

An anisotropic Dirichlet system including unbounded coefficients

Dumitru Motreanu^a, Abdolrahman Razani^b, and Elisabetta Tornatore^c

^aDepartment of Mathematics, University of Perpignan, 66860 Perpignan, France. E-Mail: motreanu@univ-perp.fr

^bDepartment of Pure Mathematics, Faculty of Science, Imam Khomeini International University, Postal code: 3414896818, Qazvin, Iran. E-Mail: razani@sci.ikiu.ac.ir

^cDepartment of Mathematics and Computer Science, University of Palermo, 90123 Palermo, Italy.
E-Mail: elisa.tornatore@unipa.it

Abstract

The paper focuses on an anisotropic system driven by (p, q) -Laplacian operators with unbounded coefficients depending on the solution and whose lower order terms exhibit full dependence on the solution and its gradient. The main results establish the existence of solutions and the uniform boundedness of the solution set. The approach is based on a suitable truncation to drop the unboundedness of the coefficients and on the solvability of the truncated system within the theory of pseudomonotone operators.

Keywords: Anisotropic p -Laplacian; unbounded coefficient; bounded solution; truncation; pseudomonotone operator.

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1 Introduction

The anisotropic phenomena represent a challenging object for the mathematical physics requiring specific analysis and techniques. This is the case for example in fluid mechanics involving anisotropic media where the conductivity depends on the direction (see [1]). Recently, Motreanu and Tornatore [8] consider a Dirichlet problem exhibiting anisotropic differential operator with unbounded coefficients and full dependence on the gradient in the lower order terms

$$\begin{cases} -\sum_{i=1}^N \partial_i(G_i(u)|\partial_i u|^{p_i-2}\partial_i u) = F(x, u, \nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where Ω is a bounded domain in \mathbb{R}^N ($N \geq 2$) with a Lipschitz boundary $\partial\Omega$, G_i are unbounded functions on \mathbb{R} , for $i = 1, \dots, N$, and F is a Carathéodory function. It is worth mentioning that anisotropic boundary value problems with full dependence of the solution and its gradient have only very recently been started to be investigated (refer to [2–4, 8]). The study of problems driven by weighted p -Laplacian operator with unbounded coefficients has been initiated in [7, 9].

Here we consider the system counterpart of equation (1.1), namely the following Dirichlet system of coupled equations

$$\begin{cases} -\sum_{i=1}^N \partial_i(G_{1i}(u)|\partial_i u|^{p_i-2}\partial_i u) = F_1(x, u, v, \nabla u, \nabla v) & \text{in } \Omega, \\ -\sum_{i=1}^N \partial_i(G_{2i}(v)|\partial_i v|^{q_i-2}\partial_i v) = F_2(x, u, v, \nabla u, \nabla v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

where Ω is as in (1.1) and

- (i) $p_i, q_i \in (1, +\infty)$ for $i = 1, \dots, N$;
- (ii) the functions $G_{ki} : \mathbb{R} \rightarrow [a_{ki}, +\infty)$ are continuous with $a_{ki} > 0$ for $i = 1, \dots, N, k = 1, 2$;
- (iii) the functions $F_1, F_2 : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ are Carathéodory functions (i.e., $F_1(\cdot, s, t, \xi, \eta)$ and $F_2(\cdot, s, t, \xi, \eta)$ are measurable on Ω for each $(s, t, \xi, \eta) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$ and $F_1(x, \cdot, \cdot, \cdot, \cdot)$ and $F_2(x, \cdot, \cdot, \cdot, \cdot)$ are continuous on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$ for almost all $x \in \Omega$).

We emphasize that generally the treatment for a system is fundamentally different with respect to an equation due to the fact that in a system problem the variables cannot be separated to verify individual equations needing to invent novel procedures keeping them together. In line with this, the main difficulty in handling system (1.2) is that the estimates obtained for equation (1.1) cannot cover the system case. For example, this is apparent in the proof that the solution set is uniformly bounded. Although we follow the reasoning plan in [8], we develop new ideas and tools to resolve the system case of (1.2).

Set $\vec{p} := (p_1, \dots, p_N)$ and $\vec{q} := (q_1, \dots, q_N)$. Denote by $W_0^{1, \vec{p}}(\Omega)$ the completion of the set $C_c^\infty(\Omega)$ of C^∞ -functions with compact support in Ω with respect to the norm

$$\|u\|_{1, \vec{p}} := \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}}$$

and by $W_0^{1, \vec{q}}(\Omega)$ the completion of the set $C_c^\infty(\Omega)$ with respect to the norm

$$\|u\|_{1, \vec{q}} := \sum_{i=1}^N \|\partial_i u\|_{L^{q_i}}.$$

It follows that $X := W_0^{1, \vec{p}}(\Omega) \times W_0^{1, \vec{q}}(\Omega)$ is a reflexive Banach space with respect to the norm

$$\|(u, v)\| := \|u\|_{1, \vec{p}} + \|v\|_{1, \vec{q}}.$$

To simplify the presentation, for any real number $r > 1$ we denote $r' := r/(r-1)$ (the Hölder conjugate of r).

Definition 1.1. *It is said that $(u, v) \in X$ is a weak solution to problem (1.2) if*

$$\begin{aligned} G_{1i}(u)|\partial_i u|^{p_i-2} \partial_i u \partial_i \phi, \quad G_{2i}(v)|\partial_i v|^{q_i-2} \partial_i v \partial_i \psi &\in L^1(\Omega), \quad \forall i = 1, \dots, N, \\ F_1(\cdot, u(\cdot), v(\cdot), \nabla u(\cdot), \nabla v(\cdot))\phi, \quad F_2(\cdot, u(\cdot), v(\cdot), \nabla u(\cdot), \nabla v(\cdot))\psi &\in L^1(\Omega), \end{aligned}$$

and

$$\left\{ \begin{array}{l} \sum_{i=1}^N \int_{\Omega} G_{1i}(u(x)) |\partial_i u(x)|^{p_i-2} \partial_i u(x) \partial_i \phi(x) dx \\ \qquad \qquad \qquad = \int_{\Omega} F_1(x, u(x), v(x), \nabla u(x), \nabla v(x)) \phi(x) dx, \\ \sum_{i=1}^N \int_{\Omega} G_{2i}(v(x)) |\partial_i v(x)|^{q_i-2} \partial_i v(x) \partial_i \psi(x) dx \\ \qquad \qquad \qquad = \int_{\Omega} F_2(x, u(x), v(x), \nabla u(x), \nabla v(x)) \psi(x) dx \end{array} \right. \quad (1.3)$$

for all $(\phi, \psi) \in W_0^{1, \vec{p}}(\Omega) \times W_0^{1, \vec{q}}(\Omega)$.

We assume that

$$\sum_{i=1}^N \frac{1}{p_i} > 1 \text{ and } \sum_{i=1}^N \frac{1}{q_i} > 1. \quad (1.4)$$

The critical exponents are defined as

$$p^* := \frac{N}{\sum_{i=1}^N \frac{1}{p_i} - 1} \text{ and } q^* := \frac{N}{\sum_{i=1}^N \frac{1}{q_i} - 1}. \quad (1.5)$$

Notice that if $p_i = p$ for all $i = 1, \dots, N$, then p^* in (1.5) becomes the ordinary Sobolev critical exponent when $N > p$. Under assumption (1.4), there are continuous embeddings

$$W_0^{1, \vec{p}}(\Omega) \hookrightarrow L^{\rho_1}(\Omega) \text{ and } W_0^{1, \vec{q}}(\Omega) \hookrightarrow L^{\rho_2}(\Omega) \quad (1.6)$$

provided $1 \leq \rho_1 \leq p^*$ and $1 \leq \rho_2 \leq q^*$, which are compact if $1 \leq \rho_1 < p^*$, $1 \leq \rho_2 < q^*$ (see [6, Theorem 1]).

