{\beta,T}$ is the set of all functions $f(x,Y,t)$ such that:

1. $f \in C^0([0,t], K^{l,\rho-\beta t,\theta-\beta t,\mu-\beta t})$ and $\partial_t \partial_x^\alpha f \in C^0([0,t], K^{0,\rho-\beta t,\theta-\beta t,\mu-\beta t})$, with $t \le T \le \min\{\rho/\beta, \theta/\beta, \mu/\beta\}$ and $|\alpha| \le l-2$;
2. $|f|_{l,\rho,\theta,\mu,\beta,T} = \sum_{\alpha_1\le 2} \sum_{\alpha_1+|\alpha_2|\le l} \sup_{0\le t\le T} |\partial_Y^{\alpha_1} \partial_x^{\alpha_2} f(\cdot,\cdot,t)|_{0,\rho-\beta t,\theta-\beta t,\mu-\beta t} + \sum_{|\alpha|\le l-2} \sup_{t\in[0,T]} |\partial_t \partial_x^\alpha f(\cdot,\cdot,t)|_{0,\rho-\beta t,\theta-\beta t,\mu-\beta t} < \infty$.

2.3. Function Spaces for the Overall Error Equation

Definition 2.8. $S^{l,\rho,m,\theta}$ is the set of all functions $f(x,Y)$ such that:

1. f is analytic inside $D(\rho) \times \Sigma(\theta, \frac{a}{\varepsilon})$ and in $L^2\left(\Gamma(\rho') \times \Gamma(\theta', \frac{a}{\varepsilon}\right)$ $\forall |\rho'| < \rho$, $\forall |\theta'| < \theta$;
2. $|f|_{l,\rho,m,\theta} = \sum_{k\le m} |\partial_Y^k f|_{l-k,\rho,\theta} < \infty$.

Definition 2.9. $S^{l,\rho,m,\theta}_{\beta,T}$ the set of all functions $f(t,x,Y)$ such that:

1. $f \in B^0_\beta\left([0,T], S^{l,\rho,m,\theta}\right)$;
2. $|f|_{l,\rho,m,\theta,\beta,T} = \sup_{t\in[0,T]} |f|_{l,\rho-\beta t,m,\theta-\beta t} < \infty$.

In the Eq. (7.1) of Sect. 7, we further decompose the error $\mathbf{e}$: the space $S^{l,\rho,1,\theta}_{\beta,T}$ is the functional setting in which the ACK theorem can be applied for the term $\mathbf{e}^*$, and the image of $\mathbf{e}^*$ under the operator $\mathcal{N}^*$ defined in Sect. 7.3 is in the space $L^{l,\rho,\theta}_{\beta,T}$, defined below.

Definition 2.10. $L^{l,\rho,\theta}_{\beta,T}$ is the set of functions $f(x,Y,t)$ such that

$f \in C^0([0,t], S^{l,\rho-\beta t,0,\theta-\beta t})$, $\partial_Y f$, $\partial_{YY} f$, $\partial_t f \in C^0([0,t], S^{l-2,\rho-\beta t,0,\theta-\beta t})$ $\forall t \le T \le \min\{\rho/\beta, 0, \theta\}$, with norm

$$\begin{aligned} |f|_{l,\rho,\theta,\beta,T} = &\sum_{j=0}^{1} \sum_{\alpha\le l-2j} \sup_{t\in[0,T]} |\partial_t^j \partial_x^\alpha f(\cdot,\cdot,t)|_{0,\rho-\beta t,\theta-\beta t} \\ &+ \sum_{1\le\alpha_1\le 2} \sum_{|\alpha_2|\le l-2} \sup_{0\le t\le T} |\partial_Y^{\alpha_1} \partial_x^{\alpha_2} f(\cdot,\cdot,t)|_{0,\rho-\beta t,\theta-\beta t}. \end{aligned} \tag{2.9}$$

2.4. Algebra Properties and Cauchy Estimates

Let d' be the dimension of x, i.e. $d' = 1$ or $d' = 2$ when solving NS equations in the half-plane or in the half-space respectively; moreover, call $d = d' + 1$

Lemma 2.1. *Let $u(x)$, $v(x)$ be in $H^{l,\rho}$, with $l > d'/2$. Then $uv \in H^{l,\rho}$, with*

$$|uv|_{l,\rho} \leq C|u|_{l,\rho}|v|_{l,\rho}. \tag{2.10}$$

Lemma 2.2. *Let $u(x,y)$, $v(x,y)$ be in $H^{l,\rho,\theta}$, with $l > d/2$. Then $uv \in H^{l,\rho,\theta}$, with*

$$|uv|_{l,\rho,\theta} \leq C|u|_{l,\rho,\theta}|v|_{l,\rho,\theta}. \tag{2.11}$$

One can easily verify both lemmas using the Paley–Wiener theorem and the usual argument used to prove the algebra properties of Sobolev spaces.

The use of complex variables allows to have simple estimates for the norms of the derivatives: using Cauchy formula for derivatives, we immediately have that

$$|\partial_x^k u|_{0,\rho'} \leq \frac{k!}{(\rho-\rho')^k}|u|_{0,\rho}. \tag{2.12}$$

For functions holomorphic in a cone or in a conoid, we cannot bound the norm of the derivatives with the norm of the function in a larger cone with the same vertex, because we cannot use the Cauchy formula for derivatives with a fixed radius in all the path of integration. If we use a radius linearly growing with $|y|$ for $Re(y) < a$, constant for $Re(y) \geq a$, we easily obtain for a function holomorphic in $\Sigma(\theta, a)$ that, $\forall p \in [1, +\infty]$

$$\sup_{|\theta'|<\theta} |\min\{a, |y|\}\partial_y f(y)|_{L^p(\Gamma(\theta',a))} \leq \frac{C}{\bar{\theta}-\theta} \sup_{|\theta'|<\bar{\theta}} |f|_{L^p(\Gamma(\theta',a))}. \tag{2.13}$$

From Eq. (2.13), we obtain the following lemma [25], which is crucial for the estimate of the nonlinear term in the Euler equations.

Lemma 2.3. *Assume that $u,v \in H^{l,\rho,\theta}$, with $u(y=0)=0$ and $l > d/2+1$. Then for any $\theta' < \theta$ we have*

$$|u\partial_y v|_{l,\rho,\theta'} \leq C|u|_{l,\rho,\theta'}\frac{|v|_{l,\rho,\theta}}{\theta-\theta'}. \tag{2.14}$$

The Prandtl equations with initial data exponentially decaying in Y need the following estimates, which can be found in [25].

Lemma 2.4. *Let $f \in K^{l,\rho',\theta'',\mu''}$. Then*

$$|min\{1, |Y|\}\partial_Y f|_{l,\rho',\theta',\mu'} \leq \frac{|f|_{l,\rho',\theta'',\mu'}}{\theta''-\theta'} + \mu'|f|_{l,\rho',\theta',\mu'} \tag{2.15}$$

$$|Y\partial_Y f|_{l,\rho',\theta',\mu'} \leq \frac{|f|_{l,\rho',\theta'',\mu'}}{\theta''-\theta'} + \mu'\frac{|f|_{l,\rho',\theta',\mu''}}{\mu''-\mu'} + |f|_{l,\rho',\theta',\mu'} \tag{2.16}$$

Notice that a different exponential decay is needed in order to estimate the product of $\partial_Y f$ with a linearly increasing function. The Prandtl equations are still well posed when the initial data decay with Y with a polynomial rate, see [23], and also [21]; in this case, using a radius linearly growing with $|Y|$ in the Cauchy formula for derivatives, it is easy to obtain

$$\sup_{Y\in\Sigma(\theta)} (1+|Y|)^\alpha |Y\partial_Y f(Y)| \leq \frac{C}{\bar{\theta}-\theta} \sup_{Y\in\Sigma(\bar{\theta})} (1+|Y|)^\alpha |f(Y)|, \tag{2.17}$$

which means that, unlike the exponential case, there is no need to change the polynomial rate of decay.

2.5. Paths of Integration

In the present paper, we shall solve equations involving heat operators in several instances: for example, boundary layer equations, where diffusion in the normal Y-variable appears, or the error equation, where diffusion is both in the x- and Y-variable is present. The method we shall use is based on representing the solutions utilizing the convolution with the appropriate Gaussian. If the integrand is holomorphic,

one can deform the integration path, choosing the most convenient one. However, when estimating the solution and passing the modulus inside the integral, the integrand is no longer holomorphic, and the integral depends on the particular path of integration; therefore, a careful a priori choice of the integration path is helpful in many estimates. Taking the convolution with respect to x, the best choice for the path is the one that makes the argument of the Gaussian real, i.e., the path $x' \in \Gamma(Im(x))$. On the other hand, for the convolution in the Y variable between a Gaussian and a function analytic in a conoid, given that the domain of analyticity shrinks near the origin, one cannot take $Im(Y') = Im(Y)$; therefore, we choose the path consisting of the segment connecting the origin and Y and the half-line from Y to $+\infty + iIm(Y)$ parallel to the real axis. Said differently, one has:

$$Y' = \begin{cases} \frac{Y}{|Y|} r & r \in [0, |Y|] \\ r + iY_{im} & r \in [Y_r, +\infty[\end{cases}. \tag{2.18}$$

In the first part of the above path of integration one has

$$\left| e^{-\frac{(Y-Y')^2}{4(t-s)}} \right| = e^{-\frac{(|Y|-r)^2}{4(t-s)} \frac{Y_r^2 - Y_{im}^2}{|Y|^2}} \leq e^{-\frac{(|Y|-r)^2}{4(t-s)} \cos(2\theta)}; \tag{2.19}$$

to get the last inequality we have used

$$\frac{Y_r^2 - Y_{im}^2}{Y_r^2 + Y_{im}^2} = \frac{Y_r^2(1 - \tan^2(\theta_Y))}{Y_r^2(1 + \tan^2(\theta_Y))} = \cos(2\theta_Y) \geq \cos(2\theta), \tag{2.20}$$

where θ_Y is the argument of the complex number Y. Analogously, one can show that

$$\left| e^{-\frac{(Y+Y')^2}{4(t-s)}} \right| \leq e^{-\frac{(|Y|+r)^2}{4(t-s)} \cos(2\theta)}, \tag{2.21}$$

$$\left| e^{-\frac{(Y')^2}{4s}} \right| \leq e^{-\frac{r^2}{4s} \cos(2\theta)}, \tag{2.22}$$

where the latter inequality will be used for estimating the singular term u^S.

In the second part of the path (2.18), we have that

$$\left| e^{-\frac{(Y-Y')^2}{4(t-s)}} \right| = e^{-\frac{(Y_r - r)^2}{4(t-s)}}, \tag{2.23}$$

while

$$\left| e^{-\frac{(Y+Y')^2}{4(t-s)}} \right| = e^{\frac{-(Y_r+r)^2 + 4Y_{im}^2}{4(t-s)}} \leq e^{-(1-\tan^2(\theta)) \frac{(Y_r+r)^2}{4(t-s)}}, \tag{2.24}$$

where we have used

$$(2Y_{im})^2 \leq \tan^2(\theta_Y)(2Y_r)^2 \leq \tan^2(\theta)(Y_r + r)^2.$$

Finally, since $(Y_{im})^2 \leq \tan^2(\theta) r^2$, we have

$$\left| e^{-\frac{(Y')^2}{4s}} \right| \leq e^{-(1-\tan^2(\theta)) \frac{r^2}{4s}}. \tag{2.25}$$

Therefore, we conclude that after a change of variable, one can deduce the estimates involving the complex gaussian on the paths (2.18) by estimates involving the real gaussian; these estimates are up constants that blow up when $\theta \to \pi/4$.

For a function analytic in a cone, we take, as the integration path, the half line which starts at the origin and passes through Y: therefore, the estimates (2.19), (2.21) and (2.22) are valid along all the path.

3. The Abstract Cauchy Kowalewski Theorem

Consider the equation

$$u + F(t, u) = 0 \quad t \in [0, T]. \tag{3.1}$$

Let $\{X_\rho\}_{\rho\in]0,\rho_0]}$ a scale of Banach spaces, with $X_{\rho'} \subseteq X_{\rho''}$ and $|\cdot|_{\rho'} \leq |\cdot|_{\rho''}$ when $\rho'' \leq \rho' \leq \rho$

Theorem 3.1. (ACK theorem) *Suppose that $\exists R > 0$, $\rho_0 > 0$, $\beta_0 > 0$ such that if $0 < \tau \leq T \leq \rho_0/\beta_0$ the following properties hold:*

1. *$\forall 0 < \rho' < \rho \leq \rho_0 - \beta_0\tau$ and $\forall u$ such that $\{u \in X_\rho : \sup_{t\in[0,\tau]} |u(t)|_\rho \leq R\}$ the map $F(u, t) : [0, \tau] \mapsto X_{\rho'}$ is continuous;*
2. *$\forall 0 < \rho \leq \rho_0 - \beta_0\tau$ the function $F(t, 0) : [0, \tau] \mapsto \{u \in X_\rho : \sup_{t\in[0,\tau]} |u(t)|_\rho \leq R\}$ is continuous and*
$$|F(t, 0)|_\rho \leq R_0 < R; \tag{3.2}$$
3. *for any $\beta \leq \beta_0$, $0 < \rho' < \rho(s) \leq \rho_0 - \beta_0 s$ and u^1 and $u^2 \in \{u : u(t) \in X_{\rho_0-\beta t} \leq R\}$*
$$|F(t, u^1) - F(t, u^2)|_{\rho'} \leq C\left[\int_0^t \frac{|u^1 - u^2|_{\rho(s)}}{\rho(s) - \rho'} + \frac{|u^1 - u^2|_{\rho'}}{(t - s)^{\alpha_1}} + \frac{1}{t^{\alpha_2}}\int_0^t |u^1 - u^2|_{\rho'} ds\right], \tag{3.3}$$
with $\alpha_1 < 1$, $\alpha_2 < 1$ and C independent of $t, \tau, u^1, u^2, \beta, \rho', \rho(s)$.

Then $\exists\beta > \beta_0$ such that Eq. (3.1) has a unique solution u such that $\forall\rho \in]0, \rho_0[$ $u(t) \in X_\rho$ $\forall t \in [0, (\rho_0 - \rho)/\beta]$; moreover $\sup_{\rho+\beta t<\rho_0} |u(t)|_\rho \leq R$.

The spaces introduced in Sect. 2 are Banach scales with respect to the parameters defining the complex domains and the exponential decay. The use of an analytic setting is mainly due to Prandtl's equations since the well-posedness results available for these equations use either an analytic setting [23] and [24] or some monotonicity assumption [16]. A key point of the ACK theorem is to prove that F is a contraction in an auxiliary Banach space: one can prove this with slight modifications of the proof in [24].

4. The Main Result

We now state the main result of the paper. The informal statement is in Theorem 1.10, in Sect. 1.2, where the reader can find a detailed explanation of the meaning of the different terms appearing in the asymptotic expansion (4.1) below.

Theorem 4.1. *Assume that $\boldsymbol{u}_0 \in H^{l,\rho,\theta}$, with $\gamma_n \boldsymbol{u}_0 = 0$, $l \geq 6$. Then, for any $\mu > 0$, there exist $\bar\rho < \rho$, $\bar\theta < \theta$, $\bar\beta > 0$, all independent of ν, such that the solution of the Navier–Stokes Eq. (1.1) can be written as*

$$\boldsymbol{u} = \boldsymbol{u}^E + \bar{\boldsymbol{u}}^P + \varepsilon[\boldsymbol{u}^E_{(1)} + \bar{\boldsymbol{u}}^P_{(1)} + \boldsymbol{e}], \tag{4.1}$$

where:

1. *The term $\boldsymbol{u}^E \in H^{l,\bar\rho,\bar\theta}_{\bar\beta,T}$ is the solution of the Euler equations (1.4).*
2. *The term $\bar{\boldsymbol{u}}^P$ is the modified Prandtl solution given by Eq. (1.8) and (1.9). The following decomposition in regular and singular part holds:*
$$\bar{\boldsymbol{u}}^P = \bar{\boldsymbol{u}}^R + \bar{\boldsymbol{u}}^S,$$
where the regular term $\bar{\boldsymbol{u}}^R \in K^{l,\bar\rho,\bar\theta,\mu}_{\bar\beta,T}$, and the singular term $\boldsymbol{u}^S$ is given by (1.14) and (1.15).

