GRADIENT HIGHER INTEGRABILITY FOR DOUBLE PHASE PROBLEMS ON METRIC MEASURE SPACES

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ABSTRACT. We study local and global higher integrability properties for quasiminimizers of a class of double phase integrals characterized by nonstandard growth conditions. We work purely on a variational level in the setting of a metric measure space with a doubling measure and a Poincaré inequality. The main novelty is an intrinsic approach to double phase Sobolev-Poincaré inequalities.

1. INTRODUCTION

Assume that (X, d, μ) is a complete metric measure space endowed with a metric d and a doubling measure μ and supporting a weak $(1, p)$ -Poincaré inequality. Let Ω be an open subset of X. This paper discusses regularity properties of the minimal p-weak upper gradient of quasiminimizers of the double phase integral

$$
\int_{\Omega} H(x, g_u) d\mu = \int_{\Omega} (g_u^p + a(x)g_u^q) d\mu,
$$
\n(1.1)

with

$$
1 < \frac{q}{p} \le 1 + \frac{\alpha}{Q}, \quad 0 < \alpha \le 1, \quad Q = \log_2 C_D,
$$

where $p > 1$ and C_D is the doubling constant of the measure. Observe that Q is a notion of dimension related to the measure μ . For example, in the Euclidean *n*-space with the Lebesgue measure we have $Q = n$. The double phase functional in [\(1.1\)](#page-0-0) is denoted by

$$
H(x, z) = |z|^p + a(x)|z|^q, \quad x \in \Omega, \quad z \in \mathbb{R}.
$$

The nonnegative coefficient function a is assumed to be α -Hölder continuous with respect to a quasi-distance related to the underlying measure μ , see [\(2.3\)](#page-2-0) below for the precise definition. This reduces to the standard Hölder continuity with the exponent α , if the measure is Q-Ahlfors–David regular.

The main feature of the functional [\(1.1\)](#page-0-0) is that it switches between two different types of growth conditions determined by the coefficient function a. When $a(x) = 0$, the variational integral in [\(1.1\)](#page-0-0) reduces to the familiar problem with p-growth and when $a(x) \geq c > 0$ we have the (p, q) -problem. Thus the zero set ${a(x) = 0}$ plays a decisive role in [\(1.1\)](#page-0-0). The main advantage of the notion of quasiminimizer of (1.1) is that it simultaneously covers a large class of problems where the variational integrand $F : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies the Carathéodory conditions and

$$
\lambda H(x, z) \le F(x, u, z) \le \Lambda H(x, z), \quad 0 < \lambda < \Lambda < \infty,
$$

for every $x \in \Omega$ and $u, z \in \mathbb{R}$. For quasiminimizers with the p-growth on Euclidean spaces, see [\[4,](#page-13-0) [14,](#page-13-1) [15\]](#page-13-2), and on metric measure spaces, see [\[1,](#page-13-3) [2,](#page-13-4) [24,](#page-13-5) [25,](#page-13-6) [26\]](#page-13-7). This paper extends the theory for quasiminimizers on metric measure space to the double phase problems.

The natural function space for a local quasiminimizer of [\(1.1\)](#page-0-0) is $u \in N^{1,1}_{loc}(\Omega)$ with $H(\cdot, g_u) \in L^1_{loc}(\Omega)$, where $N^{1,1}$ denotes the Newtonian–Sobolev space on a metric measure space, see [\[1\]](#page-13-3), [\[21\]](#page-13-8) and [\[38\]](#page-14-0). We show that if u is a local quasiminimizer of [\(1.1\)](#page-0-0), then $H(x, g_u)$ is locally integrable to a slightly higher power than one, see Theorem [4.2.](#page-9-0) We also discuss the corresponding question up to the boundary for quasiminiminimizers with boundary values, see Theorem [4.4.](#page-10-0) For this kind of local higher integrability results in the Euclidean case, see [\[3,](#page-13-9) [13,](#page-13-10) [14,](#page-13-1) [15,](#page-13-2) [16,](#page-13-11) [33,](#page-14-1) [34\]](#page-14-2). For the corresponding global results, we refer to $[18, 23]$ $[18, 23]$. For results with functionals of the type (1.1) in the Euclidean setting, we refer to $[5, 6, 7, 8, 9]$ $[5, 6, 7, 8, 9]$ $[5, 6, 7, 8, 9]$ $[5, 6, 7, 8, 9]$ $[5, 6, 7, 8, 9]$, [28,](#page-14-3) [29,](#page-14-4) [30,](#page-14-5) [35,](#page-14-6) [36\]](#page-14-7). Higher integrability questions for variational problems on metric measure spaces have

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been studied in [\[10,](#page-13-19) [11,](#page-13-20) [12,](#page-13-21) [19,](#page-13-22) [20,](#page-13-23) [27,](#page-14-8) [32,](#page-14-9) [31,](#page-14-10) [37\]](#page-14-11). Our work shows that the corresponding theory can be developed for double phase problems on metric measure spaces. The argument is based on energy estimates, double phase Sobolev-Poincaré inequalities and a self-improving property of reverse Hölder inequalities.

2. Preliminaries

Throughout the paper, positive constants are denoted by C and the dependencies on parameters are listed in the parentheses. We assume that (X, d, μ) is a complete metric measure space with a metric d and a Borel regular measure μ . The measure μ is assumed to be doubling, that is, there exists a constant $C_D \geq 1$ such that

$$
0 < \mu(B_{2r}) \le C_D \mu(B_r) < \infty,\tag{2.1}
$$

for every ball B_r in X. Here $B_r = B_r(x) = \{x \in X : d(y, x) < r\}$ is an open ball with the center $x \in X$ and the radius $0 < r < \infty$. The following result gives a notion of dimension related to a doubling measure.

Lemma 2.1 ([\[1\]](#page-13-3), Lemma 3.3). Let (X, d, μ) be a metric measure space with a doubling measure μ . Then

$$
\frac{\mu(B_r(y))}{\mu(B_R(x))} \ge C\left(\frac{r}{R}\right)^Q,\tag{2.2}
$$

for every $0 < r \le R < \infty$, $x \in X$ and $y \in B_R(x)$. Here $Q = \log_2 C_D$ and $C = C_D^{-2}$.

A complete metric measure space with a doubling measure is proper, that is, closed and bounded subsets are compact, see [\[1,](#page-13-3) Proposition 3.1]. We discuss the notion of upper gradient as a way to generalize modulus of the gradient in the Euclidean case to the metric setting. For further details, we refer to the book by Björn and Björn [\[1\]](#page-13-3).

Definition 2.2. A nonnegative Borel function g is said to be an upper gradient of function $u : X \to [-\infty, \infty]$ if, for all paths γ connecting x and y, we have

$$
|u(x) - u(y)| \le \int_{\gamma} g \, ds,
$$

whenever $u(x)$ and $u(y)$ are both finite and $\int_{\gamma} g ds = \infty$ otherwise. Here x and y are the endpoints of γ . Moreover, if a nonnegative measurable function g satisfies the inequality above for p -almost every path, that is, with the exception of a path family of zero p-modulus, then g is called a p-weak upper gradient of u .

For $1 \leq p < \infty$ and an open set $\Omega \subset X$, let

$$
||u||_{N^{1,p}(\Omega)} = ||u||_{L^p(\Omega)} + \inf ||g||_{L^p(\Omega)},
$$

where the infimum is taken over all upper gradients g of u. Consider the collection of functions $u \in L^p(\Omega)$ with an upper gradient $g \in L^p(\Omega)$ and let

$$
\widetilde{N}^{1,p}(\Omega) = \{u : ||u||_{N^{1,p}(\Omega)} < \infty\}.
$$

The Newtonian space is defined by

$$
N^{1,p}(\Omega) = \{u : ||u||_{N^{1,p}(\Omega)} < \infty\} / \sim,
$$

where $u \sim v$ if and only if $||u - v||_{N^{1,p}(\Omega)} = 0$.

The corresponding local Newtonian space is defined by $u \in N^{1,p}_{loc}(\Omega)$ if $u \in N^{1,p}(\Omega')$ for all $\Omega' \subseteq \Omega$, see [\[1,](#page-13-3) Proposition 2.29], where $\Omega' \in \Omega$ means that $\overline{\Omega'}$ is a compact subset of Ω . If u has an upper gradient $g \in L^p(\Omega)$, there exists a unique minimal p-weak upper gradient $g_u \in L^p(\Omega)$ with $g_u \leq g$ μ -almost everywhere for all p-weak upper gradients $g \in L^p(\Omega)$ of u, see [\[1,](#page-13-3) Theorem 2.5]. Moreover, the minimal p-weak upper gradient is unique up to sets of measure zero. For $u \in N^{1,p}(\Omega)$ we have

$$
||u||_{N^{1,p}(\Omega)} = ||u||_{L^p(\Omega)} + ||g_u||_{L^p(\Omega)},
$$

where g_u is the minimal p-weak upper gradient of u. The main advantage is that p-weak upper gradients behave better under L^p -convergence than upper gradients, see [\[1,](#page-13-3) Proposition 2.2]. However, the difference is relatively small, since every p-weak upper gradient can be approximated by a sequence of upper gradients in L^p , see [\[1,](#page-13-3) Lemma 1.46]. This implies that the $N^{1,p}$ -norm above remains the same if the infimum is taken over upper gradients instead of p-weak upper gradients.

