*-GRADED CAPELLI POLYNOMIALS AND THEIR ASYMPTOTICS

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Abstract. Let $F\langle Y \cup Z, *\rangle$ be the free $*$-superalgebra over a field F of characteristic zero and let $\Gamma^*_{M^\pm, L^\pm}$ be the $T^*_{\mathbb{Z}_2}$-ideal generated by the set of the $*$-graded Capelli polynomials $Cap^{(\mathbb{Z}_2,*)}_{M^+}[Y^+, X]$, $Cap^{(\mathbb{Z}_2,*)}_{M^-}[Y^-, X]$, $Cap^{(\mathbb{Z}_2,*)}_{L^+}[Z^+, X]$, $Cap^{(\mathbb{Z}_2,*)}_{L^-}[Z^-, X]$ alternating on M^+ symmetric variables of homogeneous degree zero, on M^- skew variables of homogeneous degree zero, on L^+ symmetric variables of homogeneous degree one and on L^- skew variables of homogeneous degree one, respectively. We study the asymptotic behavior of the sequence of $*$-graded codimensions of $\Gamma^*_{M^\pm, L^\pm}$. In particular we prove that the $*$-graded codimensions of the finite dimensional simple $*$-superalgebras are asymptotically equal to the $*$-graded codimensions of $\Gamma^*_{M^\pm, L^\pm}$, for some fixed natural numbers M^+, M^-, L^+ and L^-.

1. Introduction

This paper is devoted to the study of the $*$-superalgebras, i.e. superalgebras endowed with a graded involution, and the asymptotic behavior of their $*$-graded codimensions. If A is an algebra over a field F of characteristic zero an effective way of measuring the polynomial identities satisfied by A is provided by its sequence of codimensions $\{c_n(A)\}_{n\geq 1}$ whose n-th therm is the dimension of the space of multilinear polynomials in n variables in the corresponding relatively free algebra of countable rank. Such sequence was introduced by Regev in [21] and, in characteristic zero, gives a quantitative measure of the identities satisfied by a given algebra. The most important result of the sequence of codimensions proved in [21] states that if A is a PI-algebra, i.e. it satisfies a non trivial polynomial identity, then $\{c_n(A)\}_{n\geq 1}$ is exponential bounded. Later, Giambruno and Zaicev ([14], [15]) answered in a positive way to a well known conjecture of Amitsur proving the existence and the integrality of

$$\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

the exponent of A. These results, in the last years, have been extended to algebras with an additional structure as algebras with involution ([1], [12]), superalgebras ([4]) and more generally algebras graded by a group ([2], [8], [11], [16]), algebras with a generalised H-action ([19]), superalgebras with graded involution ([22]) and superalgebras with superinvolution ([20]).

Let $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$ be a $*$-superalgebra and let $c_n^{(\mathbb{Z}_2,*)}(A)$, $n = 1, 2, \ldots$, be its sequence of $*$-graded codimensions. If A is a PI-algebra it can be easily proved that the relation between codimensions and $*$-graded codimensions is given by $c_n(A) \leq c_n^{(\mathbb{Z}_2,*)}(A) \leq 4^n c_n(A)$. Hence, as in the ordinary case, the sequence of $*$-graded codimensions is exponentially bounded. Moreover, since a $*$-superalgebra can be viewed as an algebra with a generalized FG-action where $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ acts on it by automorphism and antiautomorphism, in

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the finite dimensional case, the existence of the $*$-graded exponent has been confirmed by Gordienko in [19].

Let M^+, M^-, L^+ and L^- be natural numbers and let's denote by $\Gamma^*_{M^\pm,L^\pm}$ the $T^*_{\mathbb{Z}_2}$-ideal generated by the set of the $*$-graded Capelli polynomials $Cap^{(\mathbb{Z}_2,*)}_{M^+}[Y^+,X]$, $Cap^{(\mathbb{Z}_2,*)}_{M^-}[Y^-,X]$, $Cap^{(\mathbb{Z}_2,*)}_{L^+}[Z^+,X]$, $Cap^{(\mathbb{Z}_2,*)}_{L^-}[Z^-,X]$ alternating on M^+ symmetric variables of homogeneous degree zero, on M^- skew variables of homogeneous degree zero, on L^+ symmetric variables of homogeneous degree one and on L^- skew variables of homogeneous degree one, respectively. In this paper we find a relation among the $*$-graded codimensions of the finite dimensional simple $*$-superalgebras and the $*$-graded codimensions of $\Gamma^*_{M^\pm,L^\pm}$ proving their asymptotic equality. Recall that two sequences a_n, b_n, $n = 1, 2, \ldots$, are asymptotically equal, $a_n \simeq b_n$, if $\lim_{n\to+\infty} \frac{a_n}{b_n} = 1$. In the ordinary case (see [17]) it was proved the asymptotic equality between the codimensions of the Capelli polynomials Cap_{k^2+1} and the codimensions of the matrix algebra $M_k(F)$. In [3] this result was extended to finite dimensional simple superalgebras and in [6] the authors found similar result in the case of algebras with involution (for a survey see [7]). The link between the asymptotic of the codimensions of the Amitsur's Capelli-type polynomials and the verbally prime algebras was studied in [5].

2. Preliminaries

Throughout this paper, F will be a field of characteristic zero and A an associative algebra over F. We say that A is a $\mathbb{Z}_2$-*graded algebra* or a *superalgebra* if it can be decomposed into a direct sum of subspaces $A = A_0 \oplus A_1$ such that $A_0A_0 + A_1A_1 \subseteq A_0$ and $A_0A_1 + A_1A_0 \subseteq A_1$. The elements of A_0 are called *homogeneous of degree zero* (*even elements*) and those of A_1 *homogeneous of degree one* (*odd elements*).

Recall that an *involution* $*$ on an algebra A is just an antiautomorphism on A of order at most 2. We write $A^+ = \{a \in A \mid a^* = a\}$ and $A^- = \{a \in A \mid a^* = -a\}$ for the set of *symmetric* and *skew symmetric* elements of A respectively.

Given a superalgebra $A = A_0 \oplus A_1$ endowed with an involution $*$, we say that $*$ is a *graded involution* if it preserves the homogeneous components of A, i.e. if $A_i^* \subseteq A_i$, $i = 0, 1$. A superalgebra endowed with a graded involution is called $*$-*superalgebra*. It is clear that a superalgebra A is a $*$-superalgebra if and only if the subspaces A^+ and A^- are graded subspaces, i.e. $A^+ = A_0^+ \oplus A_1^+$ and $A^- = A_0^- \oplus A_1^-$. Thus, since char F= 0, the $*$-superalgebra A can be written as

$$A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$$

where, for $i = 0, 1$, $A_i^+ = \{a \in A_i \mid a^* = a\}$ and $A_i^- = \{a \in A_i \mid a^* = -a\}$ denote the sets of homogeneous symmetric and skew elements of A_i, respectively. We remark that an algebra with involution $*$ and trivial $\mathbb{Z}_2$-grading is a $*$-superalgebra.

Let A be a $*$-superalgebra and let I be an ideal of A, we say that I is a $*$-*graded ideal* of A if it is homogeneous in the $\mathbb{Z}_2$-grading and invariant under $*$. Moreover A is called *simple* $*$-superalgebra if $A^2 \neq \{0\}$ and it has no non-zero $*$-graded ideals.

Let $X = \{x_1, x_2, \ldots\}$ be a countable set of non commutative variables and $F\langle X\rangle$ the free associative algebra on X over F. We write $X = Y \cup Z$ as the disjoint union of two countable sets of variables $Y = \{y_1, y_2, \ldots\}$ and $Z = \{z_1, z_2, \ldots\}$, then $F\langle X\rangle = F\langle Y \cup Z\rangle = \langle y_1, z_1, y_2, z_2, \ldots\rangle$ has a natural structure of free superalgebra if we require that the variables from Y have degree zero and the variables from Z have degree one. This algebra is said to be the *free superalgebra* over F. Moreover, if we write each set as the disjoint union of two other infinite sets of symmetric and skew elements, respectively, then we obtain the *free* $*$-*superalgebra*

$$F\langle Y \cup Z, *\rangle = F\langle y_1^+, y_1^-, z_1^+, z_1^-, \ldots\rangle$$

where $y_i^+ = y_i + y_i^*$ denotes a symmetric variable of even degree, $y_i^- = y_i - y_i^*$ a skew variable of even degree, $z_i^+ = z_i + z_i^*$ a symmetric variable of odd degree and $z_i^- = z_i - z_i^*$ a skew variable of odd degree.

An element $f = f(y_1^+, \ldots, y_n^+, y_1^-, \ldots, y_m^-, z_1^+, \ldots, z_p^+, z_1^-, \ldots, z_q^-)$ of $F\langle Y \cup Z, *\rangle$ is a $*$-*graded polynomial identity* for a $*$-superalgebra A if

$$f(a_{1,0}^+, \ldots, a_{n,0}^+, a_{1,0}^-, \ldots, a_{m,0}^-, a_{1,1}^+, \ldots, a_{p,1}^+, a_{1,1}^-, \ldots, a_{q,1}^-) = 0_A$$

for every $a_{1,0}^+, \ldots, a_{n,0}^+ \in A_0^+$, $a_{1,0}^-, \ldots, a_{m,0}^- \in A_0^-$, $a_{1,1}^+, \ldots, a_{p,1}^+ \in A_1^+$, $a_{1,1}^-, \ldots, a_{q,1}^- \in A_1^-$ and we write $f \equiv 0$. The set of all $*$-graded polynomial identities satisfied by A

$$Id_{\mathbb{Z}_2}^*(A) = \{f \in F\langle Y \cup Z, *\rangle \mid f \equiv 0 \text{ on } A\}$$

is an ideal of $F\langle Y \cup Z, *\rangle$ called the *ideal of $*$-graded identities of A*. It is easy to show that $Id_{\mathbb{Z}_2}^*(A)$ is a $T_{\mathbb{Z}_2}^*$-*ideal* of $F\langle Y \cup Z, *\rangle$, i.e. a two-sided ideal invariant under all endomorphisms of the free $*$-superalgebra that preserve the superstructure and commute with the graded involution $*$. Now, let

$$P_n^{(\mathbb{Z}_2,*)} = \{w_{\sigma(1)}, \ldots, w_{\sigma(n)} \mid \sigma \in S_n,\ w_i \in \{y_i^+, y_i^-, z_i^+, z_i^-\},\ i = 1, \ldots, n\}$$

be the space of multilinear polynomials of degree n in the variables y_1^+, y_1^-, z_1^+, z_1^-,…, y_n^+, y_n^-, z_n^+, z_n^-, (i.e., y_i^+, y_i^-, z_i^+ or z_i^- appears in each monomial at degree 1). Since char $F=0$, it is well known that $Id_{\mathbb{Z}_2}^*(A)$ is completely determined by its multilinear polynomials, then the study of $Id_{\mathbb{Z}_2}^*(A)$ is equivalent to that of $Id_{\mathbb{Z}_2}^*(A) \cap P_n^{(\mathbb{Z}_2,*)}$ for all $n \geq 1$. As in the ordinary case (see [21]), one defines the *n-th $*$-graded codimension* $c_n^{(\mathbb{Z}_2,*)}(A)$ of the $*$-superalgebra A as

$$c_n^{(\mathbb{Z}_2,*)}(A) = \dim_F \frac{P_n^{(\mathbb{Z}_2,*)}}{P_n^{(\mathbb{Z}_2,*)} \cap Id_{\mathbb{Z}_2}^*(A)}.$$

If A is a PI-algebra, i.e. satisfies an ordinary polynomial identity, then the sequence $\{c_n^{(\mathbb{Z}_2,*)}(A)\}_{n \geq 1}$ is exponentially bounded (see [13, Lemma 3.1]). If A is a finite dimensional PI-algebra, Gordienko in [19] proved that

$$\exp_{\mathbb{Z}_2}^*(A) = \lim_{n \to \infty} \sqrt[n]{c_n^{(\mathbb{Z}_2,*)}(A)}$$

exists and is a non-negative integer which is called the *$*$-graded exponent* of the $*$-superalgebra A. It is often more useful to study $*$-superalgebras up to $*$-graded PI-equivalence, then it is convenient to use the language of varieties. Let I be a $T_{\mathbb{Z}_2}^*$-ideal of $F\langle Y \cup Z, *\rangle$ and $\mathcal{V}_{\mathbb{Z}_2}^*$ the *variety of $*$-superalgebras* associated to I, i.e. the class of all the $*$-superalgebras A such that I is contained in $Id_{\mathbb{Z}_2}^*(A)$. We put $I = Id_{\mathbb{Z}_2}^*(\mathcal{V}_{\mathbb{Z}_2}^*)$. When $Id_{\mathbb{Z}_2}^*(\mathcal{V}_{\mathbb{Z}_2}^*) = Id_{\mathbb{Z}_2}^*(A)$ we say that the variety $\mathcal{V}_{\mathbb{Z}_2}^*$ is generated by the $*$-superalgebra A and we write $\mathcal{V}_{\mathbb{Z}_2}^* = \mathrm{var}_{\mathbb{Z}_2}^*(A)$ and set $\exp_{\mathbb{Z}_2}^*(\mathcal{V}_{\mathbb{Z}_2}^*) = \exp_{\mathbb{Z}_2}^*(A)$ the $*$-graded exponent of the variety $\mathcal{V}_{\mathbb{Z}_2}^*$, if $\exp_{\mathbb{Z}_2}^*(A)$ exists.