We also set

$$\begin{aligned} \bar{p} &:= \max\{p_1, \dots, p_N\} \quad \text{and} \quad \underline{p} := \min\{p_1, \dots, p_N\} \\ \bar{q} &:= \max\{q_1, \dots, q_N\} \quad \text{and} \quad \underline{q} := \min\{q_1, \dots, q_N\} \end{aligned}$$

and further assume

$$\bar{p} < p^* \text{ and } \bar{q} < q^*. \quad (1.7)$$

Consequently, by (1.6) there is a constant $\theta > 0$ such that

$$\|u\|_{L^{\rho_1}}^{\bar{p}} \leq \theta \|u\|_{1, \vec{p}}^{\bar{p}} \text{ and } \|v\|_{L^{\rho_2}}^{\bar{q}} \leq \theta \|v\|_{1, \vec{q}}^{\bar{q}}, \quad \forall (u, v) \in X. \quad (1.8)$$

Theorem 1.1. *Assume that conditions (1.4) and (1.7) hold, $G_{ki} : \mathbb{R} \rightarrow [a_{ki}, +\infty)$, with $a_{ki} > 0$ for $i = 1, \dots, N$ and $k = 1, 2$, are continuous functions, and $F_k : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$, for $k = 1, 2$, are Carathéodory functions satisfying hypotheses (H1) and (H2). Then problem (1.2) has at least a weak solution $(u, v) \in X \cap L^\infty(\Omega)^2$.*

The proof of Theorem 1.1 relies on an appropriate truncation destined to cut the unboundedness of coefficients G_{ki} . The right choice for the truncation is possible due to the uniform boundedness of the solution set by a bound that only depends on the lower bound a_{ki} of the functions G_{ki} . This is the contents of Theorem 3.1 below. In the proof of Theorem 3.1 we succeeded to find a way to reduce the estimates for the solution to system (1.2) to a Moser iteration addressing separately each component. The existence of solutions to the truncated system is proven in Theorem 5.1 by applying the surjectivity theorem for pseudomonotone operators. Finally, a comparison argument enables us to prove Theorem 1.1.

The rest of the paper consists of the following sections. Section 2 discusses the Nemytskii operator associated to the reaction terms in system (1.2). Section 3 contains Theorem 3.1 and its proof establishing the uniform bound for the solution set of (1.2). Section 4 examines the truncated system. Section 5 presents the existence of solutions to the truncated and original systems.

2 Associated Nemytskii operator

Proposition 2.1. *Assume (1.4), (1.7) and (H1). Then the map $\mathcal{N} : X \rightarrow L^{r'_1}(\Omega) \times L^{r'_2}(\Omega)$ given by*

$$\mathcal{N}(u, v) = (F_1(x, u, v, \nabla u, \nabla v), F_2(x, u, v, \nabla u, \nabla v)), \quad \forall (u, v) \in X,$$

is well defined, bounded (in the sense it maps bounded sets into bounded sets) and continuous.

Proof. By (H1), through a well-known convexity inequality we obtain

$$\begin{aligned} \int_{\Omega} |F_1(x, u, v, \nabla u, \nabla v)|^{r'_1} dx \leq & C \left(\sum_{i=1}^N \int_{\Omega} |\partial_i u|^{p_i} dx + \sum_{i=1}^N \int_{\Omega} |\partial_i v|^{q_i} dx \right. \\ & \left. + \int_{\Omega} |u|^{r_1} dx + \int_{\Omega} |v|^{r_2} dx + 1 \right) \end{aligned}$$

and

$$\int_{\Omega} |F_2(x, u, v, \nabla u, \nabla v)|^{r'_2} dx \leq C \left(\sum_{i=1}^N \int_{\Omega} |\partial_i u|^{p_i} dx + \sum_{i=1}^N \int_{\Omega} |\partial_i v|^{q_i} dx + \int_{\Omega} |u|^{r_1} dx + \int_{\Omega} |v|^{r_2} dx + 1 \right),$$

for all $(u, v) \in X$, with a constant $C > 0$. Since $\partial_i u \in L^{p_i}(\Omega)$ and $\partial_i v \in L^{q_i}(\Omega)$, we infer that the map $\mathcal{N} : X \rightarrow L^{r'_1}(\Omega) \times L^{r'_2}(\Omega)$ is well defined and bounded.

In order to show the continuity of \mathcal{N} , let $(u_n, v_n) \rightarrow (u, v)$ in X . By the definition of the space X and the continuous embedding $X \hookrightarrow L^{r_1}(\Omega) \times L^{r_2}(\Omega)$ in (1.6), we know that $(u_n, v_n) \rightarrow (u, v)$ in $L^{r_1}(\Omega) \times L^{r_2}(\Omega)$ and $(\partial_i(u_n), \partial_i(v_n)) \rightarrow (\partial_i u, \partial_i v)$ in $L^{p_i}(\Omega) \times L^{q_i}(\Omega)$ for $i = 1, \dots, N$. Then the growth conditions for F_1 and F_2 in assumption (H1) permit to apply Krasnoselkii's theorem obtaining

$$\begin{aligned} & (F_1(x, u_n, v_n, \nabla u_n, \nabla v_n), F_2(x, u_n, v_n, \nabla u_n, \nabla v_n)) \\ & \rightarrow (F_1(x, u, v, \nabla u, \nabla v), F_2(x, u, v, \nabla u, \nabla v)) \end{aligned}$$

in $L^{r'_1}(\Omega) \times L^{r'_2}(\Omega)$, which completes the proof. \square

Corollary 2.1. *Assume that conditions (1.4), (1.7) and (H1) are fulfilled. If $(u_n, v_n) \rightarrow (u, v)$ in X , then*

$$\lim_{n \rightarrow \infty} \langle \mathcal{N}(u_n, v_n), (u_n - u, v_n - v) \rangle = 0.$$

Proof. Hölder's inequality entails the estimate

$$\begin{aligned} & |\langle \mathcal{N}(u_n, v_n), (u_n - u, v_n - v) \rangle| \\ & = \left| \int_{\Omega} F_1(x, u_n, v_n, \nabla u_n, \nabla v_n)(u_n - u) dx + \int_{\Omega} F_2(x, u_n, v_n, \nabla u_n, \nabla v_n)(v_n - v) dx \right| \\ & \leq \|F_1(x, u_n, v_n, \nabla u_n, \nabla v_n)\|_{L^{r'_1}} \|u_n - u\|_{L^{r_1}} + \|F_2(x, u_n, v_n, \nabla u_n, \nabla v_n)\|_{L^{r'_2}} \|v_n - v\|_{L^{r_2}}. \end{aligned}$$

The compact embedding (1.6) yields $(u_n, v_n) \rightarrow (u, v)$ in $L^{r_1}(\Omega) \times L^{r_2}(\Omega)$. Since Proposition 2.1 ensures the boundedness of the sequences $F_1(x, u_n, v_n, \nabla u_n, \nabla v_n)$ and $F_2(x, u_n, v_n, \nabla u_n, \nabla v_n)$ in $L^{r'_1}(\Omega)$ and $L^{r'_2}(\Omega)$, respectively, the conclusion is achieved. \square

3 Global estimates

We first estimate the solutions to (1.1) in the space X .

Lemma 3.1. *Assume that conditions (1.4), (1.7), (H1) and (H2) hold. Then the solution set to problem (1.2) is bounded in X with a bound that depends on the function G_{ki} only through its lower bound a_{ki} for $i = 1, \dots, N$ and $k = 1, 2$.*

Proof. Let $(u, v) \in X$ be a weak solution of (1.2). Insert $(\phi, \psi) = (u, v)$ in (1.3) to get

$$\begin{cases} \sum_{i=1}^N \int_{\Omega} G_{1i}(u(x)) |\partial_i u(x)|^{p_i} dx = \int_{\Omega} F_1(x, u, v, \nabla u, \nabla v) u dx, \\ \sum_{i=1}^N \int_{\Omega} G_{2i}(v(x)) |\partial_i v(x)|^{q_i} dx = \int_{\Omega} F_2(x, u, v, \nabla u, \nabla v) v dx. \end{cases}$$

Then from hypothesis (H2) we infer

$$\begin{cases} \sum_{i=1}^N a_{1i} \|\partial_i u\|_{L^{p_i}}^{p_i} \leq \|\zeta_1\|_{L^1} + c_1 \|u\|_{L^{\underline{p}}}^{\underline{p}} + c_2 \|v\|_{L^{\underline{q}}}^{\underline{q}} + c_3 \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}}^{p_i} + c_4 \sum_{i=1}^N \|\partial_i v\|_{L^{q_i}}^{q_i}, \\ \sum_{i=1}^N a_{2i} \|\partial_i v\|_{L^{q_i}}^{q_i} \leq \|\zeta_2\|_{L^1} + d_1 \|u\|_{L^{\underline{p}}}^{\underline{p}} + d_2 \|v\|_{L^{\underline{q}}}^{\underline{q}} + d_3 \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}}^{p_i} + d_4 \sum_{i=1}^N \|\partial_i v\|_{L^{q_i}}^{q_i}. \end{cases}$$

