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POSITIVE SOLUTIONS FOR SINGULAR (p, q)-LAPLACIAN EQUATIONS WITH NEGATIVE PERTURBATION

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Abstract. We consider a nonlinear Dirichlet problem driven by the (p, q)-Laplacian and with a reaction consisting of a singular term plus a negative perturbation. Using regularization of the singular term and truncation and comparison techniques, we show that the problem has a unique positive smooth solution.

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper we study the following singular Dirichlet (p, q)-equation

$$\begin{gathered} -\Delta_p u(z) - \Delta_q u(z) = u(z)^{-\eta} - f(z, u(z)) \quad \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \quad 1 < q < p,\ 0 < \eta < 1,\ u > 0. \end{gathered} \tag{1.1}$$

For $r \in (1, +\infty)$ by Δ_r we denote the r-Laplace differential operator defined by

$$\Delta_r u = \operatorname{div}(|\nabla u|^{r-2}\nabla u) \quad \text{for all } u \in W_0^{1,r}(\Omega).$$

Equation (1.1) is driven by the sum of two such operators with different exponents (double phase problem). Therefore the differential operator of our problem is not homogeneous. In the reaction (right hand side), there is a singular term $u^{-\eta}$ and a perturbation $-f(z, u)$, with $f(z, x)$ being a Carathéodory function (that is, for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z, x)$ is continuous) with values in $\mathbb{R}_+ = [0, +\infty)$ (that is, $f \geq 0$). So, in problem (1.1) the perturbation of the singular term is negative. This is in contrast with most earlier works on singular elliptic equations, where the perturbation is positive. We refer to works of Sun-Wu-Long [17], Haitao [8], Ghergu-Rădulescu [3] (semilinear equations), Giacomoni-Schindler-Takáč [5], Papageorgiou-Winkert [14] (equations driven by the p-Laplacian), Mukherjee-Sreenadh [10] (equations driven by the fractional p-Laplacian) and of Papageorgiou-Rădulescu-Repovš [12] (equations driven by a general nonlinear nonhomogeneous differential operator). Singular equations with a negative perturbation were investigated by Godoy-Guerin [7] (semilinear equations driven by the Laplacian) and by Saoudi [16] (nonlinear equations driven by the p-Laplacian). In both works the negative perturbation of the singular term is a power of u. Here we allow a more general perturbation. In both papers the

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approach is based on the direct method of the calculus of variations. In Godoy-Guerin [7], the notion of weak solution is more restrictive, since they require that the test functions belong in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. On the other hand Saoudi [16] considers a parametric problem with the parameter $\lambda > 0$ multiplying the singular term. The author shows that there exists $\Lambda_* \geq 0$ (notation in [16]) such that for all $\lambda > \Lambda_*$ the problem has a solution. In fact as we explain in Remark 3.5 $\Lambda_* = 0$ and so the existence theorem is valid for all parameters $\lambda > 0$ and so there is no need to introduce a parameter to the problem. Finally we mention that in [7] the equation is driven by the Laplacian (semilinear equation), while in [16] is driven by the p-Laplacian.

The fact that the perturbation is negative, makes it difficult to generate a lower solution for the problem which is helpful in bypassing the singularity and dealing with C^1-functionals. The solution of the purely singular problem can not serve as a lower solution as is the case in problems with positive perturbation (see for example Papageorgiou-Rădulescu-Repovš [12]). Our approach is different and uses upper solutions and regularizations of the singular term.

2. Mathematical background - hypotheses

The main spaces in the analysis of problem (1.1) are the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega}) = \{u \in C^1(\overline{\Omega}) : u\big|_{\partial\Omega} = 0\}$. On account of the Poincaré inequality the norm of $W_0^{1,p}(\Omega)$ is given by

$$\|u\| = \|\nabla u\|_p \text{ for all } u \in W_0^{1,p}(\Omega).$$

The space $C_0^1(\overline{\Omega})$ is an ordered Banach space with positive (order) cone given by $C_+ = \{u \in C_0^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}$. This cone has a nonempty interior

$$\operatorname{int} C_+ = \Big\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega, \frac{\partial u}{\partial n}\Big|_{\partial\Omega} < 0\Big\},$$

with $n(\cdot)$ being the outward unit normal on $\partial\Omega$ and $\frac{\partial u}{\partial n} = (\nabla u, n)_{\mathbb{R}^N}$. Also ordered Banach space is the Lebesgue space $L^\infty(\Omega)$ with positive (order) cone $L^\infty(\Omega)_+ = \{u \in L^\infty(\Omega) : u(z) \geq 0 \text{ for a.a. } z \in \Omega\}$. This order cone has a nonempty interior

$$\operatorname{int} L^\infty(\Omega)_+ = \big\{u \in L^\infty(\Omega)_+ : \operatorname{ess\,inf}_\Omega u(z) > 0\big\}.$$

