On umbral properties of a family of hyperbolic-like functions appearing in magnetic transport problem

Article *in* Reports on Mathematical Physics · August 2023

DOI: 10.1016/S0034-4877(23)00053-8

4 authors:

Giuseppe Dattoli
ENEA

Subuhi Khan
Aligarh Muslim University

Mehnaz Haneef
Aligarh Muslim University

Silvia Licciardi
ENEA

ON UMBRAL PROPERTIES OF A FAMILY OF HYPERBOLIC-LIKE FUNCTIONS APPEARING IN MAGNETIC TRANSPORT PROBLEM

Giuseppe Dattoli[1], Subuhi Khan*[2], Mehnaz Haneef[2] and Silvia Licciardi[1]

[1]ENEA–Frascati Research Center, Via Enrico Fermi 45, 00044 Rome, Italy
[2]Department of Mathematics, Aligarh Muslim University, Aligarh-202001, India
(e-mails: pinodattoli@libero.it, subuhi2006@gmail.com, mehnaz272@gmail.com, silvia.licciardi@enea.it)

(*Received October 26, 2022 — Revised April 12, 2023*)

Umbral operational techniques offer sturdy mechanism in the studies of special functions and special polynomials. The techniques of umbral calculus are employed to derive properties of families of exponential-like functions and their hyperbolic forms. The generalized forms of Mittag-Leffler functions are used to solve technical problems concerning transport of a charged beam in a solenoid magnet. The proposed method is flexible and has many advantages over standard computational techniques.

Keywords: umbral methods, pseudo-exponential functions, pseudo-hyperbolic functions, Mittag-Leffler functions, hypergeometric functions.

2020 MSC: 33C10; 33C45; 33F10.

1. Introduction

The umbral notion is an ideally suited tool to study the properties of families of functions which resemble the exponential and other associated functions [15]. A restatement of the Roman-Rota formulation [21] inspired the extension of umbral approach to special functions and polynomials. It is a versatile instrument for explicit calculations that aids in the development of concepts in the setting of operational calculus. The approaches associated with umbral methods have been widely employed to investigate the characteristics of special functions. It has been shown to be beneficial in obtaining known and innovative forms of integrals including products of special functions and other functions, which are not categorised as special functions.

The function

$$\psi(n) := \psi_n = \frac{1}{(a)_n}, \quad \forall\, a \in \mathbb{R}, \tag{1.1}$$

is called the umbral vacuum with initial state $\psi_0 = 1/(a)_0 = 1$.

*Corresponding author

DEFINITION 1. *The vacuum shift operator* or *the umbral operator* $\hat{a}$ is defined as

$$\hat{a} = e^{\partial x}, \tag{1.2}$$

where the variable x lies in the domain of the function on which the operator acts.

The action of the umbral operator $\hat{a}^n$, $\forall n \in \mathbb{R}$, on the vacuum ψ_0 is given as

$$\hat{a}^n \psi_0 := \psi_n = \frac{1}{(a)_n}. \tag{1.3}$$

The Pochhammer symbol

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \tag{1.4}$$

is used to define the pseudo-exponential function $e(x;a)$ by the following series

$$e(x;a) = \sum_{n=0}^{\infty} \frac{x^n}{(a)_n\, n!}. \tag{1.5}$$

The umbral image of the pseudo-exponential function is given as

$$e(x;a) = e^{\hat{a}x}\psi_0. \tag{1.6}$$

The action of the operator $D_x := d/dx$ on Eq. (1.6) gives

$$\hat{D}_x\, e(x;a) = e_1(x;a) = \hat{a}e^{\hat{a}x}\,\psi_0 = \sum_{r=0}^{\infty} \frac{x^r}{r!\,(a)_{r+1}}. \tag{1.7}$$

It is therefore evident that the exponential in equation (1.5) is satisfying an umbral-eigenvalue equation. However, it is not an eigenfunction of the derivative operator. In general, it follows that

$$\hat{D}_x^n\, e(x;a) = e_n(x;a) = \hat{a}^n e^{\hat{a}x}\,\psi_0 = \sum_{r=0}^{\infty} \frac{x^r}{r!(a)_{r+n}}. \tag{1.8}$$