3. *The term* $\boldsymbol{u}^E_{(1)}$ *is the first order correction to the inviscid flow solving the system* (1.18)*. The following decomposition in regular and singular part holds:*
$$\boldsymbol{u}^E_{(1)} = \boldsymbol{u}^{ER}_{(1)} + \sqrt{t}\boldsymbol{w}^S_b(x,y),$$
where $\boldsymbol{u}^{ER}_{(1)} \in \tilde{H}^{l,\bar{\rho},\bar{\theta}}_{\bar{\beta},T,1}$ *and* $\boldsymbol{w}^S_b \in H^{l,\bar{\rho},\bar{\theta}}$ *is given by* (5.3)*.*
4. $\bar{\boldsymbol{u}}^P_{(1)} = \left(\bar{u}^P_{(1)}, \varepsilon\bar{v}^P_{(1)}\right)$ *is the first order correction to the boundary layer flow;* $\bar{u}^P_{(1)}$ *solves the linear heat Eq.* (1.20)*, while* $\bar{v}^P_{(1)}$ *is given by the incompressibility condition* (1.21)*. The following decomposition in regular and singular part holds:*
$$\bar{\boldsymbol{u}}^P_{(1)} = \bar{\boldsymbol{u}}^R_{(1)} + \bar{\boldsymbol{u}}^S_{(1)} \tag{4.2}$$
where the regular part $\bar{\boldsymbol{u}}^R_{(1)} \in K^{l,\bar{\rho},\bar{\theta},\mu}_{\bar{\beta},T}$ *and the singular part* $\bar{\boldsymbol{u}}^S_{(1)}$ *is given by* (6.1)*.*
5. *The term* $\boldsymbol{e}$ *is an overall error that closes the asymptotic procedure with* $\boldsymbol{e} \in L^{l,\bar{\rho},\bar{\theta}}_{\bar{\beta},T}$*.*

The result concerning the Euler flow can be found in [25] as Theorem 4.1. The result for $\bar{\mathbf{u}}^P$ can be easily obtained with a slight modification of the argument used in [21].

Remark 4.1. One can give a more detailed description of the structure of $\bar{\mathbf{u}}^P$ as follows. The solution u^P of Prandtl equations with non compatible data can be written as the sum of

1. a singular term u^S given by (1.14);
2. the solution u^D of Prandtl's equations with compatible data ($\gamma u^D = u_{00}$) and an initial datum given by the initial value of the Euler flow at the boundary, which means that u_D approaches the Euler flow for $Y \to +\infty$ at any time with any exponential rate;
3. an interaction term, which is exponentially decaying: indeed, the interaction part can be written in terms of the heat operator E_2 for the half space with zero boundary and initial condition and a forcing term, so in Y we have a convolution between a gaussian and some terms deriving from $\tilde{u}^D$ and u^S. Now we have
$$e^{\mu Re(Y)} = e^{\mu Re(Y-Y')}e^{\mu Re(Y')}, \tag{4.3}$$
with $e^{\mu Re(Y')}\tilde{u}^D$ bounded, while
$$\begin{aligned} e^{\mu Re(Y-Y')}\left|\frac{e^{-\frac{(Y-Y')^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}}\right| &\le e^{\mu Re(Y-Y')}e^{-c\frac{Re(Y-Y')^2}{8T}}\frac{e^{-\frac{cRe(Y-Y')^2}{8(t-s)}}}{\sqrt{4\pi(t-s)}} \le \\ &\le C\frac{e^{-\frac{cRe(Y-Y')^2}{8(t-s)}}}{\sqrt{4\pi(t-s)}}, \end{aligned} \tag{4.4}$$
with c depending on θ. A similar argument applies to $e^{\mu Re(Y)}u^S$.

The result for $\bar{\mathbf{u}}^P$ can be obtained with minor modifications of the argument in [21], taking into account that the decay in Y is exponential (rather than polynomial) and that we have analyticity also in the normal variable.

Remark 4.2. To prove theorem 4.1, we use the fact that when we reduce the strip of x-analyticity, we can get as much x-regularity as needed. However, while we can obtain arbitrary regularity in the tangential variable, this does not hold for the normal variable since the reduction of the cone of analyticity does not provide any additional regularity: the regularity in the normal variable does not exceed the regularity of the initial datum.

The construction of $\mathbf{u}^E$ and $\bar{\mathbf{u}}^P$ already appears in the literature, see [25] and [21] respectively; therefore, in the rest of the paper, we shall construct the remaining terms in the asymptotic expansion (4.1).

5. Correction to the Euler Flow

The Prandtl solution generates an $O(\varepsilon)$ inflow at the boundary; one introduces the first-order correction to the Euler flow to cancel this inflow; given that the Prandtl solution has a regular and a singular part, in (1.17), we have split, in (1.17), the inflow g in singular and regular parts, g^S and g^R, respectively. The first-order Euler equations (1.18) are linear; therefore, by the superposition principle, we can decompose the first order Euler correction $\bar{\mathbf{u}}^E_{(1)}$ as in (1.19), i.e., as singular and regular parts. The singular part $\boldsymbol{w}^S$ solves

$$\begin{aligned} \partial_t \boldsymbol{w}^S + \boldsymbol{w}^S \cdot \nabla \mathbf{u}^E + \mathbf{u}^E \cdot \nabla \boldsymbol{w}^S + \nabla p^{w^S} &= \mathbf{0}, \\ \nabla \cdot \boldsymbol{w}^S &= 0, \\ \gamma_n \boldsymbol{w}^S = g^S &= -\sqrt{\frac{4}{\pi}} \sqrt{t} \partial_x u_{00}(x), \\ \boldsymbol{w}^S\big|_{t=0} &= \mathbf{0}, \end{aligned} \tag{5.1}$$

while the regular part $\boldsymbol{w}^R$ solves the system obtained by (5.1) replacing g^S with g^R. The regularity of g^R is the same that the boundary value of the normal part of the Prandtl's correction would have in the compatible case; therefore, we have $\boldsymbol{w}^R \in \tilde{H}^{l-1,\bar{\rho},\bar{\theta}}_{\bar{\beta},T,1}$.

Remark 5.1. The regularity in the y-variable is greater than the one stated in [35]; however, for this term, the regularity in the normal variable depends only on "symmetric" operators, which allow a number of derivatives in y equal to the ones that have not been used in x; this enhanced regularity will allow us to prove that some terms that are zero at time $t = 0$ are $O(t)$, which will be helpful for the estimates of some nonlinear terms in the equation of $\mathbf{e}$.

We now pass to the analysis of the singular part $\boldsymbol{w}^S$. If one denotes by ξ' the Fourier variable corresponding to x, it is useful to write the solution as

$$\boldsymbol{w}^S(t, \xi', y) = \sqrt{t} \boldsymbol{w}^S_b + I_{\sqrt{t}} \boldsymbol{w}^{\boldsymbol{S}^*}, \tag{5.2}$$

where

$$\boldsymbol{w}^S_b(\xi', y) = C_1(-i\frac{\xi'}{|\xi'|}, 1) e^{-|\xi'|y} i\xi' u_{00}(\xi') \tag{5.3}$$

is divergence-free and takes into account the normal inflow. The operator $I_{\sqrt{t}}$ is a weighted integration in time:

$$I_{\sqrt{t}} \boldsymbol{w}^{\boldsymbol{S}^*} = \int_0^t \sqrt{\tau} \boldsymbol{w}^{S^*}(\tau, \xi', y) d\tau. \tag{5.4}$$

The rest of this section is devoted to constructing $\boldsymbol{w}^{\boldsymbol{S}^*}$. Call P the Leray projector (projection on the space of divergence-free L^2 functions with zero normal components at the boundary); substituting the ansatz (5.2) into (5.1), and then applying the Leray projector, one derives the following equation for $\boldsymbol{w}^{\boldsymbol{S}^*}$,

$$\boldsymbol{w}^{S^*} + P\left[\frac{I_{\sqrt{t}} \boldsymbol{w}^{\boldsymbol{S}^*}}{\sqrt{t}} \cdot \nabla \mathbf{u}^E + \mathbf{u}^E \cdot \nabla \frac{I_{\sqrt{t}} \boldsymbol{w}^{\boldsymbol{S}^*}}{\sqrt{t}} + \boldsymbol{w}^S_b \cdot \nabla \mathbf{u}^E + \mathbf{u}^E \cdot \nabla \boldsymbol{w}^S_b\right] = \mathbf{0}. \tag{5.5}$$

The form of the above equation is suitable for applying the abstract Cauchy–Kowalewski theorem. Since P is a bounded operator in $H^{l,\rho,\theta}$ (see [25]) and commutes with time derivatives, we can estimate $\mathbf{v}$ instead of $P\mathbf{v}$ for any $\mathbf{v}$. We shall show that, assuming that $u_{00} \in H^{l+1,\rho}$ and $\mathbf{u}^E \in H^{l,\rho,\theta}_{\beta,T}$, then $\boldsymbol{w}^{S*} \in \tilde{H}^{l-1,\rho,\theta}_{\bar{\beta},T,1}$ for some $\bar{\beta} > \beta$: such a regularity for u_{00} can always be obtained from the trace of a function $u_0 \in H^{l,\rho,\theta}$ by reducing the strip of analyticity.

5.1. The Forcing Term

For any $\rho' < \rho - \beta t$, $\theta' \leq \theta - \beta t$, we use the algebra property (2.11) to obtain

$$|\boldsymbol{w}_b^S \cdot \nabla \mathbf{u}^E|_{l-1,\rho',\theta'} \leq C|\boldsymbol{w}_b^S|_{l-1,\rho',\theta'}|\mathbf{u}^E|_{l,\rho',\theta'} \leq C|u_{00}|_{l,\rho}|\mathbf{u}^E|_{l,\rho,\theta,\beta,T} \tag{5.6}$$

and similarly

$$|\mathbf{u}^E \cdot \nabla \boldsymbol{w}_b^S|_{l-1,\rho',\theta'} \leq C|u_{00}|_{l+1,\rho}|\mathbf{u}^E|_{l-1,\rho,\theta,\beta,T}. \tag{5.7}$$

5.2. Quasi Contractiveness

For the term $\mathbf{u}^E \cdot \nabla \frac{I_{\sqrt{t}}\boldsymbol{w}^{S^*}}{\sqrt{t}}$, we have

$$\begin{aligned} \left| u^E \partial_x \frac{I_{\sqrt{t}}\boldsymbol{w}^{S*}}{\sqrt{t}} \right|_{l-1,\rho',\theta'} &\leq C|u^E|_{l-1,\rho',\theta'} \int_0^t \frac{\sqrt{\tau}}{\sqrt{t}} |\partial_x \boldsymbol{w}^{S^*}|_{l-1,\rho',\theta'} d\tau \leq \\ &\leq C|u^E|_{l-1,\rho,\theta,\beta,T} \int_0^t \frac{|\boldsymbol{w}^{S*}|_{l-1,\rho(s),\theta'}}{\rho(s)-\rho'} ds, \end{aligned} \tag{5.8}$$

while, with an argument similar to the one used in the proof of lemma 2.3, since $\partial_y \frac{I_{\sqrt{t}}\boldsymbol{w}^{S^*}}{\sqrt{t}}$ multiplies v^E, which goes to zero linearly as y approaches zero, we have

$$\left| v^E \partial_y \frac{I_{\sqrt{t}} w^{S^*}}{\sqrt{t}} \right|_{l-1,\rho',\theta'} \leq C|v^E|_{l-1,\rho,\theta,\beta,T} \int_0^t \frac{|\boldsymbol{w}^{S^*}|_{l-1,\rho',\theta(s)}}{\theta(s)-\theta'} ds. \tag{5.9}$$

The term $\frac{I_{\sqrt{t}}\boldsymbol{w}^{S^*}}{\sqrt{t}} \cdot \nabla \mathbf{u}^E$ is easier to estimate, since no Cauchy estimate is needed

$$|\frac{I_{\sqrt{t}}\boldsymbol{w}^{S^*}}{\sqrt{t}} \cdot \nabla \mathbf{u}^E| \leq C|\mathbf{u}^E|_{l,\rho,\theta,\beta,T} \int_0^t |\boldsymbol{w}^{S*}|_{l-1,\rho',\theta'} ds. \tag{5.10}$$

By the ACK theorem, we obtain the existence of a unique solution of Eq. (5.5) in $\tilde{H}^{l-1,\rho,\theta}_{\bar{\beta},T,0}$; taking the time derivative of that equation, it is easy to see that $\boldsymbol{w}^{S*} \in \tilde{H}^{l-1,\rho,\theta}_{\bar{\beta},T,1}$.

6. Boundary Layer Corrector

The first order correction $\bar{\mathbf{u}}^P_{(1)} = (u^P_{(1)}, \varepsilon \bar{v}^P_{(1)})$ is the sum (see (4.2)) of the solution $\bar{\mathbf{u}}^S_{(1)}$ of the system

$$\begin{aligned} (\partial_t - \partial_{YY}) u^S_{(1)} &= u^S, \\ \gamma u^S_{(1)} &= -\gamma \boldsymbol{w}^S, \\ u^S_{(1)}(t=0) &= 0, \\ \bar{v}^S_{(1)} &= \varepsilon \int_Y^{+\infty} \partial_x u^S dY', \end{aligned} \tag{6.1}$$

and the solution $\bar{\mathbf{u}}^R_{(1)}$ of the system obtained by (6.1) replacing $\boldsymbol{w}^S$ with $\boldsymbol{w}^R$ We have $\bar{\mathbf{u}}^R_{(1)} \in K^{l,\bar{\rho},\bar{\theta},\mu}_{\bar{\beta},T}$, see [35] for a proof. In Fourier terms, $u^S_{(1)}$ is given by

$$u^S_{(1)} = -C_1|\xi'|u_{00}(\xi')F_{1/2}(t,Y), \tag{6.2}$$

where

$$F_{1/2} = \int_0^t \frac{Y}{2(t-s)} \frac{e^{-\frac{Y^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}} \sqrt{s} ds = \frac{1}{\sqrt{\pi}} \int_{\frac{Y}{2\sqrt{t}}}^{+\infty} e^{-\sigma^2} \sqrt{t - \frac{Y^2}{4\sigma^2}} d\sigma. \tag{6.3}$$

We estimate $u^S_{(1)}$ in terms of F: we have

$$|F_{1/2}| \leq Ce^{-\frac{Y^2}{8t}} \sqrt{t}. \tag{6.4}$$

The derivative with respect to Y is

$$\partial_Y F_{1/2} = -\frac{1}{\sqrt{\pi}} \int_{\frac{Y}{2\sqrt{t}}}^{+\infty} e^{-\sigma^2} \frac{Y}{4\sigma^2} \frac{d\sigma}{\sqrt{t - \frac{Y^2}{4\sigma^2}}} = -\frac{2}{\sqrt{\pi}} \int_0^t e^{-\frac{Y^2}{4(t-s)}} \frac{ds}{\sqrt{s}\sqrt{(t-s)}}, \tag{6.5}$$

so

$$|\partial_Y F_{1/2}| \leq Ce^{-\frac{Y^2}{4t}}. \tag{6.6}$$

The term $\partial_{YY} F_{1/2}$ is singular near the boundary (although $Y\partial_{YY} F_{1/2}$ is not).
The normal component $\bar{v}^S_{(1)}$ has a better regularity with respect to t and Y; we have

$$|\partial_Y^j \bar{v}^S_{(1)}|_{k,\rho} \leq C|u_{00}|_{k+2,\rho} e^{-c\frac{|Y|^2}{t}} t^{1-\frac{j}{2}} \tag{6.7}$$

7. The Remainder e

The remainder $\mathbf{e}$ satisfies the system (1.22), which is Navier–Stokes type system with a source term and non-homogeneous boundary conditions. It is important to notice that the source term contains $\partial_{xx} u^S$, the most singular term. Therefore, we decompose the error $\mathbf{e}$ as

$$\mathbf{e} = \mathcal{N}^* \mathbf{e}^* + \boldsymbol{\sigma} + \mathbf{h}. \tag{7.1}$$

In the above decomposition, the tangential part of $\mathbf{h} = (h', \varepsilon h_n)$ takes care of the most singular term $\partial_{xx} u^S$; in fact, h' solves the heat equation in the half-space with forcing term $\varepsilon \partial_{xx} u^S$, and homogeneous boundary and initial conditions, see the system (7.2); imposing the incompressibility condition and the decay at infinity in Y, one obtains the normal component h_n, see (7.6). Therefore, h_n has a nonzero trace at the boundary. The construction of $\mathbf{h}$ will be accomplished in Sect. 7.1, while the fact that $\mathbf{h} \in L^{l,\bar{\rho},\theta}_{\beta,T}$ is stated in Proposition 7.2.