Let Ω be an open subset of X. We define $N_0^{1,q}(\Omega)$ to be the set of functions $u \in N^{1,q}(X)$ that are zero on $X \setminus \Omega$ μ -a.e. The space $N_0^{1,q}(\Omega)$ is equipped with the norm $\|\cdot\|_{N^{1,q}}$. Note also that if $\mu(X \setminus \Omega) = 0$, then $N_0^{1,q}(\Omega) = N^{1,q}(X)$. We shall therefore always assume that $\mu(X \setminus \Omega) > 0$.

The integral average is denoted by

$$
u_B = \int_B u \, \mathrm{d}\mu = \frac{1}{\mu(B)} \int_B u \, \mathrm{d}\mu.
$$

We assume that X supports the following Poincaré inequality.

Definition 2.3. Let $1 \leq p < \infty$. A metric measure space (X, d, μ) supports a weak $(1, p)$ -Poincaré inequality if there exist a constant C_{PI} and a dilation factor $\lambda \geq 1$ such that

$$
\int_{B_r} |u - u_{B_r}| \, \mathrm{d}\mu \leq C_{P I} r \left(\int_{B_{\lambda r}} g_u^p \, \mathrm{d}\mu \right)^{\frac{1}{p}},
$$

for every ball B_r in X and for every $u \in L^1_{loc}(X)$.

As shown in [\[22,](#page-13-24) Theorem 1.0.1] by Keith and Zhong, see also [\[1,](#page-13-3) Theorem 4.30], the Poincaré inequality is a self-improving property.

Theorem 2.4. Let (X, d, μ) be a complete metric measure space with a doubling measure μ and a weak $(1,p)$ -Poincaré inequality with $p > 1$. Then there exists $\varepsilon > 0$ such that X supports a weak $(1,q)$ -Poincaré inequality for every $q > p - \varepsilon$. Here, ε and the constants associated with the (1, q)-Poincaré inequality depend only on C_D , C_{PI} and p .

The following result shows that the Poincaré inequality implies a Sobolev–Poincaré inequality, see [\[1,](#page-13-3) Theorem 4.21 and Corollary 4.26].

Theorem 2.5. Assume that μ is a doubling measure and X supports a weak $(1, p)$ -Poincaré inequality and let $Q = \log_2 C_D$ be as in (2.2) . Let $1 \le p^* \le \frac{Q_p}{Q-p}$ for $1 \le p < Q$ and $1 \le p^* < \infty$ for $Q \le p < \infty$. Then X supports a weak (p^*, p) -Poincaré inequality, that is, there exist a constant $C = C(C_D, C_{PI}, p)$ such that

$$
\left(\oint_{B_r} |u - u_{B_r}|^{p^*} \, \mathrm{d}\mu\right)^{\frac{1}{p^*}} \le Cr \left(\oint_{B_{2\lambda r}} g_u^p \, \mathrm{d}\mu\right)^{\frac{1}{p}},
$$

for every ball B_r in X and every $u \in L^1_{loc}(X)$.

The following notation and assumptions will be used throughout the paper. For the coefficient function $a: X \to [0, \infty)$ in [\(1.1\)](#page-0-0), we assume that there exists $\alpha, 0 < \alpha \leq 1$, such that

$$
[a]_{\alpha} = \sup_{x,y \in \Omega, x \neq y} \frac{|a(x) - a(y)|}{\delta_{\mu}(x,y)^{\alpha}} < \infty,
$$
\n(2.3)

where δ_{μ} is a quasi-distance given by

$$
\delta_{\mu}(x,y) = (\mu(B_{d(x,y)}(x)) + \mu(B_{d(x,y)}(y)))^{1/Q}, \quad x, y \in X, \ x \neq y.
$$

Here $Q = \log_2 C_D$ is as in [\(2.2\)](#page-1-0) and we set $\delta_\mu(x, x) = 0$.

Remark 2.6. A measure is called Ahlfors–David regular, if there exist constants $0 < C_1 \le C_2 < \infty$ such that

$$
C_1 r^Q \le \mu(B_r(x)) \le C_2 r^Q,\tag{2.4}
$$

for every $x \in X$ and $0 < r \leq \text{diam}(X)$. If the measure μ is Ahlfors–David regular, then $\delta_{\mu}(x, y) \approx d(x, y)$ for every x, y and, consequently, $[a]_{\alpha} < \infty$ if and only if a is Hölder continuous with the exponent α .

We assume that

$$
1 < \frac{q}{p} \le 1 + \frac{\alpha}{Q},\tag{2.5}
$$

where $p > 1$, α is as in [\(2.3\)](#page-2-0) and $Q = \log_2 C_D$ is as in [\(2.2\)](#page-1-0).

By [\(2.5\)](#page-2-1), Theorem [2.4](#page-2-2) and Theorem [2.5](#page-2-3) there exists $s = s(C_D, p, q)$, with $1 < s < p < q < s^*$, such that X supports a (s^*, s) -Poincaré inequality, that is,

$$
\left(\oint_{B_r} |u - u_{B_r}|^{s^*} \, \mathrm{d}\mu\right)^{\frac{1}{s^*}} \le Cr \left(\oint_{B_{2\lambda r}} g_u^s \, \mathrm{d}\mu\right)^{\frac{1}{s}},\tag{2.6}
$$

for every ball B_r in X and every $u \in L^1_{loc}(X)$ with $C = C(C_D, C_{PI}, \lambda, p, q)$. We keep track on dependencies and denote

$$
C(\text{data}) = C(C_D, C_{PI}, \lambda, p, q, K, \alpha, [a]_{\alpha}).
$$

Here K is the quasimimizing constant in Definition [2.7](#page-3-0) below. By the structure of a double phase functional we have $g_u \in L^p(\Omega)$. However, we cannot conclude that $g_u \in L^q(\Omega)$, since the function a may be zero on a subset of Ω . Next we discuss the definition of a local quasiminimizer.

Definition 2.7. A function $u \in N^{1,1}_{loc}(\Omega)$ with $H(\cdot, g_u) \in L^1_{loc}(\Omega)$ is a local quasiminimizer on Ω , if there exists a constant $K \geq 1$ such that

$$
\int_{\Omega' \cap \{u \neq v\}} H(x, g_u) \, \mathrm{d}\mu \leq K \int_{\Omega' \cap \{u \neq v\}} H(x, g_v) \, \mathrm{d}\mu,
$$

for every open subset $\Omega' \in \Omega$ and for every function $v \in N^{1,1}(\Omega')$ with $u - v \in N_0^{1,1}(\Omega')$.

Then we give a definition of quasiminimizers with boundary values.

Definition 2.8. Let $w \in N^{1,1}(\Omega)$ with $H(\cdot, g_w) \in L^1(\Omega)$. A function $u \in N^{1,1}(\Omega)$ with $H(\cdot, g_u) \in L^1(\Omega)$ is a quasiminimizer on Ω with the boundary values w, if $u - w \in N_0^{1,1}(\Omega)$ and there exists a constant $K \geq 1$ such that

$$
\int_{\Omega' \cap \{u \neq v\}} H(x, g_u) d\mu \leq K \int_{\Omega' \cap \{u \neq v\}} H(x, g_v) d\mu,
$$

for every open subset $\Omega' \subset \Omega$ and for every function $v \in N^{1,1}(\Omega')$ with $u - v \in N_0^{1,1}(\Omega')$.

The main difference in the definitions above is that the assumption $u \in N^{1,1}_{loc}(\Omega)$ with $H(\cdot, g_u) \in L^1_{loc}(\Omega)$ in the local case is replaced with $u \in N^{1,1}(\Omega)$ with $H(\cdot, g_u) \in L^1(\Omega)$. It is obvious that a quasiminimizer with boundary values is a local quasiminimizer.

We state a local energy estimate for the double phase problem.