In view of Eq. (1.6), the pseudo-hyperbolic functions associated with pseudo-exponential function $e(x;a)$ are expressed as:

$$\mathrm{ch}(x;a) = \frac{e^{\hat{a}x} + e^{-\hat{a}x}}{2}\psi_0, \tag{1.9}$$

$$\mathrm{sh}(x;a) = \frac{e^{\hat{a}x} - e^{-\hat{a}x}}{2}\psi_0. \tag{1.10}$$

The first and second-order derivatives of pseudo-hyperbolic functions are obtained as follows:

$$\hat{D}_x\,\mathrm{ch}(x;a) := \mathrm{sh}_1(x;a) = \hat{a}\frac{e^{\hat{a}x} - e^{-\hat{a}x}}{2}\psi_0, \tag{1.11}$$

$$\hat{D}_x^2\,\mathrm{ch}(x;a) := \mathrm{ch}_2(x;a) = \hat{a}^2\frac{e^{\hat{a}x} + e^{-\hat{a}x}}{2}\psi_0, \tag{1.12}$$

and

$$\hat{D}_x \operatorname{sh}(x;a) := \operatorname{ch}_1(x;a) = \hat{a}\frac{e^{\hat{a}x} + e^{-\hat{a}x}}{2}\psi_0, \tag{1.13}$$

$$\hat{D}_x^2 \operatorname{sh}(x;a) := \operatorname{sh}_2(x;a) = \hat{a}^2\frac{e^{\hat{a}x} - e^{-\hat{a}x}}{2}\psi_0, \tag{1.14}$$

respectively.

It is to be noted that pseudo-hyperbolic functions satisfy cyclical derivative rule analogous to that of their ordinary counterparts, that is

$$\hat{D}_x^{2n-1} \operatorname{ch}(x;a) = \operatorname{sh}_{2n-1}(x;a), \tag{1.15}$$

$$\hat{D}_x^{2n} \operatorname{ch}(x;a) = \operatorname{ch}_{2n}(x;a), \tag{1.16}$$

and

$$\hat{D}_x^{2n-1} \operatorname{sh}(x;a) = \operatorname{ch}_{2n-1}(x;a), \tag{1.17}$$

$$\hat{D}_x^{2n} \operatorname{sh}(x;a) = \operatorname{sh}_{2n}(x;a), \tag{1.18}$$

where

$$\operatorname{ch}_n(x;a) := \hat{a}^n \operatorname{ch}(x;a), \tag{1.19}$$

$$\operatorname{sh}_n(x;a) := \hat{a}^n \operatorname{sh}(x;a). \tag{1.20}$$

The pseudo exponential $e(x;a)$ is essentially a Bessel type function and the relevant properties will be discussed in the concluding part of this article.

The umbral methods are flexible and there are problems that are hardly guessed within a "non-umbral" framework as for example evaluation of the Gaussian integral

$$I(a) = \int_{-\infty}^{\infty} e(-x^2;a)dx = \sqrt{\pi}\frac{\Gamma(a)}{\Gamma\left(a-\frac{1}{2}\right)}. \tag{1.21}$$

Derivation of the above integral will be discussed in the forthcoming sections, along with further examples, stressing the usefulness of the proposed formalism.

The introductory remarks in this section are aimed at clarifying the techniques, which will be used in the forthcoming parts of the paper. In Section 2, the umbral methods are employed to study the properties of a family of hyperbolic-like functions, introduced in the theory of electron beam transport in solenoid lenses [10]. In Section 3, the 3-parameter version of the Mittag-Leffler function [18], which is of pivotal importance in fractional calculus [17] is considered within the umbral framework. Finally, in concluding section, the extension of the method to hypergeometric function is discussed.