The term $\boldsymbol{\sigma}$ solves the Stokes equations with boundary conditions; see the system (7.8) below. We have introduced the term $\boldsymbol{\sigma}$ to take into account the boundary conditions of $\mathbf{e}$, deriving from the boundary layer corrector, and the boundary conditions generated by $\mathbf{h}$. The construction of $\boldsymbol{\sigma}$ will be accomplished in Sect. 7.2, while the fact that $\boldsymbol{\sigma} \in L^{l,\bar{\rho},\theta}_{\beta,T}$ is stated in Proposition 7.3.

Finally, $\mathcal{N}^* \mathbf{e}^*$ solves the Stokes equations with homogeneous boundary conditions and with forcing term $\mathbf{e}^*$, see the system (7.15). The Navier–Stokes operator $\mathcal{N}^*$ can be defined explicitly in terms of the Leray projector and heat operator; see the formula (7.16). The construction of the operator $\mathcal{N}^*$, and the necessary estimates, will be presented in Sect. 7.3

To construct $\mathbf{e}^*$ we shall use the ACK theorem in the $S^{l,\rho,1,\theta}$ setting: given that $u^S \notin S^{l,\rho,1,\theta}$, this has led us to isolate the effect of $\varepsilon \partial_{xx} u^S$ introducing the term $\mathbf{h}$. We shall construct $\mathbf{e}^*$

Finally, $\mathcal{N}^* \mathbf{e}^*$ solves the Stokes equations with homogeneous boundary conditions and with forcing term $\mathbf{e}^*$, see the system (7.15). The Navier–Stokes operator $\mathcal{N}^*$ can be defined explicitly in terms of the Leray projector and heat operator; see the formula (7.16). The construction of the operator $\mathcal{N}^*$, and the necessary estimates, will be presented in Sect. 7.3

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For the compatible case, in [35], the authors showed that $\mathbf{e} \in L^{l,\rho,\theta}_{\beta,T}$ by proving that both $\boldsymbol{\sigma}$ and $\mathbf{e}^*$ were in that space. In our case, $\boldsymbol{\sigma}$ is still in $L^{l,\rho,\theta}_{\beta,T}$; however, we shall see that $\mathbf{e}^* \in S^{l,\rho,1,\theta}$ only; nevertheless, we shall prove that the image under $\mathcal{N}^*$ is still in $L^{l,\rho,\theta}_{\beta,T}$.

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7.1. Heat Term

The tangential part h' of $\mathbf{h}$ satisfies

$$\begin{aligned} (\partial_t - \partial_{YY})h' &= \varepsilon \partial_{xx} u^S, \\ \gamma h' &= 0, \\ h'(t=0) &= 0. \end{aligned} \tag{7.2}$$

Given that u^S is given by (1.14), one can write the explicit expression of h':

$$h' = \varepsilon \partial_{xx} u_{00} F(t, Y), \tag{7.3}$$

where

$$F = \frac{2}{\sqrt{\pi}} \int_0^t ds \int_0^{+\infty} (E_0^- - E_0^+) \int_{\frac{Y'}{2\sqrt{s}}}^{+\infty} e^{-\sigma^2} d\sigma dY', \tag{7.4}$$

and E_0^- and E_0^+ are:

$$E_0^- = \frac{e^{-\frac{(Y-Y')^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}}; \; E_0^+ = \frac{e^{-\frac{(Y+Y')^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}}. \tag{7.5}$$

The normal part of $\mathbf{h}$ is obtained through the incompressibility condition expressed in the Y variable; thus, it is an $O(\varepsilon)$ with respect to h'. We shall denote the normal part of $\mathbf{h}$ as εh_n to stress this fact. Imposing the decay in Y of h_n, one writes

$$h_n = \int_Y^{+\infty} \partial_x h' dY'. \tag{7.6}$$

One can give the following estimate:

Proposition 7.1. *For $j = 0, 1, 2$, $\forall \mu > 0$, we have*

$$\begin{aligned} \sup_{Y \in \Sigma(\theta)} e^{\mu Re(Y)} |\partial_Y^j h'|_{l-2-j,\rho} &\leq C\varepsilon t^{\frac{2-j}{2}} |u_{00}|_{l-j,\rho}, \\ \sup_{Y \in \Sigma(\theta)} e^{\mu Re(Y)} |\partial_Y^j h_n|_{l-3-j,\rho} &\leq C\varepsilon t^{\frac{3-j}{2}} |u_{00}|_{l-j,\rho}. \end{aligned} \tag{7.7}$$

The proof of the proposition is given in Appendix Appendix B. The exponential decay in the Y variable, stated in proposition 7.1, implies the boundedness in L^2_Y; moreover every $u_0 \in H^{l,\rho,\theta}$ has trace $u_{00} \in H^{l-1,\rho} \subset H^{k,\bar{\rho}}$ $\forall k \geq l$, $\forall \bar{\rho} < \rho$. Therefore, we obtain the following result:

Proposition 7.2. *Assume $u_0 \in H^{l,\rho,\theta}$: then $\forall \bar{\rho} < \rho$, $\forall T$, $\forall \beta$, we have $\boldsymbol{h} \in L^{l,\bar{\rho},\theta}_{\beta,T}$.*

The estimates given in Proposition 7.1 imply an L^∞_Y boundedness, which will be useful in the treatment of the nonlinear terms in the equation of $\mathbf{e}^*$.

7.2. Boundary Value of the Error

The term $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ is needed to cancel the boundary value of h_n and $\bar{v}^P_{(1)}$; therefore, it satisfies

$$\begin{aligned} (\partial_t - \partial_{YY})\boldsymbol{\sigma} + \nabla\phi &= 0, \\ \nabla \cdot \boldsymbol{\sigma} &= 0, \\ \gamma\boldsymbol{\sigma} &= (0, \varepsilon G), \\ \boldsymbol{\sigma}(t=0) &= \mathbf{0}, \end{aligned} \tag{7.8}$$

where

$$G = -\gamma\bar{v}^R_{(1)} - \gamma\bar{v}^P_{(1)} - \gamma h_n = \partial_x \int_0^{+\infty} \left(u^R_{(1)} + u^P_{(1)} + h'\right) dY = \partial_x \tilde{G}. \tag{7.9}$$

The above system is a Stokes problem with boundary datum and homogeneous initial datum. In [35] one can find the procedure to solve such a problem; the solution writes as

$$\begin{aligned} \sigma_1 &= \varepsilon N' E_1 G - N'\varepsilon \int_0^Y \varepsilon|\xi'| e^{-\varepsilon|\xi'|(Y-Y')} E_1 G dY' - N' e^{-\varepsilon|\xi'|Y} G, \\ \sigma_2 &= \varepsilon e^{-|\xi'|\varepsilon Y} G + \varepsilon \int_0^Y \varepsilon|\xi'| e^{-\varepsilon|\xi'|(Y-Y')} E_1 G, \end{aligned} \tag{7.10}$$

where

$$N' = \frac{i\xi'}{|\xi'|}, \tag{7.11}$$

while $E_1 f(t)$ gives the solution of the heat equation with boundary datum $f(t)$ and homogeneous initial datum; the explicit expression is

$$E_1 f = \int_0^t \frac{Y}{2(t-s)} \frac{e^{-\frac{Y^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}} f(s) ds. \tag{7.12}$$

Proposition 7.3. *Let $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ with σ_i given in Eq. (7.10). Then $\boldsymbol{\sigma} \in L^{l,\bar{\rho},\bar{\theta}}_{\bar{\beta},\bar{T}}$ and, $\forall|\theta'| < \theta$, $\forall\theta \in]0, \pi/4[$, $\forall a$ the following estimates, hold:*

$$\begin{aligned} ||\boldsymbol{\sigma}|_{k,\rho}|_{L^2(\Gamma(\theta',a))} &\le C|\tilde{G}|_{k+1,\rho}, \\ ||\partial_Y\boldsymbol{\sigma}|_{k,\rho}|_{L^2(\Gamma(\theta',a))} &\le C\varepsilon\left[t^{3/4}|\partial_t G|_{k,\rho} + |G|_{\rho,k+1}\right], \\ ||\partial_{YY}\boldsymbol{\sigma}|_{k,\rho}|_{L^2(\Gamma(\theta',a))} &\le C\varepsilon\left[|G|_{k+2,\rho} + t^{1/4}|\partial_t G|_{k+1,\rho}\right], \\ ||\partial_t\boldsymbol{\sigma}|_{k,\rho}|_{L^2(\Gamma(\theta',a))} &\le C\left|\partial_t\tilde{G}\right|_{k+1,\rho}, \end{aligned} \tag{7.13}$$

$$\begin{aligned} \sup_Y |\boldsymbol{\sigma}|_{k,\rho} &\le C|G|_{k,\rho}, \\ \sup_Y |\partial_Y\boldsymbol{\sigma}|_{k,\rho} &\le C\varepsilon\left[|G|_{k+1,\rho} + t^{1/2}|\partial_t G|_{k,\rho}\right], \\ \sup_Y |\partial_{YY}\boldsymbol{\sigma}|_{k,\rho} &\le C\varepsilon\left[|G|_{k+2,\rho}|\partial_t G|_{k+1,\rho}\right]. \end{aligned} \tag{7.14}$$

The proof of the proposition is straightforward, and can be obtained by applying Young's convolution inequality to the expressions (7.10).

Remark 7.1. In the expression of $\boldsymbol{\sigma}$, the term $N' e^{-\varepsilon|\xi'|Y} G$ is the only one which is not $O(\varepsilon)$; however, $\partial_Y\boldsymbol{\sigma}$ is $O(\epsilon)$, which implies that $\partial_y\boldsymbol{\sigma}$ is $O(1)$.

Remark 7.2. The estimates for $\boldsymbol{\sigma}$ and $\partial_Y \boldsymbol{\sigma}$ show that, near $t = 0$, they are small in t; in fact, $G(t = 0) = 0$ and continuosly differentiable in t, so that G goes linearly to zero with t. The regularity of G with respect to time is due to the regularizing effect that the integration in Y, appearing in (7.9), has on "gaussian"-type singularities.

These remarks and the estimates in the L^∞_Y norm in (7.14) will be useful in the estimates for the nonlinear terms in the equation of $\mathbf{e}^*$.

7.3. The Navier–Stokes Operator

In the present subsection we shall introduce the operator $\mathcal{N}^*$ and give the estimate in Proposition 7.5, which is the main result of the subsection. The Navier–Stokes operator $\mathcal{N}^*$ solves the time-dependent Stokes equation with a forcing term: $\boldsymbol{w} = \mathcal{N}^* \boldsymbol{w}^*$ is the solution of

$$\begin{aligned} (\partial_t - \varepsilon^2 \partial_{xx} - \partial_{YY})\boldsymbol{w} + \nabla p^w &= \boldsymbol{w}^* \\ \nabla \cdot \boldsymbol{w} &= 0, \\ \gamma \boldsymbol{w} &= \mathbf{0}, \\ \boldsymbol{w}(t = 0) &= \mathbf{0}. \end{aligned} \tag{7.15}$$

One can write the explicit expression of $\mathcal{N}^*$ in terms of the projection operator $\bar{P}^\infty$, of the heat operator $\tilde{E}_2$, and of the operator $\mathcal{S}$ which solves the Stokes equations with boundary data:

$$\mathcal{N}^* = \bar{P}^\infty \tilde{E}_2 - \mathcal{S}\gamma \bar{P}^\infty \tilde{E}_2. \tag{7.16}$$

In the rest of the section, we shall give the explicit expression of $\bar{P}^\infty$, $\tilde{E}_2$, and $\mathcal{S}$, and state the necessary estimates for $\mathcal{N}^*$.

If $\mathbf{v}$ is a vector field defined on the upper plane, $\bar{P}^\infty \mathbf{v}$ is obtained in the following way: extend $\mathbf{v}$ oddly for $Y < 0$, then apply the Leray projector for functions defined on the whole space, and finally restrict the result to $Y \geq 0$. The explicit expression of the normal component is given by

$$\begin{aligned} \bar{P}^\infty_n \mathbf{v} = \frac{1}{2}\varepsilon|\xi'| \Bigg[\int_0^Y dY' \left(e^{-\varepsilon|\xi'|(Y-Y')} - e^{-\varepsilon|\xi'|(Y+Y')} \right) (-N' v_1 + v_2) + \\ + \int_Y^{+\infty} dY' \left(e^{\varepsilon|\xi'|(Y-Y')} (N' v_1 + v_2) - e^{-\varepsilon|\xi'|(Y+Y')} (-N' v_1 + v_2) \right) \Bigg], \end{aligned} \tag{7.17}$$

where the Riesz-type operator N' is defined in (7.11). The tangential component is

$$\begin{aligned} \bar{P}^{\infty\prime} \mathbf{v} = v_1 - \frac{\varepsilon|\xi'|}{2} \Bigg[\int_0^Y dY' \left(e^{-\varepsilon|\xi'|(Y-Y')} - e^{-\varepsilon|\xi'|(Y+Y')} \right) (v_1 + N' v_2) + \\ + \int_Y^{+\infty} dY' \left(e^{\varepsilon|\xi'|(Y-Y')} (v_1 - N' v_2) - e^{-\varepsilon|\xi'|(Y+Y')} (v_1 + N' v_2) \right) \Bigg]. \end{aligned} \tag{7.18}$$

Notice that, if we had extended the tangential part evenly and the normal part oddly, we would have obtained the Leray projector for the half space.

The operator $\tilde{E}_2$ is such that $\tilde{E}_2 f$ solves the heat equation in the half space with source f and homogeneous data:

$$\begin{aligned} (\partial_t - \varepsilon^2 \partial_{xx} - \partial_{YY}) u &= f \\ \gamma u &= 0 \\ u(t = 0) &= 0. \end{aligned} \tag{7.19}$$

The explicit expression can be given in terms of convolutions with gaussians:

$$\tilde{E}_2 f(x,Y,t) = \int_0^t ds \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-x')^2}{4(t-s)\varepsilon^2}}}{\sqrt{4\pi\varepsilon^2(t-s)}} dx' \\ \int_0^{+\infty} \left(\frac{e^{-\frac{(Y-Y')^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}} - \frac{e^{-\frac{(Y+Y')^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}} \right) f(x',Y',s)dY'. \tag{7.20}$$

The Stokes operator $\mathcal{S}$ is the operator such that $\mathcal{S}\mathbf{g}$ solves the Stokes equations with boundary datum $\mathbf{g}$:

$$\begin{aligned} (\partial_y - \varepsilon^2\Delta)\mathbf{u}^S + \nabla p^S &= 0, \\ \nabla\cdot\mathbf{u}^S &= 0, \\ \gamma\mathbf{u}^S &= \mathbf{g}, \\ \mathbf{u}^S(t=0) &= \mathbf{0}. \end{aligned} \tag{7.21}$$

The following proposition holds.

Proposition 7.4. *Suppose that* $\boldsymbol{g} = (g', g_n) \in K^{l,\rho}_{\beta,T}$ *with* $\boldsymbol{g}(t=0) = \boldsymbol{0}$ *and* g_n *of the form*

$$g_n = |\xi'| \int_0^{+\infty} dy' f(\xi', y', t) k(\xi', y')$$

with $|\xi'| \int_0^{+\infty} dy' |k(\varepsilon', y')| \le 1$ *and* $f \in L^{l,\rho,\theta}_{\beta,T}$. *Then* $\mathcal{S}\boldsymbol{g} \in L^{l,\rho,\theta}_{\beta,T}$ *and*

$$|\mathcal{S}\boldsymbol{g}|_{l,\rho,\theta,\beta,T} \le c(|g'|_{l,\rho,\theta,\beta,T} + |f|_{l,\rho,\theta,\beta,T}). \tag{7.22}$$

The proof can be found in [35] where the above results appears as Proposition 3.4.

