REGULARITY OF K-QUASIMINIMIZERS UNDER $L^{p\&q}\log L$ DOUBLE PHASE GROWTH

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Abstract. We study the local quasiminimizers of an integral functional exhibiting $L^{p\&q}\log L$ double phase growth. More precisely, some properties of a special double phase type function in the context of Musielak-Orlicz-Sobolev spaces are obtained. We first get Caccioppoli type and Sobolev-Poincaré type inequalities, then we establish that the gradient of a local quasiminimizer has local higher integrability on a bounded domain. We also get boundary higher integrability on a ball for local quasiminimizers. Finally, we explore the existence of weak solutions to certain double phase Dirichlet problems.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$. The paper focuses on the analysis of local quasiminimizers to the energy integral functional (for short, $\mathcal{H}_L$-energy)

$$u \mapsto \int_\Omega \mathcal{H}_L(x, |\nabla u|)\, dx, \tag{1}$$

where

$$\mathcal{H}_L(x,t) = [t^p + a(x)t^q]\log(e+t) \quad \text{for all } x \in \Omega, \text{ all } t \geq 0,\ 1 < p < q < +\infty, \tag{2}$$

depicts the $L^{p\&q}\log L$ double phase growth, via the modulating coefficient

$$a \in C^{0,\alpha}(\overline{\Omega}) \text{ with } 0 < \alpha \leq 1 \text{ and } a(x) \geq 0 \text{ for all } x \in \Omega,\ \frac{q}{p} \leq 1 + \frac{\alpha}{N}, \tag{3}$$

and e denotes the Euler constant.

The precise notions and notation, together with the hypotheses of the work, will be stated and discussed in the sequel, however for convenience of reader we include here the main results of this manuscript. The first result establishes a Sobolev-Poincaré type inequality for functions which need not vanish everywhere on the boundary, but they are zero on a set of positive measure.

Theorem 1. *Let $\mathcal{H}_L$ be the function defined by (2)-(3) with $\operatorname{ess\,inf}_{B_r} a(x) = a_0 > 0$ where $B_r \subset \Omega$. If $u \in W^{1,\mathcal{H}_L}(B_r)$ with $\|\nabla u\|_{L^{\mathcal{H}_L}(B_r)} < 1$ and $u = 0$ a.e. on a measurable set $A \subset B_r$ such that $|A| \geq c_0|B_r|$ for some $0 < c_0 < 1$, then there exists a constant $c = c(q, c_0, N, s, \mathcal{H}_L) > 0$, independent of u, such that*

$$\fint_{B_r} \mathcal{H}_L\left(x, \frac{|u|}{r}\right) dx \leq c\left[\left(\fint_{B_r} \mathcal{H}_L(x, |\nabla u|)^{\frac{1}{s}} dx\right)^s + 1\right] \tag{4}$$

for some $1 \leq s < \min\{p, \frac{N}{N-1}\}$.

Key words and phrases. $L^{p\&q}\log L$ double phase growth, higher integrability, boundary higher integrability, logarithmic perturbation, Dirichlet problem, minimizers.

2020 Mathematics Subject Classification: 49N60, 35J66, 35J20.

The second result is a theorem about the local higher integrability for the gradient of the integrand in (1), that is, for the function $\mathcal{H}_L(\cdot, |\nabla u|)$.

Theorem 2. *Let $\mathcal{H}_L$ be the function defined by (2)-(3). If $u \in W^{1,\mathcal{H}_L}_{\mathrm{loc}}(\Omega)$ is a local K-quasiminimizer of the $\mathcal{H}_L$-energy, then we can find $\varepsilon > 0$ such that*

$$\mathcal{H}_L(\cdot, |\nabla u|) \in L^{1+\epsilon}_{\mathrm{loc}}(\Omega).$$

Its proof follows the classical arguments, in the sense that it is established by using a Caccioppoli type inequality, a modular version of the Sobolev-Poincaré inequality and a reverse Hölder inequality. The third main result aims to establish higher integrability up to the boundary for the gradient of a local K-quasiminimizer to the functional in (1).

Theorem 3. *Let $\mathcal{H}_L$ be the function defined by (2)-(3) with $\operatorname{ess\,inf}_{B_{2r}} a(x) > 0$, let $B_{2r} \Subset \Omega$ and let $w \in W^{1,\mathcal{H}_L}_{\mathrm{loc}}(\Omega)$ be a function such that*

$$\int_{B_{2r}} \mathcal{H}_L(x, |\nabla w|)^{1+\delta} dx < +\infty,$$

for some $\delta > 0$. If v is a local K-quasiminimizer of the functional

$$\int_{B_r} \mathcal{H}_L(x, |\nabla u|) dx$$

with boundary values w, then there exists $0 < \varepsilon < \delta$ such that

$$\mathcal{H}_L(x, |\nabla v|)^{1+\varepsilon} \in L^1(B_r).$$

Moreover, the following estimate holds

$$\fint_{B_r} \mathcal{H}_L(x, |\nabla v|)^{1+\varepsilon} dx \leq c \left(\fint_{B_r} \mathcal{H}_L(x, |\nabla v|) dx \right)^{1+\varepsilon} + c \fint_{B_r} \mathcal{H}_L(x, |\nabla w|)^{1+\varepsilon} dx + c,$$

for some positive constant $c = c(p, q, N, K, \mathcal{H}_L)$.

In general, we do not impose that the modulating coefficient $a(\cdot)$ is bounded away from zero, hence we use the bounded condition $\operatorname{ess\,inf}_\Omega a > 0$ only to get higher integrability result up to the boundary (as shown in the statement of Theorem 3, whose proof makes use of Theorem 1). Under such setting, the double phase function $\mathcal{H}(x,t) = t^p + a(x)t^q$ has unbalanced growth as

$$t^p \leq \mathcal{H}(x,t) \leq c_0[t^p + t^q] \quad \text{for a.a. } x \in \Omega, \text{ all } t \geq 0, \text{ some } c_0 > 0.$$

We note that the double phase growth reduces to the weighted (p,q)-Laplacian growth when we assume $\operatorname{ess\,inf}_\Omega a > 0$, and to the p-Laplacian growth when we assume $a \equiv 0$. The logarithm $\log(e + t)$ gives us a perturbation of the growth condition, and we are interested to study the properties of this special class of weak Φ-functions (namely, $\mathcal{H}_L(x,t)$); we refer also to the notion of Orlicz function in the book of Musielak [31]. As usual, the qualitative study of such functions involves the Musielak-Orlicz-Sobolev space $W^{1,\mathcal{H}_L}(\Omega)$ (we give the precise definitions of weak Φ-function and related framework spaces in Sections 2-3). Integral functionals of the form

$$u \mapsto \int_\Omega f(x, \nabla u) dx, \tag{5}$$

where the integrand $f : \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a suitable function, were studied by Marcellini [27, 28, 29] and Zhikov [38, 39, 40] in the context of problems of the calculus of variations and of nonlinear elasticity theory. Marcellini and Zhikov studied the regularity of local minimizers under the $p\&q$-growth assumption

$$|t|^p \leq f(x,t) \leq c[1+|t|^q], \quad q>p>1, \text{ for a.a. } x\in\Omega, \text{ all } t\in\mathbb{R}^N, \text{ some } c>0, \tag{6}$$

which identifies the growth of the integrand with respect to the gradient variable. This task becomes relevant in mathematical models (of anisotropic materials) involving derivatives with different weights along distinct directions. We say that $u \in W^{1,1}_{\text{loc}}(\Omega)$ is a local minimizer of (5) if and only if $f(x,\nabla u)\in L^1_{\text{loc}}(\Omega)$ and satisfies