2. The umbral methods and charged beam transport problem

The solenoids are devices of electromagnetic nature, consisting of elongated coils transporting a current, which produces along the solenoid axis, a magnetic field of desired strength. They are exploited in different applications involving plasma confinement or charged beam transport. An undesired feature of these devices, when exploited in magnetic transport is the effect of fringing which is responsible for distortion of the electron trajectories. The latter effects should be carefully determined, since they are responsible for mechanisms making the transported charged beam useless for the desired applications [4].

The problem of a charged beam transport through a solenoid has been treated by using operational methods, which revealed particular usefulness for the inclusion of the effects of fringing [5]. Analogous methods were empoyed to accomplish the study of the electron beam transport through quadrupoles with the inclusion of wakefield effects [10].

The authors in [16] employed the formalism of the symplectic mechanics to find the "optical" properties of a solenoid magnet with the inclusion of the fringe field (namely the field extension outside the solenoid).

The possibility of exploiting umbral technicalities (or operational methods as well) to treat beam transport problem including magnetic fringing and wakes, is suggested in [10]. In that article the equations ruling the beam dynamics have been shown to be formally equivalent to the second-order equation accounting for the ordinary betatron motion [6]. Accordingly all the relevant dynamics can be treated using an equivalent transport matrix expressed in terms of hyperbolic-like functions, amenable for a corresponding umbral restyling. The associated mathematical problems have been shown to be simplified by the introduction of a new set of functions that generalize the ordinary hyperbolic functions. These functions occur as solutions of the following characteristic differential equation

$$\frac{d^2}{dx^2}C - x\frac{d}{dx}C - aC = 0. \tag{2.1}$$

The solution of the above equation can be cast in the form [16]

$$C = A_1\mathrm{CH}(x;a) + A_2\mathrm{SH}(x;a), \tag{2.2}$$

where A_1, A_2 are arbitrary constants and $\mathrm{CH}(x;a)$ and $\mathrm{SH}(x;a)$ are given by the series:

$$\mathrm{CH}(x;a) = 1 + \sum_{n=1}^{\infty} \frac{a(a+2)\cdots(a+2n-2)}{(2n)!}x^{2n} \tag{2.3}$$

and

$$\mathrm{SH}(x;a) = 1 + \sum_{n=1}^{\infty} \frac{(a+1)(a+3)\cdots(a+2n-1)}{(2n+1)!}x^{2n+1}, \tag{2.4}$$

respectively.

The functions in (2.3) and (2.4) show formal resemblance with the ordinary hyperbolic functions. In order to take advantage from such an analogy, the umbral operator ${}_a\hat{b}$ is defined as

$${}_a\hat{b}^n \xi_0 = (a)_n \tag{2.5}$$

Further, we introduce the associated exponential function as

$$e_0(x,a) = e^{a\hat{b}x}\,\xi_0 = \sum_{r=0}^{\infty} \frac{(a)_r\, x^r}{r!}. \tag{2.6}$$

The successive derivatives can be used to define the higher-order counterparts, namely

$$e_m(x,a) = \hat{D}_x^m e^{a\hat{b}x}\xi_0 = \sum_{r=0}^{\infty} \frac{(a)_{r+m}\, x^r}{r!}. \tag{2.7}$$

The associated hyperbolic functions are defined as:

$$\mathrm{ch}_m(x,a) = \frac{e_m(x,a) + e_m(-x,a)}{2}, \tag{2.8}$$

$$\mathrm{sh}_m(x,a) = \frac{e_m(x,a) - e_m(-x,a)}{2}, \tag{2.9}$$

which satisfy the following differential equations:

$$\hat{D}_x^2\, \mathrm{ch}_m(x,a) = \mathrm{ch}_{m+2}(x,a), \tag{2.10}$$

$$\hat{D}_x^2\, \mathrm{sh}_m(x,a) = \mathrm{sh}_{m+2}(x,a). \tag{2.11}$$

It is to be noted that this family of functions exhibits limited convergence $(0 < x < 1)$.