In Appendix C, we shall show that, when $\mathbf{v} \in S^{l,\rho,1,\theta}_{\beta,T}$ then $\bar{P}^\infty \tilde{E}_2 \mathbf{v} \in L^{l,\rho,\theta}_{\beta,T}$ and that, furthermore, $\gamma \bar{P}^\infty \tilde{E}_2$ satisfies the hypothesis of Proposition 7.4; all this leads to the following result.

Proposition 7.5. *Suppose that* $\boldsymbol{u} \in S^{l,\rho,1,\theta}_{\beta,T}$, *then* $\mathcal{N}^*\boldsymbol{u} \in L^{l,\rho,\theta}_{\beta,T}$ *and*

$$|\mathcal{N}^*\boldsymbol{u}|_{l,\rho,\theta,\beta,T} \le C|\boldsymbol{u}|_{l,\rho,1,\theta,\beta,T}. \tag{7.23}$$

We now give some bounds in a time integrated form that will be useful for the application of the ACK theorem.

Proposition 7.6. *Assume that* $\boldsymbol{u} \in S^{l,\rho,1,\theta}_{\beta,T}$. *Then we have that, for* $\rho' < \rho - \beta t$, $\theta' < \theta - \beta t$,

$$|\mathcal{N}^*\boldsymbol{u}|_{l,\rho',0,\theta'} \le C \int_0^t ds |\boldsymbol{u}(\cdot,\cdot,s)|_{l,\rho',0,\theta'}, \tag{7.24}$$

$$|\partial_Y \mathcal{N}^*\boldsymbol{u}|_{l-1,\rho',0,\theta'} \le C \int_0^t ds \frac{|\boldsymbol{u}(\cdot,\cdot,s)|_{l,\rho',1,\theta'}}{(t-s)^{1/4}}. \tag{7.25}$$

Contrary to the compatible case, the mild singularity in time cannot be completely eliminated, even when $\gamma\mathbf{u} = \mathbf{0}$: this is because the functional setting $S^{l,\rho,1,\theta}_{\beta,T}$ is more singular than the one used in the compatible case. In particular, this setting does not allow a time derivative. The presence of this singularity implies that, in order to estimate a derivative of order $l+1$, like $\partial_Y \partial_x^l$, we cannot use the Cauchy estimate, that would lead to the presence, at the denominator, of $(t-s)^{1/4}(\rho(s) - \rho')$, which is not allowed by the ACK theorem. Therefore, we need a better estimate of the derivatives of order $l+1$. A similar problem

would appear if, in order to deal with the bilinear terms appearing in the error equation, one tries to use the algebra properties of $S^{l,\rho,1,\theta}_{\beta,T}$; therefore, we need some estimates in an $L^\infty_Y L^2_x$-like setting, so that the algebra properties are used only in the tangential variable.

Next proposition shows how to treat derivatives of order $l+1$

Proposition 7.7. *For any $\boldsymbol{u} \in S^{l,\rho,1,\theta}_{\beta,T}$, we have for $\rho' < \rho - \beta t$, $\theta' < \theta - \beta t$*

$$|\mathcal{N}^* \boldsymbol{u}|_{l+1,\rho',0,\theta'} \leq C \int_0^t \frac{|\boldsymbol{u}|_{l,\rho(s),0,\theta'}}{\rho(s)-\rho'} ds, \tag{7.26}$$

$$|\mathcal{N}^* \boldsymbol{u}|_{l+1,\rho',0,\theta'} \leq \frac{C}{\varepsilon} \int_0^t \frac{|\boldsymbol{u}|_{l,\rho',0,\theta'}}{(t-s)^{1/2}} ds, \tag{7.27}$$

$$|\partial_Y \mathcal{N}^* \boldsymbol{u}|_{l,\rho',0,\theta'} \leq C \int_0^t \left[\frac{|\boldsymbol{u}|_{l,\rho',1,\theta'}}{(t-s)^{1/2}} + \frac{|\boldsymbol{u}|_{l,\rho(s),1,\theta'}}{\rho(s)-\rho'} \right] ds, \tag{7.28}$$

$$|\partial_{YY} \mathcal{N}^* \boldsymbol{u}|_{l-1,\rho',0,\theta'} \leq C \int_0^t \left[\frac{|\boldsymbol{u}|_{l,\rho',1,\theta'}}{(t-s)^{1/2}} + \varepsilon^{1/2} \frac{|\boldsymbol{u}|_{l-1/2,\rho',0,\theta'}}{(t-s)^{3/4}} \right] ds. \tag{7.29}$$

Furthermore, if $\gamma \boldsymbol{u} = \boldsymbol{0}$, then

$$|\partial_Y \mathcal{N}^* \boldsymbol{u}|_{l,\rho',0,\theta'} \leq C \int_0^t \frac{|\boldsymbol{u}|_{l,\rho',1,\theta'}}{(t-s)^{1/2}} ds. \tag{7.30}$$

The case $\gamma \mathbf{u} = \mathbf{0}$ is used to prove the quasi-contractiveness hypothesis of the ACK theorem: in this case, if we do not use Cauchy estimates, the only derivative of order $l+1$ which can be an $O(1/\varepsilon)$ is the purely tangential one, $\partial_x^{l+1} \mathcal{N}^* \mathbf{u}$.

The next proposition provides some estimates in the L^∞_Y norm.

Proposition 7.8. *For any $\boldsymbol{u} \in S^{l,\rho,1,\theta}_{\beta,T}$ we have that, $\forall \rho' < \rho - \beta t$, $\forall t \in [0,T]$*

$$\sup_Y |\mathcal{N}^* \boldsymbol{u}|_{l,\rho'} \leq C \int_0^t \frac{|\boldsymbol{u}|_{l,\rho',0,\theta'}}{(t-s)^{1/4}}, \tag{7.31}$$

$$\sup_Y |\partial_Y \mathcal{N}^* \boldsymbol{u}|_{l-1,\rho'} \leq C \int_0^t \frac{|\boldsymbol{u}|_{l,\rho',0,\theta'}}{(t-s)^{3/4}}. \tag{7.32}$$

The proof of Propositions 7.6, 7.7, and 7.8 can be found in Appendix E.

7.4. The Error Equation

Given the expression (7.1) for $\mathbf{e}$, the system (1.22) is now an equation for $\mathbf{e}^*$ that can be cast in the following form:

$$\mathbf{e}^* = \mathbf{F}(\mathbf{e}^*, t), \tag{7.33}$$

where

$$\begin{aligned} \mathbf{F}(\mathbf{e}^*, t) = \mathbf{k} - \{ [\mathbf{u}^{NS0} + \varepsilon(\mathbf{u}^E_{(1)} + \bar{\mathbf{u}}^P_{(1)} + \boldsymbol{\sigma})] \cdot \nabla \mathcal{N}^* \mathbf{e}^* + \mathcal{N}^* \mathbf{e}^* \cdot \nabla [\mathbf{u}^{NS0} + \\ + \varepsilon(\mathbf{u}^E_{(1)} + \bar{\mathbf{u}}^P_{(1)} + \boldsymbol{\sigma})] + \varepsilon \mathcal{N}^* \mathbf{e}^* \cdot \nabla \mathcal{N}^* \mathbf{e}^* \}. \end{aligned} \tag{7.34}$$

The forcing term $\mathbf{k}$ is given by

$$\mathbf{k} = \boldsymbol{\Xi}_r + \boldsymbol{\Psi}. \tag{7.35}$$

In the above expression, $\boldsymbol{\Xi}_r$ is the regular part of the source term appearing in the equation of the remainder $\mathbf{e}$, see the system (1.22). The expression for $\boldsymbol{\Xi}_r$ is in (A.5). The term $\boldsymbol{\Psi}$ derives from the introduction of $\mathbf{h}$ and $\boldsymbol{\sigma}$ in the decomposition (7.1); the explicit expression of $\boldsymbol{\Psi}$ is given in (A.7). The following proposition states that the forcing term is bounded.

Proposition 7.9. *There exists a constant R_0, independent of ε, such that*

$$|\boldsymbol{k}|_{l,\bar{\rho},1,\bar{\theta},\bar{\beta},\bar{T}} \leq R_0 \tag{7.36}$$

The key of the proof is to rearrange $\mathbf{k}$ in a way such that terms that are $O(\varepsilon^{-1})$ multiply terms that are $O(\varepsilon)$ and terms that have singular derivative with respect to Y multiply terms that go to zero with Y or t. The details are in the Appendix A.2.

7.4.1. Quasi Contractiveness Hypothesis. In this subsection, we shall prove that the right-hand side of Eq. (7.33), explicitly defined in (7.34), satisfies the quasi-contractiveness hypothesis of the ACK Theorem. It is useful to distinguish between the nonlinear and the linear terms present in (7.34).

Concerning the nonlinear term, one needs to pay special attention to the term $(\mathcal{N}^*(\mathbf{e}^{*1} - \mathbf{e}^{*2}))_n$ $\partial_Y(\mathcal{N}^*\mathbf{e}^{*2})'$: indeed, since $\gamma\mathbf{e}^{*2} \neq \mathbf{0}$, the $|\,|_{l,\rho',0,\theta'}$ norm of $\partial_Y(\mathcal{N}^*\mathbf{e}^{*2})'$ cannot be bounded by a constant, due to a term which behaves like $\partial_Y\tilde{E}_1\gamma\bar{P}^\infty\tilde{E}_2\mathbf{e}^{2*}$. However, for the properties of $\tilde{E}_1$, $Y\partial_Y\tilde{E}_1\gamma\bar{P}^\infty\tilde{E}_2\mathbf{e}^{2*}$ behaves like $\tilde{E}_1\gamma\bar{P}^\infty\tilde{E}_2\mathbf{e}^{2*}$, while $(\mathcal{N}^*(\mathbf{e}^{*1} - \mathbf{e}^{*2}))_n/Y$ can be estimated in terms of $\partial_Y(\mathcal{N}^*(\mathbf{e}^{*1} - \mathbf{e}^{*2}))_n$. Therefore, by multiplying and dividing by Y, we move the normal derivative on the term with a more favorable estimate.

The other nonlinear terms are easily estimated using Propositions 7.6, 7.7, and 7.8, so we obtain the following result.

Proposition 7.10. *Suppose $\boldsymbol{e}^{*i} \in S^{l.\rho',1,\theta'}$ for $i = 1, 2$. Then the following estimate holds:*

$$\left|\varepsilon\mathcal{N}^*\boldsymbol{e}^{*1}\cdot\nabla\mathcal{N}^*\boldsymbol{e}^{*1} - \varepsilon\mathcal{N}^*\boldsymbol{e}^{*2}\cdot\nabla\mathcal{N}^*\boldsymbol{e}^{*2}\right|_{l,\rho',1,\theta'} \leq C\int_0^t \frac{|\boldsymbol{e}^{*1} - \boldsymbol{e}^{*2}|_{l,\rho',1,\theta'}}{(t-s)^{3/4}}ds. \tag{7.37}$$

Concerning the linear terms, since $\gamma\mathbf{e}^{*1} = \gamma\mathbf{e}^{*2}$, we can apply the results of proposition 7.7 regarding the case with zero trace. The linear terms are problematic essentially for the presence of u^S and its Y-derivatives, as well for the presence of the $O(\varepsilon^{-1})$ term $\partial_y\tilde{u}^P$. In Appendix F, we show how it is possible to estimate these terms.

Proposition 7.9, the estimate of the nonlinear term (7.37), and the estimates on the linear terms given in Appendix F allow us to use the ACK theorem in the functional setting $S^{l,\bar{\rho},1,\bar{\theta}}_{\bar{\beta},T}$. This leads to the following result:

Proposition 7.11. *Assume that $\boldsymbol{u}_0 \in H^{l,\rho,\theta}$, with $\gamma_n\boldsymbol{u}_0 = 0$; then $\boldsymbol{e}^* \in S^{l,\bar{\rho},1,\bar{\theta}}_{\bar{\beta},T}$ for some $T > 0$, $\bar{\rho} < \rho$, $\bar{\theta} < \theta$, $\bar{\beta} > \beta$, with all those parameters independent of the viscosity.*

By proposition 7.5, we therefore have $\mathcal{N}^*\mathbf{e}^* \in L^{l,\bar{\rho},\bar{\theta}}_{\bar{\beta},T}$; combining with propositions 7.2 and 7.3, we have that the overall error $\mathbf{e}$ is in $L^{l,\bar{\rho},\bar{\theta}}_{\bar{\beta},T}$, and this concludes the proof of theorem 4.1.

7.5. More General Initial Conditions

The estimates performed for the forcing term in the equation of $\mathbf{e}^*$ heavily relied on the assumption that the initial condition are purely eulerian, which means that $\mathbf{e}(t=0) = \mathbf{u}^E_{(1)}(t=0) = \bar{\mathbf{u}}^P_{(1)}(t=0) = \bar{\mathbf{u}}^P(t=0) = \mathbf{0}$. However, the zero viscosity limit holds also for more general initial conditions like

$$\mathbf{u}_0 = \mathbf{u}^E_0 + \bar{\mathbf{u}}^P_0 + \varepsilon\left(\mathbf{u}^E_{(1),0} + \bar{\mathbf{u}}^P_{(1),0} + \mathbf{e}_0\right) \tag{7.38}$$

with

$$\gamma v^E_{(1),0} = -\gamma \bar{v}^P_0\,,\ \gamma u^P_{(1),0} = -\gamma u^E_{(1),0},\ \gamma \mathbf{e}_0 = (0, -\gamma \bar{v}^P_{(1),0}); \tag{7.39}$$

this kind of initial data allows a zero order incompatibility with no-slip boundary condition, since we are not assuming that $\gamma \tilde{u}^P_0 = -\gamma u^E_0$. The most challenging variation needed is to prove that the forcing term is still in $S^{l,\bar{\rho},1,\bar{\theta}}_{\bar{\beta},\bar{T}}$ and $O(1)$: we need to regroup some terms carefully in order to show the derivative $\partial_Y \mathbf{k}$ has the desired regularity. This is done in appendix A.3

8. Conclusions

In this paper, we have proved the short-time existence of analytic solutions of the Navier–Stokes equations in the half-space when the initial data are incompatible with the boundary data: the existence time of the solution we constructed is independent of the viscosity. When the viscosity goes to zero, the Navier–Stokes solution approaches the Euler solution away from the boundary, and the Prandtl solution in a boundary layer whose thickness is proportional to the square root of the viscosity ν.

Our method of constructing the Navier–Stokes solution is based on a matched asymptotic expansion: the solution is the superposition of an outer (Euler) solution and an inner (Prandtl) solution, plus correction terms that are $O(\varepsilon)$, $\varepsilon = \sqrt{\nu}$, see (1.10) or (4.1). The inner Prandtl solution decays to zero exponentially outside the boundary layer; this gives the convergence of the Navier–Stokes solutions to the Euler solution in the L^2 norm.

The incompatibility between the initial data and the no-slip condition causes the presence of singular terms. These singular terms are $O(1)$ near the boundary, are given in terms of an error function expressing the existence of the initial discontinuity, and are proportional to the initial slip, see (1.14). Interestingly, the incompatibility also causes the appearance of an initial singularity on the whole domain. This singularity shows up in the first-order correction to the Euler flow, and it is milder than the singularity appearing in the boundary layer because it is of the $\sqrt{t}$-type, see (1.19), (5.2) and (5.3). The reason for this non-localized singularity is that the boundary layer initial singularity causes a mildly singular normal inflow, which needs to be compensated by the first-order correction Euler flow. The ellipticity encoded in the incompressibility condition instantly propagates the singularity on the whole domain.

The construction of the overall error $\mathbf{e}$ that closes the asymptotic procedure, accomplished in Sect. 7, is the most challenging part of the present paper. The main reasons are the following. First, the system satisfied by $\mathbf{e}$ contains the interactions between a) terms from the boundary layer, rapidly varying in y and therefore most naturally handled using the rescaled variable Y; b) terms from the outer flow that one must handle in an L^2_y setting. Second, terms coming from the initial singularity force us to use a function setting, the $S^{l,\rho,1,\theta}_{\beta,T}$ space, that is more singular in time. However, this led to the necessity of very careful and technical estimates of the Navier–Stokes operator $\mathcal{N}^*$, which are in Propositions 7.5, 7.6, 7.7, and 7.8. For example, one has to take advantage of the regularizing properties of the Navier–Stokes operator $\mathcal{N}^*$, due to the presence of diffusion in the Y-direction. These properties allow trading a Y-derivative with the appearance of a mild singularity in time, which the ACK Theorem can nevertheless handle.