We mention that from all the Lebesgue spaces $L^p(\Omega)$, $1 \leq p \leq +\infty$ (all of which are ordered Banach spaces with the pointwise order), only $L^\infty(\Omega)$ has positive cone with a nonempty interior. This is a consequence of the fact that only the norm of $L^\infty(\Omega)$ is defined in a pointwise fashion.

For $r \in (1, +\infty)$, let $A_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega) = W_0^{1,r}(\Omega)^* (\frac{1}{r} + \frac{1}{r'} = 1)$ be defined by

$$\langle A_r(u), h\rangle = \int_\Omega |\nabla u|^{r-2}(\nabla u, \nabla h)_{\mathbb{R}^N}\, dz \quad \text{for all } u, h \in W_0^{1,r}(\Omega).$$

We know (see Gasiński-Papageorgiou [2, p. 279]) that $A_r(\cdot)$ is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (thus maximal monotone too) and of type $(S)_+$, which means that it has the following property

$u_n \xrightarrow{w} u$ in $W_0^{1,r}(\Omega)$ and $\limsup_{n\to+\infty} \langle A_r(u_n), u_n - u\rangle \leq 0$ imply
$u_n \to u$ in $W_0^{1,r}(\Omega)$ as $n \to +\infty$.

We set $V = A_p + A_q$. Then $V : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)(\frac{1}{p} + \frac{1}{p'} = 1)$ and it has the following properties:

- $V(\cdot)$ is continuous, strictly monotone (thus maximal monotone too);
- $V(\cdot)$ is of type $(S)_+$.

If $u : \Omega \to \mathbb{R}$ is a measurable function, then we set $u^{\pm}(z) = \max\{\pm u(z), 0\}$ for all $z \in \Omega$. We have $u = u^+ - u^-$, $|u| = u^+ + u^-$ and if $u \in W_0^{1,p}(\Omega)$, then $u^{\pm} \in W_0^{1,p}(\Omega)$.

Our hypotheses on the perturbation $f(z,x)$ are the following:

(H1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$ $f(z,0) = 0$, $f(z,x) \geq 0$ for all $x \geq 0$, there exists $\tau \in (1,q]$ such that $x \to f(z,x)/x^{\tau-1}$ is nondecreasing on $\mathring{R}_+ = (0,+\infty)$ and $|f(z,x)| \leq \widehat{a}(z)[1 + x^{r-1}]$ for a.a. $z \in \Omega$, all $x \geq 0$, with $\widehat{a} \in L^\infty(\Omega)_+$, and $p \leq r < p^*$.

Remark 2.1. Recall that

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } N \leq p \end{cases}$$

is the critical Sobolev exponent corresponding to p. Since we look for positive solutions and the above hypotheses concern the positive semiaxis $\mathbb{R}_+ = [0,+\infty)$, without any loss of generality we may assume that $f(z,x) = 0$ for a.a. $z \in \Omega$, all $x \leq 0$.

By a solution of (1.1) we mean a function $u \in W_0^{1,p}(\Omega)$ such that $u^{-\eta}h \in L^1(\Omega)$ for all $h \in W_0^{1,p}(\Omega)$ and

$$\int_\Omega (|\nabla u|^{p-2}\nabla u + |\nabla u|^{q-2}\nabla u, \nabla h)_{\mathbb{R}^N}\, dz = \int_\Omega u^{-\eta} h\, dz - \int_\Omega f(z,u)h\, dz$$

for all $h \in W_0^{1,p}(\Omega)$.

3. Positive solutions

First we consider the purely singular problem

$$\begin{gathered} -\Delta_p u(z) - \Delta_q u(z) = u(z)^{-\eta} \text{ in } \Omega, \\ u\big|_{\partial\Omega} = 0, \quad 1 < q < p,\ 0 < \eta < 1,\ u > 0. \end{gathered} \tag{3.1}$$

From Papageorgiou-Rădulescu-Repovš [12, Proposition 11] we have the following result.