Lemma 2.9. Assume that $u \in N_{loc}^{1,1}(\Omega)$ with $H(\cdot, g_u) \in L_{loc}^1(\Omega)$ is a local quasiminimizer in Ω and let $B_r \subset B_R \Subset \Omega$ be concentric balls. Then there exists a constant $C = C(K, q)$ such that

$$
\int_{B_r} H(x, g_u) d\mu \le C \int_{B_R} H\left(x, \frac{u - u_{B_R}}{R - r}\right) d\mu.
$$

Proof. Let η be a $(R-r)^{-1}$ -Lipschitz cutoff function such that $0 \leq \eta \leq 1$, $\eta = 1$ on B_r and $\eta = 0$ in $X \setminus B_R$. Let $v = u - \eta(u - u_{B_R})$. By the Leibniz rule for the upper gradients ([\[1\]](#page-13-3), Lemma 2.18]), we have

$$
g_v \le |u - u_{B_R}| g_{\eta} + (1 - \eta) g_u \le \frac{|u - u_{B_R}|}{R - r} + (1 - \chi_{B_r}) g_u.
$$

Since u is a local quasiminimizer and $u - v \in N_0^{1,1}(B_R)$, by Definition [2.7](#page-3-0) we obtain

$$
\int_{B_r} H(x, g_u) d\mu \le \int_{B_R} H(x, g_u) d\mu \le K \int_{B_R} H(x, g_v) d\mu
$$
\n
$$
\le 2^q K \left(\int_{B_R} H\left(x, \frac{u - u_{B_R}}{R - r}\right) d\mu + \int_{B_R \setminus B_r} H(x, g_u) d\mu \right). \tag{2.7}
$$

By adding $K2^q \int_{B_r} H(x, g_u) d\mu$ to the both sides of [\(2.7\)](#page-3-1), we get

$$
(1+K2^q)\int_{B_r} H(x,g_u) d\mu \le K2^q \left(\int_{B_R} H\left(x, \frac{u-u_{B_R}}{R-r}\right) d\mu + \int_{B_R} H(x,g_u) d\mu\right).
$$

This implies

$$
\int_{B_r} H(x, g_u) d\mu \le \theta \left(\int_{B_R} H\left(x, \frac{u - u_{B_R}}{R - r}\right) d\mu + \int_{B_R} H(x, g_u) d\mu \right)
$$
\n
$$
\le (R - r)^{-p} \int_{B_R} |u - u_{B_R}|^p d\mu + (R - r)^{-q} \int_{B_R} a |u - u_{B_R}|^q d\mu + \theta \int_{B_R} H(x, g_u) d\mu,
$$

with $\theta = \frac{K2^q}{1+K^q}$ $\frac{K2^4}{1+K2^q}$ < 1. We apply a standard iteration lemma, see [\[17,](#page-13-25) Lemma 6.1], to obtain $\left(\begin{array}{cc} u-u_{B_R}\end{array}\right)$

$$
\int_{B_r} H(x, g_u) \, \mathrm{d}\mu \le C \int_{B_R} H\left(x, \frac{a - u_{B_R}}{R - r}\right) \, \mathrm{d}\mu,
$$

where $C = C(q, K)$.

Next we discuss a global energy estimate for quasiminimizers with boundary values.

Lemma 2.10. Let $w \in N^{1,1}(\Omega)$ with $H(\cdot, g_w) \in L^1(\Omega)$. Assume that $u \in N^{1,1}(\Omega)$ with $H(\cdot, g_u) \in L^1(\Omega)$ is a quasiminimizer on Ω with $u - w \in N_0^{1,1}(\Omega)$ and let $B_r \subset B_R$ be concentric balls. Then there exists a constant $C = C(K, q)$ such that

$$
\int_{B_r \cap \Omega} H(x, g_u) d\mu \le C \left(\int_{B_R \cap \Omega} H\left(x, \frac{u - w}{R - r}\right) d\mu + \int_{B_R \cap \Omega} H(x, g_w) d\mu \right).
$$

Proof. Let η be a $(R-r)^{-1}$ -Lipschitz cutoff function such that $0 \leq \eta \leq 1$, $\eta = 1$ on B_r and $\eta = 0$ in $X \setminus B_R$. Let $v = u - \eta(u - w)$. Then $\eta(u - w) \in N_0^{1,1}(B_R \cap \Omega)$ and thus $v - u \in N_0^{1,1}(B_R \cap \Omega)$. By Definition [2.8,](#page-3-2) we obtain

$$
\int_{B_R \cap \Omega} H(x, g_u) \, \mathrm{d}\mu \le K \int_{B_R \cap \Omega} H(x, g_v) \, \mathrm{d}\mu,
$$

where $v = u + \eta(w - u)$. Since

$$
g_v \le |u - w|g_{\eta} + (1 - \eta)g_u + \eta g_w \le \frac{|u - w|}{R - r} + (1 - \chi_{B_r})g_u + g_w,
$$

we obtain

$$
\int_{B_r \cap \Omega} H(x, g_u) d\mu \le \int_{B_R \cap \Omega} H(x, g_u) d\mu \le K \int_{B_R \cap \Omega} H(x, g_v) d\mu
$$
\n
$$
\le 3^q K \left(\int_{B_R \cap \Omega} H\left(x, \frac{u - w}{R - r}\right) d\mu + \int_{(B_R \backslash B_r) \cap \Omega} H(x, g_u) d\mu + \int_{B_R \cap \Omega} H(x, g_w) d\mu \right).
$$

By filling the hole and iterating as in the proof of Lemma [2.9,](#page-3-3) we arrive at

$$
\int_{B_r \cap \Omega} H(x, g_u) d\mu \le C \left(\int_{B_R \cap \Omega} H\left(x, \frac{u - w}{R - r}\right) d\mu + \int_{B_R \cap \Omega} H(x, g_w) d\mu \right),
$$

where $C = C(q, K)$.

3. Double phase Sobolev–Poincaré inequalities

This section discusses double phase Sobolev–Poincaré inequalities. We consider interior and boundary estimates separetely. We begin with interior estimates.

Lemma 3.1. Assume that $u \in N^{1,1}_{loc}(\Omega)$ with $H(\cdot, g_u) \in L^1_{loc}(\Omega)$. Let $a_0 = \inf_{x \in B_{2\lambda r}} a(x)$. Then there exist a constant $C = C$ (data) and exponents $0 < d_2 < 1 \le d_1 < \infty$, with $d_1 = d_1(C_D, p, q)$ and $d_2 = d_2(C_D, p, q)$, such that

$$
\left(\oint_{B_r} \left(\left|\frac{u - u_{B_r}}{r}\right|^p + a_0 \left|\frac{u - u_{B_r}}{r}\right|^q\right)^{d_1} d\mu\right)^{\frac{1}{d_1}} \le C \left(\oint_{B_{2\lambda r}} (g_u^p + a_0 g_u^q)^{d_2} d\mu\right)^{\frac{1}{d_2}},\tag{3.1}
$$

whenever $B_{2\lambda r} \in \Omega$.

Proof. By [\(2.6\)](#page-2-4) there exists s, with $1 < s < p < q < s^*$, such that

$$
\left(\oint_{B_r} \left|\frac{u - u_{B_r}}{r}\right|^{s^*} \, \mathrm{d}\mu\right)^{\frac{1}{s^*}} \le C \left(\oint_{B_{2\lambda r}} g_u^s \, \mathrm{d}\mu\right)^{\frac{1}{s}}.\tag{3.2}
$$

Let $\frac{s}{p} < d_2 < 1$ and $1 \leq d_1 < \frac{s^*}{q}$ $g_{\overline{q}}^*$. Since $pd_1 < qd_1 < s^*$ and $s < pd_2 < qd_2$, by Hölder's inequality, we have

$$
\left(\oint_{B_r} \left|\frac{u - u_{B_r}}{r}\right|^{pd_1} d\mu\right)^{\frac{1}{pd_1}} \le C \left(\oint_{B_{2\lambda r}} g_u^{pd_2} d\mu\right)^{\frac{1}{pd_2}},\tag{3.3}
$$

and

$$
\left(\oint_{B_r}\left|\frac{u-u_{B_r}}{r}\right|^{qd_1} \mathrm{d}\mu\right)^{\frac{1}{qd_1}} \leq C \left(\oint_{B_{2\lambda_r}} g_u^{qd_2} \mathrm{d}\mu\right)^{\frac{1}{qd_2}}.
$$

It follows that

$$
\left(\oint_{B_r} \left(\left|\frac{u-u_{B_r}}{r}\right|^p + a_0 \left|\frac{u-u_{B_r}}{r}\right|^q\right)^{d_1} d\mu\right)^{\frac{1}{d_1}}\n\leq \left(\oint_{B_r} \left|\frac{u-u_{B_r}}{r}\right|^{pd_1} d\mu\right)^{\frac{1}{d_1}} + a_0 \left(\oint_{B_r} \left|\frac{u-u_{B_r}}{r}\right|^{qd_1} d\mu\right)^{\frac{1}{d_1}}\n\leq C \left(\left(\oint_{B_{2\lambda r}} g_u^{pd_2} d\mu\right)^{\frac{1}{d_2}} + a_0 \left(\oint_{B_{2\lambda r}} g_u^{qd_2} d\mu\right)^{\frac{1}{d_2}}\right)\n\leq C \left(\oint_{B_{2\lambda r}} g_u^{pd_2} d\mu + \oint_{B_{2\lambda r}} (a_0 g_u^q)^{d_2} d\mu\right)^{\frac{1}{d_2}}\n\leq C \left(\oint_{B_{2\lambda r}} (g_u^p + a_0 g_u^q)^{d_2} d\mu\right)^{\frac{1}{d_2}},
$$

where $C = C(C_D, C_{PI}, \lambda, p, q)$. Observe that all integrals are finite, since

$$
\left(\oint_{B_{2\lambda r}} (g_u^p + a_0 g_u^q)^{d_2} d\mu\right)^{\frac{1}{d_2}} \le \left(\oint_{B_{2\lambda r}} (g_u^p + a(x) g_u^q)^{d_2} d\mu\right)^{\frac{1}{d_2}}
$$

$$
\le \oint_{B_{2\lambda r}} (g_u^p + a(x) g_u^q) d\mu
$$

$$
= \oint_{B_{2\lambda r}} H(x, g_u) d\mu < \infty.
$$

 \Box

Then we consider an interior double phase Sobolev–Poincaré inequality.