The central hypothesis we had to impose is analyticity of the initial datum, which, as discussed in the Introduction, appears to be a natural one, given the instabilities of the Prandtl equations, see, e.g., [27,29], and of the outer-inner-flow expansion, see [26,30,31,71,72]. It would be interesting to see whether imposing analyticity close to the boundary, as in [39–41,43], is enough to control incompatible data. The vorticity formulation (that one usually employs to prove that analyticity close to the boundary is sufficient to avoid ill-posedness or instability) should deal with a more singular initial datum, i.e., a vortex sheet attached at the boundary; see [53] for the study of the interaction of vorticity measures with boundaries.

Another problem we have not addressed here is whether analyticity in the normal direction could be relaxed, as in [24] where the diffusive regularization stabilizes the Prandtl solution without analyticity in the normal direction. To show a similar result for the full Navier–Stokes equations would be more

challenging because the interactions between the boundary layer and the outer Euler terms are much more difficult to handle.

Finally, we recall that Prandtl equations are known to develop singularity, a phenomenon that is different from the ill-posedness we discussed in the Introduction, and that some believe to be related to the break-up of the boundary layer, although this seems to be disproved by the analysis of the complex singularities developed by the Navier–Stokes equations reported in [14]. Nevertheless, it would be interesting to know whether the rigorous singularity construction, see [73,74], can be repeated with Euler-type initial data considered in the present paper.

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Declarations

Conflict of interest The author's declared that they have no conflict of interest.

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Appendix A. The Source Terms

Appendix A.1. The Source Term for the Error Equation Eq. (1.22)

The source term $\boldsymbol{\Xi}$ has the following expression:

$$\begin{aligned}\boldsymbol{\Xi} = \mathbf{f} + \Big(g\partial_y \tilde{u}^P, 0\Big) - \Big[&\mathbf{u}^{NS}_{(0)} \cdot \nabla \bar{\mathbf{u}}^P_{(1)} + \bar{\mathbf{u}}^P_{(1)} \cdot \nabla \mathbf{u}^{NS}_{(0)} + \\ &+ \Big(\bar{\mathbf{u}}^P + \varepsilon \mathbf{u}^{NS}_{(1)}\Big) \cdot \nabla \mathbf{u}^E_{(1)} + \mathbf{u}^E_{(1)} \cdot \nabla \Big(\bar{\mathbf{u}}^P + \varepsilon \bar{\mathbf{u}}^P_{(1)}\Big) + \varepsilon \bar{\mathbf{u}}^P_{(1)} \cdot \nabla \bar{\mathbf{u}}^P_{(1)}\Big] \\ &+ \varepsilon^2 \Big[\Delta \mathbf{u}^E_{(1)} + \Big(\partial_{xx} \bar{u}^P_{(1)}, 0\Big)\Big] - \Big(0, \Big(\partial_t - \varepsilon^2 \Delta\Big) \bar{v}^P_{(1)}\Big)\end{aligned} \tag{A.1}$$

where $\mathbf{f}$ is given by:

$$\mathbf{f} = \mathbf{f}_r + (\varepsilon \partial_x^2 u^S, 0), \tag{A.2}$$

$$\begin{aligned} f^1_r &= -\varepsilon^{-1}\{\tilde{u}^P(\partial_x u^E - \partial_x u^E\big|_{y=0}) + \partial_x \tilde{u}^P(u^E - u^E\big|_{y=0}) + \\ &\quad + \partial_y \tilde{u}^P(v^E + y\partial_x u^E\big|_{y=0})\} - \bar{v}^P \partial_y u^E + \varepsilon \Delta u^E + \varepsilon \partial_x^2 \tilde{u}^R, \\ f^2_r &= -[\partial_t \bar{v}^R + u^{NS0} \partial_x \bar{v}^P + v^{NS0} \partial_y \bar{v}^P + \bar{v}^P \partial_y v^E] - \varepsilon^{-1} \tilde{u}^P \partial_x v^E + \\ &\quad + \varepsilon \Delta v^E + \varepsilon^2 \Delta v^R + \varepsilon^2 \partial_x^2 \tilde{v}^S. \end{aligned} \tag{A.3}$$

Writing down the expression of f^2, we used the fact that $\partial_t \bar{v}^S - \partial_Y^2 \bar{v}^S = 0$ to get rid of those singular terms. We can write $g = g^R + g^S$, with

$$g^S = -\partial_x u_0(x, y=0) \int_0^{+\infty} dY \int_Y^{+\infty} \frac{e^{\frac{-Y'^2}{4t}}}{\sqrt{t\pi}} dY' = C_1 \sqrt{t} \partial_x u_0(x, y=0). \tag{A.4}$$

In Eq. (A.2), we isolated the most singular term of $\mathbf{f}$; in a similar way, we write

$$\boldsymbol{\Xi}_r = \boldsymbol{\Xi} - \varepsilon(\partial_x^2 u^S, 0), \tag{A.5}$$

where $\boldsymbol{\Xi}$ is given in (A.1). Notice that $\mathbf{f} + \big(g\partial_y \tilde{u}^P, 0\big)$ is the forcing term in the equation of the remainder $\mathbf{R}$ in the decomposition

$$\mathbf{u}^{NS} = \mathbf{u}^E + \bar{\mathbf{u}}^P + \varepsilon \mathbf{R} \tag{A.6}$$

The term $\mathbf{f}$ is $O(1)$ with respect to ε, while $g\partial_y \tilde{u}^P$ is $O(\varepsilon^{-1})$: therefore, an additional decomposition of the remainder $\mathbf{R}$ is needed to prove the validity of the inviscid limit.

Appendix A.2. Estimates for the Forcing Term of e*

The source term $\mathbf{k}$ in (7.35) is decomposed as $\mathbf{k} = \boldsymbol{\Xi}_r + \boldsymbol{\Psi}$ where

$$\begin{aligned}
\boldsymbol{\Psi} = &-\Big(\big(\mathbf{u}^{NS0} + \varepsilon(\mathbf{u}^E_{(1)} + \bar{\mathbf{u}}^P_{(1)})\big) \cdot \nabla(\mathbf{h} + \boldsymbol{\sigma}) + \\
&+ (\mathbf{h} + \boldsymbol{\sigma}) \cdot \nabla(\mathbf{u}^{NS0} + \varepsilon(\mathbf{u}^E_{(1)} + \bar{\mathbf{u}}^P_{(1)})) + (\mathbf{h} + \boldsymbol{\sigma}) \cdot \nabla(\mathbf{h} + \boldsymbol{\sigma})\Big) + \\
&+ \varepsilon^2 \partial_{xx}(\mathbf{h} + \boldsymbol{\sigma}) - \varepsilon(0, (\partial_t - \partial_{YY})h_n) \\
= &-\Big\{\big[\big(\mathbf{u}^{NS0} + \varepsilon(\mathbf{u}^E_{(1)} + \bar{\mathbf{u}}^P_{(1)} + \mathbf{h} + \boldsymbol{\sigma})\big) \cdot \nabla(\bar{\mathbf{u}}^P_{(1)} + \mathbf{h}) + \\
&+ (\mathbf{u}^E_{(1)} \cdot \nabla \bar{\mathbf{u}}^P - (g\partial_y \tilde{u}^P, 0)) + (\bar{\mathbf{u}}^P_{(1)} + \mathbf{h} + \boldsymbol{\sigma}) \cdot \nabla \bar{\mathbf{u}}^P\big] + \\
&+ \big[(\mathbf{u}^E_{(1)} + \bar{\mathbf{u}}^P_{(1)} + \mathbf{h}) \cdot \nabla \mathbf{u}^E + \\
&+ (\bar{u}^P + \varepsilon(\mathbf{u}^E_{(1)} + \bar{\mathbf{u}}^P_{(1)} + \mathbf{h} + \boldsymbol{\sigma})) \cdot \nabla \mathbf{u}^E_{(1)} + \\
&+ (\mathbf{u}^{NS0} + \varepsilon(\mathbf{u}^E_{(1)} + \bar{\mathbf{u}}^P_{(1)} + \mathbf{h} + \boldsymbol{\sigma})) \cdot \nabla \boldsymbol{\sigma}\big]\Big\} + \\
&+ \mathbf{f}_r - (0, (\partial_t - \varepsilon^2 \Delta)\varepsilon(\bar{v}^P_{(1)} + h_n)) + \\
&+ \varepsilon^2 \big[\Delta \mathbf{u}^E_{(1)} + \partial_{xx}(u^P_{(1)} + h', 0) + \partial_{xx}\boldsymbol{\sigma}\big].
\end{aligned} \tag{A.7}$$

We want to prove that the forcing term $\mathbf{k}$ is in $S^{l,\bar{\rho},1,\bar{\theta}}_{\bar{\beta},\bar{T}}$ and that it is $O(1)$ with respect to ε. In our analysis we shall focus on the most challenging terms. We begin with:

$$[\mathbf{u}^{NS0} + \varepsilon(\mathbf{u}^E_{(1)} + \bar{\mathbf{u}}^P_{(1)} + \mathbf{h} + \boldsymbol{\sigma}))] \cdot \nabla(\bar{\mathbf{u}}^P_{(1)} + \mathbf{h}). \tag{A.8}$$

This term is $O(1)$ because $\partial_y(\bar{\mathbf{u}}^P_{(1)} + \mathbf{h})$, which has an $O(1/\varepsilon)$ L^2_Y norm, multiplies a term with an $O(\varepsilon)$ L^∞_Y norm. When we take a partial derivative with respect to Y, some singular terms appear, $\partial_Y \tilde{u}^S \partial_x(\bar{\mathbf{u}}^P_{(1)} + \mathbf{h})$ and $[\mathbf{u}^{NS0}/\varepsilon + (\mathbf{u}^E_{(1)} + \bar{\mathbf{u}}^P_{(1)} + \mathbf{h} + \boldsymbol{\sigma})]_2 \partial_{YY} u^S_1$: for the first term, since $v^E = \varepsilon Y(\partial_y v^E)(\lambda(y)y)$ for some $\lambda(y) \in [0,1]$, we have that $\bar{\mathbf{u}}^S_{(1)}$ goes to zero as $\sqrt{t}$ in the L^∞_Y norm, $\bar{\mathbf{u}}^R_{(1)}$ and $\mathbf{h}$ go to zero linearly with t, and this is enough to balance the singularity of $\partial_Y \tilde{u}^S$; for the second term, $v^E/\varepsilon + \bar{v}^S + (\mathbf{u}^E_{(1)} + \bar{\mathbf{u}}^P_{(1)} + \mathbf{h} + \boldsymbol{\sigma}))_2$ goes to zero linearly with Y, and this allows to balance the singularity of $\partial_{YY} u^S_{(1)}$. So, all the singular terms in $[\mathbf{u}^{NS0} + \varepsilon(\boldsymbol{w} + \boldsymbol{w} + \boldsymbol{\sigma})] \cdot \nabla(\bar{\mathbf{u}}^P_{(1)} + \mathbf{h})$ are balanced, and its norm is controlled through the usual arguments involving algebra properties.

We can now pass to estimate the terms:

$$\mathbf{u}^E_{(1)} \cdot \nabla \bar{\mathbf{u}}^P - (g\partial_y \tilde{u}^P, 0) \tag{A.9}$$

The most problematic term is $(v^E_{(1)} - g)\partial_y \tilde{u}^P$, which is $O(1)$ since $v^E_{(1)} - g$ goes to zero linearly in $y = \varepsilon Y$; since $\partial_y u^S_{(1)}$ is multiplied by εY times a bounded function, the product has finite $|\cdot|_{l,\rho',0,\theta'}$ norm. When we take the partial derivative with respect to Y, $\partial_Y v^E_{(1)} = \varepsilon \partial_y v^E_{(1)}$, so $\partial_Y v^E_{(1)} \partial_y \tilde{u}^P$ is still $O(1)$, and since $\partial_Y v^E_{(1)}$ has a regular part which goes to zero linearly with t and a singular part which goes to zero with t like $\sqrt{t}$, this product has finite $|\cdot|_{l,\rho',0,\theta'}$ norm. Finally, $(v^E_{(1)} - g)\partial_{yY}\tilde{u}^P = (\partial_Y v^E_{(1)})(\lambda y) Y \partial_{YY}\tilde{u}^P$, with the regular part of $\partial_Y v^E_{(1)}$ going to zero linearly with t, while the singular part goes to zero like $\sqrt{t}$.

We now estimate the terms:

$$(\bar{\mathbf{u}}^P_{(1)} + \mathbf{h} + \boldsymbol{\sigma}) \cdot \nabla \bar{\mathbf{u}}^P. \tag{A.10}$$

Both $(u^P_{(1)} + h' + \sigma_1)\partial_Y \partial_x \tilde{u}^P$ and $\big(\varepsilon \partial_Y \bar{v}^P_{(1)} + \varepsilon \partial_Y h_n + \partial_Y \sigma_2\big)\partial_y \bar{\mathbf{u}}^P = -(\partial_x u^P_{(1)} + \partial_x h' + \partial_x \sigma_1)\partial_Y \bar{\mathbf{u}}^P$ are products of terms whose L^∞ norm is $O(1)$ and goes to zero with t at least as $\sqrt{t}$ and a function whose L^2_Y behaves like $t^{-1/4}$; in $(\bar{v}^P + h_n + \sigma_2/\varepsilon)\partial_{YY}\bar{\mathbf{u}}^P$, the first factor has an $O(1)$ L^∞_Y norm which goes to zero linearly with t, and this balances the singularity of the second factor.

The term

$$\varepsilon^{-1}\tilde{u}^P(\partial_x u^E - \partial_x \gamma u^E) \tag{A.11}$$

is $O(1)$ because $\partial_x u^E - \partial_x \gamma u^E$ goes to zero linearly with $y = \varepsilon Y$, and for the same reason $\varepsilon^{-1}\partial_Y \tilde{u}^P(\partial_x u^E - \partial_x \gamma u^E)$ is $O(1)$ and with finite $|\cdot|_{l,\rho',0,\theta'}$ norm, while $\varepsilon^{-1}\tilde{u}^P \partial_Y \partial_x u^E = \tilde{u}^P \partial_y \partial_x u^E$ is of course $O(1)$ and bounded. The term $\varepsilon^{-1}\partial_x \tilde{u}^P(u^E - \gamma u^E)$ can be treated in a similar way.

Finally, the term

$$\frac{1}{\varepsilon}\partial_y \tilde{u}^P(v^E + y\partial_x \gamma u^E) = \frac{1}{\varepsilon^2}\partial_Y \tilde{u}^P(v^E + y\partial_x \gamma u^E) \tag{A.12}$$

is $O(1)$ because $(v^E + y\partial_x \gamma u^E)$ goes to zero quadratically in $y^2 = \varepsilon^2 Y^2$ due to the incompressibility condition, and for the same reason $\frac{1}{\varepsilon^2}\partial_{YY}\tilde{u}^P(v^E + y\partial_x \gamma u^E)$ is $O(1)$ and with finite $|\cdot|_{l,\rho',0,\theta'}$ norm, while $\frac{1}{\varepsilon}\partial_Y \tilde{u}^P(\partial_y v^E + \partial_x \gamma u^E)$ has $\partial_y v^E + \partial_x \gamma u^E$ which goes to zero linearly with εY

All the other terms in $\mathbf{k}$ are easier to estimate.

Appendix A.3. Forcing Term k with General Initial Conditions

When the initial conditions are not purely Eulerian, we arrange the terms so that their sum is zero at the boundary. This is done in a different way for the tangential and the normal components.

The sum of $\partial_Y u^S \partial_x w^R_1$, deriving from the derivative of $u^S \partial_x w^R_1$, and $\partial_Y u^S \partial_x u^R_{(1)}$, deriving from the derivative of $u^S \partial_x u^R_{(1)}$, has finite $|\cdot|_{l-1,\rho',0,\theta'}$ norm because $\partial_x w^R_1 + \partial_x u^R_{(1)}$ goes to zero linearly in Y (as well in t, since $\gamma w^{RR}_{1,0} = -\gamma w^{RR}_{1,0}$).

