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A DECOMPOSITION THEOREM FOR σ-$\mathcal{P}$-DIRECTIONALLY POROUS SETS IN FRÉCHET SPACES

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In this paper we study suitable notions of porosity and directional porosity in Fréchet spaces. Moreover we give a decomposition theorem for σ-$\mathcal{P}$-directionally porous sets.

1. Introduction.

Most authors tend to understand “infinite dimensional spaces”, in the context of differentiability, as Banach spaces; but starting with Mankiewicz [4], the more general context of Fréchet spaces has also been considered. The natural question of extending the Banach space results to the more general setting has been answered in various stages by several authors. In this paper we continue the programme of transferring the strongest Banach space results to Fréchet spaces.

The porosity of a set arises, in a natural way, in questions concerning the differentiability of Lipschitz maps. For subsets of the real line this notion was introduced by A. Denjoy [1] and E. P. Dolzhenko [2]. The

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family of all σ-porous subsets of the real line is properly included between the family of all Lebesgue null sets and the class of all first category sets. Many authors have been investigated on the porosity and generalized this notion to metric spaces and in particular to Banach spaces. For an exhaustive study on this argument see [8] and [9].

D. Preiss and J. Tišer [5] proved that there exist σ-porous subsets of a separable Banach space which are not "null" in any known sense. Moreover D. Preiss and L. Zajíček [6] observed that Borel σ-directionally porous subsets of a separable Banach space are null in Aronszajn's sense.

In this paper, we generalize both the notions of porosity and directional porosity. More precisely, given a Fréchet space X, a family of directions V and a sequence $\mathcal{P}$ of continuous seminorms on X, we introduce the notions of $\mathcal{P}$-porosity, σ-$\mathcal{P}$-porosity in direction V and σ-$\mathcal{P}$-directional porosity. We prove decomposition theorems for σ-$\mathcal{P}$-porous sets in direction V (Proposition 3.1 and Proposition 3.2) and for σ-$\mathcal{P}$-directionally porous sets (Theorem 3.1). These decompositions are used in [3] to study the sets of points of Gâteaux non-differentiability for Lipschitz maps.

2. Preliminaries.

Trough this paper X is a Fréchet space, i.e. a locally convex space which is metrizable and complete.

If $A \subset X$ we denote by $\overline{A}$, $Int(A)$, $\delta(A)$ and A^C, the closure, the interior, the boundary and the complementary set of A, respectively. The linear span of A is denoted by $< A >$.

If p is a continuous seminorm on X, $x \in X$ and $r > 0$, we use the notation $B_p(x, r)$ for the set $\{y \in X : p(y - x) < r\}$.

Definition 2.1. Let M be a subset of X and let x be a point of X. M is said to be *porous at the point* x if there exist a positive constant c and a continuous seminorm p on X such that, for each $\epsilon > 0$, there exist $y \in X$ and $r > c \cdot p(x - y)$, with $p(x - y) < \epsilon$ and $M \cap B_p(y, r) = \emptyset$. M is said to be *porous* if it is porous at each of its points.

Note that the constant c and the seminorm p depend on the point x and that we can assume $c < 1$.

Definition 2.2. Let M be a subset of X and let p be a continuous seminorm on X.

(i) M is said to be *p-porous* if it is porous at each point $x \in M$ and the conditions of Definition 2.1 are fulfilled by the continuous seminorm p.

(ii) M is said to be *c-porous* if it is porous at each point $x \in M$ and the conditions of Definition 2.1 are fulfilled by the constant c.

(iii) M is said to be *(c, p)-porous* if it is porous at each point $x \in M$ and the conditions of Definition 2.1 are fulfilled by the continuous seminorm p and by the constant c.

Definition 2.3. Let M and V be subsets of X and let x be a point of X. The set M is said to be *porous* (*p-porous, c-porous, (c, p)-porous*) *at x in direction V* if it is porous (p-porous, c-porous, (c, p)-porous) at x and the vector y satisfying the conditions of Definition 2.1 has the form $y = x + tv$, where $v \in V$ and $t \geq 0$ ($t > 0$, if $x \in M$). If $V = \{v\}$, then M is said to be *porous* (*p-porous, c-porous, (c, p)-porous*) *at x in the direction v.*

Note that in the above Definition, we can assume $v \neq 0$. Moreover if M is porous at x in the direction v, then there exist $c > 0$ and a continuous seminorm p on X, with $p(v) \neq 0$, such that for each $\epsilon > 0$ there exist $t \geq 0$ and $r > c \cdot tp(v)$, with $t \cdot p(v) < \epsilon$ and $M \cap B_p(x + tv, r) = \emptyset$.