Lemma 3.2. Assume that $u \in N_{loc}^{1,1}(\Omega)$ with $H(\cdot, g_u) \in L_{loc}^1(\Omega)$. Then there exists a constant $C = C(\text{data})$ and exponents $0 < d_2 < 1 \le d_1 < \infty$, with $d_1 = d_1(\text{data})$ and $d_2 = d_2(\text{data})$, such that

$$
\left(\oint_{B_r} H\left(x, \frac{u-u_{B_r}}{r}\right)^{d_1} d\mu\right)^{\frac{1}{d_1}} \le C\left(1+\|g_u\|_{L^p(B_{2\lambda r})}^{q-p} \mu(B_{2\lambda r})^{\frac{\alpha}{Q}-\frac{q-p}{p}}\right) \left(\oint_{B_{2\lambda r}} H(x, g_u)^{d_2} d\mu\right)^{\frac{1}{d_2}},
$$

whenever $B_{2\lambda r} \in \Omega$.

Proof. First assume that

$$
a_0 = \inf_{x \in B_{2\lambda r}} a(x) > 2[a]_{\alpha} (2C_D^2 \mu(B_{2\lambda r}))^{\alpha/Q}, \tag{3.4}
$$

where $Q = \log_2 C_D$ is as in Lemma [2.1.](#page-1-1) Note that, for every $x, y \in B_{2\lambda r}$, we have

$$
\delta_{\mu}(x, y) = (\mu(B_{d(x, y)}(x)) + \mu(B_{d(x, y)}(y)))^{1/Q}
$$

\n
$$
\leq (\mu(B_{4\lambda r}(x)) + \mu(B_{4\lambda r}(y)))^{1/Q}
$$

\n
$$
\leq (2\mu(B_{6\lambda r}))^{1/Q} \leq (2C_{D}^{2}\mu(B_{2\lambda r}))^{1/Q}.
$$

By [\(3.4\)](#page-5-0) we obtain

$$
2a_0 = 2a(x) - 2(a(x) - a_0) \ge a(x) + a_0 - 2(a(x) - a_0)
$$

\n
$$
\ge a(x) + 2[a]_{\alpha}(2C_D^2\mu(B_{2\lambda r}))^{\alpha/Q} - 2(a(x) - a_0)
$$

\n
$$
\ge a(x) + 2 \sup_{\substack{x,y \in B_{2\lambda r} \\ x \neq y}} \frac{|a(x) - a(y)|}{\delta_{\mu}(x,y)^{\alpha}} (2C_D^2\mu(B_{2\lambda r}))^{\alpha/Q} - 2(a(x) - a_0)
$$

\n
$$
\ge a(x) + 2 \sup_{\substack{x,y \in B_{2\lambda r} \\ x,y \in B_{2\lambda r}}} |a(x) - a(y)| - 2(a(x) - a_0)
$$

\n
$$
\ge a(x) + 2 \sup_{\substack{x,y \in B_{2\lambda r} \\ x,y \in B_{2\lambda r}}} (a(x) - a(y)) - 2(a(x) - a_0)
$$

\n
$$
\ge a(x) + 2a(x) - 2 \inf_{y \in B_{2\lambda r}} a(y) - 2(a(x) - a_0) = a(x),
$$

for every $x \in B_{2\lambda r}$. On the other hand, we have $a(x) \ge \inf_{x \in B_{2\lambda r}} a(x) = a_0$ for every $x \in B_{2\lambda r}$. This implies that $a_0 \le a(x) \le 2a_0$ for every $x \in B_{2\lambda r}$. By Lemma [3.1,](#page-4-0) we conclude that

$$
\left(\oint_{B_r} H\left(x, \frac{u - u_{B_r}}{r}\right)^{d_1} d\mu\right)^{\frac{1}{d_1}} = \left(\oint_{B_r} \left(\left|\frac{u - u_{B_r}}{r}\right|^p + a(x)\left|\frac{u - u_{B_r}}{r}\right|^q\right)^{d_1} d\mu\right)^{\frac{1}{d_1}}\n\leq C \left(\oint_{B_r} \left(\left|\frac{u - u_{B_r}}{r}\right|^p + a_0\left|\frac{u - u_{B_r}}{r}\right|^q\right)^{d_1} d\mu\right)^{\frac{1}{d_1}}\n\leq C \left(\oint_{B_{2\lambda r}} (g_u^p + a_0 g_u^q)^{d_2} d\mu\right)^{\frac{1}{d_2}}\n\leq C \left(\oint_{B_{2\lambda r}} (g_u^p + a(x)g_u^q)^{d_2} d\mu\right)^{\frac{1}{d_2}}\n\leq C \left(\oint_{B_{2\lambda r}} H(x, g_u)^{d_2} d\mu\right)^{\frac{1}{d_2}},
$$
\n(3.5)

where $C = C(\text{data})$, $d_1 = d_1(\text{data})$ and $d_2 = d_2(\text{data})$ with $0 < d_2 < 1 \leq d_1 < \infty$.

Next we consider the case which is complementary to [\(3.4\)](#page-5-0), that is,

$$
a_0 = \inf_{x \in B_{2\lambda r}} a(x) \le 2[a]_{\alpha} (2C_D^2 \mu(B_{2\lambda r}))^{\alpha/Q}.
$$
\n(3.6)

Notice that, for every $x \in B_{2\lambda r}$ and $y \in B_r$, with $y \neq x$, we have

$$
a(y) - a(x) \le |a(x) - a(y)| = \frac{|a(x) - a(y)|}{\delta_{\mu}(x, y)^{\alpha}} \delta_{\mu}(x, y)^{\alpha} \le [a]_{\alpha} \delta_{\mu}(x, y)^{\alpha}.
$$
 (3.7)

Note that, for every $x \in B_{2\lambda r}$ and $y \in B_r$, with $y \neq x$, we have

$$
\delta_{\mu}(x, y) = (\mu(B_{d(x, y)}(x)) + \mu(B_{d(x, y)}(y)))^{1/Q}
$$

\n
$$
\leq (\mu(B_{3\lambda r}(x)) + \mu(B_{3\lambda r}(y)))^{1/Q}
$$

\n
$$
\leq (2\mu(B_{5\lambda r}))^{1/Q} \leq C\mu(B_{2\lambda r})^{1/Q},
$$

where $C = C(C_D)$. By [\(3.7\)](#page-6-0), we get

$$
a(y) \le a(x) + C[a]_{\alpha} \mu(B_{2\lambda r})^{\alpha/Q},
$$

where $C = C(C_D, \alpha)$. By taking infimum over all $x \in 2\lambda B_r$, we obtain

$$
a(y) \le \inf_{x \in B_{2\lambda r}} a(x) + C[a]_{\alpha} \mu(B_{2\lambda r})^{\alpha/Q}
$$

\n
$$
\le 2[a]_{\alpha} (2C_D^2 \mu(B_{2\lambda r}))^{\alpha/Q} + C[a]_{\alpha} \mu(B_{2\lambda r})^{\alpha/Q}
$$