The use of the umbral formalism is fairly powerful in simplifying of the underlying calculations. For example all the computations involving average values yielding the beam divergences and transverse sections [6] can be worked out by the use of integral identities like that reported in (1.21). It is now interesting to see whether the formalism, we have envisaged, may be further extended to include further generalizations of special functions, whose usefulness to treat physical problems, will be discussed later in the paper. The Bessel functions $J_n(x)$ are reformulated by the following relation [15],

$$J_n(x) = -\left(\hat{c}\frac{x}{2}\right)^n e^{-\hat{c}\left(\frac{x}{2}\right)^2}\phi_0, \tag{2.12}$$

where the action of the umbral operator $\hat{c}$ on the vacuum function $\phi_n = 1/\Gamma(n+1)$ is defined as

$$\hat{c}^n \phi_0 := \phi_n = \frac{1}{\Gamma(n+1)}. \tag{2.13}$$

Further, the Bessel type functions associated with the umbral exponential $e_0(x;a)$, are defined by the following series

$$J_0(x,a) = \sum_{r=0}^{\infty} \frac{(-1)^r (a)_r \left(\frac{x}{2}\right)^{2r}}{(r!)^2}. \tag{2.14}$$

The uniform convergence of $J_0(x;a)$ holds for $|x| < \infty$ and by the use of umbral formalism, these functions are expressed in the Gaussian form as

$$J_0(x,a) = e^{-{}_a\hat{b}\hat{c}\left(\frac{x}{2}\right)^2}\xi_0\phi_0, \tag{2.15}$$

where the combined action of the operators $\hat{b}$ and $\hat{c}$ is defined as

$$({}_a\hat{b}\hat{c})^r\,\xi_0\,\phi_0 = \frac{\Gamma(a+r)}{\Gamma(a)\Gamma(r+1)}. \tag{2.16}$$

Here, we have used a double vacuum which is fully legitimate within the context of the method we are employing.

To evaluate the integral of the function $J_0(x;a)$ in Eq. (2.15), we proceed as follows,

$$\int_{-\infty}^{\infty} J_0(x,a)\,dx = \int_{-\infty}^{\infty} e^{-{}_a\hat{b}\hat{c}\left(\frac{x}{2}\right)^2}\,\xi_0\,\phi_0\,dx, \tag{2.17}$$

which on using the following well-known integral

$$\int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}\,dx = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}-c} \tag{2.18}$$

becomes

$$\int_{-\infty}^{\infty} J_0(x,a)\,dx = 2\sqrt{\pi}\hat{c}^{-\frac{1}{2}}{}_a\hat{b}^{-\frac{1}{2}}\,\xi_0\,\phi_0. \tag{2.19}$$

In view of Eq. (2.16), we obtain

$$\int_{-\infty}^{\infty} J_0(x,a)\,dx = 2\ \ (a)_{-1/2}, \quad \operatorname{Re} a > 1. \tag{2.20}$$

Further comments on the properties of this family of functions will be discussed in the concluding section.

3. The 3-parameter Mittag-Leffler functions

The 3-parameter Mittag-Leffler function (3PML) defined as [18]

$$E^{(\gamma)}_{(\alpha,\beta)}(x) = \sum_{n=0}^{\infty}\frac{(\gamma)_n x^n}{\Gamma(\alpha n+\beta)n!}, \tag{3.1}$$

is also known as Prabhakar function. For the relevant study we limit ourselves to real parameters α, β, γ and variable x. It plays a primary role in many problems associated with fractional calculus and with anomalous transport problems, for details see [20].

In a previous paper [7], the Mittag-Leffler functions of one and two parameters have been studied using the umbral formalism. Following the strategy defined in [7], we set

$$E^{(\gamma)}_{(\alpha,\beta)}(x) = \hat{c}^{\beta-1} e^{\hat{b}\hat{c}^{\alpha} x}\, \xi_0\, \phi_0. \tag{3.2}$$