Now, if $f \in F\langle Y \cup Z, *\rangle$ we denote by $\langle f\rangle_{\mathbb{Z}_2}^*$ the $T_{\mathbb{Z}_2}^*$-ideal generated by f. Also for a set of polynomials $V \subset F\langle Y \cup Z, *\rangle$ we write $\langle V\rangle_{\mathbb{Z}_2}^*$ to indicate the $T_{\mathbb{Z}_2}^*$-ideal generated by V.

In PI-theory a prominent role is played by the Capelli polynomial. Let us recall that, for any positive integer m, the *m-th Capelli polynomial* is the element of $F\langle X\rangle$ defined as

$$Cap_m[T, X] = Cap_m(t_1, \ldots, t_m; x_1, \ldots, x_{m-1}) =$$
$$= \sum_{\sigma \in S_m} (\mathrm{sgn}\sigma) t_{\sigma(1)} x_1 t_{\sigma(2)} \cdots t_{\sigma(m-1)} x_{m-1} t_{\sigma(m)}$$

where S_m is the symmetric group on $\{1, \ldots, m\}$. In particular we write

$$Cap_m^{(\mathbb{Z}_2,*)}[Y^+,X],\ Cap_m^{(\mathbb{Z}_2,*)}[Y^-,X],\ Cap_m^{(\mathbb{Z}_2,*)}[Z^+,X] \text{ and } Cap_m^{(\mathbb{Z}_2,*)}[Z^-,X]$$

to indicate the *m-th $*$-graded Capelli polynomial* alternating in the symmetric variables of degree zero $y_1^+,\dots,y_m^+$, in the skew variables of degree zero $y_1^-,\dots,y_m^-$, in the symmetric variables of degree one $z_1^+,\dots,z_m^+$ and in the skew variables of degree one $z_1^-,\dots,z_m^-$, respectively ($x_1,\dots,x_{m-1}$ are arbitrary variables). Let $\overline{Cap}_m^{(\mathbb{Z}_2,*)}[Y^+,X]$ denote the set of 2^{m-1} polynomials obtained from $Cap_m^{(\mathbb{Z}_2,*)}[Y^+,X]$ by deleting any subset of variables x_i (by evaluating the variables x_i to 1 in all possible way). In a similar way we define $\overline{Cap}_m^{(\mathbb{Z}_2,*)}[Y^-,X]$, $\overline{Cap}_m^{(\mathbb{Z}_2,*)}[Z^+,X]$ and $\overline{Cap}_m^{(\mathbb{Z}_2,*)}[Z^-,X]$. If M^+, M^-, L^+ and L^- are natural numbers, we denote by

$$\Gamma^*_{M^\pm,L^\pm} = \langle \overline{Cap}_{M^+}^{(\mathbb{Z}_2,*)}[Y^+,X], \overline{Cap}_{M^-}^{(\mathbb{Z}_2,*)}[Y^-,X], \overline{Cap}_{L^+}^{(\mathbb{Z}_2,*)}[Z^+,X], \overline{Cap}_{L^-}^{(\mathbb{Z}_2,*)}[Z^-,X]\rangle^*_{\mathbb{Z}_2}$$

the $T^*_{\mathbb{Z}_2}$-ideal generated by $\overline{Cap}_{M^+}^{(\mathbb{Z}_2,*)}[Y^+,X]$, $\overline{Cap}_{M^-}^{(\mathbb{Z}_2,*)}[Y^-,X]$, $\overline{Cap}_{L^+}^{(\mathbb{Z}_2,*)}[Z^+,X]$ and $\overline{Cap}_{L^-}^{(\mathbb{Z}_2,*)}[Z^-,X]$.

The purpose of this paper is to find a close relation among the asymptotic behavior of the $*$-graded codimensions of any finite dimensional simple $*$-superalgebra $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$ and the asymptotic behavior of the $*$-graded codimensions of $\Gamma^*_{M^\pm+1,L^\pm+1}$, where $M^+ = \dim_F A_0^+$, $M^- = \dim_F A_0^-$, $L^+ = \dim_F A_1^+$ and $L^- = \dim_F A_1^-$. More precisely, we characterize the $T^*_{\mathbb{Z}_2}$-ideal $Id^*_{\mathbb{Z}_2}(A)$ showing that

$$\Gamma^*_{M^\pm+1,L^\pm+1} = Id^*_{\mathbb{Z}_2}(A\oplus D),$$

where D is a finite dimensional $*$-superalgebra such that $\exp^*_{\mathbb{Z}_2}(D) < \exp^*_{\mathbb{Z}_2}(A)$. Moreover we obtain the asymptotic equality

$$c_n^{(\mathbb{Z}_2,*)}(\Gamma^*_{M^\pm+1,L^\pm+1}) \simeq c_n^{(\mathbb{Z}_2,*)}(A).$$

3. Basic Results

Let A be a finite dimensional $*$-superalgebra over a field F of characteristic zero. From now on we assume that F is algebraically closed. In fact, since F has characteristic zero, $Id^*_{\mathbb{Z}_2}(A) = Id^*_{\mathbb{Z}_2}(A\otimes_F L)$ for any extension field L of F then also the $*$-graded codimensions of A do not change upon extension of the base field. By the generalization of the Wedderburn-Malcev Theorem (see [13, Theorem 7.3]), we can write $A = A_1 \oplus \cdots \oplus A_s + J$, where $A_1,\dots,A_s$ are simple $*$-superalgebras and $J = J(A)$ is the Jacobson radical of A which is a $*$-graded ideal.

We say that a subalgebra $A_{i_1}\oplus\cdots\oplus A_{i_k}$ of A, where $A_{i_1},\dots,A_{i_k}$ are distinct simple components, is *admissible* if for some permutation $(l_1,\dots,l_k)$ of $(i_1,\dots,i_k)$ we have that $A_{l_1}J\cdots JA_{l_k} \neq 0$. Moreover, if $A_{i_1}\oplus\cdots\oplus A_{i_k}$ is an admissible subalgebra of A then $A' = A_{i_1}\oplus\cdots\oplus A_{i_k} + J$ is called a *reduced* algebra.

The notion of admissible $*$-superalgebra is closely linked to that of $*$-graded exponent in fact, in [19], Gordienko proved that $\exp^*_{\mathbb{Z}_2}(A) = d$ where d is the maximal dimension of an admissible subalgebra of A. It follows immediately that

Remark 1. *If A is a simple $*$-superalgebra then* $\exp^*_{\mathbb{Z}_2}(A) = \dim_F A$.

By [10, Theorem 5.3] the Gordienko's result on the existence of the $*$-graded exponent can be actually extended to any finitely generated PI-$*$-superalgebra since it satisfies the same $*$-graded polynomial identities of a finite-dimensional $*$-superalgebra.

In [17] it was showed that reduced superalgebras are building blocks of any proper variety. Here we obtain the analogous result for varieties of $*$-superalgebras.

Let's first start with the following

Lemma 1. *Let A and B be $*$-superalgebras satisfying an ordinary polynomial identity. Then*

$$c_n^{(\mathbb{Z}_2,*)}(A), c_n^{(\mathbb{Z}_2,*)}(B) \leq c_n^{(\mathbb{Z}_2,*)}(A \oplus B) \leq c_n^{(\mathbb{Z}_2,*)}(A) + c_n^{(\mathbb{Z}_2,*)}(B).$$

If A and B are finitely generated $*$-superalgebras, then

$$\exp^*_{\mathbb{Z}_2}(A \oplus B) = max\{\exp^*_{\mathbb{Z}_2}(A), \exp^*_{\mathbb{Z}_2}(B)\}.$$

Proof. The proof is the same of the proof of the Lemma 1 in [17].

We have the following

Theorem 1. *Let A be a finitely generated $*$-superalgebra satisfying an ordinary polynomial identity. Then there exists a finite number of reduced $*$-superalgebras $B_1, \dots, B_t$ and a finite dimensional $*$-superalgebra D such that*

$$\mathrm{var}^*_{\mathbb{Z}_2}(A) = \mathrm{var}^*_{\mathbb{Z}_2}(B_1 \oplus \cdots \oplus B_t \oplus D)$$

with $\exp^*_{\mathbb{Z}_2}(A) = \exp^*_{\mathbb{Z}_2}(B_1) = \cdots = \exp^*_{\mathbb{Z}_2}(B_t)$ *and* $\exp^*_{\mathbb{Z}_2}(D) < \exp^*_{\mathbb{Z}_2}(A)$.

Proof. The proof follows closely the proof given in [3, Theorem 3]. Since A is a finitely generated $*$-superalgebra, by [10], there exists a finite dimensional $*$-superalgebra B such that $Id^*_{Z_2}(A) = Id^*_{Z_2}(B)$. Therefore we may assume that $A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$ is a finite dimensional $*$-superalgebra over F satisfying an ordinary polynomial identity. Also, by [13, Theorem 7.3] we can write

$$A = A_1 \oplus \cdots \oplus A_s + J$$

where $A_1, \dots A_s$ are simple $*$-superalgebras and $J = J(A)$ is the Jacobson radical of A which is a $*$-graded ideal. Let $\exp^*_{\mathbb{Z}_2}(A) = d$. Then there exist distinct simple $*$-superalgebras $A_{j_1}, \dots A_{j_k}$ such that

$$A_{j_1} J \cdots J A_{j_k} \neq 0 \quad \text{and} \quad \dim_F(A_{j_1} \oplus \cdots \oplus A_{j_k}) = d.$$

Let $\Gamma_1, \dots, \Gamma_t$ be all possible subset of $\{1, \dots, s\}$ such that, if $\Gamma_j = \{j_1, \dots, j_k\}$ then $\dim_F(A_{j_1} \oplus \cdots \oplus A_{j_k}) = d$ and $A_{\sigma(j_1)} J \cdots J A_{\sigma(j_k)} \neq 0$ for some permutation $\sigma \in S_k$. For any such Γ_j, $j = 1, \dots, t$, then we put $B_j = A_{j_1} \oplus \cdots \oplus A_{j_k} + J$. It follows, by the characterization of the $*$-graded exponent, that

$$\exp^*_{\mathbb{Z}_2}(B_1) = \cdots = \exp^*_{\mathbb{Z}_2}(B_t) = d = \exp^*_{\mathbb{Z}_2}(A).$$

Let $D = D_1 \oplus \cdots \oplus D_p$, where $D_1, \dots, D_p$ are all $*$-graded subalgebras of A of the type $A_{i_1} \oplus \cdots \oplus A_{i_r} + J$, with $1 \leq i_1 < \cdots < i_r \leq s$ and $\dim_F(A_{i_1} \oplus \cdots \oplus A_{i_r}) < d$. Then, by the previous lemma, we have that $\exp^*_{\mathbb{Z}_2}(D) < \exp^*_{\mathbb{Z}_2}(A)$. Now, we want to prove that $\exp^*_{\mathbb{Z}_2}(B_1 \oplus \cdots \oplus B_t \oplus D) = \exp^*_{\mathbb{Z}_2}(A)$. The inclusion

$$\mathrm{var}^*_{\mathbb{Z}_2}(B_1 \oplus \cdots \oplus B_t \oplus D) \subseteq \mathrm{var}^*_{\mathbb{Z}_2}(A)$$

follows since $D, B_i \in \mathrm{var}^*_{\mathbb{Z}_2}(A)$, $\forall i = 1, \dots, t$.