Through (1.8) it turns out

$$\begin{aligned} & \sum_{i=1}^N (a_{1i} - c_3) \|\partial_i u\|_{L^{p_i}}^{p_i} \\ & \leq c_1 \|u\|_{L^{\underline{p}}}^{\underline{p}} + c_2 \|v\|_{L^{\underline{q}}}^{\underline{q}} + c_4 \sum_{i=1}^N \|\partial_i v\|_{L^{q_i}}^{q_i} + \|\zeta_1\|_{L^1} \\ & \leq N^{p-1} c_1 \theta \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}}^{p_i} + N^{q-1} c_2 \theta \sum_{i=1}^N \|\partial_i v\|_{L^{q_i}}^{q_i} + c_4 \sum_{i=1}^N \|\partial_i v\|_{L^{q_i}}^{q_i} + \|\zeta_1\|_{L^1} \\ & \leq N^{p-1} c_1 \theta (N + \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}}^{p_i}) + N^{q-1} c_2 \theta (N + \sum_{i=1}^N \|\partial_i v\|_{L^{q_i}}^{q_i}) + c_4 \sum_{i=1}^N \|\partial_i v\|_{L^{q_i}}^{q_i} + \|\zeta_1\|_{L^1} \end{aligned}$$

and

$$\begin{aligned}
& \sum_{i=1}^N (a_{2i} - d_4) \|\partial_i v\|_{L^{q_i}}^{q_i} \\
& \leq d_1 \theta \left(\sum_{i=1}^N \|\partial_i u\|_{L^{p_i}} \right)^p + d_2 \theta \left(\sum_{i=1}^N \|\partial_i v\|_{L^{q_i}} \right)^q + d_3 \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}}^{p_i} + \|\zeta_2\|_{L^1} \\
& \leq N^{p-1} d_1 \theta \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}}^p + N^{q-1} d_2 \theta \sum_{i=1}^N \|\partial_i v\|_{L^{q_i}}^q + d_3 \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}}^{p_i} + \|\zeta_2\|_{L^1} \\
& \leq N^{p-1} d_1 \theta \left(N + \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}}^{p_i} \right) + N^{q-1} d_2 \theta \left(N + \sum_{i=1}^N \|\partial_i v\|_{L^{q_i}}^{q_i} \right) + d_3 \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}}^{p_i} + \|\zeta_2\|_{L^1}.
\end{aligned}$$

These estimates imply that

$$\begin{aligned}
& \sum_{i=1}^N (a_{1i} - (c_3 + d_3)) \|\partial_i u\|_{L^{p_i}}^{p_i} + \sum_{i=1}^N (a_{2i} - (c_4 + d_4)) \|\partial_i v\|_{L^{q_i}}^{q_i} \\
& \leq N^{p-1} \theta (c_1 + d_1) \left(N + \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}}^{p_i} \right) + N^{q-1} \theta (c_2 + d_2) \left(N + \sum_{i=1}^N \|\partial_i v\|_{L^{q_i}}^{q_i} \right) \\
& \quad + \|\zeta_1\|_{L^1} + \|\zeta_2\|_{L^1}.
\end{aligned}$$

The conditions imposed in assumption (H2) ensure

$$\begin{cases} c_3 + d_3 + N^{p-1} \theta (c_1 + d_1) < a_{1i} \\ c_4 + d_4 + N^{q-1} \theta (c_2 + d_2) < a_{2i} \end{cases}$$

for all $i = 1, \dots, N$, so the stated conclusion is valid. \square

In fact, the solution set of problem (1.2) is uniformly bounded.

Theorem 3.1. *If conditions (1.4), (1.7), (H1) and (H2) are satisfied, then the solution set of problem (1.2) is uniformly bounded, that is there exists a constant $C_0 > 0$ such that $\|u\|_{L^\infty} \leq C_0$ and $\|v\|_{L^\infty} \leq C_0$ for all weak solutions $(u, v) \in X$ to problem (1.2). The uniform bound C_0 depends on the function G_{ki} only through its lower bound a_{ki} for $i = 1, \dots, N$ and $k = 1, 2$.*

Proof. Let $(u, v) \in X$ be a weak solution to problem (1.2). We can express $u = u^+ - u^-$ and $v = v^+ - v^-$, where $u^+ = \max\{u, 0\}$ (the positive part of u)

and $u^- = \max\{-u, 0\}$ (the negative part of u) and similarly for v^+ and v^- . It suffices to prove the uniform boundedness for (u^+, v^+) because for (u^-, v^-) the reasoning is similar.

For an arbitrary number $h > 0$, set $u_h := \min\{u^+, h\}$ and $v_h := \min\{v^+, h\}$. Corresponding to any $k > 0$ and $1 \leq i \leq N$, we have $(u^+(u_h)^{kp_i}, v^+(v_h)^{kq_i}) \in X$ because

$$\begin{cases} |\partial_i(u^+(u_h)^k)| = |(u_h)^k \partial_i(u^+) + k(u_h)^{k-1} u^+ \partial_i(u_h)| \\ \leq (k+1)(u_h)^k |\partial_i(u^+)|, \\ |\partial_i(v^+(v_h)^k)| = |(v_h)^k \partial_i(v^+) + k(v_h)^{k-1} v^+ \partial_i(v_h)| \\ \leq (k+1)(v_h)^k |\partial_i(v^+)|. \end{cases} \quad (3.1)$$

Testing (1.3) with $(u^+(u_h)^{kp_j}, v^+(v_h)^{kq_j})$, $1 \leq j \leq N$, we find

$$\begin{cases} \sum_{i=1}^N \int_{\Omega} G_{1i}(u) |\partial_i u|^{p_i-2} \partial_i u \partial_i (u^+(u_h)^{kp_j}) dx = \int_{\Omega} F_1(x, u, v, \nabla u, \nabla v) u^+(u_h)^{kp_j} dx, \\ \sum_{i=1}^N \int_{\Omega} G_{2i}(v) |\partial_i v|^{q_i-2} \partial_i v \partial_i (v^+(v_h)^{kq_j}) dx = \int_{\Omega} F_2(x, u, v, \nabla u, \nabla v) v^+(v_h)^{kq_j} dx. \end{cases} \quad (3.2)$$

For the left-hand side of the equations in (3.2) hold the estimates

$$\begin{aligned} & \sum_{i=1}^N \int_{\Omega} G_{1i}(u) |\partial_i u|^{p_i-2} \partial_i u \partial_i (u^+(u_h)^{kp_j}) dx \\ &= \sum_{i=1}^N \int_{\Omega} G_{1i}(u) |\partial_i u|^{p_i-2} \partial_i u (\partial_i(u^+)(u_h)^{kp_j} + kp_j (u_h)^{kp_j-1} u^+ \partial_i(u_h)) dx \quad (3.3) \\ &\geq \sum_{i=1}^N a_{1i} \int_{\Omega} (u_h)^{kp_j} |\partial_i(u^+)|^{p_i} dx \end{aligned}$$

and

$$\begin{aligned}
& \sum_{i=1}^N \int_{\Omega} G_{2i}(v) |\partial_i v|^{q_i-2} \partial_i v \partial_i (v^+ (v_h)^{kp_j}) dx \\
&= \sum_{i=1}^N \int_{\Omega} G_{2i}(v) |\partial_i v|^{q_i-2} \partial_i v (\partial_i (v^+) (v_h)^{kq_j} + kq_j (v_h)^{kq_j-1} v^+ \partial_i (v_h)) dx \quad (3.4) \\
&\geq \sum_{i=1}^N a_{2i} \int_{\Omega} (v_h)^{kq_j} |\partial_i (v^+)|^{q_i} dx,
\end{aligned}$$

for $j = 1, \dots, N$.