The condition $p(v) \neq 0$ follows from the fact that the condition $p(v) = 0$ implies $x \in B_p(x + tv, r)$.

Definition 2.4. Let M and V be subsets of X and let x be a point of X.

(i) M is said to be *porous in direction V* if it is porous in direction V at each of its points.

(ii) M is said to be *directionally porous at x* if there exists $v \in X$ such that M is porous at x in the direction v.

(iii) M is said to be *directionally porous*, if it is directionally porous at each of its points (the directions depend on the points).

Remark 2.1. From the above definitions it follows that:

(r_1) M is porous at x in the direction v if and only if M is porous at

x in direction $\{tv : t \geq 0\}$.

(r_2) M is c-porous at x in the direction v if and only if M is c-porous at x in the direction $\dfrac{v}{p(v)}$, where p is the continuous seminorm that satisfies the porosity of M at x in the direction v.

(r_3) M is (c, p)-porous at x in direction V if and only if M is (c, p)-porous at x in direction $V_s = \left\{ \dfrac{v}{p(v)} : 0 \neq v \in V \right\}$.

(r_4) If M is porous at x in direction $V = V_1 \cup V_2 \cup ... \cup V_n$, then we can write $M = \bigcup_{i=1}^{n} M_i$, where M_i is porous in direction V_i, $i = 1, ..., n$.

Definition 2.5. M is *σ-porous* (*σ-porous in direction* V, ...), if M is a countable union of porous (porous in direction V, ...) sets.

Definition 2.6. Let $\mathcal{P} = (p_i)_{i \in \mathbb{N}}$ be a sequence of continuous seminorms on X and let $\mathcal{C} = (c_i)_{i \in \mathbb{N}}$ be a sequence of positive constants and let M be a subset of X.

(i) M is said to be *σ-$\mathcal{P}$-porous* (*σ-$\mathcal{P}$-porous in direction* V,...) if $M = \bigcup_{i=1}^{\infty} M_i$, where M_i is p_i-porous (p_i-porous in direction V,...), $i = 1, 2, ...$

(ii) M is said to be *σ-$(\mathcal{C}, \mathcal{P})$-porous* (*$\sigma$-$(\mathcal{C}, \mathcal{P})$-porous in direction* V,...) if $M = \bigcup_{i=1}^{\infty} M_i$, where M_i is (c_i, p_i)-porous ((c_i, p_i)-porous in direction V, ...), $i = 1, 2, ...$

Lemma 2.1. *Let x be a point of X, let M be a subset of X and let $0 < \bar{c} < c$. Then the following assertions hold:*

(a) *If M is c-porous at x in the direction v and $0 \neq w \in X$ is such that $p(w) \neq 0$ and $p(v - w) < c - \bar{c}$, where p is the continuous seminorm which satisfies the porosity of M at x in the direction v, then M is $\bar{c}$-porous at x in the direction w.*

(b) *If M is (c, p)-porous at x in direction V, and $W \subset X$ is such that*

$p(w) \neq 0$, for each $w \in W$, and $inf\{p(v-w) : w \in W\} < c - \bar{c}$, for each $v \in V$, then M is $(\bar{c}, p)$-porous at x in direction W.

Proof. (a) Since $p(w) \neq 0$, we may assume that $p(v) = p(w) = 1$ (see Remark 2.1(r_2)). Choose $\overline{r} = r - t \cdot p(v-w)$, where $t \geq 0$ and $r > 0$ satisfy the condition of porosity of M at x in the direction v. Observe that $p(tw) = t < \epsilon$ and that

$$\overline{c} \cdot p(tw) = ct - (c - \bar{c})t < r - t \cdot p(v-w) = \bar{r}.$$

To complete the proof we need to prove that $M \cap B_p(x + tw, \overline{r}) = \emptyset$. Now $M \cap B_p(x + tv, r) = \emptyset$, by hypothesis. Therefore it is enough to prove that $B_p(x + tw, \overline{r}) \subset B_p(x + tv, r)$. Let $y \in B_p(x + tw, \overline{r})$, then

$$\begin{aligned} &p(y - (x + tv)) \\ &\quad < p(y - (x + tw)) + p(tw - tv) \\ &\quad < r - tp(w - v) + tp(w - v) \\ &\quad = r. \end{aligned}$$

This completes the proof.