$$\int_{\operatorname{supp} v} f(x,\nabla u)dx \leq \int_{\operatorname{supp} v} f(x,\nabla(u+v))dx,$$

for all $v\in W^{1,1}(\Omega)$ such that $\operatorname{supp} v\subset\Omega$ (see again Section 3 for more details). An interesting class of functionals that satisfy non-standard growth conditions are those with so-called φ-growth, given as

$$\varphi(|t|) \leq f(x,t) \leq c[1+\varphi(|t|)] \quad \text{for a.a. } x\in\Omega, \text{ all } t\in\mathbb{R}^N, \text{ some } c>0, \tag{7}$$

where $\varphi : [0,+\infty[\to [0,+\infty[$ is an appropriate Orlicz function. Clearly, if $\varphi(|t|)=|t|^p$ then we retrieve a p-growth assumption. We note that the regularity of local minimizers of functionals subject to φ-growth, with appropriate features for φ, were considered by Passarelli di Napoli and co-workers [2, 13], Diening et al. [8], Fusco and Sbordone [11], and many other authors. We recall that the notion of quasiminimizers of functionals into the context of elliptic equations, was considered in the works of Giaquinta [14] and Giusti [18], see also their joint papers [15, 16]. In these works, the authors showed that many features of weak solutions to elliptic problems generalize to appropriate elliptic quasiminimizers $u\in W^{1,1}_{\text{loc}}(\Omega)$ of the type inequality

$$\int_{\operatorname{supp} v} f(x,\nabla u)dx \leq K\int_{\operatorname{supp} v} f(x,\nabla(u+v))dx, \quad K\geq 1, \tag{8}$$

for all $v\in W^{1,1}(\Omega)$ such that $\operatorname{supp} v\subset\Omega$. In the case $K=1$, we have that u reduces to a local minimizer. Of course, there is a strong link between quasiminimizers and minimizers to nonlinear elliptic equations, but some crucial differences occur too. Among the drawbacks of quasiminimizers, we recall some facts in the context of potential theory. Namely, quasiminimizers do not obey any comparison principles, and do not give unique solution for the Dirichlet problem. However, quasiminimizers are largely used to prove regularity of minimizers to variational integrals (without the need to consider the corresponding Euler equation). Thus, we establish here a higher integrability result for an integral functional under unbalanced double phase growth condition exhibiting a logarithmic perturbation (namely $L^{p\&q}\log L$ double phase growth), that at the best of our knowledge is not yet considered in the literature. For other special classes of weak Φ-functions we refer to Example 2.3 of Harjulehto et al. [21], where the authors collect in a table the interpretation of the assumptions in each one of their special cases. A practical motivation in introducing function such $\mathcal{H}_L$ originates in dynamic modeling of mechanical systems, where certain barrier logarithmic terms are brought into the main equations for governing the output signals within appropriate constrained regions (see, for example, He et al. [22]). Further, the fact to consider the logarithmic perturbation on both the terms of the double phase function $\mathcal{H}$, links the working space here to the setting of Orlicz-Zygmund spaces (see Remark 2). However, the salient point is that we can note the equivalence between the modular function (18) and the Luxemburg norm (21) (see Proposition 3).

We note that Giaquinta and Modica [17] focused on higher integrability of the gradient, which is a basic regularity property of weak solutions for elliptic equations in divergence form, and it is a crucial ingredient in relying on regularity theory. We also mention the work of Colombo and Mingione [5] where the assumption $\frac{q}{p} < 1+\frac{\alpha}{N}$ (α being the Hölder exponent of $a(\cdot)$) is used to obtain higher integrability of the gradient. This assumption essentially says that the Hölder exponent of $a(\cdot)$ has to be balanced with the triplet N, p, q. In [1], the authors pointed out that the condition

$$a \in C^{0,\alpha}(\Omega) \text{ with } 0 < \alpha \leq 1 \text{ and } \frac{q}{p} \leq 1 + \frac{\alpha}{N}$$

is sufficient enough to ensure maximal regularity (precisely [5] already covered the case "$<$" (less than) in the last inequality, while [1] handles the case "$=$" (equal to)). In the following, we make large use of this condition. Thus, we establish our results combining a Caccioppoli type inequality together with a Sobolev-Poincaré type inequality. These two inequalities are the necessary tools to obtain a reverse Hölder type inequality which is the main step in showing that the gradient of a local quasiminimizer has local higher integrability. Precisely speaking, we benefit from the De Giorgi's techniques on the regularity of minima, which briefly derived the Hölder continuity of minima (solutions), with an exponent depending only on the constant in the right hand side of appropriate Caccioppoli type inequalities on level sets (recall that weak solutions satisfy such inequalities), and hence depending on the ellipticity ratio of the coefficients' matrix in the considered elliptic equation (see, [30], pp. 358-360). Indeed, Giaquinta, Giusti and Modica applied usefully the Widman's hole filling method, keeping advantage on the fact that the minimality property together with appropriate growth conditions, is sufficient enough to force minimizers to satisfy Caccioppoli type inequalities, and hence the Hölder continuity can be deduced as a consequence of De Giorgi's method (that is, we get the expected regularity results). Moreover, we refer to some representative works concerning double phase problems and quasiminimizer problems: Choudhuri et al. [3], Colasuonno and Squassina [4], De Filippis and Mingione [6, 7], Fang et al. [10], Gasiński and Winkert [12], Liu and Dai [25], Liu and Papageorgiou [26], Papageorgiou et al. [32], Ragusa and Tachikawa [33], Vetro [35], Vetro and Winkert [36], Zhang and Rădulescu [37], and the references therein.

The manuscript is organized as follows. In Section 2, we first introduce the notation and collect relevant facts about certain classes of weak Φ-functions (namely, $\Phi_w(\Omega)$ and $\Phi_c(\Omega)$), then we study the properties of $L^{p\&q}\log L$ double phase function $\mathcal{H}_L$, finally we collect the relevant facts about Musielak-Orlicz spaces related to function $\mathcal{H}_L$. In Section 3, we first revisit the Caccioppoli inequality in the special case of our function $\mathcal{H}_L$ and retrieve a local boundedness result of local K-quasiminimizers, then we obtain a Sobolev-Poincaré type inequality and note a Harnack inequality leading to Hölder continuity of local K-quasiminimizers. In Section 4, we establish the proofs of main results, namely Theorems 2 and 3, respectively, about local higher integrability on a bounded domain for the gradient of a local K-quasiminimizer, and about boundary higher integrability on a ball for local K-quasiminimizers. Finally, in the last section, namely Section 5, we remark that weak solutions to a Dirichlet $\mathcal{H}_L$-Laplacian problem are minimizers of the associated energy functional.

2. Weak Φ-function $\mathcal{H}_L$ and framework space

In this section, we first introduce the notion of Φ-function and list their relevant properties, then we discuss the special class of $L^{p\&q}\log L$ double phase function $\mathcal{H}_L$, hence we collect the relevant facts about the Musielak-Orlicz spaces related to $\mathcal{H}_L$. We refer to the book of Harjulehto and Hasto [19] (see also [31]), and to the works of Colasuonno and Squassina [4], Fan [9], Harjulehto et al. [21], Liu and Dai [25]. According to the notation therein, we now give some preliminaries and materials.