This form allows a significant simplification of the associated computational technicalities. By applying the m-th order derivative w.r.t. x in Eq. (3.2), it follows that

$${}_mE^{(\gamma)}_{(\alpha,\beta+m\alpha)}(x) := \hat{D}^m_x\, E^{(\gamma)}_{(\alpha,\beta)}(x) = \hat{c}^{\beta-1+m\alpha}\, \hat{b}^m\, e^{\hat{b}\,\hat{c}^{\alpha} x}\, \xi_0\, \phi_0, \tag{3.3}$$

which in view of Eq. (2.5) and (2.13) yields

$${}_mE^{(\gamma)}_{(\alpha,\beta+m\alpha)}(x) := \hat{D}^m_x\, E^{(\gamma)}_{(\alpha,\beta)}(x) = \sum_{n=0}^{\infty} \frac{(\gamma)_{n+m} x^n}{\Gamma(\alpha n + \beta + m\alpha) n!}. \tag{3.4}$$

Next, operating $e^{y\hat{D}_x{}^2}$ on both sides of Eq. (3.2), we find

$$e^{y\hat{D}_x{}^2} E^{(\gamma)}_{(\alpha,\beta)}(x) = \hat{c}^{\beta-1} e^{y(\hat{b}\hat{c}^{\alpha})^2} e^{\hat{b}\hat{c}^{\alpha} x}\, \xi_0\, \phi_0,$$

which in view of the following generating function of 2-variable Hermite Kampé de Fériet polynomials [1]

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y) \frac{t^n}{n!}, \tag{3.5}$$

yields

$$e^{y\hat{D}_x{}^2} E^{(\gamma)}_{(\alpha,\beta)}(x) = \hat{c}^{\beta-1} \sum_{n=0}^{\infty} \frac{H_n(x,y)}{n!} \left(\hat{b}\hat{c}^{\alpha}\right)^n \xi_0\, \phi_0. \tag{3.6}$$

Finally, using Eq. (2.16), we obtain

$$e^{y\hat{D}_x{}^2} E^{(\gamma)}_{(\alpha,\beta)}(x) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(n\alpha+\beta)} \frac{H_n(x,y)}{n!}. \tag{3.7}$$

It is to be noted that results in Eqs. (3.4) and (3.7) are easily obtained by the use of standard calculus. Next, we proceed for the derivation of other identities which are strenuous to derive without employing the operational-umbral technique. We consider the derivation of the "pseudo-Fourier" transform in the form of following result.

THEOREM 1. *The following "pseudo-Fourier" transform for the 3-parameter Mittag-Leffler function holds,*

$$\tilde{F}^{(\gamma)}_{(\alpha,\beta)}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} E^{(\gamma)}_{(\alpha,\beta)}(ikx)\, dx = \frac{1}{\sqrt{2}} \sum_{r=0}^{\infty} \frac{(-1)^r (\gamma)_{2r}}{r!\Gamma(2\alpha r+\beta)} \left(\frac{k}{2}\right)^{2r}. \tag{3.8}$$

Proof: Considering equation (3.2) and treating umbral operators as ordinary algebraic constants, we find

$$\tilde{F}^{(\gamma)}_{(\alpha,\beta)}(k) = \frac{\hat{c}^{\beta-1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} e^{i\hat{b}\hat{c}^{\alpha}kx}\, \xi_0\, \phi_0\, dx,$$

which in view of Eq. (2.18) gives

$$\tilde{F}^{(\gamma)}_{(\alpha,\beta)}(k) = \frac{\hat{c}^{\beta-1}}{\sqrt{2}} e^{-\frac{\hat{b}^2\hat{c}^{2\alpha}}{4}k^2}\, \xi_0\, \phi_0.$$

Finally, in view of equation (2.16), assertion (3.8) follows. □

Corollary 1. *For the 3-parameter Mittag-Leffler function, the following integral holds,*

$$\int_{-\infty}^{\infty} E^{(\gamma)}_{(\alpha,\beta)}(-x^2)dx = \sqrt{\frac{\pi}{\hat{b}\hat{c}^{\alpha}}}\hat{c}^{\beta-1}\, \xi_0\, \phi_0 = \sqrt{\pi}\frac{(\gamma)_{-\frac{1}{2}}}{\Gamma(\beta - \frac{\alpha}{2})}, \tag{3.9}$$

where $(\gamma)_{-\frac{1}{2}} = \frac{\Gamma(\gamma-\frac{1}{2})}{\Gamma(\gamma)}$.