(b) Since M is (c, p)-porous at x in direction V, then there exists $\bar{v} \in V$ such that M is (c, p)-porous at x in the direction $\bar{v}$. Moreover there exists $\bar{w} \in W$ such that $p(\bar{v} - \bar{w}) < c - \bar{c}$. Consequently, by Lemma 2.1(a) M is $(\bar{c}, p)$-porous at x in the direction $\bar{w}$. Thus M is $(\bar{c}, p)$-porous at x in direction W. □

Let M and V be subsets of X, c and ϵ be two positive real numbers and p be a continuous seminorm on X. We denote by $P(M, V, c, p, \epsilon)$ the set of all points $a \in X$ for which there exist $v \in V$ with $p(v) \neq 0$, $t \geq 0$ and $r > 0$, such that $r > c \cdot p(tv)$, $p(tv) < \epsilon$ and $B_p(a + tv, r) \cap A = \emptyset$.

Remark 2.2. By the definition of $P(M, V, c, p, \epsilon)$, it follows that:

(i) M is (c, p)-porous at $a \in X$ in direction V iff

$$a \in \bigcap_{\epsilon > 0} P(M, V, c, p, \epsilon) = \bigcap_{n=1}^{\infty} P\left(M, V, c, p, \frac{1}{n}\right);$$

(ii) M is p-porous at a in direction V iff

$$a \in \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} P\left(M, V, \frac{1}{k}, p, \frac{1}{n}\right).$$

Remark 2.3. If M is p-porous in direction V (resp. p-directionally porous), then we can write $M = \bigcup_{k=1}^{\infty} M_k$, where M_k is $\left(\frac{1}{k}, p\right)$-porous in direction V (resp. $\left(\frac{1}{k}, p\right)$-directionally porous), for $k = 1, 2, \ldots$ In fact, it is enough to set

$$M_k = \left\{ a \in M : M \text{ is } \left(\frac{1}{k}, p\right) - \text{porous at } a \text{ in direction } V \right\}.$$

Lemma 2.2. *The following assertions hold:*

(i) $P(M, V, c, p, \epsilon)$ *is an open set.*

(ii) *If* M *is* (c, p)*-porous in direction* V*, then there exists a* G_δ *set* $\tilde{M} \supset M$ *which is* (c, p)*-porous in direction* V*.*

(iii) *If* M *is* p*-porous (resp.* σ*-*$\mathcal{P}$*-porous) in direction* V*, then there exists a* $G_{\delta\sigma}$ *set* $M^* \supset M$ *which is* p*-porous (resp.* σ*-*$\mathcal{P}$*-porous) in direction* V*.*

Proof. Condition (i) follows by the definition of $P(M, V, c, p, \epsilon)$.

(ii) Put $\tilde{M} = \overline{M} \cap \bigcap_{n=1}^{\infty} P\left(M, V, c, p, \frac{1}{n}\right)$ and observe that $\tilde{M} \supset M$. In fact, since M is (c, p)-porous in direction V, then for each $a \in M$ it follows $a \in \overline{M}$, and by Remark 2.2 (i) the point a belongs to $\bigcap_{n=1}^{\infty} P\left(M, V, c, p, \frac{1}{n}\right)$. Moreover by (i) $\tilde{M}$ is a G_δ set. Observe that since M is (c, p)-porous in direction V, then any subset of $\overline{M}$ is (c, p)-porous in direction V. Therefore $\tilde{M}$ is (c, p)-porous in direction V.

(iii) Put $M^* = \overline{M} \cap \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} P\big(M, V, \frac{1}{k}, p, \frac{1}{n}\big)$. As above, each $a \in M$ belongs to $\overline{M}$ and, since M is p-porous in direction V, by Remark 2.2 (ii), we have $a \in \bigcup_{k=1}^{\infty} \bigcap_{n=1}^{\infty} P\big(M, V, \frac{1}{k}, p, \frac{1}{n}\big)$. Thus $M^* \supset M$. Moreover by (i) M^* is a $G_{\delta\sigma}$ set, and since M is p-porous in direction V, it is p-porous in direction V. □

Proposition 2.1. *Let p be a continuous seminorm on X and let c be a positive real number. Moreover let $A \subset X$ be (c, p)-porous in direction $E+F$, where E and F are subspaces of X with the property that there exists $\eta_p > 0$ such that*

$$p(e+f) \geq \eta_p \cdot \max\{p(e), p(f)\} \tag{1}$$

whenever $e \in E$, $f \in F$ and $p(e+f) \neq 0$. Then $A = A_1 \cup A_2$, where A_1 is p-porous in direction E and A_2 is σ-$\mathcal{P}$-porous in direction F, where $\mathcal{P}$ is the constant sequence (p).