2.1. **Weak Φ-functions.** As usual $B_r(x)$ means the open ball $B(x,r) = \{z \in \mathbb{R}^N : |x-z| < r\}$ centered at $x \in \mathbb{R}^N$ with radius $r > 0$. Moreover, we will use only B_r and B_{nr} $(n > 0)$ to denote two balls with radius r and nr, respectively (that is, we omit the information on the center whenever we work with concentric balls and hence there is not ambiguity in identifying the center). The Lebesgue measure of a set $A \subset \mathbb{R}^N$ is denoted by $|A|$. If $|B_1(0)| = \omega(N) > 1$, then $|B_r| = \omega(N)r^N$. When A and Ω are open sets and $\overline{A}$ is compact, by $A \Subset \Omega$ we mean that $\overline{A} \subset \Omega$. Moreover, by $(u)_A$ we mean the integral average of an integrable function $u : A \to \mathbb{R}$ with $0 < |A| < +\infty$, namely

$$(u)_A = \fint_A u(x)\,dx := \frac{1}{|A|}\int_A u(x)\,dx.$$

We recall that u is an almost increasing function, provided that we can find $\ell \geq 1$ satisfying $u(s) \leq \ell u(t)$ for all $s \leq t$. Similarly, one has the notion of almost decreasing function. Here, we use the concept of weak Φ-function (that is, a special class of Orlicz functions) pointing out on some regularities in the sense of measurability, monotonicity type conditions and weak continuity. Precisely, we have the following key definition.

Definition 1. A weak Φ-function is a function $\varphi : \Omega \times [0,+\infty[\to [0,+\infty]$ such that:

- (φ_1) $x \mapsto \varphi(x,t)$ is a measurable function for all $t \geq 0$, and $t \mapsto \varphi(x,t)$ is an increasing function for all $x \in \Omega$;
- (φ_2) $\varphi(x,0) = \lim_{t\to 0^+}\varphi(x,t) = 0$ and $\lim_{t\to+\infty}\varphi(x,t) = +\infty$ for all $x \in \Omega$;
- (φ_3) $t \mapsto t^{-1}\varphi(x,t)$ is an almost increasing function for all $t > 0$, all $x \in \Omega$;
- (φ_4) $t \mapsto \varphi(x,t)$ is a left-continuous function for all $t > 0$, all $x \in \Omega$.

Let φ^{-1} mean the left-continuous inverse of φ, defined by

$$\varphi^{-1}(x,\tau) := \inf\{t \geq 0 : \varphi(x,t) \geq \tau\}.$$

From now on, we will denote by $\Phi_w(\Omega)$ the class of weak Φ-functions. If $\varphi \in \Phi_w(\Omega)$ satisfies the following:

- (φ_5) $t \mapsto \varphi(x,t)$ is a convex function,

then we write $\varphi \in \Phi_c(\Omega)$, that is, we consider the class of convex Φ-functions. Two interesting properties of the classes $\Phi_w(\Omega)$ and $\Phi_c(\Omega)$ appear in [19] in the following form (namely, properties $(aInc)$ and $(aDec)$).

Definition 2. $\varphi \in \Phi_w(\Omega)$ possesses the property:

- $(aInc)$ if the function $t \mapsto t^{-p}\varphi(x,t)$ is almost increasing for some $p > 1$, a.a. $x \in \Omega$;
- $(aDec)$ if the function $t \mapsto t^{-q}\varphi(x,t)$ is almost decreasing for some $0 < q < +\infty$, a.a. $x \in \Omega$.

Definition 2 essentially provides finer monotonicity conditions, linking the growth of $\varphi \in \Phi_w(\Omega)$ to certain power terms t^p and t^q involved in the double phase growth $\mathcal{H}$. As already mentioned in the Introduction, we remark that appropriate growth conditions are key tools in satisfying some technical requirements to establish regularity results of functionals. Following this finding, we will benefit of the following classical properties of $\varphi \in \Phi_w(\Omega)$. Hence, we say that φ is doubling (namely, $\varphi \in \Delta_2$) if there exists a constant $c_\varphi \geq 1$ such that

$$\varphi(x, 2t) \leq c_\varphi \varphi(x, t) \quad \text{for all } x \in \Omega, \text{ all } t \geq 0. \tag{9}$$

By (9), we deduce the inequality

$$\varphi(x, t+s) \leq c_\varphi \varphi(x, t) + c_\varphi \varphi(x, s) \quad \text{for all } x \in \Omega,\ t, s \geq 0. \tag{10}$$

Moreover, we say that $\varphi \in \Phi_w$ satisfies the ∇_2-condition if there exists a constant $c^\varphi > 1$ such that

$$\varphi(x, c^\varphi t) \geq 2c^\varphi \varphi(x, t) \quad \text{for all } x \in \Omega,\ t \geq 0.$$

To denote this property we write $\varphi \in \nabla_2$. Both the conditions Δ_2 and ∇_2 were introduced to control the growth of the integrand (see (7)). Precisely, Δ_2-condition is used to avoid fast growth (for example, as for e^{t^2}), and the ∇_2-condition is used to avoid slow growth (for example, as for $t \ln(1 + t^2)$). Moreover, we refer the reader to the book [19] for linking the Δ_2-condition to property $(aDec)$ (see Lemma 2.2.6), and the ∇_2-condition to $(aInc)$ (see Corollary 2.4.11), respectively. We remark that certain inequalities as the ones obtained above, play a crucial role in obtaining useful estimates and hence establishing regularity results. Among the others, we recall the Young inequality

$$ts \leq \varphi(x, t) + \varphi^*(x, s), \tag{11}$$

where we involve the notion of complementary of a weak Φ-function $\varphi \in \Phi_w$, namely we define the function

$$\varphi^*(x, t) = \sup_{s \geq 0}(st - \varphi(x, s)) \quad \text{for all } x \in \Omega.$$

2.2. **Properties of function $\mathcal{H}_L$.** This subsection is devoted to the qualitative analysis of the $L^{p\&q} \log L$ double phase growth related to the integral functional in (1). Namely, we consider the function

$$\mathcal{H}_L(x, t) = [t^p + a(x)t^q] \log(e + t) \quad \text{for all } x \in \Omega, \text{ all } t \geq 0,\ 1 < p < q,$$

and introduce suitable hypotheses (on the exponents $p\&q$ and the weight $a(\cdot)$) to verify that this function can be seen as a special class of weak Φ-functions. Clearly, if we assume that $a(x) \geq 0$ for all $x \in \Omega$, then all the conditions of Definition 1 are fulfilled. Moreover, for all $\lambda > 1$ we get the inequalities

$$\lambda^p \mathcal{H}_L(x, t) \leq \mathcal{H}_L(x, \lambda t) = [(\lambda t)^p + a(x)(\lambda t)^q] \log(e + \lambda t) \leq \lambda^{q+1} \mathcal{H}_L(x, t).$$

These inequalities say that $\mathcal{H}_L$ is doubling with constant $c_{\mathcal{H}_L} = c_{\mathcal{H}_L}(q)$ and that $\mathcal{H}_L \in \nabla_2$ with constant $c^{\mathcal{H}_L} = c^{\mathcal{H}_L}(p)$. We remark that $c_{\mathcal{H}_L}, c^{\mathcal{H}_L}$ do not depend on $a(\cdot)$. In the following we shall denote by $c > 0$ a constant whose value may change from line to line, and any relevant dependencies will be underlined by using round parentheses (as in the case of $c_{\mathcal{H}_L}$ above, where we point out the dependence by q).