Let's consider a multilinear polynomial $f = f(y_1^+, \dots, y_n^+, y_1^-, \dots, y_m^-, z_1^+, \dots, z_p^+, z_1^-, \dots, z_q^-)$ such that $f \notin Id^*_{\mathbb{Z}_2}(A)$. We shall prove that $f \notin Id^*_{\mathbb{Z}_2}(B_1 \oplus \cdots \oplus B_t \oplus D)$. Since $f \notin Id^*_{\mathbb{Z}_2}(A)$

there exist $a^+_{1,0},\dots,a^+_{n,0}\in A^+_0$, $a^-_{1,0},\dots,a^-_{m,0}\in A^-_0$, $a^+_{1,1},\dots,a^+_{p,1}\in A^+_1$ and $a^-_{1,1},\dots,a^-_{q,1}\in A^-_1$ such that

$$f(a^+_{1,0},\dots,a^+_{n,0},a^-_{1,0},\dots,a^-_{m,0},a^+_{1,1},\dots,a^+_{p,1},a^-_{1,1},\dots,a^-_{q,1})\neq 0.$$

From the linearity of f we can assume that $a^+_{i,0},a^-_{i,0},a^+_{i,1},a^-_{i,1}\in A_1\cup\dots\cup A_s\cup J$. Since $A_iA_j=0$ for $i\neq j$, from the property of the $*$-graded exponent we have

$$a^+_{1,0},\dots,a^+_{n,0},a^-_{1,0},\dots,a^-_{m,0},a^+_{1,1},\dots,a^+_{p,1},a^-_{1,1},\dots,a^-_{q,1}\in A_{j_1}\oplus\dots\oplus A_{j_k}+J$$

for some $A_{j_1},\dots,A_{j_k}$ such that $\dim_F(A_{j_1}\oplus\dots\oplus A_{j_k})\leq d$. Thus f is not an identity for one of the algebras $B_1,\dots,B_t,D$. Hence $f\notin Id^*_{\mathbb{Z}_2}(B_1\oplus\dots\oplus B_t\oplus D)$. In conclusion

$$\mathrm{var}^*_{\mathbb{Z}_2}(A)\subseteq \mathrm{var}^*_{\mathbb{Z}_2}(B_1\oplus\dots\oplus B_t\oplus D)$$

and the proof is complete.

An application of Theorem 1 is given in terms of $*$-graded codimensions.

Corollary 1. *Let A be a finitely generated PI-$*$-superalgebra. Then there exists a finite number of reduced $*$-superalgebras $B_1,\dots,B_t$ such that*

$$c_n^{(\mathbb{Z}_2,*)}(A)\simeq c_n^{(\mathbb{Z}_2,*)}(B_1\oplus\dots\oplus B_t)$$

Proof. By Theorem 1 there is a finite number of reduced $*$-superalgebras $B_1,\dots,B_t$ and a finite dimensional $*$-superalgebra D such that

$$\mathrm{var}^*_{\mathbb{Z}_2}(A)=\mathrm{var}^*_{\mathbb{Z}_2}(B_1\oplus\dots\oplus B_t\oplus D)$$

with $\exp^*_{\mathbb{Z}_2}(A)=\exp^*_{\mathbb{Z}_2}(B_1)=\dots=\exp^*_{\mathbb{Z}_2}(B_t)$ and $\exp^*_{\mathbb{Z}_2}(D)<\exp^*_{\mathbb{Z}_2}(A)$. By Lemma 1,

$$c_n^{(\mathbb{Z}_2,*)}(B_1\oplus\dots\oplus B_t)\leq c_n^{(\mathbb{Z}_2,*)}(B_1\oplus\dots\oplus B_t\oplus D)\leq c_n^{(\mathbb{Z}_2,*)}(B_1\oplus\dots\oplus B_t)+c_n^{(\mathbb{Z}_2,*)}(D).$$

Recalling that $\exp^*_{\mathbb{Z}_2}(D)<\exp^*_{\mathbb{Z}_2}(B_1)=\exp^*_{\mathbb{Z}_2}(B_1\oplus\dots\oplus B_t)$ we have that

$$c_n^{(\mathbb{Z}_2,*)}(A)\simeq c_n^{(\mathbb{Z}_2,*)}(B_1\oplus\dots\oplus B_t)$$

and the proof of the corollary is complete.

The following results give us a characterization of the varieties of $*$-superalgebras satisfying a Capelli identity. Let's start with the following lemma

Lemma 2. *Let M^+, M^-, L^+ and L^- be natural numbers. If A is a $*$-superalgebra satisfying the $*$-graded Capelli polynomials $Cap^{(\mathbb{Z}_2,*)}_{M^+}[Y^+,X]$, $Cap^{(\mathbb{Z}_2,*)}_{M^-}[Y^-,X]$, $Cap^{(\mathbb{Z}_2,*)}_{L^+}[Z^+,X]$ and $Cap^{(\mathbb{Z}_2,*)}_{L^-}[Z^-,X]$, then A satisfies the Capelli identity $Cap_k(x_1,\dots,x_k;\bar{x}_1,\dots,\bar{x}_{k-1})$, where $k=M^++M^-+L^++L^-$.*

Proof. Let $k=M^++M^-+L^++L^-$, then we obtain immediately the thesis if we observe that

$$Cap_k(x_1,\dots,x_k;\bar{x}_1,\dots,\bar{x}_{k-1})=$$
$$Cap_k(\frac{y_1^++y_1^-}{2}+\frac{z_1^++z_1^-}{2},\dots,\frac{y_k^++y_k^-}{2}+\frac{z_k^++z_k^-}{2};\bar{x}_1,\dots,\bar{x}_{k-1})$$

is a linear combinations of $*$-graded Capelli polynomials alternating or in $m^+\geq M^+$ symmetric variables of zero degree, or in $m^-\geq M^-$ skew variables of zero degree, or in $l^+\geq L^+$ symmetric variables of one degree or in $l^-\geq L^-$ skew variables of one degree.

Theorem 2. *Let $\mathcal{V}^*_{\mathbb{Z}_2}$ be a variety of $*$-superalgebras. If $\mathcal{V}^*_{\mathbb{Z}_2}$ satisfies the Capelli identity of some rank, then $\mathcal{V}^*_{\mathbb{Z}_2}=\mathrm{var}^*_{\mathbb{Z}_2}(A)$, for some finitely generated $*$-superalgebra A.*

Proof. The proof follows very closely the proof given in [18, Theorem 11.4.3] for superalgebras.

4. The $*$-superalgebra $UT^*_{\mathbb{Z}_2}(A_1,\dots,A_m)$

In this section we recall the construction of the $*$-superalgebra $UT^*_{\mathbb{Z}_2}(A_1,\dots,A_m)$ given in section 3 of [9] and we investigate the link between the degrees of the $*$-graded Capelli polynomials and the $*$-graded identities of this $*$-superalgebra.

If F is an algebraically closed field of characteristic zero, then, up to graded isomorphisms, the only finite dimensional simple $*$-superalgebras are the following (see [13, Theorem 7.6])

(1) $(M_{h,l},\diamond)$, with $h\geq l\geq 0$, $h\neq 0$;
(2) $(M_{h,l}\oplus M_{h,l}^{op}, exc)$, with $h\geq l\geq 0$, $h\neq 0$, and induced grading;
(3) $(M_n+cM_n,\star)$, with involution given by $(a+cb)^\star=a^\diamond-cb^\diamond$;
(4) $(M_n+cM_n,\dagger)$, with involution given by $(a+cb)^\dagger=a^\diamond+cb^\diamond$;
(5) $((M_n+cM_n)\oplus(M_n+cM_n)^{op}, exc)$, with grading $(M_n\oplus M_n^{op}, c(M_n\oplus M_n^{op}))$;

where $\diamond=t,s$ denotes the transpose or symplectic involution and exc is the exchange involution. Remember that the symplectic involution can occur only when $h=l$. Moreover $M_h=M_h(F)$ is the superalgebra of $h\times h$ matrices over F with trivial grading, $M_{h,l}=M_{h+l}(F)$ is the superalgebra with grading $\left(\begin{pmatrix} F_{11} & 0\\ 0 & F_{22}\end{pmatrix},\begin{pmatrix} 0 & F_{12}\\ F_{21} & 0\end{pmatrix}\right)$, where F_{11}, F_{12}, F_{21}, F_{22} are $h\times h$, $h\times l$, $l\times h$ and $l\times l$ matrices respectively, $h\geq l\geq 0$, $h\neq 0$ and $M_n+cM_n=M_n(F\oplus cF)$ denotes the simple superalgebra with grading $(M_n(F),cM_n(F))$, where $c^2=1$.

Let $(A_1,\dots,A_m)$ be a m-tuple of finite dimensional simple $*$-superalgebras. For every $k=1,\dots,m$, the size of A_k is given by

$$s_k=\begin{cases} h_k+l_k & \text{if } A_k=M_{h_k,l_k} \text{ or } A_k=M_{h_k,l_k}\oplus M^{op}_{h_k,l_k};\\ 2n_k & \text{if } A_k=M_{n_k}+cM_{n_k} \text{ or } A_k=(M_{n_k}+cM_{n_k})\oplus(M_{n_k}+cM_{n_k})^{op}\end{cases}$$

and, set $\eta_0=0$, let $\eta_k=\Sigma_{i=1}^k s_i$ and $\mathrm{B}l_k=\{\eta_{k-1}+1,\dots,\eta_k\}$. Moreover, we denote by γ_m the orthogonal involution defined on the matrix algebra M_m by sending each $a\in M_m$ into the element $a^{\gamma_m}\in M_m$ obtained reflecting a along its secondary diagonal. In particular for any matrix unit $e_{i,j}$ of M_m, $e_{i,j}^{\gamma_m}=e_{m-j+1,m-i+1}$.

Then, we have a monomorphism of $*$-algebra

$$\Delta:\bigoplus_{k=1}^m A_k\to(M_{2\eta_m},\gamma_{2\eta_m})$$

defined by

$$(c_1,\dots,c_m)\to\begin{pmatrix} \bar a_1 &&&&& \\ & \ddots &&&& \\ && \bar a_m &&& \\ &&& \bar b_m && \\ &&&& \ddots & \\ &&&&& \bar b_1 \end{pmatrix}$$

where the elements $\bar a_i$ and $\bar b_i$ are defined as follows:

- if $c_i\in(M_{h,l};\diamond)$, then $\bar a_i=c_i$ and $\bar b_i=(c_i^\diamond)^{\gamma_{h+l}}$;

• if $c_i = (a_i, b_i) \in (M_{h,l} \oplus M_{h,l}^{op}, exc)$, then $\bar{a}_i = a_i$ and $\bar{b}_i = b_i^{\gamma_{h+l}}$;

• if $c_i = a_i + cb_i \in (M_n + cM_n, \star)$, then $\bar{a}_i = \begin{pmatrix} a_i & b_i \\ b_i & a_i \end{pmatrix}$ and $\bar{b}_i = (\bar{a}_i^{\perp})^{\gamma_{2n}}$ where $\begin{pmatrix} x & y \\ y & x \end{pmatrix}^{\perp} = \begin{pmatrix} x^{\diamond} & -y^{\diamond} \\ -y^{\diamond} & x^{\diamond} \end{pmatrix}$;

• if $c_i = a_i + cb_i \in (M_n + cM_n, \dagger)$, then $\bar{a}_i = \begin{pmatrix} a_i & b_i \\ b_i & a_i \end{pmatrix}$ and $\bar{b}_i = (\bar{a}_i^{\top})^{\gamma_{2n}}$ where $\begin{pmatrix} x & y \\ y & x \end{pmatrix}^{\top} = \begin{pmatrix} x^{\diamond} & y^{\diamond} \\ y^{\diamond} & x^{\diamond} \end{pmatrix}$;

• if $c_i = (a_i + cb_i, u_i + cv_i) \in ((M_n + cM_n) \oplus (M_n + cM_n)^{op}, exc)$, then $\bar{a}_i = \begin{pmatrix} a_i & b_i \\ b_i & a_i \end{pmatrix}$ and $\bar{b}_i = \begin{pmatrix} u_i & v_i \\ v_i & u_i \end{pmatrix}^{\gamma_{2n}}$.