Proof. Define

$$A_1 = A \cap \bigcap_{m=1}^{\infty} P\left(A, E, \frac{c\eta_p}{2}, p, \frac{1}{m}\right)$$

and

$$A_2 = A \setminus A_1 = \bigcup_{m=1}^{\infty} A \setminus P\left(A, E, \frac{c\eta_p}{2}, p, \frac{1}{m}\right).$$

By Remark 2.2 (i), A_1 is $\left(\frac{c\eta_p}{2}, p\right)$-porous in direction E. We are now showing that A_2 is σ-$\mathcal{P}$-porous in direction F, where $\mathcal{P}$ is the constant sequence (p). To this end we shall prove that the set $D_m = A \setminus P\left(A, E, \frac{c\eta_p}{2}, p, \frac{1}{m}\right)$ is $\left(\frac{c\eta_p}{2}, p\right)$-porous in direction F. By Remark 2.2 (ii) it is enough to show that, for $x \in D_m$ and for every $0 < \epsilon < \frac{1}{m}$, we have $x \in P\left(D_m, F, \frac{c\eta_p}{2}, p, \epsilon\right)$. Since $x \in D_m$, it is $x \in A$. By hypothesis, A is (c, p)-porous at x in direction $E+F$. Then there exist $e \in E$, $f \in F$, with $p(e+f) \neq 0$, $t > 0$, $r > 0$ such that $p(te+tf) < \epsilon \cdot \eta_p$, $B_p(x+t(e+f), r) \cap A = \emptyset$ and $r > c \cdot p(te+tf)$. Now we prove that $p(f) \neq 0$. Assume, by contradiction, that $p(f) = 0$. Then by (1) we have

$$\eta_p \cdot p(e) \leq p(e+f).$$

Hence

$$\frac{c\eta_p}{2} \cdot t \cdot p(e) \leq \frac{c}{2} \cdot t \cdot p(e+f) < \frac{r}{2}.$$

Since $\eta_p \cdot p(te) \leq p(te+tf) < \epsilon \cdot \eta_p$, then

$$p(te) < \epsilon < \frac{1}{m}.$$

So for $y \in B_p(x+te, \frac{r}{2})$, we have

$$p(y-(x+t(e+f)) \\ \leq p(y-(x+te)) + p(tf) \leq \frac{r}{2} < r.$$

Therefore $B_p(x+te, \frac{r}{2}) \subset B_p(x+t(e+f), r)$.

Since $B_p(x+t(e+f), r) \cap A = \emptyset$, then $B_p(x+te, \frac{r}{2}) \cap A = \emptyset$. Hence $x \in P\big(A, E, \frac{c\eta_p}{2}, p, \frac{1}{m}\big)$, which contradicts the fact that $x \in D_m$. This completes the proof of the condition $p(f) \neq 0$.

By (1) we have

$$\max(p(te), p(tf)) \leq \frac{1}{\eta_p} \cdot p(te+tf) < \epsilon < \frac{1}{m}. \tag{2}$$

So we obtain $p(tf) < \epsilon$ and $p(tf) \leq \frac{1}{\eta_p} \cdot p(te+tf)$. Thus

$$\frac{c\eta_p}{2} \cdot p(tf) < \frac{c\eta_p}{2} \cdot \frac{1}{\eta_p} \cdot \frac{r}{c} = \frac{r}{2}.$$

Now we prove that

$$B_p\Big(x+tf, \frac{r}{2}\Big) \cap D_m = \emptyset.$$

We may assume that $p(e) \neq 0$. In fact the condition $p(e) = 0$, implies

$$B_p\Big(x+tf, \frac{r}{2}\Big) \subset B_p(x+t(e+f), r).$$

But $B_p(x+t(e+f), r) \cap A = \emptyset$. Therefore $B_p\big(x+tf, \frac{r}{2}\big) \cap A = \emptyset$ and

$$B_p\Big(x+tf, \frac{r}{2}\Big) \cap D_m = \emptyset.$$

Assume, by contradiction, that there is $y \in B_p\Big(x+tf, \frac{r}{2}\Big) \cap D_m$. Note that $B_p\big(y+te, \frac{r}{2}\big) \subset B_p(x+t(e+f), r)$ and therefore $B_p\big(y+te, \frac{r}{2}\big) \cap A = \emptyset$. Moreover by (2) we obtain

$$p(te) < \frac{1}{m} \quad \text{and} \quad \frac{c\eta_p}{2} \cdot p(te) \leq \frac{c}{2} \cdot p(te+tf) < \frac{r}{2}.$$