For the right-hand side of the equations in (3.2), hypothesis (H1) enables to derive the estimates

$$\begin{aligned}
& \int_{\Omega} F_1(x, u, v, \nabla u, \nabla v) u^+ (u_h)^{kp_j} dx \\
&\leq a_1 \int_{\Omega} |u|^{r_1-1} (u_h)^{kp_j} u^+ dx + a_2 \int_{\Omega} |v|^{\frac{r_2}{r_1}} (u_h)^{kp_j} u^+ dx \\
&\quad + a_3 \int_{\Omega} \left(\left(\sum_{i=1}^N |\partial_i u|^{p_i} \right)^{1/r_1'} (u_h)^{\frac{kp_j}{r_1'}} \right) \left((u_h)^{\frac{kp_j}{r_1}} u^+ \right) dx \\
&\quad + a_4 \int_{\Omega} \left(\sum_{i=1}^N |\partial_i v|^{q_i} \right)^{1/r_1'} (u_h)^{kp_j} u^+ dx + a_5 \int_{\Omega} (u_h)^{kp_j} u^+ dx
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\Omega} F_2(x, u, v, \nabla u, \nabla v) v^+ (v_h)^{kq_j} dx \\
&\leq b_1 \int_{\Omega} |u|^{\frac{r_1}{r_2}} (v_h)^{kq_j} v^+ dx + b_2 \int_{\Omega} |v|^{r_2-1} (v_h)^{kq_j} v^+ dx \\
&\quad + b_3 \int_{\Omega} \left(\sum_{i=1}^N |\partial_i u|^{p_i} \right)^{1/r_2'} (v_h)^{kq_j} v^+ dx \\
&\quad + b_4 \int_{\Omega} \left(\left(\sum_{i=1}^N |\partial_i v|^{q_i} \right)^{1/r_2'} (v_h)^{\frac{kq_j}{r_2'}} \right) \left((v_h)^{\frac{kq_j}{r_2}} v^+ \right) dx + b_5 \int_{\Omega} (v_h)^{kq_j} v^+ dx.
\end{aligned}$$

For any $\varepsilon > 0$, Young's inequality provides constants $c(\varepsilon), d(\varepsilon) > 0$ such that

$$\begin{aligned} & \int_{\Omega} \left(\left(\sum_{i=1}^N |\partial_i u|^{p_i} \right)^{1/r'_1} (u_h)^{\frac{kp_j}{r'_1}} \right) \left((u_h)^{\frac{kp_j}{r_1}} u^+ \right) dx \\ & \leq \varepsilon \sum_{i=1}^N \int_{\Omega} |\partial_i(u^+)|^{p_i} (u_h)^{kp_j} dx + c(\varepsilon) \int_{\Omega} (u_h)^{kp_j} (u^+)^{r_1} dx \end{aligned}$$

and

$$\begin{aligned} & \int_{\Omega} \left(\left(\sum_{i=1}^N |\partial_i v|^{q_i} \right)^{1/r'_2} (v_h)^{\frac{kq_j}{r'_2}} \right) \left((v_h)^{\frac{kq_j}{r_2}} v^+ \right) dx \\ & \leq \varepsilon \sum_{i=1}^N \int_{\Omega} |\partial_i(v^+)|^{q_i} (v_h)^{kq_j} dx + d(\varepsilon) \int_{\Omega} (v_h)^{kq_j} (v^+)^{r_2} dx. \end{aligned}$$

By Hölder's inequality, Lemma 3.1 and (1.5) there exists a constant $C_1 > 0$ such that

$$\begin{aligned} \int_{\Omega} |v|^{\frac{r_2}{r_1}} (u_h)^{kp_j} u^+ dx & \leq \left(\int_{\Omega} |v|^{r_2} dx \right)^{\frac{1}{r_1}} \| (u_h)^{kp_j} u^+ \|_{L^{r_1}} \\ & \leq C_1 \| (u_h)^{kp_j} u^+ \|_{L^{r_1}} \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega} |u|^{\frac{r_1}{r_2}} (v_h)^{kq_j} v^+ dx & \leq \left(\int_{\Omega} |u|^{r_1} dx \right)^{\frac{1}{r_2}} \| (v_h)^{kq_j} v^+ \|_{L^{r_2}} \\ & \leq C_1 \| (v_h)^{kq_j} v^+ \|_{L^{r_2}}. \end{aligned}$$

Again by Hölder's inequality, Lemma 3.1 and (1.5) we obtain a constant

$C_2 > 0$ with

$$\begin{aligned} & \int_{\Omega} \left(\sum_{i=1}^N |\partial_i v|^{q_i} \right)^{1/r_1'} (u_h)^{kp_j} u^+ dx \\ & \leq \left(\sum_{i=1}^N \int_{\Omega} |\partial_i v|^{q_i} dx \right) \|(u_h)^{kp_j} u^+\|_{L^{r_1}} \leq C_2 \|(u_h)^{kp_j} u^+\|_{L^{r_1}} \end{aligned}$$

and

$$\begin{aligned} & \int_{\Omega} \left(\sum_{i=1}^N |\partial_i u|^{p_i} \right)^{1/r_2'} (v_h)^{kq_j} v^+ dx \\ & \leq \left(\sum_{i=1}^N \int_{\Omega} |\partial_i u|^{p_i} dx \right) \|(v_h)^{kq_j} v^+\|_{L^{r_2}} \leq C_2 \|(v_h)^{kq_j} v^+\|_{L^{r_2}}. \end{aligned}$$

There are also the estimates

$$\int_{\Omega} (u_h)^{kp_j} u^+ dx \leq \int_{\Omega} (u_h)^{kp_j} (u^+)^{r_1} dx + |\Omega|$$

and

$$\int_{\Omega} (v_h)^{kq_j} v^+ dx \leq \int_{\Omega} (v_h)^{kq_j} (v^+)^{r_2} dx + |\Omega|,$$

where $|\Omega|$ denotes the Lebesgue measure of Ω .

The preceding estimates lead to

$$\begin{aligned}
& \int_{\Omega} F_1(x, u, v, \nabla u, \nabla v) u^+ (u_h)^{kp_j} dx \\
& \leq a_1 \int_{\Omega} (u_h)^{kp_j} (u^+)^{r_1} dx + a_2 C_1 \| (u_h)^{kp_j} u^+ \|_{L^{r_1}} + a_3 \varepsilon \sum_{i=1}^N \int_{\Omega} |\partial_i(u^+)|^{p_i} (u_h)^{kp_j} dx \\
& \quad + a_3 c(\varepsilon) \int_{\Omega} (u_h)^{kp_j} (u^+)^{r_1} dx + a_4 C_2 \| (u_h)^{kp_j} u^+ \|_{L^{r_1}} \\
& \quad + a_5 \left(\int_{\Omega} (u_h)^{kp_j} (u^+)^{r_1} dx + |\Omega| \right)
\end{aligned}$$

and

$$\begin{aligned}
& \int_{\Omega} F_2(x, u, v, \nabla u, \nabla v) v^+ (v_h)^{kq_j} dx \\
& \leq b_1 C_1 \| (v_h)^{kq_j} v^+ \|_{L^{r_2}} + b_2 \int_{\Omega} (v_h)^{kq_j} (v^+)^{r_2} dx + b_3 C_2 \| (v_h)^{kq_j} v^+ \|_{L^{r_2}} \\
& \quad + b_4 \varepsilon \sum_{i=1}^N \int_{\Omega} |\partial_i(v^+)|^{q_i} (v_h)^{kq_j} dx + b_4 d(\varepsilon) \int_{\Omega} (v_h)^{kq_j} (v^+)^{r_2} dx \\
& \quad + b_5 \left(\int_{\Omega} (v_h)^{kq_j} (v^+)^{r_2} dx + |\Omega| \right).
\end{aligned}$$

Let us note that

$$\begin{aligned}
& \| (u_h)^{kp_j} u^+ \|_{L^{r_1}} = \left(\int_{\Omega} ((u_h)^{kp_j} u^+)^{r_1} dx \right)^{\frac{1}{r_1}} \leq \left(\int_{\Omega} ((u_h)^{kp_j r_1} (u^+)^{p_j r_1} + 1) dx \right)^{\frac{1}{r_1}} \\
& = \left(\int_{\Omega} ((u_h)^k u^+)^{p_j r_1} dx + |\Omega| \right)^{\frac{1}{r_1}} \leq \left(\int_{\Omega} ((u_h)^k u^+)^{p_j r_1} dx \right)^{\frac{1}{r_1}} + |\Omega|^{\frac{1}{r_1}} \\
& = \| (u_h)^k u^+ \|_{L^{p_j r_1}}^{p_j} + |\Omega|^{\frac{1}{r_1}}.
\end{aligned}$$

and similarly

$$\| (v_h)^{kq_j} v^+ \|_{L^{r_2}} \leq \| (v_h)^k v^+ \|_{L^{q_j r_2}}^{q_j} + |\Omega|^{\frac{1}{r_2}}.$$

Consequently, we obtain

$$\left\{ \begin{array}{l} \int_{\Omega} F_1(x, u, v, \nabla u, \nabla v) u^+ (u_h)^{kp_j} dx \leq a_3 \varepsilon \sum_{i=1}^N \int_{\Omega} |\partial_i(u^+)|^{p_i} (u_h)^{kp_j} dx \\ \quad + C \left(\int_{\Omega} (u_h)^{kp_j} (u^+)^{r_1} dx + \|(u_h)^k u^+\|_{L^{p_j r_1}}^{p_j} + 1 \right), \\ \\ \int_{\Omega} F_2(x, u, v, \nabla u, \nabla v) v^+ (v_h)^{kq_j} dx \leq b_4 \varepsilon \sum_{i=1}^N \int_{\Omega} |\partial_i(v^+)|^{q_i} (v_h)^{kq_j} dx \\ \quad + C \left(\int_{\Omega} (v_h)^{kq_j} (v^+)^{r_2} dx + \|(v_h)^k v^+\|_{L^{q_j r_2}}^{q_j} + 1 \right), \end{array} \right. \quad (3.5)$$

with a constant $C > 0$.