The sum of $\partial_Y u^S \varepsilon \partial_x \bar{v}^R_{(1)}$, deriving from the derivative of $u^S \partial_x \bar{v}^R_{(1)}$, and $\partial_Y \tilde{u}^S \partial_x \sigma_2$, deriving from the derivative of $\tilde{u}^S \partial_x \sigma_2$, has finite $|\cdot|_{l-1,\rho',0,\theta'}$ norm because $\varepsilon \partial_x \bar{v}^R_{(1)} + \partial_x \sigma_2$ goes to zero linearly in Y (as well in t).

The sum of $w^R_1 \partial_x \partial_Y u^S$, deriving from the derivative of $w^R_1 \partial_x u^S$, and $u^R_{(1)} \partial_x \partial_Y u^S$, deriving from the derivative of $w^{RR}_1 \partial_x u^S$, has finite $|\cdot|_{l-1,\rho',0,\theta'}$ norm because $w^R_1 + u^R_{(1)}$ goes to zero linearly in Y (and also in t).

The sum of $\partial_y w^R_2 \partial_Y u^S = -\partial_x w^R_1 \partial_Y u^S$, deriving from the derivative of $(w^R_2 - g^R)\partial_y u^S$, and $(\partial_Y \bar{v}^R_{(1)} + \varepsilon^{-1}\partial_Y \sigma_2)\partial_Y u^S = -(\partial_x u^R_{(1)} + \partial_x \sigma_1)\partial_Y u^S$, deriving from the derivative of $(\bar{v}^R_{(1)} + \varepsilon^{-1}\sigma_2)\partial_Y u^S$, has finite $|\cdot|_{l-1,\rho',0,\theta'}$ norm because $\partial_x w^R_1 + \partial_x u^R_{(1)}$ goes to zero linearly in Y (and in t) and $\partial_x \sigma_1$ goes to zero linearly with t.

The sum of $-w^R_2 \partial_{YY} \bar{v}^S$, deriving from the derivative of $-w^{RR}_2 \partial_Y \bar{v}^S$, and $\varepsilon^{-1} v^{NS0}\ \partial_{YY} \bar{v}^S$, deriving from the derivative of $\varepsilon v^{NS0} \partial_Y \bar{v}^S$, has finite $|\cdot|_{l-1,\rho',0,\theta'}$ norm: indeed, $v^{NS0} = v^E + \varepsilon \bar{v}^R + \varepsilon \bar{v}^S$, where

v^E goes to zero linearly with $y = \varepsilon Y$, $\bar{v}^S$ goes to zero like $\sqrt{t}$ and $-w_2^R + \bar{v}^R$ goes to zero linearly in Y (and in t).

For all the other terms there is no need to change the argument used in the case $\mathbf{u}_0 = \mathbf{u}_0^E$.

Appendix B. Proof of Proposition 7.1

The estimate on h' is a direct consequence of the expression (7.3), and of the following estimates on F and on its Y-derivatives. We begin with F.

$$\sup_{Y\in\Sigma(\theta)} e^{\mu Y}\,|F| =$$

$$= \sup_{Y}\left|\frac{2}{\sqrt{\pi}}\int_0^t ds\int_0^{+\infty} e^{\mu Re(Y-Y')}(E_0^- - E_0^+)e^{\mu Re(Y')}\int_{\frac{Y'}{2\sqrt{t}}}^{+\infty} e^{-\sigma^2}d\sigma dY'\right|$$

$$\leq \sup_{Y}\left|\frac{2}{\sqrt{\pi}}\int_0^t ds\int_0^{+\infty} e^{\mu Re(Y-Y')}(E_0^- - E_0^+)\right| \leq ct \leq C$$

To pass from the first to the second line we have taken into account the argument of Sect. 2.5 about the complex gaussian and the fact that

$$\int_{\frac{Y'}{2\sqrt{s}}}^{+\infty} e^{-\sigma^2}d\sigma \leq e^{-\frac{Y'^2}{8s}}\int_0^{+\infty} e^{-\frac{\sigma^2}{2}}d\sigma,$$

while, to get the last inequality, we have used that for $z \in \mathbb{R}$

$$e^{\mu z}e^{-c\frac{z^2}{t}} \leq e^{\mu z}e^{-c\frac{z^2}{2T}}e^{-c\frac{z^2}{2t}} \leq C(\mu,T)e^{-c\frac{z^2}{2t}}.$$

A similar argument shows that

$$\sup_{Y\in\Sigma(\theta)} e^{\mu Y}\,|\partial_Y F| \leq c\sqrt{t}.$$

To estimate $\partial_{YY}F$, we first compute ∂_Y. Integrating by parts, we have

$$\partial_Y F = -\int_0^t \frac{e^{-\frac{Y^2}{4(t-s)}}}{\sqrt{\pi(t-s)}}ds + \int_0^t ds\int_0^{+\infty}(E_0^- + E_0^+)\frac{e^{-\frac{Y'^2}{4s}}}{\sqrt{\pi s}}dY'. \tag{B.1}$$

We can now estimate $\partial_{YY}F$ as follows:

$$
\begin{aligned}
e^{\mu Re(Y)}|\partial_{YY}F| = \\
&= e^{\mu Re(Y)}\left|\int_0^t \frac{Ye^{-\frac{Y^2}{4(t-s)}}}{2\sqrt{\pi}(t-s)^{3/2}}ds + \int_0^t ds \int_0^{+\infty} \partial_Y(E_0^- + E_0^+)\frac{e^{-\frac{Y'^2}{4s}}}{\sqrt{\pi s}}dY'\right| \\
&\leq e^{\mu Re(Y)} \int_{\frac{Y}{2\sqrt{t}}}^{+\infty} \frac{e^{-\sigma^2}}{\sqrt{\pi}}d\sigma + e^{\mu Re(Y-Y')} + \\
&\quad + \int_0^t ds \int_0^{+\infty} e^{\mu Re(Y-Y')}\partial_Y(E_0^- + E_0^+)e^{\mu Re(Y')}\frac{e^{-\frac{Y'^2}{4s}}}{\sqrt{\pi s}}dY' \\
&\leq C + C\int_0^t \frac{ds}{\sqrt{s(t-s)}} \leq C
\end{aligned}
$$

In the first of the inequalities above, we have used a standard change of variable in the first integral. To pass from the third line to the fourth, in the first integral, we have used the exponential decay of the integral, while in the second integral, the fact that the Gaussian in Y' dominates the exponential $e^{\mu Re(Y')}$, that the Gaussian in $Y - Y'$ dominates the exponential $e^{\mu Re(Y-Y')}$, and a standard change of variable.

The above estimate concludes the bound on h'. The estimate on the normal component h_n is a direct consequence of the expression (7.6), and of the above estimates on F. Actually, one could say more:

$$\sup_{Y\in\Sigma(\theta)} |\partial_Y^j h_n|_{l-3-j,\rho} \leq C\varepsilon t^{\frac{3-j}{2}}|u_{00}|_{l-j,\rho}. \tag{B.2}$$

Appendix C. Properties of $\bar{P}^\infty\tilde{E}_2$

It is easy to see that

$$|\bar{P}^\infty \mathbf{v}|_{l,\rho,j,\theta} \leq C|\mathbf{v}|_{l,\rho,j,\theta}. \tag{C.1}$$

Indeed, the convolution part of $\bar{P}^\infty\mathbf{u}$ can be treated with Young's inequality for convolutions, with the L^1 norm of $\varepsilon|\xi'|e^{-|\xi'|\varepsilon Y}\chi_{Y\geq 0}$ equal to 1; furthermore, taking derivatives with respect to Y, when the derivatives hit the extreme of integration the exponentials multiplying u_1 and u_2 are bounded and the $\varepsilon|\xi'|$ outside the integral is balanced by the fact that we are taking fewer derivatives with respect to x; on the other side, when the derivatives hit the exponentials inside the integral, this simply causes the appearance of an $\varepsilon|\xi'|$. and

The same argument shows that

$$|\sup_Y |\partial_Y^j\partial_x^i\bar{P}^\infty\mathbf{v}||_{0,\rho} \leq C\sum_{h+k\leq i+j, h\leq i} |\sup_Y |\partial_Y^h\partial_x^k\mathbf{v}||_{0,\rho}. \tag{C.2}$$

Moreover, since only the non convolution part of $\bar{P}^{\infty'}$ actually sees all the partial derivatives with respect to Y, we can write

$$\sup_Y |\partial_Y^j\partial_x^i\bar{P}^\infty\mathbf{v}|_{0,\rho} \leq \sup_Y |\partial_Y^j\partial_x^i v_1|_{0,\rho} + C\sum_{h+k\leq i+j, h<j} |\sup_Y |\partial_Y^h\partial_x^k\mathbf{v}||_{0,\rho}. \tag{C.3}$$

The above two estimates imply that, instead of $|\bar{P}^\infty\tilde{E}_2\mathbf{u}|_{l,\rho,2,\theta}$, we can estimate $|\tilde{E}_2\mathbf{u}|_{l,\rho,2,\theta}$, and that, instead of $\sup_Y |\partial_Y^i\partial_x^j\bar{P}^\infty\tilde{E}_2\mathbf{u}|_{0,\rho}$, we can estimate the right hand side of (C.3).

It is also useful to notice that

$$\sup_{Y\in\Gamma(\theta')} |f|_{0,\rho} \leq \left| \sup_{Y\in\Gamma(\theta')} |f| \right|_{0,\rho} \tag{C.4}$$

that can be proven using Minkowski's integral inequality.

We now pass to the estimates involving the $\tilde{E}_2$ operator. Using Minkowski integral inequality to pass under integral sign, then Young inequality for convolutions in both x and Y, we have that

$$|\partial_x^i \tilde{E}_2 f|_{0,\rho',\theta'}(t) \leq C \int_0^t ds |\partial_x^i f|_{0,\rho',\theta'}(s) \leq C|f|_{l,\rho,0,\theta,\beta,T} \tag{C.5}$$

$\forall i = 0,\ldots,l,\ \ \forall t,\ \ \forall \rho' \leq \rho - \beta t,\ \ \forall \theta' \leq \theta - \beta t$. If we want to take an extra partial derivative, we can either use Cauchy estimates or we can let the derivative hit the kernel, the cost being the appearance of an unbalanced $\frac{1}{\varepsilon\sqrt{t-s}}$

$$|\tilde{E}_2 f|_{l+1,\rho',0,\theta'} \leq C \int_0^t \frac{|f|_{l,\rho(s),0,\theta'}}{\rho(s)-\rho'} ds, \tag{C.6}$$

$$|\tilde{E}_2 f|_{l+1,\rho',0,\theta'} \leq \frac{C}{\varepsilon} \int_0^t \frac{|f|_{l,\rho',0,\theta'}}{\sqrt{t-s}} ds. \tag{C.7}$$

When we use an L^∞ norm for $Y \in \Gamma(\theta')$, we still use Young's inequality for convolution, but this time we use the L^2 norm (with respect to Y) on the kernel, so an additional $\frac{1}{(t-s)^{1/4}}$ appears: we have that

$$\begin{aligned} |\sup_{Y\in\Gamma(\theta')} |\partial_x^i \tilde{E}_2 f||_{0,\rho} &\leq C \int_0^t \frac{||\partial_x^i f|_{L^2(\gamma(\theta'))}|_{0,\rho}}{(t-s)^{1/4}} ds = C \int_0^t \frac{||\partial_x^i f|_{0,\rho}|_{L^2(\gamma(\theta'))}}{(t-s)^{1/4}} ds \\ &\leq C|f|_{l,\rho,0,\theta,\beta,T}. \end{aligned} \tag{C.8}$$

The key point here is that the L^2 norm in the Y variable and the Hardy norm in the x variable are interchangeable, essentially because the Hardy norm is an L^2 norm on the boundary lines. We also obtain that

$$|\sup_{Y\in\Gamma(\theta')} |\partial_x^{l+1} \tilde{E}_2 f||_{l+1,\rho} \leq \frac{C}{\varepsilon} \int_0^t \frac{||f|_{l,\rho}|_{L^2(\gamma(\theta'))}}{(t-s)^{3/4}} ds. \tag{C.9}$$

The same argument works for the first partial derivative with respect to Y: while we make the derivatives with respect to x act on f, we can leave the derivative with respect to Y on the kernel, so we obtain

$$|\partial_x^i \partial_Y \tilde{E}_2 f|_{0,\rho',\theta'} \leq C \int_0^t ds \frac{|\partial_x^i f|_{0,\rho',\theta'}(s)}{\sqrt{t-s}} \leq C|f|_{l,\rho,0,\theta,\beta,T} \tag{C.10}$$

$\forall i = 0,\ldots,l-1,\ \ \forall t,\ \ \forall \rho' \leq \rho - \beta t,\ \ \forall \theta' \leq \theta - \beta t$. If $\gamma f = 0$, then we can use integration by parts to move $\partial_Y = -\partial_{Y'}$ on f, so we obtain

$$|\partial_x^i \partial_Y \tilde{E}_2 f|_{0,\rho',\theta'} \leq C \int_0^t |\partial_x^i \partial_Y f|_{0,\rho',\theta'}(s) ds. \tag{C.11}$$

If we want to take l derivatives with respect to x, we still have

$$|\partial_Y \tilde{E}_2 f|_{l,\rho',0,\theta'} \leq C \int_0^t ds \frac{|f|_{l,\rho',0,\theta'}(s)}{\sqrt{t-s}}. \tag{C.12}$$

Using an L^∞ norm on Y, we have that

$$|\sup_{Y\in\Gamma(\theta')}|\partial_x^i\partial_Y\tilde{E}_2||_{0,\rho'}\leq C\int_0^t\frac{|f|_{l,\rho',0,\theta'}}{(t-s)^{3/4}}ds. \tag{C.13}$$

$i=0,\ldots,l$. Before we take a second partial derivative with respect to Y, we first perform an integration by parts; if $\gamma f=0$, then

$$|\partial_{YY}\tilde{E}_2 f|_{i,\rho',0,\theta'}\leq C\int_0^t ds\frac{|\partial_Y f|_{i,\rho',0,\theta'}}{(t-s)^{1/2}} \tag{C.14}$$

$i=0,\ldots,l-1$, while if $\gamma f\neq 0$ we have an additional term given by

$$\begin{aligned} C\left|\left|\int_0^t ds\frac{Ye^{-\frac{Y^2}{4(t-s)}}}{2(t-s)^{3/2}}\gamma\partial_x^i\tilde{f}ds\right|_{L^2(\Gamma(\theta'))}\right|_{0,\rho'} &\leq C\left|\int_0^t\frac{|\partial_x^i\tilde{f}|_{W^{1,2}(\Gamma(\theta'))}}{(t-s)^{3/4}}ds\right|_{0,\rho'}\\ &\leq C\int_0^t\frac{|f|_{l,\rho',0,\theta'}(s)}{(t-s)^{3/4}}ds, \end{aligned} \tag{C.15}$$

where

$$\tilde{f}=\int_{-\infty}^{+\infty}\frac{e^{-\frac{(x-x')^2}{4\varepsilon^2(t-s)}}}{\sqrt{4\pi(t-s)}}f(s,x)dx'. \tag{C.16}$$

In the L^∞ norm, if $\gamma f=0$ then

$$||\partial_x^i\partial_{YY}\tilde{E}_2 f|_{L^\infty(\Gamma(\theta'))}|_{0,\rho'}\leq C\int_0^t\frac{|\partial_Y f|_{i,\rho',0,\theta'}}{(t-s)^{3/4}}ds, \tag{C.17}$$

$i=0,\ldots,l-1$; otherwise, since we have an additional term given by

$$C\int_0^t\frac{Ye^{-\frac{Y^2}{4(t-s)}}}{2(t-s)^{3/2}}\gamma\partial_x^i\tilde{f}ds \tag{C.18}$$

this term has finite $||\cdot|_{l,\rho}|_{L^\infty(\Sigma(\theta'))}$ norm, because

$$\begin{aligned} C\left|\left|\int_0^t\frac{Ye^{-\frac{Y^2}{4(t-s)}}}{2(t-s)^{3/2}}\gamma\partial_x^i\tilde{f}ds\right|_{0,\rho'}\right|_{L^\infty_Y} &\leq C\left|\int_0^t\frac{Ye^{-\frac{Y^2}{4(t-s)}}}{2(t-s)^{3/2}}\gamma|\partial_x^i f|_{0,\rho'}ds\right|_{L^\infty_Y}\\ \leq C\left|\int_0^t\frac{Ye^{-\frac{Y^2}{4(t-s)}}}{2(t-s)^{3/2}}|f|_{l,\rho',1,\theta'}ds\right|_{L^\infty_Y} &\leq C|f|_{l,\rho,1,\theta,\beta,T}\sup_Y|\int_{\frac{Y}{2\sqrt{t}}}^{+\infty}e^{-\sigma^2}d\sigma|\\ &\leq C|f|_{l,\rho,1,\theta,\beta,T}. \end{aligned} \tag{C.19}$$

Notice that, in this case, we are not able to use the $||\cdot|_{L^\infty(\Sigma(\theta'))}|_{l,\rho}$ norm, because either we have to take the supremum with respect to s before we take the $|\cdot|_{0,\rho'}$ norm (in order to take f outside the time integral), or we have to bring the L^∞_Y norm inside the time integral, with $\sup_Y|\frac{Ye^{-\frac{Y^2}{4(t-s)}}}{2(t-s)^{3/2}}|=\frac{C}{t-s}$, which is not integrable in time.