Proof: Following the same lines of proof of Theorem 1 and considering Eq. (2.5) and (2.13), assertion (3.9) follows. □

Note. Numerical check confirms the validity of Eq. (3.9) for $1 < \alpha, \beta, \gamma < 2$.

We have used the notations and rules of the umbral calculus to deal with a family of functions, which has been proven to be very useful to treat problems of applicative nature. The umbral formalism has provided, a fairly concise way of expressing these functions and offered the possibility of extending, in quite a straightforward way, the properties of the ordinary hyperbolic and circular functions.

The formalism underlying the hypergeometric functions is a powerful tool to unify different families of special functions. It is interesting to note that the umbral protocol is an effective means to be extended to hypergeometric functions to obtain further development in this field of research. In the next section, we deal with the preliminary extension of the outlined methods to the hypergeometric functions.

4. Concluding remarks

The use of the formalism, we have used so far, allows to cast the Gauss hypergeometric functions [19] in the form

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!} = \sum_{n=0}^{\infty} \hat{\kappa}^n \frac{x^n}{n!}\, \zeta_0, \tag{4.1}$$

where $\hat{\kappa}$ is the umbral operator whose action on the vacuum ζ_0 is defined as

$$\hat{\kappa}^n \zeta_0 = \frac{(a)_n (b)_n}{(c)_n}. \tag{4.2}$$

The use of the formulae established in the introductory section allows the derivation of the following well-known recurrence relation for the hypergeometric functions in a straightforward manner,

$$\partial_x^m \, {}_2F_1(a, b; c; x) = \hat{\kappa}^m \sum_{n=0}^{\infty} \hat{\kappa}^n \frac{x^n}{n!} \zeta_0 = \frac{(a)_m (b)_m}{(c)_m} \, {}_2F_1(a+m, b+m; c+m; x). \tag{4.3}$$

The above recurrence relation is obtained in view of the identity

$$\hat{\kappa}^{n+m} \zeta_0 = \frac{(a)_{n+m} (b)_{n+m}}{(c)_{n+m}}, \tag{4.4}$$

where $(s)_{n+1} = s(s+1)_n$. The pseudo exponential function $e(x; a)$ defined in Eq. (1.5) can be regarded as a particular case of hypergeometric function and indeed we find

$$e(x; a) = {}_0F_1(-; a; x), \tag{4.5}$$

a form amenable for the same umbral translation as noted for the Gauss hypergeometric functions.

Regarding possible applications, we note the following relation between arcsin function and hypergeometric functions,

$$\frac{\arcsin(x)}{x} = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; x^2\right) = e^{\hat{\kappa} x^2} \zeta_0. \tag{4.6}$$

Within the present context, the umbral image of the function on the l.h.s. of the above equation is a Gaussian function. We can use the relevant known properties, to derive the new and unknown properties relevant to $\frac{\arcsin(x)}{x}$. By using following identity [3]

$$\partial_x^n e^{ax^2} = H_n(2ax, a) e^{ax^2}, \tag{4.7}$$

in Eq. (4.6), it follows that

$$\partial_x^n \left(\frac{\arcsin(x)}{x}\right) = H_n(2\hat{\kappa}x, \hat{\kappa}) e^{\hat{\kappa}x^2} \zeta_0 = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(2x)^{n-2k} \hat{\kappa}^{n-k}}{(n-2k)!\,k!} e^{\hat{\kappa}x^2} \zeta_0. \tag{4.8}$$

Therefore, keeping the derivatives of the above equation and using the identities (4.3) and (4.8), we eventually end up with

$$\partial_x^n \left(\frac{\arcsin(x)}{x}\right) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(2x)^{n-2k} \hat{\kappa}^{n-k}}{(n-2k)!\,k!} \frac{\left(\frac{1}{2}\right)_{n-k} \left(\frac{1}{2}\right)_{n-k}}{\left(\frac{3}{2}\right)_{n-k}} \, {}_2F_1\left(\frac{1}{2}+n-k, \frac{1}{2}+n-k; \frac{3}{2}+n-k; x^2\right). \tag{4.9}$$