Now, for $B \subset \Omega$ we define the quantities:

$$\mathcal{H}^-_{L,B}(t) := \inf_{x \in B} \mathcal{H}_L(x, t) \quad \text{and} \quad \mathcal{H}^+_{L,B}(t) := \sup_{x \in B} \mathcal{H}_L(x, t),$$

and without loss of generality we identify $\mathcal{H}^\pm_{L,\Omega}$ by $\mathcal{H}^\pm_L$, that is we set $\mathcal{H}^\pm_L := \mathcal{H}^\pm_{L,\Omega}$. Among the relevant properties that the interested reader can find in the development of weak Φ-functions' theory, we point out the so-called properties $(A0)$ and $(A1)$ (according to the notation used in [20, 21]). Briefly, these technical conditions realize the requirements of maximal regularity, as mentioned in Section 1 (see page 3 herein and refer to [1, 5]). Here we are also interested to prove a third property, namely $(A1\text{-}N)$ that was introduced in the recent work of Harjulehto et al. [21]. This condition corresponds to the log-Hölder continuity in the case of variable exponent setting and to the above mentioned maximal regularity condition by Baroni et al. [1] in the case of double phase growth. We summarize these properties in the following definition.

Definition 3. $\mathcal{H}_L \in \Phi_w(\Omega)$ is said to satisfy the property:

$(A0)$ if we can find $0<\beta<1$ such that $\mathcal{H}_L^+(\beta) \le 1 \le \mathcal{H}_L^-(1)$;

$(A1)$ if we can find $0<\beta<1$ such that

$$\mathcal{H}_{L,B_r}^+(\beta t) \le \mathcal{H}_{L,B_r}^-(t) \quad \text{for all } t \in [1, (\mathcal{H}_{L,B_r}^-)^{-1}(1/|B_r|)], \text{ all balls } B_r \subset \Omega;$$

$(A1\text{-}N)$ if we can find $0<\beta<1$ such that

$$\mathcal{H}_{L,B_r}^+(\beta t) \le \mathcal{H}_{L,B_r}^-(t) \quad \text{for all } t \in [1, \tfrac{1}{\operatorname{diam} B_r}], \text{ for every ball } B_r \subset \Omega.$$

Since these properties realize the regularity requirements in [1], then we establish them by imposing the following set of assumptions:

$$a \in C^{0,\alpha}(\overline{\Omega}) \text{ with } 0<\alpha\le 1 \text{ and } a(x)\ge 0 \text{ for all } x\in\Omega, \tag{12}$$

$$\frac{q}{p} \le 1 + \frac{\alpha}{N}, \tag{13}$$

$$q \le p+\alpha. \tag{14}$$

We note that (13) implies (14) if $p<N$, and (14) implies (13) if $p \ge N$.

In a first result we prove properties $(A0)$ and $(A1)$ to functional $\mathcal{H}_L$ given in (2).

Proposition 1. *If* (3) *holds, then $\mathcal{H}_L$ defined by* (2) *satisfies both* $(A0)$ *and* $(A1)$.

Proof. We obtain the two properties in separate steps. Hence, we first establish $(A0)$.

Claim 1: $\mathcal{H}_L$ satisfies the property $(A0)$. Since $\mathcal{H}_L^-(1) \ge 1$, and for $\beta \in]0,1[$, we have the bound

$$\mathcal{H}_L^+(\beta) \le \beta^p[1+\|a\|_\infty]\log(e+1).$$

It is directly obtained that the following restriction

$$\mathcal{H}_L^+(\beta) \le 1 \le \mathcal{H}_L^-(1)$$

holds for the range of values β satisfying

$$0<\beta\le\left(\frac{1}{[1+\|a\|_\infty]\log(e+1)}\right)^{\frac{1}{p}}.$$

Consequently, property (A0) is verified herein.

Claim 2: $\mathcal{H}_L$ satisfies the property $(A1)$. For a fixed ball $B_r \subset \Omega$, we introduce the quantities

$$i(B_r) := \inf_{x\in B_r} a(x) \quad \text{and} \quad s(B_r) := \sup_{x\in B_r} a(x) \le i(B_r) + c_a(2r)^\alpha,$$

where c_a is the Hölder constant of $a(\cdot)$. Accordingly, we can write the following equalities and inequalities:

$$\mathcal{H}_{L,B_r}^-(t) = [t^p + i(B_r)t^q]\log(e+t),$$

and

$$\mathcal{H}^+_{L,B_r}(t) = [t^p + s(B_r)t^q]\log(e+t),$$

and

$$t^p \le \mathcal{H}^-_{L,B_r}(t) \Longrightarrow (\mathcal{H}^-_{L,B_r})^{-1}\left(\frac{1}{|B_r|}\right) \le \left(\frac{1}{|B_r|}\right)^{\frac{1}{p}}.$$

Therefore, if we choose $t \in \left[1, (\mathcal{H}^-_{L,B_r})^{-1}\left(\frac{1}{|B_r|}\right)\right]$, then we deduce the following

$$t \le |B_r|^{-\frac{1}{p}} = w(N)^{-\frac{1}{p}}\, r^{-\frac{N}{p}} \quad \text{with } r<1. \tag{15}$$

Since $r<1$, from the assumption (13) on the exponents, we obtain the following implication

$$\alpha - \frac{N}{p}(q-p) \ge 0 \Longrightarrow r^{\alpha-\frac{N}{p}(q-p)} \le 1. \tag{16}$$

In the case that $i(B_r) \le c_a(2r)^\alpha$, we first deduce that $s(B_r) \le 2c_a(2r)^\alpha$ and then we show that there exists a suitable value β satisfying

$$\begin{aligned}\mathcal{H}^+_{L,B_r}(\beta t) &\le \beta^p[t^p + s(B_r)t^q]\log(e+t)\\ &\le \beta^p[1+2c_a(2r)^\alpha t^{q-p}]t^p\log(e+t)\\ &\le t^p\log(e+t) \le [t^p + i(B_r)t^q]\log(e+t)\\ &= \mathcal{H}^-_{L,B_r}(t).\end{aligned}$$

To this aim, if we choose $\beta \in]0,1[$ satisfying the bound condition

$$\begin{aligned}\beta^p[1+2^{\alpha+1}c_a(r)^\alpha t^{q-p}] &\le \beta^p[1+2^{\alpha+1}c_a r^{\alpha-\frac{N}{p}(q-p)}w(N)^{-\frac{1}{p}(q-p)}] \quad \text{(by (15))}\\ &\le \beta^p[1+2^{\alpha+1}c_a w(N)^{-\frac{1}{p}(q-p)}] \quad \text{(by (16))}\\ &\le 1,\end{aligned}$$

that is, we choose β such that

$$\beta \le \left(\frac{1}{1+2^{\alpha+1}c_a w(N)^{-\frac{1}{p}(q-p)}}\right)^{\frac{1}{p}},$$

hence we obtain the inequality

$$\mathcal{H}^+_{L,B_r}(\beta t) \le \mathcal{H}^-_{L,B_r}(t) \quad \text{for all } t \in [1, (\mathcal{H}^-_{L,B_r})^{-1}(1/|B_r|)].$$