Then (3.2), (3.3), (3.4) and (3.5) show that

$$\left\{ \begin{array}{l} \sum_{i=1}^N (a_{1i} - a_3 \varepsilon) \int_{\Omega} (u_h)^{kp_j} |\partial_i(u^+)|^{p_i} dx \\ \leq C \left(\int_{\Omega} (u_h)^{kp_j} (u^+)^{r_1} dx + \|(u_h)^k u^+\|_{L^{p_j r_1}}^{p_j} + 1 \right), \\ \\ \sum_{i=1}^N (a_{2i} - b_4 \varepsilon) \int_{\Omega} (v_h)^{kq_j} |\partial_i(v^+)|^{q_i} dx \\ \leq C \left(\int_{\Omega} (v_h)^{kq_j} (v^+)^{r_2} dx + \|(v_h)^k v^+\|_{L^{q_j r_2}}^{q_j} + 1 \right). \end{array} \right.$$

For an $\varepsilon > 0$ small enough, we get

$$\left\{ \begin{array}{l} \sum_{i=1}^N \int_{\Omega} (u_h)^{kp_j} |\partial_i(u^+)|^{p_i} dx \leq C \left(\int_{\Omega} (u_h)^{kp_j} (u^+)^{r_1} dx + \|(u_h)^k u^+\|_{L^{p_j r_1}}^{p_j} + 1 \right), \\ \sum_{i=1}^N \int_{\Omega} (v_h)^{kq_j} |\partial_i(v^+)|^{q_i} dx \leq C \left(\int_{\Omega} (v_h)^{kq_j} (v^+)^{r_2} dx + \|(v_h)^k v^+\|_{L^{q_j r_2}}^{q_j} + 1 \right), \end{array} \right.$$

with a new constant $C > 0$.

From (3.1) we infer that

$$\begin{cases} \|\partial_j(u^+(u_h)^k)\|_{L^{p_j}} \leq C^{\frac{1}{p_j}}(k+1) \left(\int_{\Omega} (u_h)^{kp_j} (u^+)^{r_1} dx + \|(u_h)^k u^+\|_{L^{p_j r_1}}^{p_j} + 1 \right)^{\frac{1}{p_j}}, \\ \|\partial_j(v^+(v_h)^k)\|_{L^{q_j}} \leq C^{\frac{1}{q_j}}(k+1) \left(\int_{\Omega} (v_h)^{kq_j} (v^+)^{r_2} dx + \|(v_h)^k v^+\|_{L^{q_j r_2}}^{q_j} + 1 \right)^{\frac{1}{q_j}}. \end{cases} \quad (3.6)$$

Using that $r_1 \in (\bar{p}, p^*)$ and $r_2 \in (\bar{q}, q^*)$ as postulated in hypothesis (H1), we can find $\alpha \in (p_j, r_1)$ and $\beta \in (q_j, r_2)$ satisfying

$$\frac{(r_1 - p_j)\alpha}{\alpha - p_j} < p^* \quad \text{and} \quad \frac{(r_2 - q_j)\beta}{\beta - q_j} < q^*. \quad (3.7)$$

Hölder's inequality, (1.6), (3.7) and Lemma 3.1 provide the existence of constants $K_1 > 0$ and $K_2 > 0$ such that

$$\begin{aligned} \int_{\Omega} (u_h)^{kp_j} (u^+)^{r_1} dx &= \int_{\Omega} (u^+)^{r_1 - p_j} ((u_h)^k u^+)^{p_j} dx \\ &\leq \left(\int_{\Omega} (u^+)^{\frac{(r_1 - p_j)\alpha}{\alpha - p_j}} dx \right)^{\frac{\alpha - p_j}{\alpha}} \left(\int_{\Omega} (u^+ (u_h)^k)^{\alpha} dx \right)^{\frac{p_j}{\alpha}} \\ &\leq K_1 \|u^+ (u_h)^k\|_{L^{\alpha}}^{p_j} \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega} (v_h)^{kq_j} (v^+)^{r_2} dx &= \int_{\Omega} (v^+)^{r_2 - q_j} ((v_h)^k v^+)^{q_j} dx \\ &\leq \left(\int_{\Omega} (v^+)^{\frac{(r_2 - q_j)\beta}{\beta - q_j}} dx \right)^{\frac{\beta - q_j}{\beta}} \left(\int_{\Omega} (v^+ (v_h)^k)^{\beta} dx \right)^{\frac{q_j}{\beta}} \\ &\leq K_2 \|v^+ (v_h)^k\|_{L^{\beta}}^{q_j}. \end{aligned}$$

Note that $p_j r_1 > \alpha$ and $q_j r_2 > \beta$. From (3.6) we arrive at

$$\begin{cases} \|\partial_j(u^+(u_h)^k)\|_{L^{p_j}} \leq K(k+1)(\|u^+(u_h)^k\|_{L^{\alpha}} + 1), \\ \|\partial_j(v^+(v_h)^k)\|_{L^{q_j}} \leq K(k+1)(\|v^+(v_h)^k\|_{L^{\beta}} + 1), \end{cases}$$

with a constant $K > 0$. As $j = 1, \dots, N$ was arbitrary, this results in

$$\begin{cases} \|u^+(u_h)^k\|_{W_0^{1, \bar{p}}(\Omega)} \leq KN(k+1)(\|u^+(u_h)^k\|_{L^{\alpha}} + 1), \\ \|v^+(v_h)^k\|_{W_0^{1, \bar{q}}(\Omega)} \leq KN(k+1)(\|v^+(v_h)^k\|_{L^{\beta}} + 1). \end{cases}$$

Then the continuous embedding (1.6) implies

$$\begin{cases} \|u^+(u_h)^k\|_{L^{p^*}} \leq C(k+1)(\|u^+\|_{L^{\alpha(k+1)}}^{k+1} + 1), \\ \|v^+(v_h)^k\|_{L^{q^*}} \leq C(k+1)(\|v^+\|_{L^{\beta(k+1)}}^{k+1} + 1), \end{cases}$$

with a constant $C > 0$. Letting $h \rightarrow 0$, Fatou's lemma yields

$$\begin{aligned} \|u^+\|_{L^{p^*(k+1)}}^{k+1} &= \|(u^+)^{k+1}\|_{L^{p^*}} \leq C(k+1)(\|u^+\|_{L^{\alpha(k+1)}}^{k+1} + 1), \\ \|v^+\|_{L^{q^*(k+1)}}^{k+1} &= \|(v^+)^{k+1}\|_{L^{q^*}} \leq C(k+1)(\|v^+\|_{L^{\beta(k+1)}}^{k+1} + 1). \end{aligned} \quad (3.8)$$

At this point, since now the variables u^+ and v^+ are decoupled, the problem is reduced to the setting of [8], so we can proceed as therein. For the sake of completeness we carry out the proof. It suffices to argue in the case of u^+ . If for a sequence $k_n \rightarrow +\infty$ we have $\|u^+\|_{L^{p^*(k_n+1)}} \leq 1$ for all n , then $\|u^+\|_{L^\infty} \leq 1$ which ends the proof. Two situations remain to be discussed: (a) $\|u^+\|_{L^{p^*(k+1)}} > 1$ for all $k \geq 0$; (b) there is $k_0 > 0$ such that $\|u^+\|_{L^{p^*(k_0+1)}} \leq 1$ and $\|u^+\|_{L^{p^*(k+1)}} > 1$ for all $k > k_0$.