The derivation of the above identity does not appear trivial with ordinary means. As a further illustration of the above mentioned formalism, we establish the n-th order derivative formula for the following function [11]

$$
{}_2F_1\left[\frac{1}{3},\frac{2}{3};\frac{3}{2};-\frac{27}{4}x^2\right]=\sqrt[3]{X_+}-\sqrt[3]{\frac{1}{X_+}}, \tag{4.10}
$$

where

$$
X_+=\frac{3\sqrt{3}+\sqrt{27x^2+4}}{2}, \tag{4.11}
$$

which occupies a central role in the theory of algebraic equations.

THEOREM 2. *For the function ${}_2F_1\left[\frac{1}{3},\frac{2}{3};\frac{3}{2};-\frac{27}{4}x^2\right]$, the following n-th order derivative formula holds*

$$
\partial_x^n {}_2F_1\left[\frac{1}{3},\frac{2}{3};\frac{3}{2};\frac{-27}{4}x^2\right]=\sum_{k=0}^{[n/2]}\frac{(2x)^{n-2k}\alpha^{n-k}}{(n-2k)!k!}\frac{\left(\frac{1}{3}\right)_{n-k}\left(\frac{2}{3}\right)_{n-k}}{\left(\frac{3}{2}\right)_{n-k}}
$$

$$
\times\,{}_2F_1\left[\frac{1}{3}+n-k,\frac{2}{3}+n-k;\frac{3}{2}+n-k;\alpha x^2\right], \tag{4.12}
$$

where $\alpha:=-\frac{27}{4}x^2$.

Proof: Taking the n-th order drivative of equation (4.10) w.r.t. x and considering equation (4.6), we find

$$
\partial_x^n {}_2F_1\left[\frac{1}{3},\frac{2}{3};\frac{3}{2};-\frac{27}{4}x^2\right]=\partial_x^n\left(\sqrt[3]{X_+}-\sqrt[3]{\frac{1}{X_+}}\right)=\partial_x^n\left(e^{\hat{\kappa}\alpha x^2}\zeta_0\right),\qquad \alpha=-\frac{27}{4}x^2,
$$

which in view of Eqs. (4.7) and (4.8) yields

$$
\partial_x^n\left(\sqrt[3]{X_+}-\sqrt[3]{\frac{1}{X_+}}\right)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(2x)^{n-2k}\alpha^{n-k}\hat{\kappa}^{n-k}}{(n-2k)!k!}e^{\hat{\kappa}\alpha x^2}\zeta_0.
$$

Finally, using Eq. (4.3) in the above equation, assertion (4.12) follows. □

Before concluding this paper, we note that the inference outlined so far are consistent with our previous researches on the umbral treatment of Bessel functions, according to which the umbral image of the cylindrical Bessel function is a Gaussian function. The link between Bessel functions and hypergeometric functions is well known and indeed we have

$$
J_0(x)={}_0F_1\left(-;1;-\frac{x^2}{4}\right)=e^{-\hat{\kappa}\left(\frac{x}{2}\right)^2}\zeta_0. \tag{4.12}
$$

Before concluding this article we like to mention further points corroborating the use of this formalism for the study of physical problems. The paradigmatic tool of the umbral procedure, we have described, foresees a modification of the exponentials. Nonstandard exponential forms occurs in the theory of quantum deformations [14]. The mathematical tool to deal with these problems is the use of q-calculus [13], which is a different flavor of umbral formalism [12]. This topic has been carefully considered in [9], where it has been shown how q and umbral formalisms can be merged. In the last two sections we have extended the original goal of the article, which was that of underscoring the umbral treatment for the analysis of transport problems. We have indeed introduced Bessel-like functions of the Mittag-Leffler type. They have been defined as umbral image of the Gaussian. Even though we have treated them using a mathematical point of view, we like to mention their importance in the study of different problems going from the definition of generalized coherent states, the solution of Schrödinger type and heat-type fractional partial differential equations [8].