Let denote by $D \subseteq (M_{2\eta_m}, \gamma_{2\eta_m})$ the $*$-algebra image of $\bigoplus_{i=1}^m A_i$ by Δ and set

$$V = \begin{pmatrix} 0 & V_{12} & \cdots & V_{1m} & & & & \\ & \ddots & \ddots & \vdots & & & & \\ & & 0 & V_{m-1m} & & & & \\ & & & 0 & & & & \\ & & & & 0 & V_{mm-1} & \cdots & V_{m1} \\ & & & & & \ddots & \ddots & \vdots \\ & & & & & & 0 & V_{21} \\ & & & & & & & 0 \end{pmatrix} \subseteq M_{2\eta_m}$$

where, for $1 \le i, j \le m$, $i \ne j$, $V_{ij} = M_{s_i \times s_j} = M_{s_i \times s_j}(F)$ is the algebra of $s_i \times s_j$ matrices of F. Let define

$$UT^*(A_1, \ldots, A_m) = D \oplus V \subseteq M_{2\eta_m}.$$

It is easy to see that $UT^*(A_1, \ldots, A_m)$ is a subalgebra with involution of $(M_{2\eta_m}(F), \gamma_{2\eta_m})$ whose Jacobson radical coincides with V.

Now, for any m-tuple $\tilde{g} = (g_1, \ldots, g_m) \in \mathbb{Z}_2^m$, we consider the map

$$\alpha_{\tilde{g}} : \{1, \ldots, 2\eta_m\} \to \mathbb{Z}_2, \ \ i \to \begin{cases} \alpha_k(i - \eta_{k-1}) + g_k & 1 \le i \le \eta_m; \\ \alpha_k(2\eta_m - i + 1 - \eta_{k-1}) + g_k & \eta_m + 1 \le i \le 2\eta_m. \end{cases}$$

where $k \in \{1, \ldots, m\}$ is the (unique) integer such that $i \in \mathrm{B}l_k$ and α_k's are maps so defined:
· if $A_k \simeq M_{h,l}$ or $A_k \simeq M_{h,l} \oplus M_{h,l}$, then

$$\alpha_k : \{1, \ldots, h + l\} \to \mathbb{Z}_2, \ \ \alpha_k(i) = \begin{cases} 0 & 1 \le i \le h; \\ 1 & h + 1 \le i \le h + l. \end{cases}$$

· if $A_k \simeq M_n + cM_n$ or $A_k \simeq (M_n + cM_n) \oplus (M_n + cM_n)$, then

$$\alpha_k : \{1, \ldots, 2n\} \to \mathbb{Z}_2, \ \ \alpha_k(i) = \begin{cases} 0 & 1 \le i \le n; \\ 1 & n + 1 \le i \le 2n. \end{cases}$$

The map $\alpha_{\tilde{g}}$ induces an elementary grading on $UT^*(A_1, \ldots, A_m)$ with respect to which $\gamma_{2\eta_m}$ is a graded involution. We shall use the symbol

$$UT^*_{\mathbb{Z}_2, \tilde{g}}(A_1, \ldots, A_m)$$

to indicate the $*$-superalgebra defined by the m-tuple $\tilde{g}$. We observe that the k-th simple component of the maximal semisimple $*$-graded subalgebra of this $*$-superalgebra is isomorphic to A_k. When convenient, any such $*$-superalgebra is simply denoted by

$$UT^*_{\mathbb{Z}_2}(A_1, \dots, A_m).$$

In the next lemma we establish the link between the degrees of the $*$-graded Capelli polynomials and the $*$-graded polynomial identities of $UT^*_{\mathbb{Z}_2,\tilde{g}}(A_1, \dots, A_m)$. For all $i = 1, \dots, m$, we write

$$A_i = A^+_{i,0} \oplus A^-_{i,0} \oplus A^+_{i,1} \oplus A^-_{i,1}.$$

Let $(d^\pm_0)_i = \dim_F A^\pm_{i,0}$ and $(d^\pm_1)_i = \dim_F A^\pm_{i,1}$, if we set $d^\pm_0 := \sum_{i=1}^m (d^\pm_0)_i$ and $d^\pm_1 := \sum_{i=1}^m (d^\pm_1)_i$, then we have the following

Lemma 3. *Let $\tilde{g} = (g_1, \dots, g_m)$ be a fixed element of $\mathbb{Z}_2^m$ and $A = UT^*_{\mathbb{Z}_2,\tilde{g}}(A_1, \dots, A_m)$, with A_i finite dimensional simple $*$-superalgebra. Let $0 < \bar{m} \le m$ denote the number of the finite dimensional simple $*$-superalgebras with trivial grading.*

1. *If $\bar{m} = 0$, $Cap^{(\mathbb{Z}_2,*)}_{q^+}[Y^+, X]$, $Cap^{(\mathbb{Z}_2,*)}_{q^-}[Y^-, X]$, $Cap^{(\mathbb{Z}_2,*)}_{k^+}[Z^+, X]$ and $Cap^{(\mathbb{Z}_2,*)}_{k^-}[Z^-, X]$ are in $Id^*_{\mathbb{Z}_2}(A)$ if and only if $q^+ \ge d^+_0 + m$, $q^- \ge d^-_0 + m$, $k^+ \ge d^+_1 + m$ and $k^- \ge d^-_1 + m$;*
2. *If $0 < \bar{m} \le m$, let $\tilde{m}$ be the number of blocks of consecutive $*$-superalgebras with trivial grading that appear in $(A_1, \dots, A_m)$. Then $Cap^{(\mathbb{Z}_2,*)}_{q^+}[Y^+, X]$, $Cap^{(\mathbb{Z}_2,*)}_{q^-}[Y^-, X]$, $Cap^{(\mathbb{Z}_2,*)}_{k^+}[Z^+, X]$ and $Cap^{(\mathbb{Z}_2,*)}_{k^-}[Z^-, X]$ are in $Id^*_{\mathbb{Z}_2}(A)$ if and only if $q^+ > d^+_0 + (m - \bar{m}) + (\tilde{m} - 1) + r_0$, $q^- > d^-_0 + (m - \bar{m}) + (\tilde{m} - 1) + r_0$, $k^+ > d^+_1 + (m - \bar{m}) + (\tilde{m} - 1) + r_1$ and $k^- > d^-_1 + (m - \bar{m}) + (\tilde{m} - 1) + r_1$, where r_0, r_1 are two non negative integers depending on the grading $\tilde{g}$, with $r_0 + r_1 = \bar{m} - \tilde{m}$.*

Proof. We will prove the statement only for $Cap^{(\mathbb{Z}_2,*)}_{q^+}[Y^+, X]$ the $*$-graded Capelli polynomial alternating on q^+ symmetric variables of degree zero since on the other cases the proofs are similar.

1. Let $\bar{m} = 0$. To prove the necessary condition of the statement for the symmetric variables of degree zero it is sufficient to prove that $Cap^{(\mathbb{Z}_2,*)}_{q^+}[Y^+, X]$ is not in $Id^*_{\mathbb{Z}_2}(A)$ when $q^+ = d^+_0 + m - 1$.

We start considering separately the components A_i of A. In each $*$-superalgebra A_i we can take $(d^+_0)_i$ symmetric elements of homogeneous degree zero

$$S_i = \{s_{\alpha_{i-1}+i}, \dots, s_{\alpha_i+i-1}\}$$

for $i = 1, \dots, m$, where $\alpha_0 = 0$ and $\alpha_i = \sum_{j=0}^i (d^+_0)_j$ and a set of elements of A_i

$$U_i = \{a_{\alpha_{i-1}+i}, \dots, a_{\alpha_i+i-2}\}$$

such that

$$Cap^{(\mathbb{Z}_2,*)}_{(d^+_0)_i}(s_{\alpha_{i-1}+i}, \dots, s_{\alpha_i+i-1}; a_{\alpha_{i-1}+i}, \dots, a_{\alpha_i+i-2}) =$$

$$\begin{cases} e_{r_i,s_i} & \text{if } (M_{h_i,l_i}, \diamond); \\ (e_{r_i,s_i}, 0) & \text{if } (M_{h_i,l_i} \oplus M^{op}_{h_i,l_i}, exc); \\ e_{r_i,s_i} & \text{if } (M_{n_i} + cM_{n_i}, \star) \text{ or } (M_{n_i} + cM_{n_i}, \dagger); \\ ((e_{r_i,s_i}, 0), (0,0)) & \text{if } ((M_{n_i} + cM_{n_i}) \oplus (M_{n_i} + cM_{n_i})^{op}, exc), \end{cases}$$

where $\diamond = t, s$ denotes the transpose or symplectic involution, exc is the exchange involution, $(a + cb)^\star = a^\diamond - cb^\diamond$ and $(a + cb)^\dagger = a^\diamond + cb^\diamond$.

For any $1 \leq i \leq m$, if ϕ_i is the $*$-embedding of A_i in A, then let

$$\bar{S}_i = \{\bar{s}_{\alpha_{i-1}+i}, \dots, \bar{s}_{\alpha_i+i-1}\}$$

and

$$\bar{U}_i = \{\bar{a}_{\alpha_{i-1}+i}, \dots, \bar{a}_{\alpha_i+i-2}\}$$

denote the images of S_i and U_i by ϕ_i, respectively.

Let observe that in A we can consider appropriate symmetric elements of homogeneous degree zero in J_0^+

$$\bar{s}_{\alpha_i+i} = e_{h,k} + e_{h,k}^*$$

and elementary matrices of A

$$\bar{a}_{\alpha_i+i-1} = e_{s_i,h} \text{ and } \bar{a}_{\alpha_i+i} = e_{k,r_{i+1}}$$

such that

$$Cap^{(\mathbb{Z}_2,*)}_{(d_0^+)_i}(\bar{s}_{\alpha_{i-1}+i}, \dots, \bar{s}_{\alpha_i+i-1}; \bar{a}_{\alpha_{i-1}+i}, \dots, \bar{a}_{\alpha_i+i-2})\bar{a}_{\alpha_i+i-1}\bar{s}_{\alpha_i+i}\bar{a}_{\alpha_i+i}$$

$$Cap^{(\mathbb{Z}_2,*)}_{(d_0^+)_{i+1}}(\bar{s}_{\alpha_i+(i+1)}, \dots, \bar{s}_{\alpha_{i+1}+i}; \bar{a}_{\alpha_i+(i+1)}, \dots, \bar{a}_{\alpha_{i+1}+(i-1)}) \neq 0.$$

From now on, we will put $Cap^{(\mathbb{Z}_2,*)}_{(d_0^+)_i} = Cap^{(\mathbb{Z}_2,*)}_{(d_0^+)_i}(\bar{s}_{\alpha_{i-1}+i}, \dots, \bar{s}_{\alpha_i+i-1}; \bar{a}_{\alpha_{i-1}+i}, \dots, \bar{a}_{\alpha_i+i-2})$. It follows that

$$Cap^{(\mathbb{Z}_2,*)}_{q^+}(\bar{s}_1, \dots, \bar{s}_{\alpha_m+(m-1)}; \bar{a}_1, \dots, \bar{a}_{\alpha_m+(m-2)}) =$$

$$Cap^{(\mathbb{Z}_2,*)}_{(d_0^+)_1}\bar{a}_{\alpha_1}\bar{s}_{\alpha_1+1}\bar{a}_{\alpha_1+1}Cap^{(\mathbb{Z}_2,*)}_{(d_0^+)_2}\cdots\cdots Cap^{(\mathbb{Z}_2,*)}_{(d_0^+)_{m-1}}\bar{a}_{\alpha_{m-1}}\bar{s}_{\alpha_{m-1}+1}\bar{a}_{\alpha_{m-1}+1}Cap^{(\mathbb{Z}_2,*)}_{(d_0^+)_m} \neq 0.$$

Conversely, let $q^+ \geq d_0^+ + m$. We observe that any monomial of elements of A containing at least m elements of J_0^+ must be zero. Then we claim that any multilinear polynomial $\tilde{f} = \tilde{f}(y_1, \dots, y_{d_0^+ + m}; x_1, x_2, \dots)$ alternating on $d_0^+ + m$ symmetric variables of degree zero must vanish in A. In fact, by multilinearity, we can consider only substitutions $\varphi : y_i^+ \to \bar{s}_i$, $x_i \to \bar{a}_i$ such that $\bar{s}_i \in D_0^+ \cup J_0^+$ for $1 \leq i \leq d_0^+ + m$.