Proposition 3.1. *Problem* (3.1) *has a unique positive solution* $\overline{u} \in \operatorname{int} C_+$.

Next let $\varepsilon > 0$ and consider the following regularized version of problem (1.1),

$$\begin{gathered} -\Delta_p u(z) - \Delta_q u(z) = [u(z) + \varepsilon]^{-\eta} - f(z,u(z)) \quad \text{in } \Omega, \\ u\big|_{\partial\Omega} = 0, \quad 1 < q < p,\ 0 < \eta < 1,\ u > 0. \end{gathered} \tag{3.2}$$

Proposition 3.2. *If hypotheses* (H1) *hold, then for every* $\varepsilon > 0$ *problem* (3.2) *has a unique positive solution* $\widetilde{u}_\varepsilon \in \operatorname{int} C_+$.

Proof. Consider the Carathéodory function

$$k_\varepsilon(z,x) = \begin{cases} [x^+ + \varepsilon]^{-\eta} - f(z,x^+) & \text{if } x \leq \overline{u}(z), \\ [\overline{u}(z) + \varepsilon]^{-\eta} - f(z,\overline{u}(z)) & \text{if } \overline{u}(z) < x. \end{cases} \tag{3.3}$$

Let $K_\varepsilon(z,x) = \int_0^x k_\varepsilon(z,s)ds$ and consider the C^1-functional $\psi_\varepsilon : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi_\varepsilon(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega K_\varepsilon(z,u)\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

From (3.3) it is clear that $\psi_\varepsilon(\cdot)$ is coercive. Also using the Sobolev embedding theorem, we see that $\psi_\varepsilon(\cdot)$ is sequentially weakly lower semicontinuous. So, by the Weierstrass-Tonelli theorem, we can find $\widetilde{u}_\varepsilon \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} &\psi_\varepsilon(\widetilde{u}_\varepsilon) = \inf[\psi_\varepsilon(u) : u \in W_0^{1,p}(\Omega)], \\ &\Rightarrow\ \psi_\varepsilon'(\widetilde{u}_\varepsilon) = 0 \quad \text{in } W^{-1,p'}(\Omega), \\ &\Rightarrow\ \langle V(\widetilde{u}_\varepsilon), h\rangle = \int_\Omega k_\varepsilon(z,\widetilde{u}_\varepsilon)h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \end{aligned} \tag{3.4}$$

In (3.4) by using the test function $h = -\widetilde{u}_\varepsilon^- \in W_0^{1,p}(\Omega)$, we have $\|\nabla\widetilde{u}_\varepsilon^-\|_p^p \le 0$ which implies

$$\widetilde{u}_\varepsilon \ge 0 \text{ and } \widetilde{u}_\varepsilon \ne 0 \text{ (since } \varepsilon > 0).$$

Next in (3.4) we choose the test function $h = (\widetilde{u}_\varepsilon - \overline{u})^+ \in W_0^{1,p}(\Omega)$. We obtain

$$\begin{aligned} \langle V(\widetilde{u}_\varepsilon), (\widetilde{u}_\varepsilon - \overline{u})^+\rangle &= \int_\Omega \big([\overline{u}+\varepsilon]^{-\eta} - f(z,\overline{u})\big)(\widetilde{u}_\varepsilon - \overline{u})^+\,dz \quad \text{(see (3.3))} \\ &\le \int_\Omega \overline{u}^{-\eta}(\widetilde{u}_\varepsilon - \overline{u})^+\,dz \quad \text{(since } f \ge 0) \\ &= \langle V(\overline{u}), (\widetilde{u}_\varepsilon - \overline{u})^+\rangle \quad \text{(see Proposition 3.1)} \end{aligned}$$

which implies

$$\widetilde{u}_\varepsilon \le \overline{u} \quad \text{(from the monotonicity of } V(\cdot)).$$

So, we have proved that

$$\widetilde{u}_\varepsilon \in [0,\overline{u}], \widetilde{u}_\varepsilon \ne 0,\ \Rightarrow\ \widetilde{u}_\varepsilon \text{ is a positive solution of (3.2).}$$