When (a) holds, (3.8) gives

$$\|u^+\|_{L^{p^*(k+1)}} \leq (2C)^{\frac{1}{k+1}} (k+1)^{\frac{1}{k+1}} \|u^+\|_{L^{\alpha(k+1)}}. \quad \forall k \geq 0.$$

The function $k \mapsto (k+1)^{1/\sqrt{k+1}}$ is bounded on $(0, +\infty)$, so one has

$$\|u^+\|_{L^{p^*(k+1)}} \leq C^{1/\sqrt{k+1}} \|u^+\|_{L^{\alpha(k+1)}}, \quad \forall k \geq 0, \quad (3.9)$$

with a new constant $C > 0$. We can build the following Moser iteration

$$(k_n + 1)\alpha = (k_{n-1} + 1)p^*, \quad \forall n \geq 2,$$

starting with $(k_1 + 1)\alpha = p^*$. Applying repeatedly (3.9) renders

$$\|u^+\|_{L^{p^*(k_n+1)}} \leq C^{\sum_{i=1}^n \frac{1}{\sqrt{k_i+1}}} \|u^+\|_{L^{p^*}}.$$

The inductive definition of (k_n) ensures that $k_n \rightarrow +\infty$ and the series $\sum_{n=1}^{\infty} 1/\sqrt{k_n+1}$ converges. Then Lemma 3.1 shows that letting $n \rightarrow \infty$ we obtain $\|u^+\|_{L^\infty} \leq C_0$, with a constant $C_0 > 0$ independent on the solution u .

For case (b), a similar reasoning can be developed giving

$$\|u^+\|_{L^{p^*(k+1)}} \leq (2C)^{\frac{1}{k+1}} (k+1)^{\frac{1}{k+1}} \|u^+\|_{L^{\alpha(k+1)}}, \quad \forall k \geq k_0,$$

The above argument shows

$$\|u^+\|_{L^{p^*(k+1)}} \leq C^{1/\sqrt{k+1}} \|u^+\|_{L^{\alpha(k+1)}}, \quad \forall k \geq k_0. \quad (3.10)$$

with a constant $C > 0$. The same Moser iteration applies, this time starting with $(k_1 + 1)\alpha = p^*(k_0 + 1)$. Then (3.10) entails

$$\|u^+\|_{L^{p^*(k_n+1)}} \leq C^{\sum_{i=1}^n \frac{1}{\sqrt{k_i+1}}} \|u^+\|_{L^{p^*(k_0+1)}}.$$

As in case (a) we prove that $\|u^+\|_{L^\infty(\Omega)} \leq C_0$ with a constant $C_0 > 0$ independent of the solution (u, v) . Similarly, we can establish the bound $\|v^+\|_{L^\infty(\Omega)} \leq C_0$.

A careful inspection of the proof reveals that the constant C_0 depends on G_{1_i} and G_{2_i} only through their lower bounds a_{ki} for $i = 1, \dots, N$, $k = 1, 2$, respectively. This completes the proof. \square

4 Auxiliary truncated system

Here we extend to system problem (1.2) the idea of truncation used in the case of isotropic equations (see [7] and [9]) to overcome the difficulty of unbounded coefficients G_{ki} for $i = 1, \dots, N$ and $k = 1, 2$.

Corresponding to any real number $R > 0$, we truncate the functions G_{ki} in the following way

$$G_{1i,R}(t) = \begin{cases} G_{1i}(t) & \text{if } |t| \leq R \\ G_{1i}(R) & \text{if } t > R \\ G_{1i}(-R) & \text{if } t < -R. \end{cases} \quad \text{and} \quad G_{2i,R}(t) = \begin{cases} G_{2i}(t) & \text{if } |t| \leq R \\ G_{ki}(R) & \text{if } t > R \\ G_{ki}(-R) & \text{if } t < -R. \end{cases} \quad (4.1)$$

In accordance with (4.1) we state the auxiliary system

$$\begin{cases} -\sum_{i=1}^N \partial_i(G_{1i,R}(u)|\partial_i u|^{p_i-2}\partial_i u) = F_1(x, u, v, \nabla u, \nabla v) & \text{in } \Omega, \\ -\sum_{i=1}^N \partial_i(G_{2i,R}(v)|\partial_i v|^{q_i-2}\partial_i v) = F_2(x, u, v, \nabla u, \nabla v) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.2)$$

The driving operator in problem (4.2) is the map $\mathcal{A}_R : X \rightarrow X^*$ defined by

$$\begin{aligned} & \langle \mathcal{A}_R(u, v), (\phi, \psi) \rangle \\ &= \sum_{i=1}^N \int_{\Omega} G_{1i,R}(u)|\partial_i u|^{p_i-2}\partial_i u \partial_i \phi dx + \sum_{i=1}^N \int_{\Omega} G_{2i,R}(v)|\partial_i v|^{q_i-2}\partial_i v \partial_i \psi dx \end{aligned} \quad (4.3)$$

for all $(u, v), (\phi, \psi) \in X$.

Proposition 4.1. *Assume conditions (1.4) and (1.7). Then, for each $R > 0$, the map $\mathcal{A}_R : X \rightarrow X^*$ in (4.3) is well defined, bounded, continuous, and fulfills the S_+ -property, that is, $(u_n, v_n) \rightharpoonup (u, v)$ in X and*

$$\limsup_{n \rightarrow \infty} \langle \mathcal{A}_R(u_n, v_n), (u_n - u, v_n - v) \rangle \leq 0 \quad (4.4)$$

imply $(u_n, v_n) \rightarrow (u, v)$ in X .

Proof. By Hölder's inequality, for all $(u, v), (\phi, \psi) \in X$ and $i = 1, \dots, N$, we get

$$\begin{aligned} & \left| \int_{\Omega} G_{1i,R}(u) |\partial_i u|^{p_i-2} \partial_i u \partial_i \phi dx \right| + \left| \int_{\Omega} G_{2i,R}(v) |\partial_i v|^{q_i-2} \partial_i v \partial_i \psi dx \right| \\ & \leq \max_{|t| \leq R} G_{1i,R}(t) \|\partial_i u\|_{L^{p_i}}^{p_i-1} \|\partial_i \phi\|_{L^{p_i}} + \max_{|t| \leq R} G_{2i,R}(t) \|\partial_i v\|_{L^{q_i}}^{q_i-1} \|\partial_i \psi\|_{L^{q_i}}. \end{aligned} \quad (4.5)$$

We infer for all $(u, v) \in X$ that $\mathcal{A}_R(u, v) \in X^*$, so \mathcal{A}_R is well defined. Moreover, (4.5) shows that the mapping \mathcal{A}_R is bounded.

In order to prove that \mathcal{A}_R is continuous, let $(u_n, v_n) \rightarrow (u, v)$ in X . We note that

$$\begin{aligned} & \|\mathcal{A}_R(u_n, v_n) - \mathcal{A}_R(u, v)\|_{X^*} \\ & \leq \sum_{i=1}^N \|(G_{1i,R}(u_n) - G_{1i,R}(u)) |\partial_i(u_n)|^{p_i-2} \partial_i(u_n)\|_{L^{\frac{p_i}{p_i-1}}} \\ & \quad + \sum_{i=1}^N \|G_{1i,R}(u) (|\partial_i(u_n)|^{p_i-2} \partial_i(u_n) - |\partial_i(u)|^{p_i-2} \partial_i(u))\|_{L^{\frac{p_i}{p_i-1}}} \\ & \quad + \sum_{i=1}^N \|(G_{2i,R}(v_n) - G_{2i,R}(v)) |\partial_i(v_n)|^{q_i-2} \partial_i(v_n)\|_{L^{\frac{q_i}{q_i-1}}} \\ & \quad + \sum_{i=1}^N \|G_{2i,R}(v) (|\partial_i(v_n)|^{q_i-2} \partial_i(v_n) - |\partial_i(v)|^{q_i-2} \partial_i(v))\|_{L^{\frac{q_i}{q_i-1}}}. \end{aligned} \quad (4.6)$$