Thus $y \in P\big(A, E, \frac{c\eta_p}{2}, p, \frac{1}{m}\big)$, which contradicts the fact that y belongs to D_m. □

In the previous Proposition we have proved, in particular, that the set A_1 is $\left(\frac{c\eta_p}{2}, p\right)$-porous in direction E and the set A_2 is σ-$(\mathcal{C}, \mathcal{P})$-porous in direction F, where $\mathcal{C}$ and $\mathcal{P}$ are the constant sequences $\left(\frac{c\eta_p}{2}\right)$ and (p), respectively. Hence the constant of porosity is the same for all points of the sets A_1 and A_2 and it depends on the constant c of porosity of A and on the constant η_p.

Proposition 2.2. *Let $\mathcal{P} = (p_i)_{i\in\mathbb{N}}$ be a sequence of continuous seminorms on X. Moreover let $A \subset X$ be a σ-$\mathcal{P}$-porous set in direction $E + F$, where E and F are two subspaces of X with the property that there exists $\eta_{p_i} > 0$ such that*

$$p_i(e + f) \geq \eta_{p_i} \cdot \max\{p_i(e), p_i(f)\}$$

whenever $e \in E$, $f \in F$ and $p_i(e + f) \neq 0$, for each $i \in \mathbb{N}$. Then $A = A_1 \cup A_2$, where A_1 is σ-$\mathcal{P}$-porous in direction E and A_2 is σ-$\mathcal{P}$-porous in direction F.

Proof. Since A is a σ-$\mathcal{P}$-porous set in direction $E + F$, then $A = \bigcup_{i=1}^{\infty} A_i$, where A_i is p_i-porous in direction $E + F$, $i = 1, 2, \ldots$ By Remark 2.3 it follows that $A_i = \bigcup_{k=1}^{\infty} A_i^k$, where A_i^k is $\left(\frac{1}{k}, p_i\right)$-porous in direction $E + F$, $k = 1, 2, \ldots.$ Moreover, by Proposition 2.1, for each i and k, we have $A_i^k = (A_i^k)_E \cup (A_i^k)_F$, where $(A_i^k)_E$ is p_i-porous in direction E and $(A_i^k)_F$ is a countable union of sets which are p_i-porous in direction F. Therefore $\bigcup_{k=1}^{\infty}(A_i^k)_E$ is p_i-porous in direction E and $\bigcup_{k=1}^{\infty}(A_i^k)_F$ is p_i-porous in direction F. We set

$$A_1 = \bigcup_{i=1}^{\infty}\left(\bigcup_{k=1}^{\infty}(A_i^k)_E\right) \quad \text{and} \quad A_2 = \bigcup_{i=1}^{\infty}\left(\bigcup_{k=1}^{\infty}(A_i^k)_F\right).$$

Then $A = A_1 \cup A_2$, where A_1 is σ-$\mathcal{P}$-porous in direction E and A_2 is σ-$\mathcal{P}$-porous in direction F. □

Lemma 2.3. *Let p be a continuous seminorm on X and let V be the subspace of X spanned by the linearly independent vectors $v_1, v_2, ..., v_n$. If $E =< v_1, v_2, ..., v_{n-1} >$ and $F =< v_n >$, then there exists a constant $\eta_p > 0$ such that*

$$p(e + f) \geq \eta_p \cdot \max\{p(e), p(f)\}$$

for each $e \in E,\ f \in F$.