\n
$$
= C[a]_{\alpha} \mu(B_{2\lambda r})^{\alpha/Q},
$$

where $C = C(C_D, \alpha)$. By taking supremum over $y \in B_r$, we conclude that

$$
\sup_{y \in B_r} a(y) \le C[a]_{\alpha} \mu(B_{2\lambda r})^{\alpha/Q}.
$$

It follows that

$$
\left(\oint_{B_r} H\left(x, \frac{u - u_{B_r}}{r}\right)^{d_1} d\mu\right)^{\frac{1}{d_1}} = \left(\oint_{B_r} \left(\left|\frac{u - u_{B_r}}{r}\right|^p + a(x)\left|\frac{u - u_{B_r}}{r}\right|^q\right)^{d_1} d\mu\right)^{\frac{1}{d_1}}
$$
\n
$$
\leq \left(\oint_{B_r} \left(\left|\frac{u - u_{B_r}}{r}\right|^p + C[a]_{\alpha}\mu(B_{2\lambda r})^{\alpha/Q} \left|\frac{u - u_{B_r}}{r}\right|^q\right)^{d_1} d\mu\right)^{\frac{1}{d_1}}
$$
\n
$$
\leq \left(\oint_{B_r} \left|\frac{u - u_{B_r}}{r}\right|^{pd_1} d\mu\right)^{\frac{1}{d_1}} + C[a]_{\alpha}\mu(B_{2\lambda r})^{\alpha/Q} \left(\oint_{B_r} \left|\frac{u - u_{B_r}}{r}\right|^{qd_1} d\mu\right)^{\frac{1}{d_1}}
$$
\n
$$
\leq C \left(\oint_{B_r} \left|\frac{u - u_{B_r}}{r}\right|^{qd_1} d\mu\right)^{\frac{1}{d_1}\frac{p}{d_1}}
$$
\n
$$
\leq C \left(\oint_{B_r} \left|\frac{u - u_{B_r}}{r}\right|^{qd_1} d\mu\right)^{\frac{1}{d_1}\frac{p}{d_1}}
$$
\n
$$
\left(1 + [a]_{\alpha}\mu(B_{2\lambda r})^{\alpha/Q} \left(\oint_{B_r} \left|\frac{u - u_{B_r}}{r}\right|^{qd_1} d\mu\right)^{\frac{q-p}{d_1}}\right),
$$
\n
$$
\left(1 + [a]_{\alpha}\mu(B_{2\lambda r})^{\alpha/Q} \left|\frac{u - u_{B_r}}{r}\right|^{qd_1} d\mu\right)^{\frac{q-p}{d_1}}
$$
\n
$$
\left(1 + [a]_{\alpha}\mu(B_{2\lambda r})^{\alpha/Q} \left|\frac{u - u_{B_r}}{r}\right|^{qd_1} d\mu\right)^{\frac{q-p}{d_1}}
$$
\n
$$
\left(1 + [a]_{\alpha}\mu(B_{2\lambda r})^{\alpha/Q} \left|\frac{u - u_{B_r}}{r}\right|^{qd_1} d\mu\right)^
$$

where $C = C(C_D, \alpha)$. Since $qd_1 < s^*$ and $s < p$, [\(3.2\)](#page-4-1) and Hölder's inequality imply

$$
\left(\oint_{B_r} \left|\frac{u - u_{B_r}}{r}\right|^{q d_1} \mathrm{d}\mu\right)^{\frac{1}{q d_1}} \le C \left(\oint_{B_{2\lambda r}} g_u^p \mathrm{d}\mu\right)^{\frac{1}{p}},
$$

where $C = C(\text{data})$. Thus we have

$$
\left(\oint_{B_r}\left|\frac{u-u_{B_r}}{r}\right|^{qd_1}\,\mathrm{d}\mu\right)^{\frac{q-p}{qd_1}}\leq C\left(\oint_{B_{2\lambda r}}g_u^p\,\mathrm{d}\mu\right)^{\frac{q-p}{p}}=C\|g_u\|_{L^p(B_{2\lambda r})}^{q-p}\mu(B_{2\lambda r})^{-\frac{q-p}{p}},
$$

where $C = C(\text{data})$. By [\(3.8\)](#page-7-0) we obtain

$$
\left(\oint_{B_r} H\left(x, \frac{u - u_{B_r}}{r}\right)^{d_1} d\mu\right)^{\frac{1}{d_1}} \le C \left(\oint_{B_r} \left|\frac{u - u_{B_r}}{r}\right|^{q d_1} d\mu\right)^{\frac{1}{d_1} \frac{p}{q}} \left(1 + \|g_u\|_{L^p(B_{2\lambda r})}^{q - p} \mu(B_{2\lambda r})^{\frac{\alpha}{Q} - \frac{q - p}{p}}\right),
$$

where $C = C(\text{data})$. Since $qd_1 < s^*$ and $s < pd_2$, by [\(3.2\)](#page-4-1) we have

$$
\left(\oint_{B_r} \left| \frac{u - u_{B_r}}{r} \right|^{q d_1} d\mu \right)^{\frac{1}{d_1} \frac{p}{q}} \le C \left(\oint_{B_{2\lambda r}} g_u^{p d_2} d\mu \right)^{\frac{1}{d_2}}\n\le C \left(\oint_{B_{2\lambda r}} (g_u^p + a(x)g_u^q)^{d_2} d\mu \right)^{\frac{1}{d_2}}\n= C \left(\oint_{B_{2\lambda r}} H(x, g_u)^{d_2} d\mu \right)^{\frac{1}{d_2}}.
$$

This completes the proof. \Box

Next we discuss a Sobolev inequality for functions which vanish on a large set, see [\[26\]](#page-13-7).

Lemma 3.3. Assume that $u \in N_{\text{loc}}^{1,1}(\Omega)$ with $H(\cdot, g_u) \in L_{\text{loc}}^1(\Omega)$. Let B_r be a ball and $a_0 = \inf_{x \in B_{2\lambda r}} a(x)$. Assume that there exists γ , $0 < \gamma < 1$, such that

$$
\mu({x \in B_r : |u(x)| > 0}) \le \gamma \mu(B_r).
$$

Then there exist a constant $C = C(\text{data}, \gamma)$ and exponents $0 < d_2 < 1 \le d_1 < \infty$, with $d_1 = d_1(\text{data})$ and $d_2 = d_2$ (data), such that

$$
\left(\oint_{B_r} \left(\left|\frac{u}{r}\right|^p + a_0\left|\frac{u}{r}\right|^q\right)^{d_1} d\mu\right)^{\frac{1}{d_1}} \le C \left(\oint_{B_{2\lambda r}} \left(g_u^p + a_0 g_u^q\right)^{d_2} d\mu\right)^{\frac{1}{d_2}}.
$$

Proof. As in the proof of Lemma [3.1,](#page-4-0) there exists s, with $1 < s < p < q < s^*$, such that [\(3.2\)](#page-4-1) holds. Let $A = \{x \in B_r : |u(x)| > 0\}.$ We observe that

$$
\left(\oint_{B_r} \left|\frac{u}{r}\right|^{s^*} \mathrm{d}\mu\right)^{\frac{1}{s^*}} \le \left(\oint_{B_r} \left|\frac{u - u_{B_r}}{r}\right|^{s^*} \mathrm{d}\mu\right)^{\frac{1}{s^*}} + \left|\frac{u_{B_r}}{r}\right|.
$$
\n(3.9)

,

By Hölder's inequality we obtain

$$
|u_{B_r}| \leq \frac{1}{\mu(B_r)} \int_A |u| \, \mathrm{d}\mu \leq \left(\frac{\mu(A)}{\mu(B_r)}\right)^{1-\frac{1}{s^*}} \left(\int_{B_r} |u|^{s^*} \, \mathrm{d}\mu\right)^{\frac{1}{s^*}} \leq \gamma^{1-\frac{1}{s^*}} \left(\int_{B_r} |u|^{s^*} \, \mathrm{d}\mu\right)^{\frac{1}{s^*}}.
$$

By [\(3.9\)](#page-8-0) and [\(3.2\)](#page-4-1) we conclude that

$$
(1 - \gamma^{1 - \frac{1}{s^*}}) \left(\int_{B_r} \left| \frac{u}{r} \right|^{s^*} d\mu \right)^{\frac{1}{s^*}} \le \left(\int_{B_r} \left| \frac{u - u_{B_r}}{r} \right|^{s^*} d\mu \right)^{\frac{1}{s^*}} \le C \left(\int_{B_{2\lambda r}} g_u^s d\mu \right)^{\frac{1}{s}}, \tag{3.10}
$$

where $C = C(\text{data})$. Since $pd_1 < qd_1 < s^*$ and $s < pd_2 < qd_2$, by Hölder's inequality, we have

$$
\left(\oint_{B_r} \left|\frac{u}{r}\right|^{pd_1} \mathrm{d} \mu\right)^{\frac{1}{pd_1}} \leq C \left(\oint_{B_{2\lambda r}} g_u^{pd_2} \mathrm{d} \mu\right)^{\frac{1}{pd_2}},
$$

and

$$
\left(\oint_{B_r} \left|\frac{u}{r}\right|^{qd_1} \mathrm{d}\mu\right)^{\frac{1}{qd_1}} \le C \left(\oint_{B_{2\lambda r}} g_u^{qd_2} \mathrm{d}\mu\right)^{\frac{1}{qd_2}}
$$

where $C = C(\text{data}, \gamma)$. The rest of the proof follows as in the proof of Lemma [3.1.](#page-4-0)

Then we consider a local double phase Sobolev–Poincaré inequality.