In the upcoming more extended note, we will study in detail the importance of the umbral formalism for the study of hypergeometric functions and their generalizations.

Conflict of Interest: The authors declare that they have no conflict of interest.

Acknowledgements

The authors are thankful to the reviewers for several useful comments and suggestions towards the improvement of the paper.

REFERENCES

[1] P. Appell and J. Kampé de Fériet: *Fonctions Hypergéométriques et Hypersphériques Polynômes d'Hermite*, Gauthier-Villars, Paris, 1926.

[2] D. Babusci and G. Dattoli: On Ramanujan master theorem, *arXiv:1103.3947 [math-ph]*, (2011), arXiv.org.

[3] D. Babusci, G. Dattoli, S. Licciardi and E. Sabia: *Mathematical Methods for Physicists*, World Scientific Press, 2019.

[4] C. Baumgartel and S. Maher: A charged particle model based on weber electrodynamics for electron beam trajectories in coil and solenoid elements, *Prog. Electromagn. Res. C* **123**, 151 (2022).

[5] F. Ciocci, G. Dattoli and M. Migliorati: Generalized twiss coefficients including transverse coupling and e-beam growth, *EPAC Edinburgh*, 1966 (2006).

[6] G. Dattoli, A. Doria, E. Sabia and M. Artioli: Charged beam dynamics, *Particle accelerators and free electron lasers - iopscience.iop.org.* (2017).

[7] G. Dattoli, K. Górska and A. Horzela: Comments on the properties of Mittag-Leffler function, *Eur. Phys. J. Spec. Top.* **226**, 3427 (2017).

[8] G. Dattoli, K. Górska, A. Horzela, S. Licciardi and R. M. Pidatella: *Eur. Phys. J. Spec. Top.* **226** (16), 3443 (2017).

[9] G. Dattoli, S. Licciardi, B. Germano and M. R. Martinelli: q-Functions and distributions, operational and umbral methods, *Mathematics* **9** (21), 2664 (2021).

[10] G. Dattoli, L. Mezi and Mauro Migliorati: Operational methods for integro-differential equations and applications to problems in particle accelerator physics, *Taiwan. J. Math.* **11** (2), 407 (2007).

[11] G. Dattoli, M. Migliorati and P. E. Ricci: The parabolic trigonometric functions and the Chebyshev radicals, *arXiv:1102.1560 [math-ph]*, (2011), arXiv.org.

[12] T. Ernst: A Method for q-calculus, *J. Nonlinear Math. Phys.* **10** (4), 487 (2003).

[13] M. E. H. Ismail: *Classical and Quantum Orthogonal Polynomials in One Variable*, Cambridge University Press, 2005.

[14] A. Lavagno and G. Gervino: Quantum mechanics in q-deformed calculus, *Journal of Physics: Conference Series* **174**, 01 (2009).
[15] S. Licciardi: Umbral calculus, A different mathematical language, Ph.D. Thesis, *arXiv:1803.03108 [math.CA]*, 2018, arXiv.org.
[16] M. Migliorati and G. Dattoli: Transport matrix of a solenoid with linear fringe field, *Nuovo Cimento B.* **124** (4), 385 (2009).
[17] K. B. Oldham and J. Spanier: *The Fractional Calculus Theory and Applications of Differentiation and Integration to arbitrary order*, Academic Press Inc., London, 1974.
[18] T. R. Prabhakar: A singular integral equation with a generalized Mittag-Leffler function in the kernel, *Yokohama Math. J.* **19**, 7 (1971).
[19] E. D. Rainville: *Special Functions*, Macmillan, New York 1960.
[20] Roberto Garra and Roberto Garrappa: The Prabhakar or three parameter Mittag–Leffler function: Theory and application, *Commun. Nonlinear Sci. Numer. Simul.* **56**, 314 (2018).
[21] S. M. Roman and G. C. Rota: The umbral calculus, *Adv. Math.* **27**, 95 (1978).