Proof. Let $\bar{X}$ be the quotient space of X with respect to the p-null set $\{x \in X : \ p(x) = 0\}$. For each $\bar{x} \in \bar{X}$, define $\bar{p}(\bar{x}) := p(x)$. It is easy to see that $\bar{p}$ is a norm on $\bar{X}$. Set $\bar{E} = \{\bar{e} \in \bar{X} : e \in E\}$, $\bar{F} = \{\bar{f} \in \bar{X} : f \in F\}$, and consider the space

$$\bar{E} \oplus \bar{F} = \{\bar{x} \in \bar{X} : \bar{x} = \bar{e} + \bar{f} \text{ with } e \in E \ \ ed \ \ f \in F\}.$$

Now define

$$|\bar{x}| = |\bar{e} + \bar{f}| = \max\{\bar{p}(\bar{e}), \bar{p}(\bar{f})\},$$

for each $\bar{x} = \bar{e} + \bar{f} \in \bar{E} \oplus \bar{F}$. We see that $|\cdot|$ is well defined. Let $\bar{x} = \bar{e_1} + \bar{f_1}$ and $\bar{y} = \bar{e_2} + \bar{f_2} \in \bar{E} \oplus \bar{F}$ such that $\bar{x} = \bar{y}$. Then $\bar{x} - \bar{y} = \bar{0}$; i.e. $\bar{p}\big(\bar{e_1} + \bar{f_1} - \bar{e_2} - \bar{f_2}\big) = 0$. Moreover, since

$$|\bar{p}(\bar{e_1} - \bar{e_2}) - \bar{p}(\bar{f_2} - \bar{f_1})| \leq \bar{p}\big(\bar{e_1} + \bar{f_1} - \bar{e_2} - \bar{f_2}\big) = 0,$$

it follows that $\bar{p}(\bar{e_1} - \bar{e_2}) = \bar{p}(\bar{f_1} - \bar{f_2})$. Hence $\bar{e_1} - \bar{e_2} = \bar{f_1} - \bar{f_2} = \bar{0}$. Therefore $\bar{p}(\bar{e_1} - \bar{e_2}) = 0$ and $\bar{p}(\bar{f_1} - \bar{f_2}) = 0$. It follows that $\bar{p}(\bar{e_1}) = \bar{p}(\bar{e_2})$ and $\bar{p}(\bar{f_1}) = \bar{p}(\bar{f_2})$. Thus $|\bar{x}| = |\bar{y}|$.

Now we see that $|\cdot|$ is subadditive. Let $\bar{x} = \bar{e_1} + \bar{f_1}$ and $\bar{y} = \bar{e_2} + \bar{f_2}$. Then

$$\begin{aligned}
&|\bar{x} + \bar{y}| \\
&\quad = |\bar{e_1} + \bar{f_1} + \bar{e_2} + \bar{f_2}| \\
&\quad = \max\{\bar{p}(\bar{e_1} + \bar{e_2}), \bar{p}(\bar{f_1} + \bar{f_2})\} \\
&\quad \leq \max\{\bar{p}(\bar{e_1}) + \bar{p}(\bar{e_2}), \ \ \bar{p}(\bar{f_1}) + \bar{p}(\bar{f_2})\} \\
&\quad \leq \max\{\bar{p}(\bar{e_1}), \bar{p}(\bar{f_1})\} \ + \ \max\{\bar{p}(\bar{e_2}), \bar{p}(\bar{f_2})\} \\
&\quad = |\bar{x}| \ + \ |\bar{y}|.
\end{aligned}$$

So $|\cdot|$ is a norm on $\bar{E} \oplus \bar{F}$. Since $\bar{E} \oplus \bar{F}$ has finite dimension, there exists a positive constant $\eta_{\bar{p}}$ such that

$$\bar{p}(\bar{e} + \bar{f}) \geq \eta_{\bar{p}} \cdot \max\{\bar{p}(\bar{e}), \bar{p}(\bar{f})\},$$

for each $\bar{e} \in \bar{E}$, $\bar{f} \in \bar{F}$. Moreover

$$p(e+f) = \bar{p}(\overline{e+f}) = \bar{p}(\bar{e}+\bar{f}), \quad p(e) = \bar{p}(\bar{e}) \quad \text{and} \quad p(f) = \bar{p}(\bar{f}),$$

for each $e \in E$ and $f \in F$. Then

$$p(e+f) \geq \eta_p \cdot \max\{p(e), p(f)\},$$

with $\eta_p = \eta_{\bar{p}}$. □

3. Main results.

Proposition 3.1. Let $\mathcal{P} = (p_i)_{i\in\mathbb{N}}$ be a sequence of continuous seminorms on X and let V be the subspace spanned by the vectors $v_1, v_2, ..., v_n$. If $A \subset X$ is σ-$\mathcal{P}$-porous in direction V, then we can write

$$A = \bigcup_{j=1}^{n} \left(A_j^+ \cup A_j^-\right)$$

where A_j^+ and A_j^- are σ-$\mathcal{P}$-porous sets in direction v_j and $-v_j$, respectively.