Now, we consider the other case, that is we assume that $i(B_r) > c_a(2r)^\alpha$, then we show that there exists a suitable value β satisfying

$$\begin{aligned}\mathcal{H}^+_{L,B_r}(\beta t) &\le \beta^p[t^p + s(B_r)t^q]\log(e+t)\\ &\le \beta^p[t^p + 2i(B_r)t^q]\log(e+t)\\ &\le [t^p + \beta^p 2i(B_r)t^q]\log(e+t)\\ &\le [t^p + i(B_r)t^q]\log(e+t)\\ &= \mathcal{H}^-_{L,B_r}(t).\end{aligned}$$

If we choose $\beta \in]0,1[$ satisfying the restriction $\beta \le 2^{-\frac{1}{p}}$, then we obtain also this time the inequality

$$\mathcal{H}^+_{L,B_r}(\beta t) \le \mathcal{H}^-_{L,B_r}(t) \quad \text{for all } t \in [1, (\mathcal{H}^-_{L,B_r})^{-1}(1/|B_r|)].$$

Considering together the restrictions for β in the two above cases, we conclude that property $(A1)$ holds true provided that

$$\beta \leq \min\left\{(1+2^{\alpha+1}c_a w(N)^{-\frac{1}{p}(q-p)})^{-\frac{1}{p}}, 2^{-\frac{1}{p}}\right\}.$$

□

Remark 1. Note that $\mathcal{H}_L$ defined by (2) satisfies the property $(A0)$ whenever $a \in L^\infty(\Omega)$. That is, the proof of Claim 1 in Proposition 1 remains true if we assume $a \in L^\infty(\Omega)$ instead of $a \in C^{0,\alpha}(\overline{\Omega})$ with $0 < \alpha \leq 1$.

In the next result we give the precise proof of the property $(A1\text{-}N)$ for the function $\mathcal{H}_L$.

Proposition 2. *If* (12) *and* (14) *hold, then* $\mathcal{H}_L$ *defined by* (2) *satisfies* $(A1\text{-}N)$.

Proof. Similar to the proof of Claim 2 in Proposition 1, the starting point of the proof is in considering a ball. Different from Claim 2, here we require that the diameter of the ball is not greater than 1 (that is, $\operatorname{diam} B_r \leq 1$). Moreover, we again set

$$i(B_r) := \inf_{x\in B_r} a(x) \quad \text{and} \quad s(B_r) := \sup_{x\in B_r} a(x) \leq i(B_r) + c_a(2r)^\alpha,$$

where c_a is the Hölder constant of $a(\cdot)$. Consequently we have

$$\mathcal{H}^-_{L,B_r}(t) = [t^p + i(B_r)t^q]\log(e+t)$$

and

$$\mathcal{H}^+_{L,B_r}(t) = [t^p + s(B_r)t^q]\log(e+t)$$

for all $1 \leq t \leq (2r)^{-1}$. Since $\operatorname{diam} B_r \leq 1$, or equivalently $2r \leq 1$, then by (14) we deduce that

$$\alpha - (q-p) \geq 0 \Longrightarrow (2r)^{\alpha-(q-p)} \leq 1. \tag{17}$$

Now, we distinguish two cases. When $i(B_r) \leq c_a(2r)^\alpha$, we immediately have that $s(B_r) \leq 2c_a(2r)^\alpha$, and then we show that there exists a suitable value β satisfying the following

$$\begin{aligned}
\mathcal{H}^+_{L,B_r}(\beta t) &\leq \beta^p[t^p + s(B_r)t^q]\log(e+t)\\
&\leq \beta^p[1 + 2c_a(2r)^\alpha t^{q-p}]t^p\log(e+t)\\
&\leq t^p\log(e+t)\\
&\leq [t^p + i(B_r)t^q]\log(e+t)\\
&= \mathcal{H}^-_{L,B_r}(t).
\end{aligned}$$

Hence, if we choose $\beta \in]0,1[$ to verify the following bound condition

$$\begin{aligned}
\beta^p[1+2c_a(2r)^\alpha t^{q-p}] &\leq \beta^p[1+2c_a(2r)^{\alpha-(q-p)}] \quad (\text{as } t \leq (2r)^{-1})\\
&\leq \beta^p[1+2c_a] \quad (\text{by (17)})\\
&\leq 1,
\end{aligned}$$

then we obtain that the following restriction on the values of β is required

$$\beta \leq \left(\frac{1}{1+2c_a}\right)^{\frac{1}{p}}.$$

Thus, we conclude the inequality

$$\mathcal{H}^+_{L,B_r}(\beta t) \leq \mathcal{H}^-_{L,B_r}(t) \quad \text{for all } t \in [1, \tfrac{1}{\operatorname{diam} B_r}].$$

On the other hand, we consider the other case, that is, we assume that $i(B_r) > c_a(2r)^\alpha$. Consequently, we get

$$\begin{aligned}
\mathcal{H}^+_{L,B_r}(\beta t) &\leq \beta^p[t^p + s(B_r)t^q]\log(e+t)\\
&\leq \beta^p[t^p + 2i(B_r)t^q]\log(e+t)\\
&\leq [t^p + \beta^p 2i(B_r)t^q]\log(e+t)\\
&\leq [t^p + i(B_r)t^q]\log(e+t)\\
&= \mathcal{H}^-_{L,B_r}(t),
\end{aligned}$$

whenever $\beta \in]0,1[$ is such that $\beta \leq 2^{-\frac{1}{p}}$. This implies

$$\mathcal{H}^+_{L,B_r}(\beta t) \leq \mathcal{H}^-_{L,B_r}(t) \quad \text{for all } t \in [1, \tfrac{1}{\operatorname{diam} B_r}].$$

Considering together the restrictions for β in the two above cases, we conclude that property ($A1$-N) holds true provided that

$$\beta \leq \min\left\{(1+2c_a)^{-\frac{1}{p}}, 2^{-\frac{1}{p}}\right\}.$$

□

2.3. **Musielak-Orlicz spaces associated to $\mathcal{H}_L$.** Let $\mathcal{M}$ denote the space of all measurable functions $u : \Omega \to \mathbb{R}$. We identify two such functions differing only on a Lebesgue-null subset of Ω. Let $\rho_{\mathcal{H}_L}(\cdot)$ be the modular function given as

$$\rho_{\mathcal{H}_L}(u) = \int_\Omega \mathcal{H}_L(x, |u|)\,dx. \tag{18}$$

The Musielak-Orlicz space $L^{\mathcal{H}_L}(\Omega)$ is defined by

$$L^{\mathcal{H}_L}(\Omega) = \{u \in \mathcal{M}(\Omega) : \rho_{\mathcal{H}_L}(u) < +\infty\},$$

and furnished with the following norm

$$\|u\|_{\mathcal{H}_L} = \inf\left\{\lambda > 0 : \rho_{\mathcal{H}_L}\left(\frac{u}{\lambda}\right) \leq 1\right\} \quad \text{(Luxemburg norm)}. \tag{19}$$

We know that $L^{\mathcal{H}_L}(\Omega)$ is complete with respect to the Luxemburg norm, namely $(L^{\mathcal{H}_L}(\Omega), \|\cdot\|_{\mathcal{H}_L})$ is a Banach space (see [31], Theorem 7.7). Inequality (11) leads to the following Hölder inequality

$$\int_\Omega |u||v|dx \leq 2\|u\|_{\mathcal{H}_L}\|v\|_{\mathcal{H}_L^*} \quad \text{for all } u \in L^{\mathcal{H}_L}(\Omega), \text{ all } v \in L^{\mathcal{H}_L^*}(\Omega).$$

We point out that $\rho_{\mathcal{H}_L}(\cdot)$ and the Luxemburg norm above are closely related to each other, as stated in the following proposition with complete proof.