The nonlinear regularity theory by Lieberman [9] implies that

$$\widetilde{u}_\varepsilon \in C_+ \setminus \{0\}.$$

Hypotheses (H1) imply that there exists $c_1 > 0$ such that

$$[x+\varepsilon]^{-\eta} - f(z,x) \ge -c_1 x^{r-1} \quad \text{for a.a. } z \in \Omega, \text{ all } x \ge 0.$$

So, we have

$$\begin{aligned} &\Delta_p\widetilde{u}_\varepsilon + \Delta_q\widetilde{u}_\varepsilon \le c_1\|\overline{u}\|_\infty^{r-p}\,\widetilde{u}_\varepsilon^{p-1} \text{ in } \Omega, \\ &\Rightarrow\ \widetilde{u}_\varepsilon \in \operatorname{int} C_+ \quad \text{(see Pucci-Serrin [15] (pp. 111, 120)).} \end{aligned}$$

Now we show the uniqueness of this positive solution. To this end we consider the integral functional $j : L^1(\Omega) \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ defined by

$$j(u) = \begin{cases} \frac{1}{p}\|\nabla u^{1/\tau}\|_p^p + \frac{1}{q}\|\nabla u^{1/\tau}\|_q^q & \text{if } u \ge 0\ ,\ u^{1/\tau} \in W^{1,p}(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

We define $\operatorname{dom} j = \{u \in L^1(\Omega) : j(u) < +\infty\}$ (the effective domain of $j(\cdot)$). Also let $\ell_0 : \mathbb{R}_+ \to \mathbb{R}_+$ be the function defined by

$$\ell_0(t) = \frac{1}{p}t^p + \frac{1}{q}t^q \quad \text{for all } t \ge 0.$$

The function $\ell_0(\cdot)$ is strictly increasing, strictly convex and since $\tau \in (1, q]$ (see hypotheses (H1)), we see that $t \to \ell_0(t^{1/\tau})$ is convex on $\mathbb{R}_+$. We define $\ell(y) = \ell_0(|y|)$ for all $y \in \mathbb{R}^N$. Then $\ell : \mathbb{R}^N \to \mathbb{R}_+$ is convex. Suppose $u_1, u_2 \in \operatorname{dom} j$ and set $u = [tu_1 + (1-t)u_2]^{1/\tau}$ with $t \in [0, 1]$. From Díaz-Saá [1] (Lemme 1), we have that

$$|\nabla u| \leq \big[t|\nabla u_1^{1/\tau}|^\tau + (1-t)|\nabla u_2^{1/\tau}|^\tau\big]^{1/\tau}$$

which implies

$$\begin{aligned}\ell_0(|\nabla u|) &\leq \ell_0\Big(\big[t|\nabla u_1^{1/\tau}|^\tau + (1-t)|\nabla u_2^{1/\tau}|^\tau\big]^{1/\tau}\Big) \quad \text{(since } \ell_0(\cdot) \text{ is increasing)},\\ &\leq t\ell_0(|\nabla u_1^{1/\tau}|) + (1-t)\ell_0(|\nabla u_2^{1/\tau}|) \quad \text{(since } t \to \ell_0(t^{1/\tau}) \text{ is convex)},\end{aligned}$$

This in turn implies

$$\ell(\nabla u) \leq t\ell(\nabla u_1^{1/\tau}) + (1-t)\ell(\nabla u_2^{1/\tau}),$$

Thus $j(\cdot)$ is convex.

Suppose that $\widetilde{v}_\varepsilon(\cdot)$ is another positive solution of problem (3.2). Again we have $\widetilde{v}_\varepsilon \in \operatorname{int} C_+$. For $\delta > 0$, we set

$$\widetilde{u}_\varepsilon^\delta = \widetilde{u}_\varepsilon + \delta, \quad \widetilde{v}_\varepsilon^\delta = \widetilde{v}_\varepsilon + \delta.$$

Evidently $\widetilde{u}_\varepsilon^\delta, \widetilde{v}_\varepsilon^\delta \in \operatorname{int} L^\infty(\Omega)_+$. So [11, Proposition 4.1.22, p. 274] implies that

$$\frac{\widetilde{u}_\varepsilon^\delta}{\widetilde{v}_\varepsilon^\delta} \in L^\infty(\Omega), \quad \frac{\widetilde{v}_\varepsilon^\delta}{\widetilde{u}_\varepsilon^\delta} \in L^\infty(\Omega). \tag{3.5}$$