Hölder's inequality yields

$$\left\{ \begin{array}{l} \|(G_{1i,R}(u_n) - G_{1i,R}(u))|\partial_i(u_n)|^{p_i-2}\partial_i(u_n)\|_{L^{\frac{p_i}{p_i-1}}} \\ \leq \int_{\Omega} |G_{1i,R}(u_n) - G_{1i,R}(u)|^{\frac{p_i}{p_i-1}} |\partial_i(u_n)|^{p_i} dx, \\ \\ \|(G_{2i,R}(v_n) - G_{2i,R}(v))|\partial_i(v_n)|^{q_i-2}\partial_i(v_n)\|_{L^{\frac{q_i}{q_i-1}}} \\ \leq \int_{\Omega} |G_{2i,R}(v_n) - G_{2i,R}(v)|^{\frac{q_i}{q_i-1}} |\partial_i(v_n)|^{q_i} dx. \end{array} \right.$$

The continuity and boundedness of the functions $G_{1i,R}$ and $G_{2i,R}$ enable us to apply Lebesgue's dominated convergence theorem on the basis of the strong convergence $(u_n, v_n) \rightarrow (u, v)$ in X that provides

$$\left\{ \begin{array}{l} \lim_{n \rightarrow \infty} \|(G_{1i,R}(u_n) - G_{1i,R}(u))|\partial_i(u_n)|^{p_i-2}\partial_i(u_n)\|_{L^{\frac{p_i}{p_i-1}}} = 0, \\ \lim_{n \rightarrow \infty} \|(G_{2i,R}(v_n) - G_{2i,R}(v))|\partial_i(v_n)|^{q_i-2}\partial_i(v_n)\|_{L^{\frac{q_i}{q_i-1}}} = 0, \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \lim_{n \rightarrow \infty} \|G_{1i,R}(u)(|\partial_i(u_n)|^{p_i-2}\partial_i(u_n) - |\partial_i(u)|^{p_i-2}\partial_i(u))\|_{L^{\frac{p_i}{p_i-1}}} = 0, \\ \lim_{n \rightarrow \infty} \|G_{2i,R}(v)(|\partial_i(v_n)|^{q_i-2}\partial_i(v_n) - |\partial_i(v)|^{q_i-2}\partial_i(v))\|_{L^{\frac{q_i}{q_i-1}}} = 0. \end{array} \right.$$

Then from (4.6) we derive that $\mathcal{A}_R(u_n, v_n) \rightarrow \mathcal{A}_R(u, v)$ in X^* , whence the continuity of \mathcal{A}_R .

Now we focus on the S_+ -property of \mathcal{A}_R . Let $(u_n, v_n) \rightharpoonup (u, v)$ in X and (4.4) be satisfied, which takes the form

$$\limsup_{n \rightarrow \infty} \langle \mathcal{A}_R(u_n, v_n) - \mathcal{A}_R(u, v), (u_n - u, v_n - v) \rangle \leq 0. \quad (4.7)$$

It is seen that

$$\begin{aligned}
& \langle \mathcal{A}_R(u_n, v_n) - \mathcal{A}_R(u, v), (u_n - u, v_n - v) \rangle \\
& \geq \sum_{i=1}^N a_{1i} \int_{\Omega} (|\partial_i(u_n)|^{p_i-2} \partial_i(u_n) - |\partial_i(u)|^{p_i-2} \partial_i(u)) \partial_i(u_n - u) dx \\
& \quad + \sum_{i=1}^N \int_{\Omega} (G_{1i,R}(u_n) - G_{1i,R}(u)) |\partial_i(u)|^{p_i-2} \partial_i(u) \partial_i(u_n - u) dx \\
& \quad + \sum_{i=1}^N a_{2i} \int_{\Omega} (|\partial_i(v_n)|^{q_i-2} \partial_i(v_n) - |\partial_i(v)|^{q_i-2} \partial_i(v)) \partial_i(v_n - v) dx \\
& \quad + \sum_{i=1}^N \int_{\Omega} (G_{2i,R}(v_n) - G_{2i,R}(v)) |\partial_i(v)|^{q_i-2} \partial_i(v) \partial_i(v_n - v) dx.
\end{aligned} \tag{4.8}$$

As before we can prove that

$$\begin{aligned}
& \lim_{n \rightarrow \infty} \int_{\Omega} (G_{1i,R}(u_n) - G_{1i,R}(u)) |\partial_i(u)|^{p_i-2} \partial_i(u) \partial_i(u_n - u) dx = 0, \\
& \lim_{n \rightarrow \infty} \int_{\Omega} (G_{2i,R}(v_n) - G_{2i,R}(v)) |\partial_i(v)|^{q_i-2} \partial_i(v) \partial_i(v_n - v) dx = 0.
\end{aligned} \tag{4.9}$$

Then from (4.7), (4.8), (4.9), and Hölder's inequality it turns out

$$\begin{aligned}
& \lim_{n \rightarrow \infty} (\|\partial_i(u_n)\|_{L^{p_i}} - \|\partial_i(u)\|_{L^{p_i}}) \left(\|\partial_i(u_n)\|_{L^{p_i}}^{p_i-1} - \|\partial_i(u)\|_{L^{p_i}}^{p_i-1} \right) = 0, \\
& \lim_{n \rightarrow \infty} (\|\partial_i(v_n)\|_{L^{q_i}} - \|\partial_i(v)\|_{L^{q_i}}) \left(\|\partial_i(v_n)\|_{L^{q_i}}^{q_i-1} - \|\partial_i(v)\|_{L^{q_i}}^{q_i-1} \right) = 0,
\end{aligned}$$

for all $i = 1, \dots, N$. This results in

$$\lim_{n \rightarrow \infty} \|\partial_i(u_n)\|_{L^{p_i}} = \|\partial_i u\|_{L^{p_i}} \text{ and } \lim_{n \rightarrow \infty} \|\partial_i(v_n)\|_{L^{q_i}} = \|\partial_i v\|_{L^{q_i}}, \quad \forall i = 1, \dots, N.$$

Since the space $L^{p_i}(\Omega)$ and $L^{q_i}(\Omega)$ are uniformly convex, we infer the strong convergence $u_n \rightarrow u$ and $v_n \rightarrow v$ in $W_0^{1, \vec{p}}(\Omega)$, which completes the proof. \square

The properties of the operator $\mathcal{A}_R - \mathcal{N}$, with \mathcal{A}_R and \mathcal{N} introduced in (4.3) and Proposition 2.1, respectively, are listed in the next statement.

Proposition 4.2. *Assume (1.4), (1.7), (H1) and (H2). Then, for each $R > 0$, the map $\mathcal{A}_R - \mathcal{N} : X \rightarrow X^*$ satisfies:*

(i) $\mathcal{A}_R - \mathcal{N}$ is bounded;

(ii) $\mathcal{A}_R - \mathcal{N}$ is pseudomonotone, that is, if $(u_n, v_n) \rightharpoonup (u, v)$ in X and

$$\limsup_{n \rightarrow \infty} \langle (\mathcal{A}_R - \mathcal{N})(u_n, v_n), (u_n - u, v_n - v) \rangle \leq 0, \quad (4.10)$$

then

$$\begin{aligned} & \liminf_{n \rightarrow \infty} \langle (\mathcal{A}_R - \mathcal{N})(u_n, v_n), (u_n - u, v_n - v) \rangle \\ & \geq \liminf_{n \rightarrow \infty} \langle (\mathcal{A}_R - \mathcal{N})(u, v), (u, v) - (\phi, \psi) \rangle, \quad \forall (\phi, \psi) \in X; \end{aligned} \quad (4.11)$$

(iii) $\mathcal{A}_R - \mathcal{N}$ is coercive, that is,

$$\lim_{\|(u,v)\| \rightarrow \infty} \frac{\langle (\mathcal{A}_R - \mathcal{N})(u, v), (u, v) \rangle}{\|(u, v)\|} = +\infty. \quad (4.12)$$

Proof. (i) The assertion follows from Propositions 2.1 and 4.1.

(ii) Let $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ in $W_0^{1, \vec{p}}(\Omega)$ and $W_0^{1, \vec{q}}(\Omega)$, respectively, and suppose that (4.10) is satisfied. From Corollary 2.1 and (4.10) we derive (4.4). Thanks to Proposition 4.1, the S_+ -property of the operator \mathcal{A}_R implies $u_n \rightarrow u$ and $v_n \rightarrow v$ in $W_0^{1, \vec{p}}(\Omega)$ and $W_0^{1, \vec{q}}(\Omega)$, respectively. Then Proposition 2.1 ensures that $\mathcal{N}(u_n, v_n) \rightarrow \mathcal{N}(u, v)$ in X^* , while Proposition 4.1 entails $\mathcal{A}_R(u_n, v_n) \rightarrow \mathcal{A}_R(u, v)$ in X^* . Therefore (4.11) holds true.