However, since $\dim_F D_0^+ = d_0^+$, if we substitute at least $d_0^+ + 1$ variables in elements of D_0^+ the polynomial vanishes. On the other hands, if we substitute at least m elements of J_0^+, we also get that $\tilde{f}$ vanishes in A. The outcome of this is that A satisfies $Cap^{(\mathbb{Z}_2,*)}_{d_0^+ + m}[Y^+, X]$ and so $Cap^{(\mathbb{Z}_2,*)}_{q^+}[Y^+, X]$, with $q^+ \geq d_0^+ + m$.

2. First let assume that $\bar{m} = m$. We recall that

$$UT^*(A_1, \dots, A_m) = D \oplus V \subseteq M_{2\eta_m},$$

where $D \subseteq (M_{2\eta_m}, \gamma_{2\eta_m})$ the $*$-algebra image of $\bigoplus_{i=1}^m A_i$ by Δ and

$$V = \begin{pmatrix} 0 & V_{12} & \cdots & V_{1m} & & & & \\ & \ddots & \ddots & \vdots & & & & \\ & & 0 & V_{m-1m} & & & & \\ & & & 0 & & & & \\ & & & & 0 & V_{mm-1} & \cdots & V_{m1} \\ & & & & & \ddots & \ddots & \vdots \\ & & & & & & 0 & V_{21} \\ & & & & & & & 0 \end{pmatrix} \subseteq M_{2\eta_m}$$

Notice that, for a fixed $\tilde{g} = (g_1, \dots, g_m) \in \mathbb{Z}_2^m$, if $g_i = g_j$, $1 \leq i, j \leq m$, then the elements of the blocks $V_{i,j}$ are homogeneous of degree zero, otherwise, if $g_i \neq g_j$, they are homogeneous of degree one. Suppose that in $\tilde{g} = (g_1, \dots, g_m)$ there are $p \geq 1$ different string of zero and one, i.e.

$$\tilde{g} = (g_1, \dots, g_{t_1}, g_{t_1+1}, \dots, g_{t_1+t_2}, \dots, g_{t_1+\dots+t_{p-1}+1}, \dots, g_{t_1+\dots+t_p}),$$

where $t_1 + \dots + t_p = m$,

$$g_1 = \dots = g_{t_1},$$
$$g_{t_1+1} = \dots = g_{t_1+t_2},$$
$$\dots\dots$$
$$g_{t_1+\dots+t_{p-1}+1} = \dots = g_{t_1+\dots+t_p}$$

and

$$g_{t_1+\dots+t_i} \neq g_{t_1+\dots+t_i+1},$$

$\forall\ i = 1, \dots, p-1$.

As in the previous case we can find in A symmetric elements of degree zero

$$\bar{S}_i = \{\bar{s}_{\alpha_{i-1}+i}, \dots, \bar{s}_{\alpha_i+i-1}, \bar{s}_{\alpha_i+i}\}$$

and generic elements

$$\bar{U}_i = \{\bar{a}_{\alpha_{i-1}+i}, \dots, \bar{a}_{\alpha_i+i-2}, \bar{a}_{\alpha_i+i-1}, \bar{a}_{\alpha_i+i}\}$$

such that, $\forall\ i = 1, \dots p$,

$$Cap_{q_i}^{(\mathbb{Z}_2,*)}(\bar{s}_{\alpha_{\tilde{t}_{i-1}}+(\tilde{t}_{i-1}+1)}, \dots, \bar{s}_{\alpha_{\tilde{t}_i}+(\tilde{t}_i-1)}; \bar{a}_{\alpha_{\tilde{t}_{i-1}}+(\tilde{t}_{i-1}+1)}, \dots, \bar{a}_{\alpha_{\tilde{t}_i}+(\tilde{t}_i-2)}) =$$
$$Cap^{(\mathbb{Z}_2,*)}_{(d_0^+)_{\tilde{t}_{i-1}+1}} \bar{a}_{\alpha_{(\tilde{t}_{i-1}+1)}+\tilde{t}_{i-1}} \bar{s}_{\alpha_{(\tilde{t}_{i-1}+1)}+(\tilde{t}_{i-1}+1)} \bar{a}_{\alpha_{(\tilde{t}_{i-1}+1)}+(\tilde{t}_{i-1}+1)} Cap^{(\mathbb{Z}_2,*)}_{(d_0^+)_{\tilde{t}_{i-1}+2}}$$
$$\dots\dots\dots\dots Cap^{(\mathbb{Z}_2,*)}_{(d_0^+)_{\tilde{t}_i}} = b_i \neq 0,$$

where $\tilde{t}_0 = t_0 = 0$, $\tilde{t}_i = \sum_{j=0}^{i} t_j$ and $q_i = (d_0^+)_{\tilde{t}_{i-1}+1} + \dots + (d_0^+)_{\tilde{t}_i} + (t_i - 1)$.

Furthermore we can find in A elementary matrices $E_1, \dots, E_{p-1}$, such that

$$Cap^{(\mathbb{Z}_2,*)}_{d_0^++m-p} = Cap^{(\mathbb{Z}_2,*)}_{q_1} E_1 Cap^{(\mathbb{Z}_2,*)}_{q_2} E_2 \cdots Cap^{(\mathbb{Z}_2,*)}_{q_{p-1}} E_{p-1} Cap^{(\mathbb{Z}_2,*)}_{q_p} =$$
$$b_1 E_1 b_2 E_2 \cdots b_{p-1} E_{p-1} b_p \neq 0.$$

This implies that, for $r_0 = m - p$,

$$Cap^{(\mathbb{Z}_2,*)}_{d_0^++r_0}[Y^+, X] \notin Id^*_{\mathbb{Z}_2}(A).$$

Moreover, let's observe that any monomial of elements of A containing at least $r_0 + 1 = (m - p) + 1$ elements of J_0 must be zero. Then, similarly to the previous case, we obtain that A satisfies $Cap^{(\mathbb{Z}_2,*)}_{d_0^++r_0+1}[Y^+, X]$.

If $0 < \bar{m} < m$, let $\tilde{m}$ be the number of blocks of consecutive $*$-superalgebras with trivial grading that appear in $(A_1, \dots, A_m)$. By considering separately the blocks of consecutive $*$-superalgebras with trivial and non-trivial grading and by using arguments similar to those of the proof of case 1, it easily follows that $Cap^{(\mathbb{Z}_2,*)}_{q^+}[Y^+, X]$, $Cap^{(\mathbb{Z}_2,*)}_{q^-}[Y^-, X]$, $Cap^{(\mathbb{Z}_2,*)}_{k^+}[Z^+, X]$ and $Cap^{(\mathbb{Z}_2,*)}_{k^-}[Z^-, X]$ are in $Id^*_{\mathbb{Z}_2}(A)$ if and only if $q^+ > d_0^+ + (m - \bar{m}) + (\tilde{m} - 1) + r_0$, $q^- > d_0^- + (m - \bar{m}) + (\tilde{m} - 1) + r_0$, $k^+ > d_1^+ + (m - \bar{m}) + (\tilde{m} - 1) + r_1$ and $k^- > d_1^- + (m - \bar{m}) + (\tilde{m} - 1) + r_1$, where r_0, r_1 are two non negative integers depending on the grading $\tilde{g}$, with $r_0 + r_1 = \bar{m} - \tilde{m}$.

5. Asymptotics for $*$-graded Capelli identities

In this section we shall study $\mathcal{U} = \mathrm{var}^*_{\mathbb{Z}_2}(\Gamma^*_{M^\pm+1,L^\pm+1})$ and we shall find a close relation among the asymptotics of $c_n^*(\Gamma^*_{M^\pm+1,L^\pm+1})$ and $c_n^*(A)$, where A is a finite dimensional simple $*$-superalgebra. Let

$$R = A \oplus J$$

where A is a finite dimensional simple $*$-superalgebra and $J = J(R)$ is its Jacobson radical.

From now on we put $M^\pm = \dim_F {A_0}^\pm$ and $L^\pm = \dim_F {A_1}^\pm$.

Let's begin with some technical lemmas that hold for any finite dimensional simple $*$-superalgebra A.

Lemma 4. *The Jacobson radical J can be decomposed into the direct sum of four A-bimodules*

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11}$$

where, for $p, q \in \{0,1\}$, J_{pq} is a left faithful module or a 0-left module according to $p = 1$, or $p = 0$, respectively. Similarly, J_{pq} is a right faithful module or a 0-right module according to $q = 1$ or $q = 0$, respectively. Moreover, for $p, q, i, l \in \{0,1\}$, $J_{pq}J_{ql} \subseteq J_{pl}$, $J_{pq}J_{il} = 0$ for $q \neq i$ and there exists a finite dimensional nilpotent $*$-superalgebra N such that N commutes with A and $J_{11} \cong A \otimes_F N$ (isomorphism of A-bimodules and of $*$-superalgebras).

Proof. It follows from Lemma 2 in [17] and Lemmas 1,6 in [5].

Notice that J_{00} and J_{11} are stable under the involution whereas $J_{01}^* = J_{10}$.