We set $h = ((\widetilde{u}_\varepsilon^\delta)^\tau - (\widetilde{v}_\varepsilon^\delta)^\tau) \in C_0^1(\overline{\Omega})$. From (3.5) it follows that for $t \in (0, 1)$ small, we have

$$(\widetilde{u}_\varepsilon^\delta)^\tau + th \in \operatorname{dom} j, \quad (\widetilde{v}_\varepsilon^\delta)^\tau + th \in \operatorname{dom} j.$$

Then the convexity of $j(\cdot)$ implies that the directional derivatives of $j(\cdot)$ at $(\widetilde{u}_\varepsilon^\delta)^\tau$ and at $(\widetilde{v}_\varepsilon^\delta)^\tau$ in the direction h exist and using the chain rule and Green's identity (see [11, p. 35]), we have

$$\begin{aligned}j'((\widetilde{u}_\varepsilon^\delta)^\tau)(h) &= \frac{1}{\tau}\int_\Omega \frac{-\Delta_p \widetilde{u}_\varepsilon - \Delta_q \widetilde{u}_\varepsilon}{(\widetilde{u}_\varepsilon^\delta)^{\tau-1}} h\, dz\\ &= \frac{1}{\tau}\int_\Omega \frac{[\widetilde{u}_\varepsilon + \varepsilon]^{-\eta} - f(z, \widetilde{u}_\varepsilon)}{(\widetilde{u}_\varepsilon^\delta)^{\tau-1}} h\, dz,\\ j'((\widetilde{v}_\varepsilon^\delta)^\tau)(h) &= \frac{1}{\tau}\int_\Omega \frac{-\Delta_p \widetilde{v}_\varepsilon - \Delta_q \widetilde{v}_\varepsilon}{(\widetilde{v}_\varepsilon^\delta)^{\tau-1}} h\, dz\\ &= \frac{1}{\tau}\int_\Omega \frac{[\widetilde{v}_\varepsilon + \varepsilon]^{-\eta} - f(z, \widetilde{v}_\varepsilon)}{(\widetilde{v}_\varepsilon^\delta)^{\tau-1}} h\, dz.\end{aligned}$$

The convexity of $j(\cdot)$ implies the monotonicity of the directional derivative. So, we have

$$\begin{aligned}0 &\leq \int_\Omega \Big(\frac{[\widetilde{u}_\varepsilon + \varepsilon]^{-\eta}}{(\widetilde{u}_\varepsilon^\delta)^{\tau-1}} - \frac{[\widetilde{v}_\varepsilon + \varepsilon]^{-\eta}}{(\widetilde{v}_\varepsilon^\delta)^{\tau-1}}\Big)((\widetilde{u}_\varepsilon^\delta)^\tau - (\widetilde{v}_\varepsilon^\delta)^\tau)\, dz\\ &\quad - \int_\Omega \Big(\frac{f(z, \widetilde{u}_\varepsilon)}{(\widetilde{u}_\varepsilon^\delta)^{\tau-1}} - \frac{f(z, \widetilde{v}_\varepsilon)}{(\widetilde{v}_\varepsilon^\delta)^{\tau-1}}\Big)((\widetilde{u}_\varepsilon^\delta)^\tau - (\widetilde{v}_\varepsilon^\delta)^\tau)\, dz.\end{aligned}$$

We let $\delta \to 0$ and use the dominated convergence theorem. Then on account of hypotheses (H1), we obtain

$$0 \leq \int_\Omega \Big(\frac{1}{\widetilde{u}_\varepsilon^{\tau+\eta-1}} - \frac{1}{\widetilde{v}_\varepsilon^{\tau+\eta-1}}\Big)(\widetilde{u}_\varepsilon^\tau - \widetilde{v}_\varepsilon^\tau)\, dz \leq 0;$$

thus $\widetilde{u}_\varepsilon = \widetilde{v}_\varepsilon$. This proves the uniqueness of the positive solution $\widetilde{u}_\varepsilon \in \operatorname{int} C_+$ of problem (3.2). $\square$

Next we show a monotonicity property of the map $\varepsilon \mapsto \widetilde{u}_\varepsilon$.