Proposition 3. *Let $\mathcal{H}_L$ be the function defined by (2)-(3), then we have:*

(i) *if $u \in L^{\mathcal{H}_L}(\Omega) \setminus \{0\}$, then $\|u\|_{\mathcal{H}_L} = \lambda$ if and only if $\rho_{\mathcal{H}_L}\left(\frac{u}{\lambda}\right) = 1$;*
(ii) *$\|u\|_{\mathcal{H}_L} < 1$ (resp. $> 1, = 1$) if and only if $\rho_{\mathcal{H}_L}(u) < 1$ (resp. $> 1, = 1$);*
(iii) *If $\|u\|_{\mathcal{H}_L} < 1$, then $\|u\|^{q+1}_{\mathcal{H}_L} \leq \rho_{\mathcal{H}_L}(u) \leq \|u\|^p_{\mathcal{H}_L}$;*
(iv) *If $\|u\|_{\mathcal{H}_L} > 1$, then $\|u\|^p_{\mathcal{H}_L} \leq \rho_{\mathcal{H}_L}(u) \leq \|u\|^{q+1}_{\mathcal{H}_L}$.*

Proof. Given $u \in L^{\mathcal{H}_L}(\Omega)$, we know that the function $\mathbb{R} \ni \lambda \mapsto \rho_{\mathcal{H}_L}(\lambda u)$ is continuous, convex and even. Since $\lambda > 0$, it is strictly increasing too. By definitions, we easily get

$$\|u\|_{\mathcal{H}_L} = \lambda \quad \text{if and only if} \quad \rho_{\mathcal{H}_L}\left(\frac{u}{\lambda}\right) = 1.$$

It follows that item (i) is fulfilled, and hence also item (ii) is proved.

Next, depending on the values of ξ, we know that

$$\begin{aligned} &\xi^p \rho_{\mathcal{H}_L}(u) \le \rho_{\mathcal{H}_L}(\xi u) \le \xi^{q+1}\rho_{\mathcal{H}_L}(u) \quad \text{if } \xi > 1,\\ &\xi^{q+1} \rho_{\mathcal{H}_L}(u) \le \rho_{\mathcal{H}_L}(\xi u) \le \xi^{p}\rho_{\mathcal{H}_L}(u) \quad \text{if } 0 < \xi < 1. \end{aligned} \tag{20}$$

Further, if $\|u\|_{\mathcal{H}_L} = \lambda$ with $0 < \lambda < 1$, by (i) we know that $\rho_{\mathcal{H}_L}\left(\frac{u}{\lambda}\right) = 1$. For $\xi = \frac{1}{\lambda} > 1$, we have

$$\frac{\rho_{\mathcal{H}_L}(u)}{\lambda^p} \le \rho_{\mathcal{H}_L}\left(\frac{u}{\lambda}\right) = 1 \le \frac{\rho_{\mathcal{H}_L}(u)}{\lambda^{q+1}},$$

by the first line of formula (20). This establishes item (iii). Using the second line of formula (20), item (iv) follows easily. This concludes the proof. □

Remark 2. Referring to Iwaniec and Verde [23], we recall the Orlicz-Zygmund space $L^p \log^\alpha L(\Omega)$ with $0 < \alpha$ and $1 < p < +\infty$, defined by

$$L^p \log^\alpha L(\Omega) = \left\{ u \in L^1(\Omega) : \int_\Omega |u|^p \log^\alpha(e + |u|)dx < +\infty \right\},$$

and furnished with the Luxemburg norm

$$\|u\|_{L^p \log^\alpha L} = \inf\left\{ \lambda > 0 : \int_\Omega \frac{|u|^p}{\lambda^p} \log^\alpha\left(e + \frac{|u|}{\lambda}\right) dx \le 1 \right\}. \tag{21}$$

Clearly, in the case $\alpha = 1$ we identify $L^{\mathcal{H}_L}(\Omega)$ with $L^p \log L(\Omega)$ on the set $\{x \in \Omega : a(x) = 0\}$. As in the other settings, a salient point is the equivalence between the Luxemburg norm (21) and the corresponding modular function in $L^p \log L(\Omega)$.

Referring to $C_0^\infty(\Omega) = \{u \in C^\infty(\Omega) : \text{supp}\, u \text{ is compact in } \Omega\}$, by [19, Theorem 3.7.15], we know that this space is dense in $L^{\mathcal{H}_L}(\Omega)$ (note that $\mathcal{H}_L$ fulfils the properties $(A0)$ and (aDec)). We introduce the Musielak-Orlicz-Sobolev space $W^{1,\mathcal{H}_L}(\Omega)$ defined by

$$W^{1,\mathcal{H}_L}(\Omega) = \{u \in L^{\mathcal{H}_L}(\Omega) : |\nabla u| \in L^{\mathcal{H}_L}(\Omega)\}.$$

Now, we endow $W^{1,\mathcal{H}_L}(\Omega)$ with the following norm

$$\|u\|_{1,\mathcal{H}_L} = \|u\|_{\mathcal{H}_L} + \|\nabla u\|_{\mathcal{H}_L},$$

where we assume $\|\nabla u\|_{\mathcal{H}_L} := \||\nabla u|\|_{\mathcal{H}_L}$. Further, we introduce the space $W_0^{1,\mathcal{H}_L}(\Omega)$ defined by

$$W_0^{1,\mathcal{H}_L}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,\mathcal{H}_L}},$$

that is the completion of $C_0^\infty(\Omega)$ in $W^{1,\mathcal{H}_L}(\Omega)$. Since $\mathcal{H}_L$ satisfies the property (aDec), we know that the Musielak-Orlicz-Sobolev spaces $W^{1,\mathcal{H}_L}(\Omega)$ and $W_0^{1,\mathcal{H}_L}(\Omega)$ are separable Banach spaces. Further, $\mathcal{H}_L$ fulfils also the property (aInc), hence we have that these spaces are uniformly convex and reflexive (we refer to [19, Theorem 6.1.4]).

Working in $W_0^{1,\mathcal{H}_L}(\Omega)$, we have the Poincaré inequality, namely, there exists $c > 0$ such that

$$\|u\|_{\mathcal{H}_L} \le c\,\|\nabla u\|_{\mathcal{H}_L} \quad \text{for all } u \in W_0^{1,\mathcal{H}_L}(\Omega). \tag{22}$$

A proof of inequality (22) is given in [34, Proposition 6], see also [19, Theorem 6.2.8] (since the function $\mathcal{H}_L$ fulfils $(A0)$ and $(A1)$ of Definition 3). As a consequence of this Poincaré inequality, we deduce that two involved norms (namely, $\|u\|_{1,\mathcal{H}_L} = \|u\|_{\mathcal{H}_L} + \|\nabla u\|_{\mathcal{H}_L}$ and $\|\nabla u\|_{\mathcal{H}_L}$) are equivalent in $W_0^{1,\mathcal{H}_L}(\Omega)$.

3. Inequalities involving $\mathcal{H}_L$

In this section, we note Sobolev-Poincaré type inequality and Harnack inequality for the special function $\mathcal{H}_L$. As mentioned in the introduction, the basic tool to obtain our main results is the Caccioppoli inequality for a local K-quasiminimizer (see also (8)). The Caccioppoli estimates are the inequalities that most strongly emphasize the reverse form of the Sobolev-Poincaré inequality (see Proposition 1). We first recall the definition of local K-quasiminimizer related to $\mathcal{H}_L$ that will be used in the sequel.