(iii) By assumption (H2) and (1.8) we infer that

$$\begin{aligned}
& \langle (\mathcal{A}_R - \mathcal{N})(u, v), (u, v) \rangle \\
& \geq \sum_{i=1}^N \int_{\Omega} G_{1i,R}(u) |\partial_i u|^{p_i} dx - c_1 \|u\|_{L^p}^p - c_2 \|v\|_{L^q}^q - c_3 \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}}^{p_i} \\
& \quad - c_4 \sum_{i=1}^N \|\partial_i v\|_{L^{q_i}}^{q_i} - \|\zeta_1\|_{L^1} \\
& + \sum_{i=1}^N \int_{\Omega} G_{2i,R}(v) |\partial_i v|^{q_i} dx - d_1 \|u\|_{L^p}^p - d_2 \|v\|_{L^q}^q - d_3 \sum_{i=1}^N \|\partial_i u\|_{L^{p_i}}^{p_i} \\
& \quad - d_4 \sum_{i=1}^N \|\partial_i v\|_{L^{q_i}}^{q_i} - \|\zeta_2\|_{L^1} \\
& \geq \sum_{i=1}^N (a_{1i} - (c_3 + d_3) - N^{p-1}(c_1 + d_1)\theta) \|\partial_i u\|_{L^{p_i}}^{p_i} \\
& \quad + \sum_{i=1}^N (a_{2i} - (c_4 + d_4) - N^{q-1}(c_2 + d_2)\theta) \|\partial_i v\|_{L^{q_i}}^{q_i} \\
& \quad - N^p(c_1 + d_1)\theta - N^q(c_2 + d_2)\theta - \|\zeta_1\|_{L^1} - \|\zeta_2\|_{L^1}.
\end{aligned}$$

From hypothesis (H2) we know that $a_{1i} - (c_3 + d_3) - N^{p-1}(c_1 + d_1)\theta > 0$ and $a_{2i} - (c_4 + d_4) - N^{q-1}(c_2 + d_2)\theta > 0$. Taking into account that $p_i > 1$ and $q_i > 1$ for all $i = 1, \dots, N$, the preceding inequality implies (4.12), thus completing the proof. \square

5 Solutions to the anisotropic systems

Now we are able to prove the existence of solutions to auxiliary system (4.2).

Theorem 5.1. *Assume that conditions (1.4) and (1.7) hold, the functions $G_{ki} : \mathbb{R} \rightarrow [a_{ki}, +\infty)$, with $a_{ki} > 0$ for $i = 1, \dots, N$, $k = 1, 2$, are continuous, and $F_k : \Omega \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$, $k = 1, 2$, are Carathéodory functions satisfying hypotheses (H1) and (H2). Then, for every $R > 0$, problem (4.2)*

has at least a weak solution $(u_R, v_R) \in X$ which means

$$\begin{cases} \sum_{i=1}^N \int_{\Omega} G_{1i,R}(u_R(x)) |\partial_i u(x)|^{p_i-2} \partial_i u(x) \partial_i \phi(x) dx = \int_{\Omega} F_1(x, u, v, \nabla u, \nabla v) \phi dx, \\ \sum_{i=1}^N \int_{\Omega} G_{2i,R}(v_R(x)) |\partial_i v(x)|^{q_i-2} \partial_i v(x) \partial_i \psi(x) dx = \int_{\Omega} F_2(x, u, v, \nabla u, \nabla v) \psi dx \end{cases} \quad (5.1)$$

for all $(\phi, \psi) \in X$. Moreover, the solution set of system (4.2) is uniformly bounded with the bound $C_0 > 0$ in Theorem 3.1. In particular, one has $\|(u_R, v_R)\|_{L^\infty} \leq C_0$.

Proof. For a fixed $R > 0$, the auxiliary system (4.2) is equivalent to resolving in X the operator equation

$$(\mathcal{A}_R - \mathcal{N})(u, v) = 0. \quad (5.2)$$

By Proposition 4.2 it is known that the operator $\mathcal{A}_R - \mathcal{N} : X \rightarrow X^*$ is pseudomonotone, bounded and coercive. Hence we are entitled to apply the main theorem for pseudomonotone operators (see, e.g., [5, Theorem 2.99]) from which we deduce that equation (5.2) possesses at least a solution $(u_R, v_R) \in X$. Due to the mentioned equivalence, (u_R, v_R) is a weak solution of auxiliary problem (4.2) in the sense of (5.1).

On the other hand, Theorem 3.1 is applicable with $G_{ki,R}$ in place of G_{ki} for every $i = 1, \dots, N$ and $k = 1, 2$. This is true because the range of $G_{ki,R}$ is contained in the interval $[a_{ki}, +\infty)$ as it is the case for G_{ki} . Consequently, the solution (u_R, v_R) of (4.2) fulfills the a priori estimate $\|(u_R, v_R)\|_{L^\infty} \leq C_0$, where $C_0 > 0$ is the uniform bound given in Theorem 3.1. This completes the proof. \square

Now we prove that $(u_R, v_R) \in X$ obtained in Theorem 5.1 is a weak solution of the original problem (1.2) provided $R > 0$ is sufficiently large.

Proof of Theorem 1.1. Theorem 3.1 provides the existence of a positive constant C_0 such that $\|(u, v)\|_{L^\infty} \leq C_0$ for all weak solutions $(u, v) \in X$ to problem (1.2). As known from the statement of Theorem 3.1, the constant C_0 depends on the function G_{ki} in (1.2) only through its lower bound a_{ki} for $i = 1, \dots, N$ and $k = 1, 2$. From (4.1) it is clear that a_{ki} is a lower bound for each truncation $G_{ki,R}$. Therefore the same constant C_0 bounds

the solution set of each auxiliary problem (4.2) whenever $R > 0$, so for the solution $(u_R, v_R) \in X$ to problem (4.2) given by Theorem 5.1 it holds $\|(u_R, v_R)\|_{L^\infty} \leq C_0$.

In view of the fact that C_0 does not depend on R , we are allowed to choose $R \geq C_0$ getting that $|u_R(x)| \leq R$ and $|v_R(x)| \leq R$ almost everywhere on Ω . Owing to (4.1) we have

$$G_{1i,R}(u_R(x)) = G_{1i}(u_R(x)) \text{ and } G_{2i,R}(v_R(x)) = G_{2i}(v_R(x))$$

for a.e. $x \in \Omega$, and $i = 1, \dots, N$. It follows that $(u_R, v_R) \in X$ is a bounded weak solution for the original problem (1.2), thus completing the proof of Theorem 1.1. \square

References

- [1] S. N. Antontsev, J. I. Daz and S. D. Shmarev, *Energy methods for free boundary problems. Applications to nonlinear PDEs and fluid mechanics*, Progress in Nonlinear Differential Equations and their Applications, 48, Birkhuser Boston, Inc., Boston, MA, 2002.
- [2] B. Brandolini and F. C. Cirstea, Anisotropic elliptic equations with gradient-dependent lower order terms and L^1 data, *Math. Eng.*, **5** (2023), 1-33.
- [3] B. Brandolini and F. C. Cirstea, Singular anisotropic elliptic equations with gradient-dependent lower order terms, *NoDEA Nonlinear Differential Equations Appl.* 30 (2023), no. 5, 58 pp.
- [4] B. Brandolini and F. C. Cirstea, Boundedness of solutions to singular anisotropic elliptic equations, arXiv preprint arXiv:2307.08369, 2023 - arxiv.org.
- [5] S. Carl, V. K. Le and D. Motreanu, *Nonsmooth variational problems and their inequalities. Comparison principles and applications*, Springer, New York, 2007.
- [6] I. Fragala, F. Gazzola and B. Kawohl, Existence and nonexistence results for anisotropic quasilinear elliptic equations, *Ann. Inst. H. Poincaré C Anal. Non Linéaire*, **21** (2007), 715–734.

- [7] D. Motreanu, Nonhomogeneous Dirichlet problems with unbounded coefficient in the principal part, *Axioms*, **11** (2022), Paper No. 739.
- [8] D. Motreanu and E. Tornatore, Dirichlet problems with anisotropic principal part involving unbounded coefficients, *Electron. J. Differential Equations*, **2024** (2024), No. 11, 13 pp..
- [9] D. Motreanu, E. Tornatore; Nonhomogeneous degenerate quasilinear problems with convection,, *Nonlinear Anal. Real World Appl.*, **71** (2023), Paper No. 103800.