Proposition 3.3. *If hypotheses* (H1) *hold and* $0 < \varepsilon' \le \varepsilon$, *then* $0 \le \widetilde{u}_\varepsilon \le \widetilde{u}_{\varepsilon'}$.

Proof. We have

$$\begin{aligned} -\Delta_p \widetilde{u}_{\varepsilon'} - \Delta_q \widetilde{u}_{\varepsilon'} &= [\widetilde{u}_{\varepsilon'} + \varepsilon']^{-\eta} - f(z, \widetilde{u}_{\varepsilon'}) \\ &\ge [\widetilde{u}_{\varepsilon'} + \varepsilon]^{-\eta} - f(z, \widetilde{u}_{\varepsilon'}) \quad \text{in } \Omega. \end{aligned} \tag{3.6}$$

We introduce the Carathéodory function

$$\ell_\varepsilon(z,x) = \begin{cases} [x^+ + \varepsilon]^{-\eta} - f(z, x^+) & \text{if } x \le \widetilde{u}_{\varepsilon'}(z), \\ [\widetilde{u}_{\varepsilon'}(z) + \varepsilon]^{-\eta} - f(z, \widetilde{u}_{\varepsilon'}(z)) & \text{if } \widetilde{u}_{\varepsilon'}(z) < x. \end{cases} \tag{3.7}$$

We set $L_\varepsilon(z,x) = \int_0^x \ell_\varepsilon(z,s)ds$ and consider the C^1-functional $\sigma_\varepsilon : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\sigma_\varepsilon(u) = \frac{1}{p}\|\nabla u\|_p^p + \frac{1}{q}\|\nabla u\|_q^q - \int_\Omega L_\varepsilon(z,u)\,dz \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

From (3.7) it is clear that $\sigma_\varepsilon(\cdot)$ is coercive. Also it is sequentially weakly lower semicontinuous. So, we can find $u_\varepsilon \in W_0^{1,p}(\Omega)$ such that

$$\sigma_\varepsilon(u_\varepsilon) = \inf[\sigma_\varepsilon(u) : u \in W_0^{1,p}(\Omega)],$$

which implies $\sigma'_\varepsilon(u_\varepsilon) = 0$ in $W^{-1,p'}(\Omega)$, and this implies

$$\langle V(u_\varepsilon), h\rangle = \int_\Omega \ell_\varepsilon(z, u_\varepsilon) h\,dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.8}$$

Let $h = -u_\varepsilon^- \in W_0^{1,p}(\Omega)$. We have $\|\nabla u_\varepsilon^-\|_p^p \le 0$; therefore,

$$u_\varepsilon \ge 0, \quad u_\varepsilon \ne 0 \quad (\text{since } \varepsilon > 0.$$

Also, in (3) we choose the test function $h = (u_\varepsilon - \widetilde{u}_{\varepsilon'})^+ \in W_0^{1,p}(\Omega)$. We obtain

$$\begin{aligned} \langle V(u_\varepsilon), (u_\varepsilon - \widetilde{u}_{\varepsilon'})^+\rangle &= \int_\Omega \left([\widetilde{u}_{\varepsilon'} + \varepsilon]^{-\eta} - f(z, \widetilde{u}_{\varepsilon'})\right)(u_\varepsilon - \widetilde{u}_{\varepsilon'})^+\,dz \quad (\text{see } (3.7)) \\ &\le \langle V(\widetilde{u}_{\varepsilon'}), (u_\varepsilon - \widetilde{u}_{\varepsilon'})^+\rangle \quad (\text{see } (3.6)). \end{aligned}$$

This implies $u_\varepsilon \le \widetilde{u}_{\varepsilon'}$. So, we have proved that

$$u_\varepsilon \in [0, \widetilde{u}_{\varepsilon'}], \quad u_\varepsilon \ne 0. \tag{3.9}$$

Then (3.9), (3.7), and (3) imply that u_ε is a positive solution of (3.2), which implies $u_\varepsilon = \widetilde{u}_\varepsilon$ (see Proposition 3.2), and $0 \le \widetilde{u}_\varepsilon \le \widetilde{u}_{\varepsilon'}$ (see (3.9)). $\square$

Finally we pass to the limit as $\varepsilon \to 0^+$ to produce a positive solution for problem (1.1). Consider the Dirichlet problem

$$-\Delta_p u(z) - \Delta_q u(z) = [u(z) + \varepsilon]^{-\eta} \text{ in } \Omega, \quad u\big|_{\partial\Omega} = 0,\ u > 0.$$