Definition 4. Let $\mathcal{H}_L$ be the function defined by (2) and (12), and let $K \geq 1$. A function $u \in W^{1,\mathcal{H}_L}_{\mathrm{loc}}(\Omega)$ is a local K-quasiminimizer of the $\mathcal{H}_L$-energy in Ω if

$$\int_{B_r} \mathcal{H}_L(x, |\nabla u|)dx \leq K \int_{B_r} \mathcal{H}_L(x, |\nabla v|)dx$$

for all $B_r \Subset \Omega$ and all $v \in W^{1,\mathcal{H}_L}(\Omega)$ with $\operatorname{supp}(u - v) \subset B_r$.

Next we give a definition of local K-quasiminimizers on a ball with boundary values.

Definition 5. Let $\mathcal{H}_L$ be the function defined by (2) and (12), $K \geq 1$ and let $B_r \subset \Omega$. A function $u \in W^{1,\mathcal{H}_L}(B_r)$ is a local K-quasiminimizer of the $\mathcal{H}_L$-energy on B_r with the boundary values $w \in W^{1,\mathcal{H}_L}(B_r)$, if $u - w \in W_0^{1,\mathcal{H}_L}(B_r)$ and

$$\int_{B_s(z)\cap B_r} \mathcal{H}_L(x, |\nabla u|)dx \leq K \int_{B_s(z)\cap B_r} \mathcal{H}_L(x, |\nabla v|)dx$$

for every $B_s(z)$ with $z \in \overline{B_r}$ and for all $v \in W^{1,\mathcal{H}_L}(B_r)$ with $\operatorname{supp}(u - v) \subset B_s(z)$.

Then, we establish the following Caccioppoli inequality for local K-quasiminimizer of (1), by using the Widman's hole filling method.

Theorem 4. *Let $\mathcal{H}_L$ be the function defined by* (2) *and* (12). *If u is a local K-quasiminimizer of the $\mathcal{H}_L$-energy in Ω and $B_{2r} \Subset \Omega$, then we have*

$$\int_{B_r} \mathcal{H}_L(x, |\nabla u|)dx \leq c \int_{B_{2r}} \mathcal{H}_L\left(x, \frac{|u - (u)_{B_{2r}}|}{r}\right) dx, \tag{23}$$

for some constant $c = c(q, K) > 0$.

Proof. Without any loss of generality, we consider concentric balls satisfying the radius constraints $r < t < s < 2r \leq 1$, and we introduce a cut-off function $\eta \in C_0^\infty(B_{2r})$ with

$$0 \leq \eta \leq 1,\ \operatorname{supp}(\eta) \subset B_s,\ \eta \equiv 1 \text{ on } B_t,\ |\nabla \eta| < \frac{2}{s-t}.$$

Based on this cut-off function, we introduce the auxiliary function

$$w(x) = \eta(x)(u(x) - (u)_{B_{2r}}),$$

and consider test function $v = u - w$. Consequently, we deduce that

$$\operatorname{supp}(u - v) \subset \operatorname{supp}(\eta) \subset B_s \quad \text{and} \quad v = (u)_{B_{2r}} \text{ on } B_t,$$

which leads to $\nabla v = 0$ on B_t. For the above choice of v, using the properties of $\eta(\cdot)$ and involving the fact that

$$\begin{aligned}|\nabla v| &\leq (1-\eta)|\nabla u| + |\nabla \eta||u-(u)_{B_{2r}}|\\ &\leq 2|\nabla u| + 2\frac{|u-(u)_{B_{2r}}|}{s-t},\end{aligned}$$

then we obtain the chain of inequalities

$$\begin{aligned}&\int_{B_t}[|\nabla u|^p + a(x)|\nabla u|^q]\log(e+|\nabla u|)dx\\
&\leq \int_{B_s}[|\nabla u|^p + a(x)|\nabla u|^q]\log(e+|\nabla u|)dx\\
&\leq K\int_{B_s}[|\nabla v|^p + a(x)|\nabla v|^q]\log(e+|\nabla v|)dx\\
&= K\int_{B_s\setminus B_t}[|\nabla v|^p + a(x)|\nabla v|^q]\log(e+|\nabla v|)dx \quad (\text{recall } \nabla v = 0 \text{ on } B_t)\\
&\leq c\int_{B_s\setminus B_t}\left[\left(|\nabla u| + \frac{|u-(u)_{B_{2r}}|}{s-t}\right)^p + a(x)\left(|\nabla u| + \frac{|u-(u)_{B_{2r}}|}{s-t}\right)^q\right]\\
&\qquad \times \log\left(e + |\nabla u| + \frac{|u-(u)_{B_{B_{2r}}}|}{s-t}\right)dx\\
&\leq c\int_{B_s\setminus B_t}[|\nabla u|^p + a(x)|\nabla u|^q]\log\left(e+|\nabla u|\right)dx\\
&\quad + c\int_{B_s}\left(\frac{|u-(u)_{B_{2r}}|^p}{(s-t)^p} + a(x)\frac{|u-(u)_{B_{2r}}|^q}{(s-t)^q}\right)\\
&\qquad \times \log\left(e + \frac{|u-(u)_{B_{2r}}|}{s-t}\right)dx \quad (\text{recall } (10))\\
&\leq c\int_{B_s\setminus B_t}[|\nabla u|^p + a(x)|\nabla u|^q]\log\left(e+|\nabla u|\right)dx\\
&\quad + c\int_{B_{2r}}\left(\frac{|u-(u)_{B_{2r}}|^p}{(s-t)^p} + a(x)\frac{|u-(u)_{B_{2r}}|^q}{(s-t)^q}\right)\log\left(e + \frac{|u-(u)_{B_{2r}}|}{s-t}\right)dx\end{aligned} \tag{24}$$

for some constant $c = c(q, K) > 0$. Involving the Widman's hole filling method, that is, by adding to both sides of the final inequality in formula (24) the quantity

$$c\int_{B_t}(|\nabla u|^p + a(x)|\nabla u|^q)\log(e+|\nabla u|)dx,$$

and dividing the obtained inequality by $c+1$, we get

$$\begin{aligned}&\int_{B_t}[|\nabla u|^p + a(x)|\nabla u|^q]\log(e+|\nabla u|)dx\\
\leq&\vartheta\int_{B_s}[|\nabla u|^p + a(x)|\nabla u|^q]\log(e+|\nabla u|)dx\\
&+ c\int_{B_{2r}}\left[\frac{|u-(u)_{B_{2r}}|^p}{(s-t)^p} + a(x)\frac{|u-(u)_{B_{2r}}|^q}{(s-t)^q}\right]\log\left(e + \frac{|u-(u)_{B_{2r}}|}{s-t}\right)dx,\end{aligned} \tag{25}$$

where $0 < \vartheta < 1$. Next, we construct a sequence $\{t_i\}$ iteratively by

$$t_0 = r \quad \text{and} \quad t_{i+1} - t_i = (1-\lambda)\lambda^i r$$

for some $\lambda \in]0,1[$. By (25) and involving a standard iteration as given in [18] (see Lemma 6.1), we obtain the following estimate