Lemma 5. *If $\Gamma^*_{M^\pm+1,L^\pm+1} \subseteq Id^*_{\mathbb{Z}_2}(R)$, then $J_{10} = J_{01} = (0)$.*

Proof. By Lemma 3 we have that A does not satisfy $Cap^{(\mathbb{Z}_2,*)}_{M^+}[Y^+, X]$. Then there exist elements $a_1^+, \dots, a_{M^+}^+ \in {A_0}^+$ and $b_1, \dots, b_{M^+-1} \in A$ such that

$$Cap^{(\mathbb{Z}_2,*)}_{M^+}(a_1^+, \dots, a_{M^+}^+; b_1, \dots, b_{M^+-1}) =$$

$$\begin{cases} e_{1,h+l} & \text{if } A = (M_{h,l}, \diamond), \diamond = t, s; \\ \tilde{e}_{1,h+l} & \text{if } A = (M_{h,l} \oplus M_{h,l}^{op}, exc); \\ e_{1,n} & \text{if } A = (M_n + cM_n, \star) \text{ or} A = (M_n + cM_n, \dagger); \\ \tilde{e}_{1,n} & \text{if } A = ((M_n + cM_n) \oplus (M_n + cM_n)^{op}, exc) \end{cases}$$

where the $e_{i,j}$'s are the usual matrix units and $\tilde{e}_{i,j} = (e_{i,j}, e_{j,i})$. We write $J_{10} = (J_{10})_0 \oplus (J_{10})_1$ and $J_{01} = (J_{01})_0 \oplus (J_{01})_1$. Let $d_0 \in (J_{01})_0$, then $d_0^* \in (J_{10})_0$ and $d_0 + d_0^* \in (J_{01} \oplus J_{10})_0^+$. Since $\Gamma^*_{M^\pm+1,L^\pm+1} \subseteq Id^*_{\mathbb{Z}_2}(R)$ it follows that there exists $b_{M^+} \in A$ such that

$$0 = Cap^{(\mathbb{Z}_2,*)}_{M^++1}(a_1^+, \dots, a_{M^+}^+, d_0 + d_0^*; b_1, \dots, b_{M^+-1}, b_{M^+}) =$$

$$\begin{cases} e_{1,h+l}d_0^* \pm d_0 e_{1,h+l} & \text{if } A = (M_{h,l}, \diamond), \diamond = t, s; \\ \tilde{e}_{1,h+l}d_0^* \pm d_0 \tilde{e}_{1,h+l} & \text{if } A = (M_{h,l} \oplus M_{h,l}^{op}, exc); \\ e_{1,n}d_0^* \pm d_0 e_{1,n} & \text{if } A = (M_n + cM_n, \star) \text{ or } A = (M_n + cM_n, \dagger); \\ \tilde{e}_{1,n}d_0^* \pm d_0 \tilde{e}_{1,n} & \text{if } A = ((M_n + cM_n) \oplus (M_n + cM_n)^{op}, exc). \end{cases}$$

If $A = (M_{h,l}, \diamond)$, then $e_{1,h+l}d_0^* \pm d_0 e_{1,h+l} = 0$ and, so, $e_{1,h+l}d_0^* = \mp d_0 e_{1,h+l} \in (J_{01})_0 \cap (J_{10})_0 = (0)$. Hence $d_0 = 0$, for all $d_0 \in (J_{01})_0$. Thus $(J_{01})_0 = (0)$ and $(J_{10})_0 = (0)$. Similarly for the other finite dimensional simple $*$-superalgebras we obtain that $(J_{01})_0 = (J_{10})_0 = (0)$. Analogously it easy to show that $(J_{01})_1 = (J_{10})_1 = (0)$ and the lemma is proved.

Lemma 6. *Let* $J_{11} \cong A \otimes_F N$*, as in Lemma 4. If* $\Gamma^*_{M^\pm+1,L^\pm+1} \subseteq Id^*_{\mathbb{Z}_2}(R)$*, then* N *is commutative.*

Proof. Let N be the finite dimensional nilpotent $*$-superalgebra of Lemma 4. Write $N = N_0^+ \oplus N_0^- \oplus N_1^+ \oplus N_1^-$, where N_0^+, N_0^-, N_1^+ and N_1^- denote the subspaces of symmetric and skew symmetric elements of N of homogeneous degree 0 and 1 respectively.

We shall prove that N is commutative when $A = (M_{h,l}, \diamond)$, with $\diamond = t$ or s. Similar calculations for the other finite dimensional simple $*$-superalgebras lead to the same conclusion.

Let's start by proving that $N_0^\pm$ commutes with $N_i^\pm, i = 0, 1$. Let $e_1^+, \dots, e_{M^+}^+$ be a basis of ${A_0}^+$ with

$$e_1^+ = \begin{cases} e_{1,2} + e_{2,1} & \text{if } A = (M_{h,l}, t)\,; \\ e_{1,2} + e_{h+2,h+1} & \text{if } A = (M_{h,h}, s) \end{cases}$$

and let $a_0 = a_1 = e_{2,1},\ a_2, \dots, a_{M^+-1} \in A$ such that $a_0 e_1^+ a_1 e_2^+ \cdots a_{M^+-1} e_{M^+}^+ = e_{2,h+l}$ and $a_0 e_{\sigma(1)}^+ a_1 \cdots a_{M^+-1} e_{\sigma(M^+)}^+ = 0$ for any $\sigma \in S_{M^+}$, $\sigma \neq id$. Let $d_1 \in N_0^\pm$ and $e_0^+ = (e_{1,2} \pm e_{1,2}^\diamond) d_1$, with $\diamond = t$ or s. Since N commutes with A we obtain that $e_0^+ \in R_0^+$. If we put $\bar{a}_0 = a_0 d_2 = e_{2,1} d_2$, with $d_2 \in N_i^\pm$, $i = 0, 1$, then

$$0 = Cap^{(\mathbb{Z}_2,*)}_{M^++1}(e_0^+, e_1^+, \dots, e_M^+; \bar{a}_0, a_1, \dots, a_{M^+-1}) = [d_1, d_2] e_{1,h+l}$$

and so $[d_1, d_2] = 0$ for all $d_1 \in N_0^\pm, d_2 \in N_i^\pm$, $i = 0, 1$.

Let's now prove that $N_1^\pm$ commutes with $N_1^\pm$. Let $e_1^+, \dots, e_{M^+}^+$ be a basis of ${A_0}^+$, with

$$e_1^+ = \begin{cases} e_{1,1} & \text{if } A = (M_{h,l}, t)\,; \\ e_{1,1} + e_{h+1,h+1} & \text{if } A = (M_{h,h}, s) \end{cases}$$

and let $a_0 = e_{h+l,1}, a_1, a_2, \dots, a_{M^+-1} \in A$ such that $a_0 e_1^+ a_1 \cdots a_{M^+-1} e_{M^+}^+ = e_{h+l,1}$ (if $\diamond =$s then $h = l$) and $a_0 e_{\sigma(1)}^+ a_1 \cdots a_{M^+-1} e_{\sigma(M^+)}^+ = 0$ for any $\sigma \in S_{M^+}$, $\sigma \neq id$.
Let $(e_{1,h+l} \pm e_{1,h+l}^\diamond) \in A_1^\pm$ and $d_1, d_2 \in N_1^\pm$ such that, for $i = 1, 2$, $c_i^+ = (e_{1,h+l} \pm e_{1,h+l}^\diamond) d_i$. Since N commutes with A then $c_i^+ \in R_0^+$, $i = 1, 2$. If $a_M = e_{1,1}$ then

$$0 = Cap^{(\mathbb{Z}_2,*)}_{M^++2}(c_1^+, e_1^+, \dots, e_M^+, c_2^+; \bar{a}_0, a_1, \dots, a_{M^+-1}, a_M) = [d_1, d_2] e_{1,h+l}$$

($h = l$ for $\diamond = s$) and so $[d_1, d_2] = 0$, for all $d_1, d_2 \in N_1^\pm$ and we are done.

Lemma 7. $\exp^*_{\mathbb{Z}_2}(\mathcal{U}) = M^+ + M^- + L^+ + L^- = M + L = \exp^*_{\mathbb{Z}_2}(A)$.

Proof. By the definition of minimal variety (see Definition 2.1 in [9]) the $*$-graded exponent of $\mathcal{U}$ is equal to the $*$-graded exponent of some minimal variety of $*$-superalgebras lying in $\mathcal{U}$. Moreover, by the classification of minimal varieties of PI-$*$-superalgebras of finite basic rank given in [9, Theorem 2.2], we have

$$\exp^*_{\mathbb{Z}_2}(\mathcal{U}) = \max\{\exp^*_{\mathbb{Z}_2}(UT^*_{\mathbb{Z}_2}(A_1, \dots, A_m)) \,|\, UT^*_{\mathbb{Z}_2}(A_1, \dots, A_m) \in \mathcal{U}\}.$$

Then, by Lemma 3,

$$\exp^*_{\mathbb{Z}_2}(\mathcal{U}) \geq \exp^*_{\mathbb{Z}_2}(UT^*_{\mathbb{Z}_2}(A)) = M + L.$$

On the other hand, since $\exp^*_{\mathbb{Z}_2}(UT^*_{\mathbb{Z}_2}(A_1, \dots, A_m)) = d_0^\pm + d_1^\pm$, we have that

$$\exp^*_{\mathbb{Z}_2}(\mathcal{U}) \leq M + L$$

and the proof is completed.

Now we are able to prove the main result.

Theorem 3. *For suitable natural numbers M^+, M^-, L^+, L^- there exists a finite dimensional simple $*$-superalgebra A such that*

$$\mathcal{U} = \mathrm{var}^*_{\mathbb{Z}_2}(\Gamma^*_{M^\pm+1,L^\pm+1}) = \mathrm{var}^*_{\mathbb{Z}_2}(A \oplus D),$$

where D is a finite dimensional $*$-superalgebra such that $\exp^*_{\mathbb{Z}_2}(D) < M + L$, with $M = M^+ + M^-$ and $L = L^+ + L^-$. In particular

1) *If $M^\pm = \frac{h(h\pm1)}{2} + \frac{l(l\pm1)}{2}$ and $L^\pm = hl$, with $h \geq l > 0$, then $A = (M_{h,l}, t)$;*
2) *If $M^\pm = h^2$ and $L^\pm = h(h \mp 1)$, with $h > 0$, then $A = (M_{h,h}, s)$;*
3) *If $M^\pm = h^2 + l^2$ and $L^\pm = 2hl$, with $h \geq l > 0$, then $A = (M_{h,l} \oplus M^{op}_{h,l}, exc)$;*
4) *If $M^+ = L^\pm = \frac{n(n+1)}{2}$, $M^- = L^\mp = \frac{n(n-1)}{2}$, with $n > 0$, then $A = (M_n + cM_n, *)$, where $(a + cb)^* = a^t \pm cb^t$;*
5) *If $M^+ = L^\pm = \frac{n(n-1)}{2}$, $M^- = L^\mp = \frac{n(n+1)}{2}$, with $n > 0$, then $A = (M_n + cM_n, *)$, where $(a + cb)^* = a^s \pm cb^s$;*
6) *If $M^\pm = L^\pm = n^2$, with $n > 0$, then $A = ((M_n + cM_n) \oplus (M_n + cM_n)^{op}, exc)$.*

Proof. By Lemma 7 we have that $\exp^*_{\mathbb{Z}_2}(\mathcal{U}) = M + L$. Let B be a generating $*$-superalgebra of $\mathcal{U}$. From Theorem 2 and by [10], since any finitely generated $*$-superalgebra satisfies the same $*$-graded polynomial identities of a finite-dimensional $*$-superalgebra, we can assume that B is finite dimensional. Thus, by Theorem 1, there exists a finite number of reduced $*$-superalgebras $B_1, \dots, B_t$ and a finite dimensional $*$-superalgebra D such that

$$\mathcal{U} = \mathrm{var}^*_{\mathbb{Z}_2}(B) = \mathrm{var}^*_{\mathbb{Z}_2}(B_1 \oplus \cdots \oplus B_t \oplus D). \tag{1}$$

Moreover

$$\exp^*_{\mathbb{Z}_2}(B_1) = \cdots = \exp^*_{\mathbb{Z}_2}(B_t) = \exp^*_{\mathbb{Z}_2}(\mathcal{U}) = M + L$$

and

$$\exp^*_{\mathbb{Z}_2}(D) < \exp^*_{\mathbb{Z}_2}(\mathcal{U}) = M + L.$$

Let's now analyze the structure of a finite dimensional reduced $*$-superalgebra R such that $\exp^*_{\mathbb{Z}_2}(R) = M + L = \exp^*_{\mathbb{Z}_2}(\mathcal{U})$ and $\Gamma^*_{M^\pm+1,L^\pm+1} \subseteq Id^*_{\mathbb{Z}_2}(R)$. We have that

$$R = R_1 \oplus \cdots \oplus R_m + J, \tag{2}$$

where R_i are simple $*$-graded subalgebras of R, $J = J(R)$ is the Jacobson radical of R and $R_1 J \cdots J R_m \neq 0$. By [9, Theorem 4.3] there exists a $*$-superalgebra $\overline{R}$ isomorphic to the $*$-superalgebra $UT^*_{\mathbb{Z}_2,\tilde{g}}(R_1, \dots, R_m)$, for some $\tilde{g} = (g_1, \dots, g_m) \in \mathbb{Z}_2^m$, such that $Id(R) \subseteq Id(\overline{R})$ and

$$\exp^*_{\mathbb{Z}_2}(R) = \exp^*_{\mathbb{Z}_2}(\overline{R}) = \exp^*_{\mathbb{Z}_2}(UT^*_{\mathbb{Z}_2,\tilde{g}}(R_1, \dots, R_m)).$$

It follows that

$$M + L = \exp^*_{\mathbb{Z}_2}(R) = \exp^*_{\mathbb{Z}_2}(\overline{R}) =$$
$$\exp^*_{\mathbb{Z}_2}(UT^*_{\mathbb{Z}_2,\tilde{g}}(R_1, \dots, R_m)) = \dim_F R_1 + \cdots + \dim_F R_m = d_0^+ + d_0^- + d_1^+ + d_1^-$$

where $d_i^\pm = \dim_F (R_1 \oplus \cdots \oplus R_m)^\pm_{(i)}$, for $i = 0, 1$.