Lemma 3.4. Assume that $u \in N_{loc}^{1,1}(\Omega)$ with $H(\cdot, g_u) \in L_{loc}^1(\Omega)$. Let B_r be a ball and assume that there exist γ , $0 < \gamma < 1$, such that

$$
\mu({x \in B_r : |u(x)| > 0}) \le \gamma \mu(B_r).
$$

Then there exists a constant $C = C(\text{data}, \gamma)$ and exponents $0 < d_2 < 1 \le d_1 < \infty$, with $d_1 = d_1(\text{data})$ and $d_2 = d_2$ (data), such that

$$
\left(\int_{B_r} H\left(x,\frac{u}{r}\right)^{d_1} d\mu\right)^{\frac{1}{d_1}} \leq C\left(1+\|g_u\|_{L^p(B_{2\lambda r})}^{q-p} \mu(B_{2\lambda r})^{\frac{\alpha}{Q}-\frac{q-p}{p}}\right) \left(\int_{B_{2\lambda r}} H(x,g_u)^{d_2} d\mu\right)^{\frac{1}{d_2}}.
$$

Proof. As in the proof of Lemma [3.2,](#page-5-1) we consider two cases [\(3.4\)](#page-5-0) and [\(3.6\)](#page-6-1). If [\(3.4\)](#page-5-0) holds, then as in [\(3.5\)](#page-6-2) with $\frac{u}{r}$ instead of $\frac{u-u_{B_r}}{r}$ and Lemma [3.3,](#page-7-1) we obtain

$$
\left(\oint_{B_r} H\left(x,\frac{u}{r}\right)^{d_1} d\mu\right)^{\frac{1}{d_1}} \leq C \left(\oint_{B_{2\lambda r}} H(x,g_u)^{d_2} d\mu\right)^{\frac{1}{d_2}},
$$

where $C = C(\text{data}, \gamma)$, $d_1 = d_1(\text{data})$ and $d_2 = d_2(\text{data})$ with $0 < d_2 < 1 \leq d_1 < \infty$. On the other hand, if (3.6) holds, then as in (3.8) , we obtain

$$
\left(\oint_{B_r} H\left(x, \frac{u}{r}\right)^{d_1} d\mu\right)^{\frac{1}{d_1}} \le \left(\oint_{B_r} \left(\left|\frac{u}{r}\right|^p + C[a]_{\alpha} \mu(B_{2\lambda r})^{\alpha/Q} \left|\frac{u}{r}\right|^q\right)^{d_1} d\mu\right)^{\frac{1}{d_1}}\n\n\le \left(\oint_{B_r} \left|\frac{u}{r}\right|^{pd_1} d\mu\right)^{\frac{1}{d_1}} + C[a]_{\alpha} \mu(B_{2\lambda r})^{\alpha/Q} \left(\oint_{B_r} \left|\frac{u}{r}\right|^{qd_1} d\mu\right)^{\frac{1}{d_1}}\n\n\le C \left(\oint_{B_r} \left|\frac{u}{r}\right|^{qd_1} d\mu\right)^{\frac{p}{qd_1}} \left(1 + [a]_{\alpha} \mu(B_{2\lambda r})^{\alpha/Q} \left(\oint_{B_r} \left|\frac{u}{r}\right|^{qd_1} d\mu\right)^{\frac{q-p}{qd_1}}\right),
$$

where $C = C(C_D, \alpha)$. Since $qd_1 < s^*$ and $s < pd_2$, [\(3.10\)](#page-8-1) and Hölder's inequality imply

$$
\left(\oint_{B_r} \left|\frac{u}{r}\right|^{qd_1} \mathrm{d}\mu\right)^{\frac{1}{qd_1}} \leq C \left(\oint_{B_{2\lambda r}} g_u^p \mathrm{d}\mu\right)^{\frac{1}{p}},
$$

where $C = C(\text{data}, \gamma)$. As in the proof of Lemma [3.2,](#page-5-1) we have

$$
\left(\oint_{B_r} \left|\frac{u}{r}\right|^{qd_1} d\mu\right)^{\frac{q-p}{qd_1}} \le C \left(\oint_{B_{2\lambda r}} g_u^p d\mu\right)^{\frac{q-p}{p}} = C\|g_u\|_{L^p(B_{2\lambda r})}^{q-p} \mu(B_{2\lambda r})^{-\frac{q-p}{p}},
$$

and thus

$$
\left(\oint_{B_r} H\left(x,\frac{u}{r}\right)^{d_1} d\mu\right)^{\frac{1}{d_1}} \leq C \left(\oint_{B_r} \left|\frac{u}{r}\right|^{qd_1} d\mu\right)^{\frac{p}{q \cdot d_1}} \left(1 + \|g_u\|_{L^p(B_{2\lambda r})}^{q-p} \mu(B_{2\lambda r})^{\frac{\alpha}{Q} - \frac{q-p}{p}}\right),
$$

where $C = C(\text{data}, \gamma)$. Since $qd_1 < s^*$ and $s < pd_2$, applying [\(3.10\)](#page-8-1) as in the proof of Lemma [3.3,](#page-7-1) we have

$$
\left(\oint_{B_r} \left| \frac{u}{r} \right|^{qd_1} d\mu \right)^{\frac{p}{qd_1}} \leq C \left(\oint_{B_{2\lambda r}} g_u^{pd_2} d\mu \right)^{\frac{1}{d_2}}\n\leq C \left(\oint_{B_{2\lambda r}} (g_u^p + a(x)g_u^q)^{d_2} d\mu \right)^{\frac{1}{d_2}}\n= C \left(\oint_{B_{2\lambda r}} H(x, g_u)^{d_2} d\mu \right)^{\frac{1}{d_2}},
$$

where $C = C(\text{data}, \gamma)$. This completes the proof.

4. Local and global higher integrability results

The main goal of this work is to get global higher integrability for quasiminimizers. In the metric setting, the improvement of integrability is obtained by using a metric space version of Gehring's lemma, whose proof can be found, for example, in [\[1\]](#page-13-3) or [\[39\]](#page-14-12).

Lemma 4.1. Let $f \in L^1_{loc}(\Omega)$ and $g \in L^{\sigma}_{loc}(X)$, $\sigma > 1$, be non-negative functions and let $\lambda > 1$. Assume that there exist a constant C_1 and an exponent $0 < d < 1$ such that

$$
\int_{B_R} f d\mu \leq C_1 \left(\left(\int_{B_{\lambda R}} f^d d\mu \right)^{\frac{1}{d}} + \int_{B_{\lambda R}} g d\mu \right),
$$

for every ball B_R with $B_{\lambda R} \in \Omega$. Then there exist a constant $C_2 = C_2(C_D, C_1, d, \lambda)$ and an exponent $\varepsilon = \varepsilon(C_D, C_1, d, \lambda) > 0$ such that

$$
\left(\oint_{B_R} f^{1+\varepsilon} d\mu\right)^{\frac{1}{1+\varepsilon}} \leq C_2 \left(\oint_{B_{\lambda R}} f d\mu + \left(\oint_{B_{\lambda R}} g^{\sigma} d\mu\right)^{\frac{1}{\sigma}}\right),
$$

for every ball B_R with $B_{\lambda R} \in \Omega$.

Next we discuss local higher integrability of the upper gradient of a local quasiminimizer.

Theorem 4.2. Let Ω be an open subset of X and let $\Omega' \in \Omega'' \in \Omega$. Assume that $u \in N_{loc}^{1,1}(\Omega)$ with $H(\cdot, g_u) \in L^1_{loc}(\Omega)$ is a local quasiminimizer in Ω . Then there exist a constant $C = C(\text{data}, \Omega'', ||g_u||_{L^p(\Omega'')})$ and an exponent $\varepsilon = \varepsilon(\text{data}, \Omega'', ||g_u||_{L^p(\Omega'')}) > 0$ such that

$$
\left(\int_{\Omega'} H(x,g_u)^{1+\varepsilon} \, \mathrm{d}\mu\right)^{\frac{1}{1+\varepsilon}} \le C \int_{\Omega''} H(x,g_u) \, \mathrm{d}\mu.
$$

Proof. Let $B_{2\lambda r} \in \Omega''$. By Lemma [2.9,](#page-3-3) there exists a constant $C = C(K, q)$ such that

$$
\int_{B_{\frac{r}{2}}} H(x, g_u) d\mu \le C \int_{B_r} H\left(x, \frac{u - u_{B_r}}{r}\right) d\mu.
$$

On the other hand, by Lemma [3.2,](#page-5-1) we obtain

$$
\int_{B_r} H\left(x, \frac{u - u_{B_r}}{r}\right) d\mu \le C\left(1 + \|g_u\|_{L^p(B_{2\lambda r})}^{q-p} \mu(B_{2\lambda r})^{\frac{\alpha}{Q} - \frac{q-p}{p}}\right) \left(\int_{B_{2\lambda r}} H(x, g_u)^d d\mu\right)^{\frac{1}{d}},
$$

where $0 < d = d(\text{data}) < 1$ and $C = C(\text{data})$. This implies that

$$
\int_{B_{\frac{r}{2}}} H(x,g_u) d\mu \le C\left(1+\|g_u\|_{L^p(B_{2\lambda r})}^{q-p} \mu(B_{2\lambda r})^{\frac{\alpha}{Q}-\frac{q-p}{p}}\right) \left(\int_{B_{2\lambda r}} H(x,g_u)^d d\mu\right)^{\frac{1}{d}},
$$

where $C = C(\text{data})$.