Appendix D. Proof of Proposition 7.5

The results of the previous section, and the obvious equality $\partial_t \tilde{E}_2 \mathbf{u} = \mathbf{u} + \varepsilon^2 \partial_{xx} \tilde{E}_2 \mathbf{u} + \partial_{YY} \tilde{E}_2 \mathbf{u}$, show that $\bar{P}^\infty \tilde{E}_2 \mathbf{u} \in L^{l,\rho,\theta}_{\beta,T}$, with

$$|\bar{P}^\infty \tilde{E}_2 \mathbf{u}|_{l,\rho,\theta,\beta,T} \leq C|\mathbf{u}|_{l,\rho,1,\theta,\beta,T}. \tag{D.1}$$

Now we want to prove that the trace of $\bar{P}^\infty \tilde{E}_2 \mathbf{u}$ is regular enough for the application of proposition 3.3 in [35]. We have

$$\gamma \bar{P}^\infty_n (\tilde{E}_2 \mathbf{u}) = \varepsilon |\xi'| \int_0^{+\infty} e^{-\varepsilon|\xi'|Y'} \frac{i\xi'}{|\xi'|} \tilde{E}_2 u_1 dY', \tag{D.2}$$

$$\gamma \bar{P}^{\infty\prime} (\tilde{E}_2 \mathbf{u}) = \varepsilon |\xi'| \int_0^{+\infty} e^{-\varepsilon|\xi'|Y'} \frac{i\xi'}{|\xi'|} \tilde{E}_2 u_2 dY'. \tag{D.3}$$

We shall prove that, if $\mathbf{u} \in S^{l,\rho,1,\theta}_{\beta,T}$, the above two terms are both in $K'^{l,\rho}_{\beta,T}$. Since the terms in (D.2) and (D.3) are formally identical, it is enough to prove the statement for the normal component. Using (C.4), we have that

$$|\partial^i_x \gamma \bar{P}^\infty_n (\tilde{E}_2 \mathbf{u})|_{0,\rho'} \leq |\sup_{Y \in \Gamma(\theta')} |\partial^i_x \tilde{E}_2 u_1||_{0,\rho'}, \tag{D.4}$$

where the last term is bounded by (C.8). We also have that

$$\begin{aligned} |\partial^i_x \partial_t \gamma \bar{P}^\infty_n (\tilde{E}_2 \mathbf{u})|_{0,\rho'} = |\partial^i_x \gamma \bar{P}^\infty_n \partial_t (\tilde{E}_2 \mathbf{u})|_{0,\rho'} \leq \\ \leq |\partial^i_x \gamma \bar{P}^\infty_n \mathbf{u}|_{0,\rho'} + \varepsilon^2 |\gamma \bar{P}^\infty_n \partial^{i+2}_x \tilde{E}_2 \mathbf{u}|_{0,\rho'} + |\gamma \bar{P}^\infty_n \partial^i_x \partial_{YY} \tilde{E}_2 \mathbf{u}|_{0,\rho'}, \end{aligned} \tag{D.5}$$

with $i = 0, \ldots, l-2$. For the first term, we have

$$\begin{aligned} |\partial^i_x \gamma \bar{P}^\infty_n \mathbf{u}|_{0,\rho'} \leq C|\sup_Y |\partial^i_x u_1||_{0,\rho'} \leq C||\partial^i_x u_1|_{W^{1,2}(\Gamma(\theta'))}|_{0,\rho'} = \\ C||\partial^i_x u_1|_{0,\rho'}|_{W^{1,2}(\Gamma(\theta'))} \leq C|u_1|_{l,\rho',\theta'}. \end{aligned} \tag{D.6}$$

For the second term, we use Holder's inequality and (C.8). For the third term, $\partial_{YY} \tilde{E}_2 \mathbf{u}$ can be decomposed in two parts: a non-boundary part, which can be treated with Holder's inequality (with exponents 1 on the exponential, ∞ on $\tilde{E}_2$) and with (C.17); and the boundary part, which can be treated with Holder inequality (this time with exponents 2 and 2) and with (C.15). In this last term, we have $|\varepsilon|\xi'|e^{-\varepsilon|\xi'|Y}|_{L^2_Y(\mathbb{R}^+)} = C\sqrt{\varepsilon|\xi'|}$, but the additional $|\xi'|^{1/2}$ can be balanced by the fact that we have

$$|\xi'|^{1/2} |\gamma \partial^i_x u_1|_{0,\rho'} \leq (1 + |\xi'|)|u_1|_{l-1,\rho',1,\theta} \leq |u_1|_{l,\rho',1,\theta'}. \tag{D.7}$$

So the terms in (D.2) and (D.3) are in $K^{l,\rho}_{\beta,T}$, their value is zero for $t = 0$ and can be expressed as $|\xi'| \int_0^{+\infty} dy' f(\xi', y', t) k(\xi', y')$, with $|\xi'| \int_0^{+\infty} dy' |k(\xi', y')| \leq 1$: thus, all the hypothesis of proposition 3.4 in [35] are satisfied.

Appendix E. Properties of $\mathcal{S}\gamma\bar{P}^\infty\tilde{E}_2$

In this appendix, we shall obtain some estimates on the operator $\mathcal{S}\gamma\bar{P}^\infty\tilde{E}_2$; combining these estimates with those obtained in appendix C for $\bar{P}^\infty\tilde{E}_2$, we shall obtain the proof of propositions 7.6, 7.7, and 7.8.

More specifically: the bounds in an L^2_{xY}-like setting for derivatives of order up to l provide the proof of proposition 7.6; the estimates of the derivatives of order $l+1$ (i.e. with an extra derivative), in an

L^2_{xY}-like setting, are used to prove proposition 7.7; finally, the estimates in $L^\infty_Y L^2_x$ are used for proposition 7.8.

For notational simplicity we shall introduce the notation

$$\mathbf{g} = \gamma \bar{P}^\infty \tilde{E}_2 \mathbf{u}. \tag{E.1}$$

The Stokes operator, see [35], can be written explicitly as

$$\mathcal{S}\mathbf{g} = \begin{pmatrix} -N' e^{-\varepsilon Y|\xi'|} g_n + N'(1-\bar{U})\tilde{E}_1 V_1 \mathbf{g} \\ e^{-\varepsilon|\xi'|Y} g_n + \bar{U}\tilde{E}_1 V_1 \mathbf{g} \end{pmatrix} \tag{E.2}$$

where N' and V_1 are defined as

$$N' = \frac{i\xi'}{|\xi'|}, \qquad V_1\mathbf{g} = g_n - N' g',$$

$\bar{U}$ is the Ukai operator

$$\bar{U} f = \varepsilon|\xi'| \int_0^Y e^{-\varepsilon|\xi'|(Y-Y')} f(\xi', Y') dY',$$

and $E_1 f$ solves the heat equation on the semi-space with boundary condition f and homogeneous initial datum; the explicit expression of E_1 is

$$\tilde{E}_1 f = \int_0^t ds \frac{Y}{t-s} \frac{e^{-\frac{Y^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}} \int_{-\infty}^{+\infty} f(x', s) \frac{e^{-\frac{(x-x')^2}{4\varepsilon^2(t-s)}}}{\sqrt{4\pi(t-s)\varepsilon^2}} dx'.$$

It is useful introduce the decomposition $\mathcal{S}\mathbf{g} = \mathcal{S}^E\mathbf{g} + \bar{\mathcal{S}}\mathbf{g} + \mathcal{S}^*\mathbf{g}$, where

$$\mathcal{S}^E\mathbf{g} = \begin{pmatrix} -N' e^{-\varepsilon Y|\xi'|} g_n \\ e^{-\varepsilon|\xi'|Y} g_n \end{pmatrix}, \qquad \bar{\mathcal{S}}\mathbf{g} = \begin{pmatrix} -N'\bar{U}\tilde{E}_1 V_1 \mathbf{g} \\ \bar{U}\tilde{E}_1 V_1 \mathbf{g} \end{pmatrix},$$
$$\mathcal{S}^*\mathbf{g} = \begin{pmatrix} N'\tilde{E}_1 V_1 \mathbf{g} \\ 0 \end{pmatrix}.$$

In what follows we shall estimate $\mathcal{S}^E\mathbf{g}$, $\bar{\mathcal{S}}\mathbf{g}$ and $\mathcal{S}^*\mathbf{g}$ separately. The general ideas to bound these terms are the following. The estimates of $\mathcal{S}^E\mathbf{g}$ are easy to achieve since every normal derivative is essentially transformed into ε times a tangential derivative. Concerning $\bar{\mathcal{S}}\mathbf{g}$, since all terms appear inside the Ukai operator $\bar{U}$, ∂_Y operates essentially as ε times a tangential derivative; therefore, the estimates of $\partial_Y \partial_x^j \bar{\mathcal{S}}\mathbf{g}$ are achieved if one is able to estimate $\varepsilon\partial_x^{j+1}\mathcal{S}^*\mathbf{g}$. Concerning the second normal derivative ∂_{YY}, one has that $\partial_{YY}\partial_x^j\bar{\mathcal{S}}\mathbf{g}$ has a part that behaves like $\varepsilon\partial_Y\partial_x^{j+1}\mathcal{S}^*\mathbf{g}$, and a part that behaves like $\varepsilon^2\partial_x^{j+2}\mathcal{S}^*\mathbf{g}$. This means that the estimates of $\bar{\mathcal{S}}\mathbf{g}$ follow from the estimates of $\mathcal{S}^*\mathbf{g}$.

Appendix E.1. Proof of Proposition 7.6

We begin with the estimates of $\mathcal{S}^E\mathbf{g}$

$$\begin{aligned}
&\left|\partial_x^i\partial_Y^j\mathcal{S}^E\mathbf{g}\right|^2_{0,\rho',\theta'}\\
&\le \varepsilon^{2j}\int\limits_{\Gamma(\theta')} dY\int d\xi' e^{2\rho|\xi'|}\left[\int\limits_0^{+\infty} dY'\varepsilon|\xi'|e^{-\varepsilon|\xi'|(Y+Y')}|\xi'|^{i+j}|\tilde{E}_2u_1|\right]^2\\
&\le \varepsilon^{2j}\int\limits_{\Gamma(\theta')} dY\int d\xi' e^{2\rho|\xi'|}\int\limits_0^{+\infty} dY'\varepsilon|\xi'|e^{-\varepsilon|\xi'|(2Y+Y')}|\xi'|^{2(i+j)}|\tilde{E}_2u_1|^2\\
&\le C\varepsilon^{2j}\int d\xi' e^{2\rho|\xi'|}\int\limits_0^{+\infty} dy' e^{-\varepsilon|\xi'|Y'}|\xi'|^{2(i+j)}|\tilde{E}_2u_1|^2dy'\\
&\le C\varepsilon^{2j}|\tilde{E}_2u_1|^2_{i+j,\rho',0,\theta'}.
\end{aligned}\tag{E.3}$$

In the second inequality we have used Cauchy-Schwarz inequality; in the third inequality we have integrated in Y, and in the last inequality we have used $|e^{-\varepsilon|\xi'|Y'}|\le 1$.

Now we pass to the estimates of $\mathcal{S}^*\mathbf{g}$. First, we consider the case when one takes the tangential derivatives ∂_x^i, with $i\le l$:

$$\begin{aligned}
&|\partial_x^i N'\tilde{E}_1V_1\mathbf{g}|_{0,\rho',\theta'}\le\\
&C\int\limits_0^t ds\frac{|\gamma\bar{P}^\infty\tilde{E}_2\mathbf{u}|_{i,\rho'}}{(t-s)^{3/4}}\le C\int\limits_0^t\frac{ds}{(t-s)^{3/4}}\int\limits_0^s\frac{|u(\cdot,\cdot,s')|_{i,\rho',0,\theta'}}{(s-s')^{1/4}}ds'=\\
&C\int\limits_0^t|u(\cdot,\cdot,s')|_{i,\rho',0,\theta'}ds'\int\limits_{s'}^t\frac{ds}{(t-s)^{3/4}(s-s')^{1/4}}\le C\int\limits_0^t|u(\cdot,\cdot,s')|_{i,\rho',0,\theta'}ds',
\end{aligned}\tag{E.4}$$

where we used the fact that

$$\int\limits_{s'}^t\frac{ds}{(t-s)^{3/4}(s-s')^{1/4}}=\int\limits_0^{t-s'}\frac{ds''}{(t-s'-s'')^{3/4}(s'')^{1/4}}=\int\limits_0^1\frac{d\tau}{(1-\tau)^{3/4}\tau^{1/4}}<C.$$

Second, one has to estimate the normal derivatives of $\mathcal{S}^*\mathbf{g}$; define

$$f=N'V_1\gamma\bar{P}^\infty\tilde{E}_2\mathbf{u}.\tag{E.5}$$

Some lengthy but straightforward calculations lead to write

$$\begin{aligned}
\partial_Y\tilde{E}_1f=\int\limits_0^t ds\frac{e^{-\frac{y^2}{4\nu(t-s)}}}{\sqrt{\pi\nu(t-s)}}\int\limits_{\mathbb{R}^{d-1}}dx''\frac{e^{-\frac{|x-x'|^2}{4\nu(t-s)}}}{(4\pi\nu(t-s))^{(d-1)/2}}(\nu\partial_{xx}f-\partial_tf)=\\
=\tilde{D}_1(\varepsilon^2\partial_{xx}f-\partial_tf).
\end{aligned}\tag{E.6}$$

In the functional setting of Proposition 7.6, we are not allowed to take a time derivative: however, we can use the properties of the operator $\tilde{E}_2$ to express the time derivative as a function of the normal and tangential derivatives. In fact

$$\partial_tf=N'V_1\gamma\bar{P}^\infty\partial_t\tilde{E}_2\mathbf{u},\tag{E.7}$$

with

$$\gamma\bar{P}^\infty\partial_t\tilde{E}_2\mathbf{u}=\gamma\bar{P}^\infty\mathbf{u}+\gamma\bar{P}^\infty\varepsilon^2\partial_{xx}\tilde{E}_2\mathbf{u}+\gamma\bar{P}^\infty\partial_{YY}\tilde{E}_2\mathbf{u}=\mathbf{h}_1+\mathbf{h}_2+\mathbf{h}_3\tag{E.8}$$

and

$$\varepsilon^2 \partial_{xx} f = N' V_1 \varepsilon^2 \partial_{xx} \gamma \bar{P}^\infty \tilde{E}_2 \mathbf{u}, \tag{E.9}$$

with

$$\varepsilon^2 \gamma \partial_{xx} \bar{P}^\infty \tilde{E}_2 \mathbf{u} = \mathbf{h}_2. \tag{E.10}$$

Although the contribution of $\mathbf{h}_2$ is canceled, it is still useful to perform an estimate for this term, since it is used in the estimate of $\mathbf{h}_3$.
As regards $\mathbf{h}_1$, we have that, for $i \leq l-1$

$$||\sup Y |\partial_x^i \mathbf{u}|||_{\rho'} \leq C ||\mathbf{u}||_{l,\rho',1,\theta'}, \tag{E.11}$$

with the trace which satisfies a similar bound. Therefore

$$|\tilde{D}_1 N' \gamma u_1|_{i,\rho',0,\theta'} \leq C \int_0^t \frac{|\mathbf{u}|_{l,\rho',0,\theta'}}{(t-s)^{1/4}} ds. \tag{E.12}$$

In order to bound $\mathbf{h}_2 = \gamma \bar{P}^\infty \varepsilon^2 \partial_{xx} \tilde{E}_2 \mathbf{u}$, we let one of the tangential derivatives of $\partial_{xx} \tilde{E}_2 \mathbf{u}$ hit the kernel $\tilde{E}_2$ and one hit $\mathbf{u}$, so that, for $i \leq l-1$

$$\begin{aligned} |\tilde{D}_1 V_1 \mathbf{h}_2|_{i,\rho',0,\theta'} \leq C\varepsilon \int_0^t \frac{ds}{(t-s)^{1/4}} \int_0^s \frac{|u|_{i+1,\rho',0,\theta'}(s')}{(s-s')^{3/4}} ds' = \\ C\varepsilon \int_0^t |u|_{l,\rho',0,\theta'}(s') ds'. \end{aligned} \tag{E.13}$$

We first estimate $\mathbf{h}_3$ assuming that $\gamma \mathbf{u} = \mathbf{0}$: under this assumption, in $\partial_{YY} \tilde{E}_2 \mathbf{u}$, we can move one of the normal derivatives on $\mathbf{u}$, without boundary terms. Therefore, for $i \leq l-1$ we have, in the L^2_Y setting of the proposition 7.6

$$|\tilde{D}_1 V_1 \mathbf{h}_3|_{i,\rho',0,\theta'} \leq C \int_0^t \frac{ds}{(t-s)^{1/4}} \int_0^s \frac{\partial_Y \mathbf{u}|(s')_{i,\rho',0,\theta'}}{(s-s')^{3/4}} |ds \leq C \int_0^t |\mathbf{u}|_{l,\rho',1,\theta'}(s) ds, \tag{E.14}$$

If $\gamma \mathbf{u} \neq \mathbf{0}$, an additional term appears in the expression of $\bar{P}^\infty \partial_{YY} \tilde{E}_2 \mathbf{u}$, which roughly behaves like $\tilde{E}_1 \gamma u_1$; the loss of half derivative due to the trace operator is not a problem for the tangential derivatives of order up to $l-1$, so inequality (E.14) still holds. This concludes the estimates of the normal derivative of $\bar{\mathcal{S}}$.