Let $0 \leq \bar{m} \leq m$ denote the number of the $*$-superalgebras R_i with trivial grading appearing in (2). We want to prove that $\bar{m} = 0$.

Let's suppose $\bar{m} > 0$. By Lemma 3, $\overline{R}$ does not satisfy the $*$-graded Capelli polynomials

$$Cap^{(\mathbb{Z}_2,*)}_{d_0^+ + (m-\bar{m}) + (\bar{m}-1) + r_0}[Y^+, X], \quad Cap^{(\mathbb{Z}_2,*)}_{d_0^- + (m-\bar{m}) + (\bar{m}-1) + r_0}[Y^-, X],$$
$$Cap^{(\mathbb{Z}_2,*)}_{d_1^+ + (m-\bar{m}) + (\bar{m}-1) + r_1}[Z^+, X], \quad Cap^{(\mathbb{Z}_2,*)}_{d_1^- + (m-\bar{m}) + (\bar{m}-1) + r_1}[Z^-, X],$$

where r_0, r_1 are two non negative integers dependent on the grading $\tilde{g}$ with $r_0 + r_1 = \bar{m} - \tilde{m}$. However $\overline{R}$ satisfies $Cap^{(\mathbb{Z}_2,*)}_{M^++1}[Y^+, X]$, $Cap^{(\mathbb{Z}_2,*)}_{M^-+1}[Y^-, X]$, $Cap^{(\mathbb{Z}_2,*)}_{L^++1}[Z^+, X]$ and $Cap^{(\mathbb{Z}_2,*)}_{L^-+1}[Z^-, X]$, then

$$d_0^+ + (m-\bar{m}) + (\tilde{m}-1) + r_0 + d_0^- + (m-\bar{m}) + (\tilde{m}-1) + r_0 +$$
$$d_1^+ + (m-\bar{m}) + (\tilde{m}-1) + r_1 + d_1^- + (m-\bar{m}) + (\tilde{m}-1) + r_1 \le M + L.$$

Since $d_0^+ + d_0^- + d_1^+ + d_1^- = M + L$ we obtain that $4(m-\bar{m}) + 4(\tilde{m}-1) + 2(r_0 + r_1) = 0$ and so $2(m-1) + \tilde{m} - \bar{m} = 0$ and this implies that $m \geq 2$. If $m = 2$ then we easily obtain a contradiction. Thus $m = \bar{m} = \tilde{m} = 1$.

Hence $R = R_1 \oplus J$ where $R_1 \simeq (M_{h_1}(F), t)$ or $R_1 \simeq (M_{2h_1}(F), s)$ or $R_1 \simeq (M_{h_1}(F) \oplus M_{h_1}(F)^{op}, exc)$ with $h_1 > 0$.

Now, let's analyze all possible cases as M and L vary.

1. Let $M^\pm = \frac{h(h\pm 1)}{2} + \frac{l(l\pm 1)}{2}$ and $L^\pm = hl$, with $h \geq l > 0$.
If $R \simeq (M_{h_1}(F), t) + J$ then $\exp^*_{\mathbb{Z}_2}(R) = h_1^2$. Since $\exp^*_{\mathbb{Z}_2}(R) = M + L = (h+l)^2$ we obtain that $h_1 = h + l$. By hypotesis, R satisfies $Cap^{(\mathbb{Z}_2,*)}_{M^++1}[Y^+; X]$ but, since $Id^*_{\mathbb{Z}_2}(R) \subseteq Id^*_{\mathbb{Z}_2}(UT^*_{\mathbb{Z}_2,\tilde{g}}(R_1, \dots, R_q))$, R does not satisfy $Cap^{(\mathbb{Z}_2,*)}_{d_0^+}[Y^+; X]$. Hence, for $h \geq l > 0$, we have

$$M^+ + 1 = \frac{h(h+1)}{2} + \frac{l(l+1)}{2} + 1 = \frac{h^2 + l^2 + (h+l) + 2}{2} \le$$
$$\frac{h^2 + l^2 + (h+l) + 2hl}{2} = \frac{(h+l)(h+l+1)}{2} = \frac{h_1(h_1+1)}{2} = d_0^+$$

and this is impossible.
If $R \simeq (M_{2h_1}(F), s) + J$ then $\exp^*_{\mathbb{Z}_2}(R) = 4h_1^2$. Since $\exp^*_{\mathbb{Z}_2}(R) = M + L = (h+l)^2$ we have that $2h_1 = h + l$. Moreover R satisfies $Cap^{(\mathbb{Z}_2,*)}_{M^-+1}[Y^-; X]$ but does not satisfy $Cap^{(\mathbb{Z}_2,*)}_{d_0^-}[Y^-; X]$ and so we get a contradiction since

$$M^- + 1 = \frac{h(h-1)}{2} + \frac{l(l-1)}{2} + 1 = \frac{h^2 + l^2 - (h+l) + 2}{2} <$$
$$\frac{h^2 + l^2 + (h+l) + 2hl}{2} = \frac{(h+l)^2 + (h+l)}{2} = \frac{4h_1^2 + 2h_1}{2} = 2h_1^2 + h_1 = d_0^-.$$

Finally, let $R \simeq (M_{h_1}(F) \oplus M_{h_1}(F)^{op}, exc) + J$, with $h_1 > 0$. Then $(h+l)^2 = M + L = \exp^*_{\mathbb{Z}_2}(R) = 2h_1^2$, a contradiction.
2. Let $M^\pm = h^2$ and $L^\pm = h(h \mp 1)$, with $h > 0$.
If $R \simeq (M_{h_1}(F), t) + J$ then, as in the previous case, we obtain that $2h = h_1$. By hypothesis R satisfies $Cap^{(\mathbb{Z}_2,*)}_{M^++1}[Y^+; X]$ but it does not satisfy $Cap^{(\mathbb{Z}_2,*)}_{d_0^+}[Y^+; X]$, thus we have

$$M^+ + 1 = h^2 + 1 = (\frac{h_1}{2})^2 + 1 = \frac{h_1^2}{4} + 1 \le \frac{h_1^2}{2} + \frac{h_1}{2} = d_0^+$$

a contradiction.
If $R \simeq (M_{2h_1}(F), s) + J$ then $h = h_1$. Since R satisfies $Cap^{(\mathbb{Z}_2,*)}_{M^-+1}[Y^-; X]$ but does not satisfy $Cap^{(\mathbb{Z}_2,*)}_{d_0^-}[Y^-; X]$ we get the contradiction $M^- + 1 = h^2 + 1 = h_1^2 + 1 < 2h_1^2 + h_1 = d_0^-$.
Finally, if $R \simeq (M_{h_1}(F) \oplus M_{h_1}(F)^{op}, exc) + J$ with $h_1 > 0$, then we have $4h^2 = 2h_1^2$, a contradiction.
3. Let $M^\pm = h^2 + l^2$ and $L^\pm = 2hl$, with $h \geq l > 0$.
If $R \simeq (M_{h_1}(F), t) + J$ then we get the contradiction $2(h+l)^2 = M + L = \exp^*_{\mathbb{Z}_2}(R) = h_1^2$. The same occurs if $R \simeq (M_{2h_1}(F), s) + J$.

Now, let $R \simeq (M_{h_1}(F) \oplus M_{h_1}(F)^{op}, exc) + J$, with $h_1 > 0$. Then $2(h+l)^2 = M + L = \exp^*_{\mathbb{Z}_2}(R) = 2h_1^2$ and so $h_1 = h + l$. Since $d_0^+ = h_1^2$ we get that $M^+ + 1 = h^2 + l^2 + 1 < h^2 + l^2 + 2hl = (h+l)^2 = h_1^2 = d_0^+$ and this is impossible.
4., **5**. We consider the case $M^+ = L^+ = \frac{n(n+1)}{2}$ and $M^- = L^- = \frac{n(n-1)}{2}$. The proof of the other cases is very similar.
If $R \simeq (M_{h_1}(F), t) + J$ then $2n^2 = M + L = \exp^*_{\mathbb{Z}_2}(R) = h_1^2$, and if $R \simeq (M_{2h_1}(F), s) + J$ then $2n^2 = M + L = \exp^*_{\mathbb{Z}_2}(R) = 4h_1^2$, a contradiction.
Let $R \simeq (M_{h_1}(F) \oplus M_{h_1}(F)^{op}, exc) + J$, with $h_1 > 0$. Then $2n^2 = M + L = \exp^*_{\mathbb{Z}_2}(R) = 2h_1^2$ so $h_1 = n$. Since R satisfies $Cap^{(\mathbb{Z}_2,*)}_{M^-+1}[Y^-; X]$ but it does not satisfy $Cap^{(\mathbb{Z}_2,*)}_{d_0^-}[Y^-; X]$ we have again a contradiction indeed $M^- + 1 = \frac{n(n-1)}{2} + 1 \leq n(n-1) + 1 \leq n^2 = h_1^2 = d_0^-$.
6. Let $M^\pm = L^\pm = n^2$, with $n > 0$.
If $R \simeq (M_{h_1}(F) \oplus M_{h_1}(F)^{op}, exc) + J$ then $4n^2 = M + L = \exp^*_{\mathbb{Z}_2}(R) = 2h_1^2$ a contradiction.
If $R \simeq (M_{h_1}(F), t) + J$ then $4n^2 = M + L = \exp^*_{\mathbb{Z}_2}(R) = h_1^2$ and so $h_1 = 2n$. R satisfies $Cap^{(\mathbb{Z}_2,*)}_{M^++1}[Y^+; X]$ but does not satisfy $Cap^*_{d_0^+}[Y^+; X]$ then we obtain a contradiction in fact $M^+ + 1 = n^2 + 1 = \frac{h_1^2}{4} + 1 \leq \frac{h_1^2}{2} + \frac{h_1}{2} = \frac{h_1(h_1+1)}{2} = d_0^+$.
Finally, let $R \simeq (M_{2h_1}(F), s) + J$. Hence $4n^2 = M + L = \exp^*_{\mathbb{Z}_2}(R) = 4h_1^2$ and so $n = h_1$. Also in this case we get the contradiction $M^- + 1 = n^2 + 1 < 2n^2 + 1 < 2n^2 + n = 2h_1^2 + h_1 = d_0^-$.

So we obtained that $\bar{m} = 0$.

Let $R = R_1 \oplus \cdots \oplus R_m + J$, where R_i are simple $*$-superalgebras with non trivial grading. Let's prove that $m = 1$. By Lemma 3, $\overline{R}$ does not satisfy the $*$-graded Capelli polynomials $Cap^{(\mathbb{Z}_2,*)}_{d_0^++m-1}[Y^+, X]$, $Cap^{(\mathbb{Z}_2,*)}_{d_0^-+m-1}[Y^-, X]$, $Cap^{(\mathbb{Z}_2,*)}_{d_1^++m-1}[Z^+, X]$ and $Cap^{(\mathbb{Z}_2,*)}_{d_1^-+m-1}[Z^-, X]$ but satisfies $Cap^{(\mathbb{Z}_2,*)}_{M^++1}[Y^+, X]$, $Cap^{(\mathbb{Z}_2,*)}_{M^-+1}[Y^-, X]$, $Cap^{(\mathbb{Z}_2,*)}_{L^++1}[Z^+, X]$ and $Cap^{(\mathbb{Z}_2,*)}_{L^-+1}[Z^-, X]$ thus $d_0^+ + m - 1 \leq M^+$, $d_0^- + m - 1 \leq M^-$, $d_1^+ + m - 1 \leq L^+$ and $d_1^- + m - 1 \leq L^-$. Hence we have that

$$d_0^+ + (m-1) + d_0^- + (m-1) + d_1^+ + (m-1) + d_1^- + (m-1) \leq M^+ + M^- + L^+ + L^- = M + L.$$

Since $d_0^+ + d_0^- + d_1^- + d_1^- = M + L$ we obtain that $4(m-1) = 0$ and so $m = 1$.