Proof. We proceed by induction. If $n = 1$, then

$$V = \{tv_1 : t \geq 0\} \cup \{t(-v_1) : t \geq 0\}.$$

As A is σ-$\mathcal{P}$-porous in direction V, then $A = \bigcup_{i=1}^{\infty} A_i$, where A_i is p_i-porous in direction V. So we can write $A_i = A_i^+ \cup A_i^-$, where A_i^+ is p_i-porous in direction $\{tv_1 : t \geq 0\}$ and A_i^- is p_i-porous in direction $\{-tv_1 : t \geq 0\}$ (see Remark 2.1 (r_1)). Hence A_i^+ is p_i-porous in direction $+v_1$ and A_i^- is p_i-porous in direction $-v_1$ (see Remark 2.1 (r_4)). We set $A_1^+ = \bigcup_{i=1}^{\infty} A_i^+$ and $A_1^- = \bigcup_{i=1}^{\infty} A_i^-$. Then

$$A = \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \left(A_i^+ \cup A_i^-\right) = \left(\bigcup_{i=1}^{\infty} A_i^+\right) \cup \left(\bigcup_{i=1}^{\infty} A_i^-\right) = A_1^+ \cup A_1^-.$$

Assume now that $n > 1$ and the statement holds for each $k < n$. Without any loss of generality, we may also assume that $v_1, v_2, ..., v_n$ are linearly independent.

Set $E = < v_1, v_2, ..., v_{n-1} >$ and $F =< v_n >$. Then $V = E + F$, and by Lemma 2.3, there exists $\eta_{p_i} > 0$ such that

$$p_i(e + f) \geq \eta_{p_i} \cdot \max\{p_i(e), p_i(f)\}$$

for each $e \in E$, $f \in F$ and for each $i \in \mathbb{N}$. By Proposition 2.2 it follows that $A = A_E \cup A_F$, where A_E is σ-$\mathcal{P}$-porous in direction $E =< v_1, ..., v_{n-1} >$ and A_F is σ-$\mathcal{P}$-porous in direction $F = < v_n >$. Now, by the induction assumption, we have

$$A_E = \bigcup_{h=1}^{n-1} \left(A_h^+ \cup A_h^-\right)$$

where A_h^+ is σ-$\mathcal{P}$-porous in direction $+v_h$ and A_h^- is σ-$\mathcal{P}$-porous in direction $-v_h$, with $h = 1, ..., n-1$. Moreover $A_F = A_n^+ \cup A_n^-$ where A_n^+ is σ-$\mathcal{P}$-porous in direction $+v_n$ and A_n^- is σ-$\mathcal{P}$-porous in direction $-v_n$. Thus

$$A = A_E \cup A_F = \left(\bigcup_{h=1}^{n-1} (A_h^+ \cup A_h^-)\right) \cup \left(A_n^+ \cup A_n^-\right). \qquad \square$$

Proposition 3.2. *Let p be a continuous seminorm on X and let $V = \overline{< v_1, v_2, ... >}$ be a subspace of X. Moreover let $A \subset X$ be a set with the property that for each point $a \in A$ there exists $v_a \in V$ such that A is p-porous at a in direction v_a. Then we can write*

$$A = \bigcup_{j=1}^{infty} \left(A_j^+ \cup A_j^-\right)$$

where A_j^+ and A_j^- are σ-$\mathcal{P}$-porous in direction v_j and $-v_j$, respectively and $\mathcal{P}$ is the constant sequence (p).

Proof. By hypothesis, for each $a \in A$ there exists $v_a \in V$ such that A is p-porous at a in the direction v_a. Without any loss of generality, we can assume $p(v_a) = 1$. In fact, since $p(v_a) \neq 0$, we can take $\overline{v_a} = \dfrac{v_a}{p(v_a)}$, with $p(\overline{v_a}) = 1$ and, by Remark 2.1 (r_2), A is p-porous at a in direction $\overline{v_a}$.

Let $W_n =< v_1, ..., v_n > \cap \{v \in X : p(v) = 1\}$ and let $A_{k,n}$ be the set of all points $a \in A$ at which A is $\left(\dfrac{1}{k}, p\right)$-porous in the direction v_a

and $\inf_{w\in W_n}\{p(v_a - w)\} < \dfrac{1}{2k}$. Then $A = \bigcup_{k,n=1}^{\infty} A_{k,n}$. In fact it is clear that $\bigcup_{k,n=1}^{\infty} A_{k,n} \subset A$. Assume that $a \in A$. Since A is p-porous at a in the direction v_a, then there exists $k \in \mathbb{N}$ such that A is $\left(\dfrac{1}{k}, p\right)$-porous at a in the direction v_a.