By [\(2.5\)](#page-2-1) we have $\frac{\alpha}{Q} - \frac{q-p}{p} \ge 0$ and thus we obtain

$$
\int_{B_{r/2}} H(x, g_u) d\mu \le C \left(1 + \|g_u\|_{L^p(\Omega'')}^{q-p} \mu(\Omega'')^{\frac{\alpha}{Q} - \frac{q-p}{p}}\right) \left(\int_{B_{2\lambda r}} H(x, g_u)^d d\mu\right)^{\frac{1}{d}},
$$

where $C = C(\text{data})$. This implies that

$$
\int_{B_{r/2}} H(x, g_u) d\mu \le C \left(\int_{B_{2\lambda r}} H(x, g_u)^d d\mu \right)^{\frac{1}{d}}, \tag{4.1}
$$

for every ball with $B_{2\lambda r} \in \Omega''$ with $C = C(\text{data}, \Omega'', ||g_u||_{L^p(\Omega'')})$. The constant C depends on Ω'' and on $||g_u||_{L^p(\Omega'')}$, but once u and Ω'' are fixed, the obtained reverse Hölder inequality is uniform over all balls with $B_{2\lambda r} \in \Omega''$. By Lemma [4.1,](#page-9-1) there exist a constant $C = C(\text{data}, \Omega'', ||g_u||_{L^p(\Omega'')})$ and an exponent $\varepsilon = \varepsilon(\text{data}, \Omega'', ||g_u||_{L^p(\Omega'')}) > 0$ such that

$$
\left(\int_{B_{r/2}} H(x,g_u)^{1+\varepsilon} \, \mathrm{d}\mu\right)^{\frac{1}{1+\varepsilon}} \leq C \int_{B_{2\lambda r}} H(x,g_u) \, \mathrm{d}\mu,
$$

for every ball with $B_{2\lambda r} \in \Omega''$. Since $\overline{\Omega'}$ is compact, we can cover it by a finite number of such balls and conclude that

$$
\left(\int_{\Omega'} H(x, g_u)^{1+\varepsilon} d\mu\right)^{\frac{1}{1+\varepsilon}} \le C \int_{\Omega''} H(x, g_u) d\mu.
$$

Remark 4.3. If the measure is Ahlfors–David regular, see [\(2.4\)](#page-2-5), we have

$$
\mu(B_{2\lambda r})^{\frac{\alpha}{Q} - \frac{q-p}{p}} \le Cr^{\alpha - Q(\frac{q}{p} - 1)},
$$

where $\alpha - Q(\frac{q}{p} - 1) \ge 0$ by [\(2.5\)](#page-2-1). As in the proof of Theorem [4.2,](#page-9-0) there exist a constant $C = C(\text{data})$ and an exponent $0 < d = d(\text{data}) < 1$ such that

$$
\int_{B_{\frac{r}{2}}} H(x,g_u) d\mu \le C\left(1+\|g_u\|_{L^p(B_{2\lambda r})}^{q-p}\right) \left(\int_{B_{2\lambda r}} H(x,g_u)^d d\mu\right)^{\frac{1}{d}},
$$

whenever $B_{2\lambda r} \in \Omega$ with $0 < r \leq 1$. A similar argument can also be applied in Lemma [3.2](#page-5-1) and Lemma [3.4.](#page-8-2)

Finally, we are ready to prove the main result of the paper, which states higher integrability for the weak upper gradient of a quasiminimizer over the entire domain under the assumption that the domain satisfies a uniform measure density property.

Theorem 4.4. Assume that Ω is a bounded open set in X with the property that there exists a constant γ , $0 < \gamma < 1$, for which

$$
\mu(B_R(x) \cap \Omega) \le \gamma \mu(B_R(x)),
$$

for every $x \in X \setminus \Omega$ and $R > 0$. Assume that $w \in N^{1,1}(\Omega)$ such that $H(\cdot, g_w) \in L^{\sigma}(\Omega)$ for some $\sigma > 1$. Assume that $u \in N^{1,1}(\Omega)$ with $H(\cdot, g_u) \in L^1(\Omega)$ is a quasiminimizer in Ω with $u - w \in N_0^{1,1}(\Omega)$. Then there exist a constant $C = C(\text{data}, \gamma, \Omega, \|g_{u-w}\|_{L^p(\Omega)})$ and an exponent $\varepsilon = \varepsilon(\text{data}, \gamma, \Omega, \|g_{u-w}\|_{L^p(\Omega)}) > 0$ such that

$$
\left(\int_{\Omega} H(x,g_u)^{1+\varepsilon} d\mu\right)^{\frac{1}{1+\varepsilon}} \leq C \left(\int_{\Omega} H(x,g_u) d\mu + \left(\int_{\Omega} H(x,g_w)^{\sigma} d\mu\right)^{\frac{1}{\sigma}}\right).
$$

Proof. Let B_r be a ball with $B_r \cap \Omega \neq \emptyset$ and $0 < r \leq 1$. Then there exist two alternatives: either $B_{3\lambda r} \subset \Omega$ or $B_{3\lambda r} \setminus \Omega \neq \emptyset$. If $B_{3\lambda r} \subset \Omega$, then $B_{2\lambda r} \Subset \Omega$ and, as in [\(4.1\)](#page-10-1), we have

$$
\int_{B_{r/2}} H(x, g_u) d\mu \le C \left(\int_{B_{2\lambda r}} H(x, g_u)^d d\mu \right)^{\frac{1}{d}}, \tag{4.2}
$$

where $C = C(\text{data}, \Omega, ||g_u||_{L^p(\Omega)})$.

Then we discuss the case $B_{3\lambda r} \setminus \Omega \neq \emptyset$. Let $x_0 \in B_{3\lambda r} \setminus \Omega$ and consider $B_R(x_0)$ with $R = 8\lambda r$. Since the center of B_r is contained in $B_{3\lambda r}(x_0)$, we have $B_r \subset B_{4\lambda r}(x_0) = B_{\frac{R}{2}}(x_0)$. Let $B_R = B_R(x_0)$. We note that $B_{3\lambda r} \subset B_R$ and $\mu(B_R \cap \Omega) \leq \gamma \mu(B_R)$, with $0 < \gamma < 1$. Since $u - w = 0$ μ -almost everywhere in $X \setminus \Omega$, we obtain

$$
\mu({x \in B_R : |u(x) - w(x)| > 0}) \le \gamma \mu(B_R).
$$

By Lemma [2.10](#page-4-2) there exists a constant $C = C(K, q)$ such that

$$
\int_{B_{\frac{R}{2}} \cap \Omega} H(x, g_u) \, \mathrm{d}\mu \le C \left(\int_{B_R \cap \Omega} H\left(x, \frac{u - w}{R}\right) \, \mathrm{d}\mu + \int_{B_R \cap \Omega} H(x, g_w) \, \mathrm{d}\mu \right). \tag{4.3}
$$

We consider the first term on the right-hand side of [\(4.3\)](#page-11-0). Since $u - w = 0$ μ -almost everywhere in $X \setminus \Omega$ and $g_{u-w} = 0$ µ-almost everywhere in $X \setminus \Omega$, by applying Lemma [3.4](#page-8-2) with $u - w$, we obtain

$$
\frac{1}{\mu(B_R)} \int_{B_R \cap \Omega} H\left(x, \frac{u - w}{R}\right) d\mu = \int_{B_R} H\left(x, \frac{u - w}{R}\right) d\mu
$$

\n
$$
\leq C\left(1 + \|g_{u-w}\|_{L^p(B_{2\lambda R})}^{q - p} \mu(B_{2\lambda R})^{\frac{\alpha}{Q} - \frac{q - p}{p}}\right) \left(\int_{B_{2\lambda R}} H(x, g_{u-w})^d d\mu\right)^{\frac{1}{d}},
$$

where $C = C(\text{data}, \gamma)$ and $0 < d = d(\text{data}) < 1$. Since $B_r \subset B_{\frac{R}{2}}(x_0)$, $B_r \cap \Omega \neq \emptyset$ and $0 < r \leq 1$, we have $B_R \cap \Omega \neq \emptyset$ and $2\lambda R = 16\lambda^2 r \leq 16\lambda^2$. This implies that

$$
B_{2\lambda R} \subset \Omega^* = \{ y \in X : \text{dist}(y, \Omega) < 24\lambda^2 \}.
$$