From Papageorgiou-Rădulescu-Zhang [13] (see the proof of Proposition 3.3), we know that this problem has a unique solution $\overline{u}_\varepsilon \in \operatorname{int} C_+$ and $\overline{u}_\varepsilon \uparrow \overline{u}$ in $C_0^1(\overline{\Omega})$ as $\varepsilon \to 0^+$. Moreover, since $f \ge 0$, as in the proof of Proposition 3.3, we show that $0 \le \widetilde{u}_\varepsilon \le \overline{u}_\varepsilon$. Therefore

$$0 \le \widetilde{u}_\varepsilon \le \overline{u} \quad \text{for all } \varepsilon > 0 \tag{3.10}$$

Theorem 3.4. *If hypotheses* (H1) hold*, then problem* (1.1) *has a unique positive solution* $\widehat{u} \in \operatorname{int} C_+$.

Proof. Let $\varepsilon_n \to 0^+$ and let $\widetilde{u}_n = \widetilde{u}_{\varepsilon_n} \in \operatorname{int} C_+$ as in Proposition 3.2. We have

$$\langle V(\widetilde{u}_n), h\rangle = \int_\Omega \big([\widetilde{u}_n + \varepsilon_n]^{-\eta} - f(z, \widetilde{u}_n)\big)\, h\, dz \quad \text{for all } h \in W_0^{1,p}(\Omega), \tag{3.11}$$

$$0 \le \widetilde{u}_1 \le \widetilde{u}_n \le \overline{u} \quad \text{for all } n \in \mathbb{N} \tag{3.12}$$

(see Proposition 3.3, (3.10) and assume $\varepsilon_n \le 1$).

In (3.11) we use the test function $h = \widetilde{u}_n \in W_0^{1,p}(\Omega)$. Using (3.12) and that $f \ge 0$, we obtain

$$\|\nabla \widetilde{u}_n\|_p^p \le \int_\Omega \widetilde{u}_n^{1-\eta}\, dz \le \int_\Omega \overline{u}^{1-\eta}\, dz;$$

therefore, $\{\widetilde{u}_n\}_{n\in\mathbb{N}} \subseteq W_0^{1,p}(\Omega)$ is bounded. So, we can assume that

$$\widetilde{u}_n \xrightarrow{w} \widehat{u} \text{ in } W_0^{1,p}(\Omega), \quad \widetilde{u}_n \to \widehat{u} \text{ in } L^r(\Omega). \tag{3.13}$$

Let $\widehat{d}(z) = d(z, \partial\Omega)$ for all $z \in \overline{\Omega}$. From Gilbarg-Trudinger [6, Lemma 14.16, p. 355] we have that $\widehat{d} \in \operatorname{int} C_+$. Since $\widetilde{u}_1 \in \operatorname{int} C_+$, using [11, Proposition 4.1.22, p. 274], we can find $c_2 > 0$ such that

$$c_2 \widehat{d} \le \widetilde{u}_1. \tag{3.14}$$

Then for $h \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}
&\int_\Omega \Big(\frac{|h|}{[\widetilde{u}_n + \varepsilon_n]^\eta}\Big)^p dz \\
&\le \int_\Omega \Big(\frac{|h|}{\widetilde{u}_1^\eta}\Big)^p dz \quad \text{(see (3.12))} \\
&= \int_\Omega \Big(\widetilde{u}_1^{1-\eta} \frac{|h|}{\widetilde{u}_1}\Big)^p dz \\
&\le c_3 \int_\Omega \Big(\frac{|h|}{\widehat{d}}\Big)^p dz \quad \text{for some } c_3 > 0 \quad (\text{since } \widetilde{u}_1 \in \operatorname{int} C_+ \text{ and using (3.14)}) \\
&\le c_4 \|\nabla h\|_p^p \quad \text{for some } c_4 > 0, \text{ all } n \in \mathbb{N}
\end{aligned}$$

(using Hardy's inequality, see [11, p. 66]) Therefore,

$$\Big\{\frac{h}{[\widetilde{u}_n + \varepsilon_n]^\eta}\Big\}_{n\in\mathbb{N}} \subseteq L^p(\Omega) \text{ is bounded for all } h \in W_0^{1,p}(\Omega). \tag{3.15}$$

From (3.13) and by passing to a subsequence if necessary, we have that

$$\frac{h}{[\widetilde{u}_n + \varepsilon_n]^\eta} \to \frac{h}{\widehat{u}^\eta} \quad \text{a.e.} \tag{3.16}$$

(note that $\widetilde{u}_1 \le \widehat{u}$, see (3.12)).