It follows that $R = R_1 \oplus J$ where R_1 is a simple $*$-superalgebra with non trivial grading. Now let's analyze the cases corresponding to the different values of M and L.

1. Let $M^\pm = \frac{h(h\pm1)}{2} + \frac{l(l\pm1)}{2}$ and $L^\pm = hl$, with $h \geq l > 0$.
If $R \simeq (M_{h_1,h_1}(F), s) + J$ then $(h+l)^2 = M + L = \exp^*_{\mathbb{Z}_2}(R) = 4h_1^2$ so we have $2h_1 = h + l$. By hypothesis R satisfies $Cap^{(\mathbb{Z}_2,*)}_{L^-+1}[Z^-; X]$ but does not satisfy $Cap^{(\mathbb{Z}_2,*)}_{d_1^-}[Z^+; X]$, where $d_1^- = h_1(h_1+1)$. Since $h + l = 2h_1$ and $h \geq l > 0$ we have that $h_1^2 \geq hl$ and so

$$L^- + 1 = hl + 1 \leq h_1^2 + 1 \leq h_1(h_1+1) = d_1^-$$

a contradiction.
If $R \simeq (M_{h_1,l_1}(F) \oplus M_{h_1,l_1}(F)^{op}, exc) + J$, with $h_1 \geq l_1 > 0$, then $(h+l)^2 = M + L = \exp^*_{\mathbb{Z}_2}(R) = 2(h_1 + l_1)^2$ and so we have again a contradiction.
If $R \simeq (M_n(F + cF), *) + J$, where $(a + cb)^* = a^\diamond \pm cb^\diamond$ and $\diamond = t, s$, then we obtain the contradiction $(h+l)^2 = 2n^2$.
If $R \simeq (M_n(F + cF) \oplus M_n(F + cF)^{op}, exc) + J$ with $n > 0$, then $(h+l)^2 = M + L = \exp^*_{\mathbb{Z}_2}(R) = 4n^2$ and so $2n = h + l$. As before we can easily obtain a contradiction. It follows that $R \simeq (M_{h,l}(F), t) + J$.

2. Let now $M^\pm = h^2$ and $L^\pm = h(h \mp 1)$, with $h > 0$.

If $R \simeq (M_{h_1,l_1}(F), t) + J$, then, since $M + L = \exp^*_{\mathbb{Z}_2}(R)$, we have $4h^2 = (h_1 + l_1)^2$ and so $h_1 + l_1 = 2h^2$. By hypothesis R satisfies $Cap^{(\mathbb{Z}_2,*)}_{M^+ +1}[Y^+; X]$ but does not satisfy $Cap^{(\mathbb{Z}_2,*)}_{d_0^+}[Y^+; X]$ where $d_0^+ = \frac{h_1(h_1+1)}{2} + \frac{l_1(l_1+1)}{2}$. Since $h_1 + l_1 = 2h$ and $h_1 \geq l_1 > 0$ we have $h^2 \geq h_1 l_1$ and so it follows that

$$M^+ + 1 = h^2 + 1 < h(2h + 1) - h_1 l_1 = \frac{h_1 + l_1}{2}(h_1 + l_1 + 1) - h_1 l_1 =$$

$$\frac{h_1(h_1 + 1)}{2} + \frac{l_1(l_1 + 1)}{2} = d_0^+$$

a contradiction.
If $R \simeq (M_{h_1,l_1}(F) \oplus M_{h_1,l_1}(F)^{op}, exc) + J$, with $h_1 \geq l_1 > 0$, or $R \simeq (M_n(F + cF), *) + J$ where $(a + cb)^* = a^\diamond \pm cb^\diamond$ and $\diamond = t, s$ then easily we get a contradiction.
If $R \simeq (M_n(F+cF) \oplus M_n(F+cF)^{op}, exc)+J$ with $n > 0$, then $4h^2 = M+L = \exp^*_{\mathbb{Z}_2}(R) = 4n^2$ and so $n = h$. R satisfies $Cap^{(\mathbb{Z}_2,*)}_{L^- +1}[Z^-; X]$ but R does not satisfy $Cap^{(\mathbb{Z}_2,*)}_{d_1^-}[Z^+; X]$, where $d_1^- = n^2$ and we obtain the following contradiction $L^- + 1 = h(h-1) = h^2 - h - 1 \leq h^2 = n^2 = d_1^-$. So, in this case, $R \simeq (M_{h,h}(F), s) + J$.

3. Let $M^\pm = h^2 + l^2$ and $L^\pm = 2hl$, with $h \geq l > 0$.
If $R \simeq (M_{h_1,l_1}(F), t)+J$, $R \simeq (M_{h_1,h_1}(F), s)+J$ or $R \simeq (M_n(F+cF) \oplus M_n(F+cF)^{op}, exc)+J$ easily we get a contradiction.
If $R \simeq (M_n(F + cF), *) + J$ where $(a + cb)^* = a^\diamond \pm cb^\diamond$ and $\diamond = t, s$ then we have that $2(h+l)^2 = 2n^2$ and so $h+l = n$. Let consider the case when $R \simeq (M_n(F+cF), *)+J$ with $(a+cb)^* = a^t - cb^t$, the other cases are very similar. Since R satisfies $Cap^{(\mathbb{Z}_2,*)}_{L^- +1}[Z^-; X]$ but R does not satisfy $Cap^{(\mathbb{Z}_2,*)}_{d_1^-}[Z^+; X]$ we obtain

$$L^- + 1 = 2hl + 1 < \frac{(h + l + 1)(h + l)}{2} = \frac{(n + 1)n}{2} = d_1^-$$

a contradiction. It follows that $R \simeq (M_{h,l}(F) \oplus M_{h_1,l_1}(F)^{op}, exc) + J$.

4., **5**. Let consider the case $M^+ = L^+ = \frac{n(n+1)}{2}$ and $M^- = L^- = \frac{n(n-1)}{2}$. The proof of the other cases is very similar. As before let $R \simeq (M_n(F + cF) \oplus M_n(F + cF)^{op}, exc) + J$, then $2n^2 = 2(h_1 + l_1)^2$ and so $n = h_1 + l_1$ with $h_1 \geq l_1 > 0$. R satisfies $Cap^{(\mathbb{Z}_2,*)}_{M^- +1}[Y^-; X]$ but does not satisfy $Cap^{(\mathbb{Z}_2,*)}_{d_0^-}[Y^-; X]$ then

$$\frac{n(n-1)}{2} + 1 = \frac{n^2 - n + 2}{2} \leq \frac{n^2 - 1}{2} = \frac{(h_1 + l_1)^2 - 1}{2} < h_1^2 + l_1^2 = d_0^-$$

a contradiction. In all other cases we obtain a contradiction except when $R \simeq (M_n(F + cF), *) + J$ and $(a + cb)^* = a^t + cb^t$.

6. Let $M^\pm = L^\pm = n^2$, with $n > 0$.
If $R \simeq (M_{h_1,l_1}(F) \oplus M_{h_1,l_1}(F)^{op}, exc)+J$ or $R \simeq (M_n(F+cF), *)+J$ with $(a+cb)^* = a^\diamond \pm cb^\diamond$ and $\diamond = t, s$, then easily we get a contradiction.
If $R \simeq (M_{h_1,l_1}(F), t) + J$, then $h_1 + l_1 = 2n$ and with analogous reasoning to that of case 2 we obtain a contradiction.
So let assume that $R \simeq (M_{h_1,h_1}(F), s) + J$, then $4n^2 = 4h_1^2$ and so $h_1 = n$. Because R satisfies $Cap^{(\mathbb{Z}_2,*)}_{L^- +1}[Z^-; X]$ but it does not satisfy $Cap^{(\mathbb{Z}_2,*)}_{d_1^-}[Z^-; X]$ we obtain $L^- + 1 = n^2 + 1 \leq n(n + 1) = h_1(h_1 + 1) = d_1^-$ and this is impossible. It follows that $R \simeq (M_n(F + cF) \oplus M_n(F + cF)^{op}, exc) + J$.

Thus we have proved that $R \simeq A + J$ where A is a simple $*$-superalgebra with non trivial grading. Then, from Lemmas 4, 5, 6 we obtain that

$$R \cong (A + J_{11}) \oplus J_{00} \cong (A \otimes N^\sharp) \oplus J_{00}$$

where $N^\sharp$ is the algebra obtained from N by adjoining a unit element. Since $N^\sharp$ is commutative, it follows that $A + J_{11}$ and A satisfy the same $*$-graded identities. Thus $\mathrm{var}^*_{\mathbb{Z}_2}(R) = \mathrm{var}^*_{\mathbb{Z}_2}(A \oplus J_{00})$ with J_{00} finite dimensional nilpotent $*$-superalgebra. Hence, from the decomposition (1), we get

$$\mathcal{U} = \mathrm{var}^*_{\mathbb{Z}_2}(\Gamma^*_{M^\pm+1,L^\pm+1}) = \mathrm{var}^*_{\mathbb{Z}_2}(A \oplus D),$$

where D is a finite dimensional $*$-superalgebra with $\exp^*_{\mathbb{Z}_2}(D) < M + L$ and the theorem is proved.

From Corollary 1 we easily obtain the following

Corollary 2. 1) *If* $M^\pm = \frac{h(h\pm1)}{2} + \frac{l(l\pm1)}{2}$ *and* $L^\pm = hl$*, with* $h \geq l > 0$*, then*

$$c_n^{(\mathbb{Z}_2,*)}(\Gamma^*_{M^\pm+1,L^\pm+1}) \simeq c_n^{(\mathbb{Z}_2,*)}((M_{h,l}(F), t));$$

2) *If* $M^\pm = h^2$ *and* $L^\pm = h(h \mp 1)$*, with* $h > 0$*, then*

$$c_n^{(\mathbb{Z}_2,*)}(\Gamma^*_{M^\pm+1,L^\pm+1}) \simeq c_n^{(\mathbb{Z}_2,*)}((M_{h,h}(F), s));$$

3) *If* $M^\pm = h^2 + l^2$ *and* $L^\pm = 2hl$*, with* $h \geq l > 0$*, then*

$$c_n^{(\mathbb{Z}_2,*)}(\Gamma^*_{M^\pm+1,L^\pm+1}) \simeq c_n^{(\mathbb{Z}_2,*)}((M_{h,l}(F) \oplus M_{h,l}(F)^{op}, exc));$$

4) *If* $M^+ = L^\pm = \frac{n(n+1)}{2}$, $M^- = L^\mp = \frac{n(n-1)}{2}$*, with* $n > 0$*, then*

$$c_n^{(\mathbb{Z}_2,*)}(\Gamma^*_{M^\pm+1,L^\pm+1}) \simeq c_n^{(\mathbb{Z}_2,*)}((M_n(F + cF), *))$$

where $(a + cb)^* = a^t \pm cb^t$;

5) *If* $M^+ = L^\pm = \frac{n(n-1)}{2}$, $M^- = L^\mp = \frac{n(n+1)}{2}$*, with* $n > 0$*, then*

$$c_n^{(\mathbb{Z}_2,*)}(\Gamma^*_{M^\pm+1,L^\pm+1}) \simeq c_n^{(\mathbb{Z}_2,*)}((M_n(F + cF), *)),$$

where $(a + cb)^* = a^s \pm cb^s$*;*

6) *If* $M^\pm = L^\pm = n^2$*, with* $n > 0$*, then*

$$c_n^{(\mathbb{Z}_2,*)}(\Gamma^*_{M^\pm+1,L^\pm+1}) \simeq c_n^{(\mathbb{Z}_2,*)}(M_n(F + cF) \oplus M_n(F + cF)^{op}, exc)).$$

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