Moreover, since $v_a \in V = \overline{< v_1, v_2, ... >}$, there exists $n \in \mathbb{N}$ such that $\inf_{w\in W_n}\{p(v_a - w)\} < \dfrac{1}{2k}$. Therefore $a \in \bigcup_{k=1}^{\infty} A_{k,n}$. Since $A_{k,n}$ is $\left(\dfrac{1}{k}, p\right)$-porous in the direction v_a, by Lemma 2.1 (b) applied to $W = W_n$, $V = \{v_a\}$, $c = \dfrac{1}{k}$ and $\bar{c} = \dfrac{1}{2k}$, we have that $A_{k,n}$ is $(\dfrac{1}{2k}, p)$-porous in direction W_n. Hence $A_{k,n}$ is $\left(\dfrac{1}{2k}, p\right)$-porous in direction $< v_1, v_2, ..., v_n >$ (see Remark 2.1 (r_3)) and so $\bigcup_{k=1}^{\infty} A_{k,n}$ is σ-$\mathcal{P}$-porous in direction $< v_1, v_2, ..., v_n >$, where $\mathcal{P}$ is the constant sequence (p). By Proposition 3.1 it follows that $\bigcup_{k=1}^{\infty} A_{k,n} = \bigcup_{j=1}^{n} \left(A_j^+ \cup A_j^-\right)$, where A_j^+ is σ-$\mathcal{P}$-porous in the direction $+v_j$ and A_j^- is σ-$\mathcal{P}$-porous in the direction $-v_j$. Thus

$$A = \bigcup_{n=1}^{\infty}\left(\bigcup_{k=1}^{\infty} A_{k,n}\right) = \bigcup_{n=1}^{\infty}\left(\bigcup_{j=1}^{n}(A_j^+ \cup A_j^-)\right) = \bigcup_{j=1}^{\infty}\left(A_j^+ \cup A_j^-\right). \qquad \square$$

Theorem 3.1. *Let X be a separable Fréchet space and let $(v_n)_{n\in\mathbb{N}}$ be a complete sequence in X. Moreover let $\mathcal{P} = (p_i)_{i\in\mathbb{N}}$ be a sequence of continuous seminorms on X. If $A \subset X$ is a σ-$\mathcal{P}$-directionally porous set, then*

$$A = \bigcup_{n=1}^{\infty}\left(A_n^+ \cup A_n^-\right),$$

where A_n^+ and A_n^- are σ-$\mathcal{P}$-porous in direction v_n and $-v_n$, respectively.

Proof. By hypothesis $\overline{< v_1, v_2, ... >} = X$. Since A is a σ-$\mathcal{P}$-directionally porous, then $A = \bigcup_{i=1}^{\infty} A_i$, where the A_i are p_i-directionally porous, for each $i = 1, 2, ...$ Therefore by Proposition 3.2 we have $A_i = \bigcup_{n=1}^{\infty} \left(A_{i,n}^+ \cup A_{i,n}^-\right)$, where $A_{i,n}^+$ is p_i-porous in the direction $+v_n$ and $A_{i,n}^-$ is p_i-porous in the direction $-v_n$.

We set $A_n^+ = \bigcup_{i=1}^{\infty} A_{i,n}^+$ and $A_n^- = \bigcup_{i=1}^{\infty} A_{i,n}^-$. Then

$$A = \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} \left(\left(\bigcup_{n=1}^{\infty} A_{i,n}^+ \right) \cup \left(\bigcup_{n=1}^{\infty} A_{i,n}^- \right) \right) = \bigcup_{n=1}^{\infty} \left(A_n^+ \cup A_n^-\right). \quad \square$$

Remark 3.1. If A is a Borel set, we can require that the sets A_n^+, A_n^- are Borel. Indeed since A_n^+ and A_n^- are σ-$\mathcal{P}$-porous, by Lemma 2.2 (iii) we infer that there exist $G_{\delta\sigma}$ sets $A_n^{+*} \supset A_n^+$ and $A_n^{-*} \supset A_n^-$ which are σ-$\mathcal{P}$-porous. Hence in the above decomposition we can replace A_n^+ and A_n^- by $A_n^{+*} \cap A$ and $A_n^{-*} \cap A$, respectively.

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