Then (3.15), (3.16) and [12, Problem 1.44] imply that

$$\int_\Omega \frac{h}{[\widetilde{u}_n + \varepsilon_n]^\eta}\, dz \to \int_\Omega \frac{h}{\widehat{u}^\eta}\, dz \quad \text{for all } h \in W_0^{1,p}(\Omega). \tag{3.17}$$

In (3.11) we choose $h = (\widetilde{u}_n - \widehat{u}) \in W_0^{1,p}(\Omega)$ and pass to the limit as $n \to +\infty$. We obtain

$$\lim_{n\to+\infty} \langle V(\widetilde{u}_n), \widetilde{u}_n - \widehat{u}\rangle = 0,$$

which implies

$$\begin{gathered}\widetilde{u}_n \to \widehat{u} \quad \text{in } W_0^{1,p}(\Omega) \quad \text{(see Section 2)},\\ \widetilde{u}_1 \le \widehat{u} \le \overline{u} \quad \text{(see (3.12))}.\end{gathered} \tag{3.18}$$

In (3.11) we pass to the limit as $n \to +\infty$ and use (3.17), (3.18). We obtain

$$\begin{gathered}\langle V(\widehat{u}), h\rangle = \int_\Omega [\widehat{u}^{-\eta} - f(z,\widehat{u})]h\,dz \text{ for all } h \in W_0^{1,p}(\Omega),\\ \widetilde{u}_1 \le \widehat{u} \le \overline{u}.\end{gathered}$$

Therefore, $\widehat{u}$ is a positive solution of (1.1). Recall that $\widehat{d} \in \operatorname{int} C_+$. So, as before, using [11, Proposition 4.1.22, p. 274], we can find $c_5 > 0$ such that

$$\overline{u} \le c_5\widehat{d}, \Rightarrow \widehat{u} \le c_5\widehat{d} \quad \text{(see (3.18))}.$$

So, we can apply [4, Theorem 1.7] and conclude that $\widehat{u} \in \operatorname{int} C_+$.

Finally reasoning as in the proof of Proposition 3.2, we show that $\widehat{u} \in \operatorname{int} C_+$ is the unique positive solution of (1.1). □

Remark 3.5. In [16] the author considers the parametric singular Dirichlet problem ($\lambda > 0$ is the parameter)

$$\begin{gathered}-\Delta_p u(z) = \lambda k(z)u(z)^{-\eta} - h(z)u(z)^{r-1} \quad \text{in } \Omega,\\ u\big|_{\partial\Omega} = 0, \quad 1 < p < r < p^*,\ 0 < \eta < 1,\ u > 0,\end{gathered} \tag{3.19}$$

with $k, h \in \operatorname{int} L^\infty(\Omega)_+$. So, the perturbation of the singular term is a special case of our perturbation $f(z,x)$. In [16] the author proves the following existence result (see [16, Theorem 1.5]):

There exists $\Lambda_* > 0$ such that

- for all $\lambda > \Lambda_*$ problem (3.19) has at least one positive solution $u \in W_0^{1,p}(\Omega)$ and for all $K \subseteq \Omega$ compact
$$0 < c_K \le u(z) \quad \text{for a.a. } z \in K;$$
- for each $\lambda < \Lambda_*$ problem (3.19) has no positive solution.

Our work in this paper shows that $\Lambda_* = 0$ and so problem (3.19) has a positive solution for all $\lambda > 0$; therefore the presence of the parameter $\lambda > 0$ in the problem is inconsequential and it can be omitted. Moreover, we show that the solution is unique and belongs in $\operatorname{int} C_+$.

Conclusion. In this article we considered a singular problem driven by the (p,q)-Laplacian and a negative perturbation. Using truncations and regularizations to accommodate the singularity, we prove the existence of a nontrivial solution. Our approach allows us to avoid restrictive definition of the solution and the introduction of a parameter. Moreover, our existence theorem provides regularity information for the positive solution.

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