As we already said at the beginning of this section, the estimates of $\bar{\mathcal{S}}\mathbf{g}$ can be derived from the ones of $\mathcal{S}^*\mathbf{g}$; therefore, the proof of the proposition is complete.

Appendix E.2. Proof of Proposition 7.7

The estimate (E.3) obtained for $\mathcal{S}^E \mathbf{g}$ still holds for derivatives of order $l+1$; the extra derivative can be treated either using the Cauchy estimate or giving a tangential derivative to the kernel of $\tilde{E}_2$.

The same applies for $\partial_x^{l+1} \mathcal{S}^* \mathbf{g}$: the only difference is that we change the order of integration in time. When we use the Cauchy estimate, this is necessary to eliminate the mild singularity in time deriving from the L^2_Y norm of the operator $\tilde{E}_1$. In contrast, when we let one tangential derivative hit the kernel of $\tilde{E}_2$, changing the order of integration allows us to reduce the singularity in time. We have

$$|N' \tilde{E}_1 V_1 \mathbf{g}|_{l+1,\rho',0,\theta'} \leq \frac{C}{\varepsilon} \int_0^t |u|_{l,\rho',0,\theta'} ds' \int_{s'}^t \frac{ds}{(t-s)^{3/4}(s-s')^{3/4}}, \tag{E.15}$$

with

$$\int_{s'}^{t} \frac{ds}{(t-s)^{3/4}(s-s')^{3/4}} = \int_{0}^{t-s'} \frac{ds''}{(t-s-s'')^{3/4}(s-s')^{3/4}} = \frac{1}{(t-s')^{1/2}} \int_{0}^{1} \frac{d\tau}{(1-\tau)^{3/4}\tau^{3/4}}, \tag{E.16}$$

so that

$$|N'\tilde{E}_1 V_1 \mathbf{g}|_{l+1,\rho',0,\theta'} \leq \frac{C}{\varepsilon} \int_{0}^{t} \frac{|u|_{l,\rho',0,\theta'}(s')}{(t-s')^{1/2}} ds'. \tag{E.17}$$

We pass to estimate $\partial_Y \mathcal{S}'\mathbf{g}$. First, we consider the term $\mathbf{h}_1$, introduced in (E.8), under the assumption that $\gamma\mathbf{u} = 0$: in this case,

$$|\mathbf{h}_1| \leq C\sqrt{\varepsilon|\xi'|}|\mathbf{u}|_{L^2_Y}. \tag{E.18}$$

Using

$$(\varepsilon|\xi'|)^{1/2} e^{-\varepsilon^2(t-s)|\xi'|^2} \leq \frac{C}{(t-s)^{1/4}} \tag{E.19}$$

we have that

$$|\tilde{D}_1 V_1 \mathbf{h}_1|_{l,\rho',0,\theta'} \leq C \int_{0}^{t} \frac{|\mathbf{u}|_{l,\rho',0,\theta'}}{(t-s)^{1/2}} ds. \tag{E.20}$$

If $\gamma\mathbf{u} \neq 0$, then $V_1\mathbf{h}_1$ contains $N'\gamma u_1$ as additional term, with

$$|\gamma u_1| \leq |u_1|^{1/2}_{L^2_Y} |\partial_Y u_1|^{1/2}_{L^2_Y}. \tag{E.21}$$

We use the interpolation inequality (E.21), together with the Cauchy estimates, to obtain

$$|\tilde{D}_1 N'\gamma u_1|_{l,\rho',0,\theta'} \leq C \int_{0}^{t} \frac{|\mathbf{u}|^{1/2}_{l,\rho',0,\theta'} |\mathbf{u}|^{1/2}_{l,\rho(s),1,\theta'}}{(t-s)^{1/4}(\rho(s)-\rho')^{1/2}} ds \leq C \int_{0}^{t} \frac{|\mathbf{u}|_{l,\rho',0,\theta'}|}{(t-s)^{1/2}} + \frac{|\mathbf{u}|_{l,\rho(s),1,\theta'}}{(\rho(s)-\rho')} ds. \tag{E.22}$$

This concludes the estimates of $\mathbf{h}_1$.

The bounds of $\mathbf{h}_2 = \gamma\bar{P}^\infty \varepsilon^2 \partial_{xx} \tilde{E}_2 \mathbf{u}$ are obtained moving one tangential derivative on the kernel of D_1 and one tangential derivative on the kernel of $\tilde{E}_2$, so

$$|\tilde{D}_1 V_1 \mathbf{h}_2|_{l,\rho',0,\theta'} \leq C \int_{0}^{t} \frac{ds}{(t-s)^{3/4}} \int_{0}^{s} \frac{|u|_{l,\rho',0,\theta'}(s')}{(s-s')^{3/4}} ds' = C \int_{0}^{t} \frac{|u|_{l,\rho',0,\theta'}(s')}{(t-s')^{1/2}} ds'. \tag{E.23}$$

We begin the estimates of $\mathbf{h}_3$, starting from the case $\gamma\mathbf{u} = \mathbf{0}$: in this case, $\gamma\partial_Y Y \tilde{E}_2 \mathbf{u} = \mathbf{0}$, so only the integral part of $\gamma\bar{P}^\infty$ is nonzero, when we apply this operator to $\partial_Y Y \tilde{E}_2 \mathbf{u}$. We can integrate by parts two times, moving the normal derivatives on the kernel of $\bar{P}^\infty$: the result is a term which behaves like $\varepsilon^2 \partial_{xx} \bar{P}^\infty \tilde{E}_2 \mathbf{u}$ (and therefore can be bounded exactly like $\mathbf{h}_2$) plus an additional term, given by

$$Ci\xi'\varepsilon \int_{0}^{s} ds' e^{-\varepsilon^2(s-s')\xi'^2} \int_{0}^{+\infty} \frac{Y'}{(s-s')^{3/2}} e^{-\frac{Y'^2}{4(s-s')}} (u_2, u_1)(\xi', Y, s) dY'. \tag{E.24}$$

This term is inside D_1: we can move $i\xi'$ on the kernel of D_1, so that the singularities in time are better distributed. Therefore, we obtain

$$|\tilde{D}_1 V_1 \mathbf{h}_3|_{l,\rho',0,\theta'} \leq C \int_0^t \frac{|u|_{l,\rho',0,\theta'}(s')}{(t-s')^{1/2}} ds'. \tag{E.25}$$

If $\gamma \mathbf{u} \neq \mathbf{0}$, an additional term appears in the expression of $\bar{P}^\infty \partial_{YY} \tilde{E}_2 \mathbf{u}$, which roughly behaves like $\tilde{E}_1 \gamma u_1$; as for $\mathbf{h}_1$, in this case we use the interpolation inequality (E.21), the Cauchy estimate and Young's inequality, obtaining

$$|\tilde{D}_1 V_1 \mathbf{h}_3|_{l,\rho',0,\theta'} \leq C \int_0^t \frac{|u|_{l,\rho',0,\theta'}}{(t-s')^{1/2}} + \frac{|u|_{l,\rho(s'),1,\theta'}(s')}{\rho(s')-\rho'} ds'. \tag{E.26}$$

Finally, when we take one more derivative with respect to Y, we obtain, for a generic f with $f(t=0)=0$, that

$$\begin{aligned} &\partial_{YY} \tilde{E}_1 f = \\ &\int_0^t \frac{Y e^{-\frac{Y^2}{4(t-s)}}}{4\sqrt{\pi}(t-s)^{3/2}} ds \int_{-\infty}^{+\infty} \left((\partial_t f)(x',s) - \varepsilon^2 (\partial_{xx} f)(x',s)\right) \frac{e^{-\frac{(x-x')^2}{4\varepsilon^2(t-s)}}}{\sqrt{4\pi\varepsilon^2(t-s)}} dx' \\ &= \tilde{E}_1(\partial_t f - \varepsilon^2 \partial_{xx} f). \end{aligned} \tag{E.27}$$

Therefore, we have to evaluate $\tilde{E}_1 \gamma V_1 \mathbf{h}_1$ and $\tilde{E}_1 \gamma V_1 \mathbf{h}_3$: this time, since two normal derivatives have been taken, only up to $l-1$ tangential derivatives can be taken. Therefore, the loss of half derivative due to the trace operator is not a problem, and the case $\gamma \mathbf{u} \neq \mathbf{0}$ does not need to be treated separately. For $\mathbf{h}_1$, we have

$$|\tilde{E}_1 V_1 \mathbf{h}_1|_{i,\rho',0,\theta'} \leq C \int_0^t \frac{|\mathbf{u}|_{i+i,\rho',0,\theta'}}{(t-s)^{3/4}} ds. \tag{E.28}$$

For $\tilde{E}_1 \gamma V_1 \mathbf{h}_3$, a similar estimate holds: therefore, the estimates of $\mathcal{S}^*$ are complete.

The estimates of $\bar{\mathcal{S}}\mathbf{g}$ can be derived from the ones of $\mathcal{S}^*\mathbf{g}$.

Appendix E.3. Proof of Proposition 7.8

In the $L^\infty_Y L^2_x$-like setting of proposition 7.8, the estimates differ from the ones of 7.7 by an additional $(t-s)^{-1/4}$, which implies a stronger (but still integrable) singularity in time. Indeed, for the estimates of $\mathcal{S}^*\mathbf{g}$ (and therefore of $\bar{\mathcal{S}}\mathbf{g}$), we pass the L^∞_Y norm to the kernel of $\mathbf{E}_1$, which gives an extra term $(t-s)^{-1/4}$ as compared to its L^2_Y norm. For the estimates of $\mathcal{S}^E\mathbf{g}$, instead, we pass to $e^{-\varepsilon|\xi'|Y}$ the L^∞_Y norm, which is 1, while its L^2_Y norm is $C(\varepsilon|\xi'|)^{-1/2}$: moving the additional $(\varepsilon|\xi'|)^{1/2}$ on the kernel of $\tilde{E}_2$, we can use the bound given by Eq. (E.19), and an additional $(t-s)^{-1/4}$ appears.

Appendix F. Estimate of the Linear Terms in (7.34)

In $[\mathbf{u}^{NS0} + \varepsilon(\boldsymbol{w} + \boldsymbol{w} + \boldsymbol{\sigma})] \cdot \nabla \mathcal{N}^* \mathbf{e}^*$ the most problematic term is $u^S \partial_x \mathcal{N}^* \mathbf{e}^*$: using propositions 7.6 and 7.7, we have that

$$|u^S \partial_x \ \mathcal{N}^* \mathbf{e}^*|_{l,\rho',0,\theta'} \leq C \int_0^t \frac{|\mathbf{e}^*|_{l,\rho(s),0,\theta}}{\rho(s) - \rho'} ds, \tag{F.1}$$

$$|\partial_Y u^S \partial_x \mathcal{N}^* \mathbf{e}^*|_{l-1,\rho',0,\theta'} \leq \frac{C}{\sqrt{t}} \int_0^t |\mathbf{e}^*|_{l,\rho',0,\theta} ds, \tag{F.2}$$

$$|u^S \partial_Y \partial_x \mathcal{N}^* \mathbf{e}^*|_{l-1,\rho',0,\theta'} \leq C \int_0^t \frac{|\mathbf{e}^*|_{l,\rho',1,\theta}}{(t-s)^{1/2}} ds. \tag{F.3}$$

In $\mathcal{N}^* \mathbf{e}^* \cdot \nabla [\mathbf{u}^{NS0} + \varepsilon(\boldsymbol{\sigma} + \boldsymbol{w} + \boldsymbol{\sigma})]$, the most problematic term is $\frac{1}{\varepsilon}(\mathcal{N}^* \mathbf{e}^*)_2 \partial_Y \tilde{u}^P$ (the problem with $\tilde{u}^R$ is to prove that the term is $O(1)$). Since $\mathcal{N}^* \mathbf{e}^*$ is divergence free and zero at the boundary, we have

$$(\mathcal{N}^* \mathbf{e}^*)_2 = -\varepsilon \int_0^Y \partial_x \left(\mathcal{N}^* \mathbf{e}^*\right)_1 dY', \tag{F.4}$$

so

$$\begin{aligned} \left| \frac{1}{\varepsilon} (\mathcal{N}^* \mathbf{e}^*)_2 \partial_Y \tilde{u}^P \right|_{l,\rho',0,\theta'} &\leq C \left| \frac{1}{Y} \int_0^Y \partial_x \left(\mathcal{N}^* \mathbf{e}^*\right)_1 dY' \right|_{l,\rho',0,\theta'} \sup_Y \left| Y \partial_Y \tilde{u}^P \right|_{l,\rho'} \leq \\ & C \left| \partial_x \left(\mathcal{N}^* \mathbf{e}^*\right)_1 \right|_{l,\rho',0,\theta'} \leq C \int_0^t \frac{|\mathbf{e}^*|_{l,\rho(s),0,\theta'}}{\rho(s) - \rho'} ds. \end{aligned} \tag{F.5}$$

In the first inequality we used the algebra property 2.1, in the second inequality we used the boundedness of $\sup_Y \left| Y \partial_Y \tilde{u}^P \right|_{l,\rho'}$ (both for the regular and for the singular part) and the fact that the average operator is bounded from L^2 to L^2 (is bounded by the maximal function, so its trivial), and in the last inequality we used proposition 7.8. With a similar argument we have that

$$\left| \frac{1}{\varepsilon} (\mathcal{N}^* \mathbf{e}^*)_2 \partial_{YY} \tilde{u}^P \right|_{l-1,\rho',0,\theta'} \leq C \left(1 + \frac{1}{t^{1/2}}\right) \int_0^t |\mathbf{e}^*|_{l,\rho',0,\theta'} ds \tag{F.6}$$

and

$$\begin{aligned} \left| \frac{1}{\varepsilon} \partial_Y (\mathcal{N}^* \mathbf{e}^*)_2 \partial_Y \tilde{u}^P \right|_{l-1,\rho',0,\theta'} &= \left| \partial_x (\mathcal{N}^* \mathbf{e}^*)_1 \partial_Y \tilde{u}^P \right|_{l-1,\rho',0,\theta'} \leq \\ & C \left(1 + \frac{1}{t^{1/2}}\right) \int_0^t |\mathbf{e}^*|_{l,\rho',0,\theta'} ds \end{aligned}. \tag{F.7}$$

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