By [\(2.5\)](#page-2-1) we have $\frac{\alpha}{Q} - \frac{q-p}{p} \ge 0$ and thus we obtain

$$
\frac{1}{\mu(B_R)} \int_{B_R \cap \Omega} H\left(x, \frac{u - w}{R}\right) d\mu \le C \left(1 + \|g_{u-w}\|_{L^p(\Omega)}^{q-p} \mu(\Omega^*)^{\frac{\alpha}{Q} - \frac{q-p}{p}}\right) \left(\int_{B_{2\lambda R}} H(x, g_{u-w})^d d\mu\right)^{\frac{1}{d}}.
$$

This implies that

$$
\frac{1}{\mu(B_R)} \int_{B_R \cap \Omega} H\left(x, \frac{u - w}{R}\right) d\mu \le C \left(\int_{B_{2\lambda R}} H(x, g_{u-w})^d d\mu\right)^{\frac{1}{d}}
$$

= $C \left(\frac{1}{\mu(B_{2\lambda R})} \int_{B_{2\lambda R} \cap \Omega} H(x, g_{u-w})^d d\mu\right)^{\frac{1}{d}},$

where $C = C(\text{data}, \gamma, \Omega, ||g_{u-w}||_{L^p(\Omega)})$. Thus we have

$$
\frac{1}{\mu(B_R)} \int_{B_R \cap \Omega} H\left(x, \frac{u - w}{R}\right) d\mu \leq C \left(\frac{1}{\mu(B_{2\lambda R})} \int_{B_{2\lambda R} \cap \Omega} H(x, g_{u-w})^d d\mu\right)^{\frac{1}{d}}
$$
\n
$$
\leq C \left(\frac{1}{\mu(B_{2\lambda R})} \int_{B_{2\lambda R} \cap \Omega} H(x, g_u + g_w)^d d\mu\right)^{\frac{1}{d}}
$$
\n
$$
\leq C \left(\frac{1}{\mu(B_{2\lambda R})} \int_{B_{2\lambda R} \cap \Omega} \left(H(x, g_u)^d + H(x, g_w)^d\right) d\mu\right)^{\frac{1}{d}}
$$
\n
$$
\leq C \left(\left(\frac{1}{\mu(B_{2\lambda R})} \int_{B_{2\lambda R} \cap \Omega} H(x, g_u)^d d\mu\right)^{\frac{1}{d}} + \left(\frac{1}{\mu(B_{2\lambda R})} \int_{B_{2\lambda R} \cap \Omega} H(x, g_w)^d d\mu\right)^{\frac{1}{d}}\right)
$$
\n
$$
\leq C \left(\left(\frac{1}{\mu(B_{2\lambda R})} \int_{B_{2\lambda R} \cap \Omega} H(x, g_u)^d d\mu\right)^{\frac{1}{d}} + \frac{1}{\mu(B_{2\lambda R})} \int_{B_{2\lambda R} \cap \Omega} H(x, g_w) d\mu\right),
$$
\n(4.4)

where $C = C(\text{data}, \gamma, \Omega, ||g_{u-w}||_{L^p(\Omega)})$. By [\(4.3\)](#page-11-0), [\(4.4\)](#page-11-1) and the doubling property [\(2.1\)](#page-1-2), we obtain

$$
\begin{split} & \frac{1}{\mu(B_{\frac{R}{2}})}\int_{B_{\frac{R}{2}}\cap\Omega}H(x,g_{u})\,\mathrm{d}\mu \\ & \leq C\left(\frac{1}{\mu(B_{R})}\int_{B_{R}\cap\Omega}H\left(x,\frac{u-w}{R}\right)\,\mathrm{d}\mu+\frac{1}{\mu(B_{2\lambda R})}\int_{B_{2\lambda R}\cap\Omega}H(x,g_{w})\,\mathrm{d}\mu\right) \\ & \leq C\left(\left(\frac{1}{\mu(B_{2\lambda R})}\int_{B_{2\lambda R}\cap\Omega}H(x,g_{u})^{d}\,\mathrm{d}\mu\right)^{\frac{1}{d}}+\frac{1}{\mu(B_{2\lambda R})}\int_{B_{2\lambda R}\cap\Omega}H(x,g_{w})\,\mathrm{d}\mu\right), \end{split}
$$

where $C = C(\text{data}, \gamma, \Omega, ||g_{u-w}||_{L^p(\Omega)})$.

Let $f = H(x, g_u) \chi_{\Omega}$ and $g = H(x, g_w) \chi_{\Omega}$. Since $B_{\frac{r}{2}} \subset B_r \subset B_{\frac{R}{2}}$, we obtain

$$
\int_{B_{\frac{r}{2}}} f d\mu \le C \int_{B_{\frac{R}{2}}} f d\mu \le C \left(\left(\int_{B_{2\lambda R}} f^d d\mu \right)^{\frac{1}{d}} + \int_{B_{2\lambda R}} g d\mu \right)
$$

= $C \left(\left(\int_{B_{16\lambda^2 r}} f^d d\mu \right)^{\frac{1}{d}} + \int_{B_{16\lambda^2 r}} g d\mu \right),$

whenever $B_{3\lambda r} \setminus \Omega \neq \emptyset$. If $B_{3\lambda r} \subset \Omega$, by [\(4.2\)](#page-11-2) we have

$$
\int_{B_{\frac{r}{2}}} f d\mu \le C \left(\int_{B_{2\lambda r}} f^d d\mu \right)^{\frac{1}{d}} \le C \left(\left(\int_{B_{16\lambda^2 r}} f^d d\mu \right)^{\frac{1}{d}} + \int_{B_{16\lambda^2 r}} g d\mu \right).
$$

It follows that

$$
\int_{B_{\frac{r}{2}}} f d\mu \leq C \left(\left(\int_{B_{16\lambda^2 r}} f^d d\mu \right)^{\frac{1}{d}} + \int_{B_{16\lambda^2 r}} g d\mu \right)
$$

for every ball B_r in X with $0 < r \leq 1$. Note that this is trivially true for balls with $B_r \cap \Omega = \emptyset$, since the left-hand side is zero. By a straight forward covering argument we have

$$
\int_{B_{\frac{r}{2}}} f d\mu \le C \left(\left(\int_{B_r} f^d d\mu \right)^{\frac{1}{d}} + \int_{B_r} g d\mu \right)
$$

for every ball B_r in X with $0 < r \leq 1$. Here $C = C(\text{data}, \Omega, \gamma, ||g_{u-w}||_{L^p(\Omega)})$ and $0 < d = d(\text{data}) < 1$. Note carefully, that the constant C depends on the underlying set Ω and on $||g_{u-w}||_{L^p(\Omega)}$, but once the domain Ω and the boundary function w are fixed, the obtained reverse Hölder inequality is uniform over balls B_r in X with $0 < r \leq 1$. By an application of Lemma [4.1,](#page-9-1) there exist a constant $C = C(\text{data}, \gamma, \Omega, ||g_{u-w}||_{L^p(\Omega)})$ and an exponent $\varepsilon = \varepsilon(\text{data}, \Omega, \gamma, ||g_{u-w}||_{L^p(\Omega)}) > 0$ such that

$$
\left(\oint_{B_{\frac{r}{2}}} f^{1+\varepsilon} d\mu\right)^{\frac{1}{1+\varepsilon}} \leq C \left(\oint_{B_r} f d\mu + \left(\oint_{B_r} g^{\sigma} d\mu\right)^{\frac{1}{\sigma}}\right),
$$

for every ball B_r in X with $0 < r \leq 1$. Thus, we have

$$
\begin{aligned} & \left(\frac{1}{\mu(B_{\frac{r}{2}})} \int_{B_{\frac{r}{2}} \cap \Omega} H(x, g_u)^{1+\varepsilon} \, \mathrm{d}\mu \right)^{\frac{1}{1+\varepsilon}} \\ &\leq C \left(\frac{1}{\mu(B_r)} \int_{B_r \cap \Omega} H(x, g_u) \, \mathrm{d}\mu + \left(\frac{1}{\mu(B_r)} \int_{B_r \cap \Omega} H(x, g_w)^{\sigma} \, \mathrm{d}\mu \right)^{\frac{1}{\sigma}} \right), \end{aligned}
$$

for every ball with B_r in X with $0 < r \leq 1$. Since Ω is bounded, we may cover it by a finite number of balls $B_{r_j}(x_j)$, $j = 1, 2, \ldots, N$, with $0 < r_j \leq 1$. By summing over $j = 1, \ldots, N$, we obtain

$$
\left(\int_{\Omega} H(x,g_u)^{1+\varepsilon} d\mu\right)^{\frac{1}{1+\varepsilon}} \leq C \left(\int_{\Omega} H(x,g_u) d\mu + \left(\int_{\Omega} H(x,g_w)^{\sigma} d\mu\right)^{\frac{1}{\sigma}}\right),
$$

where $C = C(\text{data}, \gamma, \Omega, \|q_{u-w}\|_{L^p(\Omega)})$.

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