$$\begin{aligned}
&\int_{B_r} [|\nabla u|^p + a(x)|\nabla u|^q] \log(e+|\nabla u|)dx \\
&\le \vartheta^n \int_{B_{t_n}} [|\nabla u|^p + a(x)|\nabla u|^q] \log(e+|\nabla u|)dx \\
&\quad + c\sum_{i=1}^{n} \vartheta^{i-1} \int_{B_{2r}} \left[\frac{|u-(u)_{B_{2r}}|^p}{(t_i - t_{i-1})^p} + a(x)\frac{|u-(u)_{B_{2r}}|^q}{(t_i - t_{i-1})^q}\right] \log\left(e + \frac{|u-(u)_{B_{2r}}|}{t_i - t_{i-1}}\right) dx \\
&\le \lim_{n\to+\infty} \vartheta^n \int_{B_{t_n}} (|\nabla u|^p + a(x)|\nabla u|^q) \log(e+|\nabla u|)dx \\
&\quad + c\sum_{n=1}^{+\infty} \vartheta^{n-1} \int_{B_{2r}} \left[\frac{|u-(u)_{B_{2r}}|^p}{(t_n - t_{n-1})^p} + a(x)\frac{|u-(u)_{B_{2r}}|^q}{(t_n - t_{n-1})^q}\right] \log\left(e + \frac{|u-(u)_{B_{2r}}|}{t_n - t_{n-1}}\right) dx \\
&\le \frac{c}{(1-\lambda)^{q+1}} \int_{B_{2r}} \left[\frac{|u-(u)_{B_{2r}}|^p}{r^p} + a(x)\frac{|u-(u)_{B_{2r}}|^q}{r^q}\right] \\
&\qquad \times \log\left(e + \frac{|u-(u)_{B_{2r}}|}{r}\right) dx \sum_{n=1}^{+\infty} \frac{\vartheta^{n-1}}{(\lambda^{q+1})^{n-1}}.
\end{aligned}$$

Finally, given λ satisfying the constraints $\vartheta < \lambda^{q+1} < 1$, we conclude that

$$\sum_{n=1}^{+\infty} \frac{\vartheta^{n-1}}{(\lambda^{q+1})^{n-1}} < +\infty$$

and hence the Caccioppoli inequality (23) is proved, as $\mathcal{H}_L(x,t) = [t^p + a(x)t^q]\log(e+t)$ for all $x \in \Omega$, all $t \ge 0$. $\square$

The following local boundedness result is analogous to Theorem 1.2 of [21], and hence we omit the proof.

Theorem 5. *Let $\mathcal{H}_L$ be the function defined by* (2)-(3)*, then each local K-quasiminimizer of the $\mathcal{H}_L$-energy is locally bounded.*

Remark 3. In establishing Theorem 5, the function $\mathcal{H}_L$ needs to satisfy the properties $(A0)$ and $(A1)$ (as follows by Proposition 1), $(aDec)$ and $(aInc)$. Now, referring to Definition 2, $\mathcal{H}_L$ given by (2)-(3) satisfies $(aDec)$ with respect to $q+1$ and the almost decreasing constant $\ell = 1$. It also satisfies $(aInc)$ with respect to p and the almost increasing constant $\ell = 1$.

We know that certain so-called energy estimates are crucial tools in establishing the regularity properties of the solutions to partial differential equations (see also Section 5). From [20, Proposition 3.6], we derive the following modular version of the Sobolev-Poincaré inequality for the function $\mathcal{H}_L : B_r \times [0,+\infty[\to [0,+\infty[$ defined by

$$\mathcal{H}_L(x,t) = [t^p + a(x)t^q]\log(e+t) \quad \text{for all } x \in B_r, \text{ all } t \ge 0,\ 1 < p < q,$$

where $a \in C^{0,\alpha}(\overline{B_r})$ is non-negative with $0<\alpha\le 1$ and $q/p \le 1+\alpha/N$ and $B_r \subset \mathbb{R}^N$ is a ball of radius r and let $s\in[1,p]$ with $s<N/(N-1)$. Then there exists a constant $c=c(N,s,\mathcal{H}_L)$ such that

$$\fint_{B_r} \mathcal{H}_L\left(x, \frac{|u-(u)_{B_r}|}{r}\right)dx \le c\left[\left(\fint_{B_r} \mathcal{H}_L(x,|\nabla u|)^{\frac{1}{s}}dx\right)^s + 1\right] \tag{26}$$

for all $u\in W^{1,1}(B_r)$ with $\|\nabla u\|_{L^{\mathcal{H}_L}(B_r)}<1$.

As usual, this Sobolev-Poincaré inequality says that the mean oscillation of a smooth perturbed function is controlled by the average of its perturbed gradient. Moreover, the above inequality yields an exponent strictly less than 1 necessary for the use of reverse Hölder inequality where the integral averages are taken over balls $B_r\subset B_{2r}\subset\Omega$ (see Lemma 1 below). The last result of this section, due to Giaquinta and Modica [17], is well-known and allows to derive higher integrability from reverse Hölder inequality.

Lemma 1. *Let $f\in L^1(B_{2r})$ be non-negative. Assume that there exists $0<s<1$ such that*

$$\fint_{B_\gamma} f dx \le c\left[\left(\fint_{B_{2\gamma}} f^s dx\right)^{\frac{1}{s}} + \fint_{B_{2r}} g dx\right]$$

for all $B_{2\gamma}\Subset B_{2r}$, some $c>0$. If $g\in L^m(B_{2r})$ for some $m>1$, then there exists $1<t<m$ such that

$$\fint_{B_r} f^t dx \le c\left[\left(\fint_{B_{2r}} f dx\right)^t + \fint_{B_{2r}} g^t dx\right] \quad \textit{for some } c>0.$$

Next, we are going to establish the proof of Theorem 1, namely we obtain a Sobolev-Poincaré type inequality for functions which need not vanish everywhere on the boundary, but they are zero on a set of positive measure. This result is constructed by the Sobolev-Poincaré inequality (26), involving the assumption (13) (namely the last inequality in (3)).

Proof of Theorem 1. The assumption $u=0$ a.e. on a measurable set $A\subset B_r$ implies that

$$\begin{aligned}
&\int_A \left[\frac{|u-(u)_{B_r}|^p}{r^p} + \|a\|_\infty \frac{|u-(u)_{B_r}|^q}{r^q}\right]\log\left(e+\frac{|u-(u)_{B_r}|}{r}\right)dx\\
&= |A|\left[\frac{|(u)_{B_r}|^p}{r^p} + \|a\|_\infty\frac{|(u)_{B_r}|^q}{r^q}\right]\log\left(e+\frac{|(u)_{B_r}|}{r}\right).
\end{aligned}$$

Dividing by $|A|$ both the sides of this equality, since $A\subset B_r$ we get that

$$\begin{aligned}
&\left[\frac{|(u)_{B_r}|^p}{r^p} + \|a\|_\infty\frac{|(u)_{B_r}|^q}{r^q}\right]\log\left(e+\frac{|(u)_{B_r}|}{r}\right)\\
&\le \frac{1}{|A|}\int_{B_r}\left[\frac{|u-(u)_{B_r}|^p}{r^p} + \|a\|_\infty \frac{|u-(u)_{B_r}|^q}{r^q}\right]\log\left(e+\frac{|u-(u)_{B_r}|}{r}\right)dx\\
&\le \frac{c}{|A|}\int_{B_r}\left[\frac{|u-(u)_{B_r}|^p}{r^p} + a_0 \frac{|u-(u)_{B_r}|^q}{r^q}\right]\log\left(e+\frac{|u-(u)_{B_r}|}{r}\right)dx\\
&\le \frac{c}{|A|}\int_{B_r}\left[\frac{|u-(u)_{B_r}|^p}{r^p} + a(x) \frac{|u-(u)_{B_r}|^q}{r^q}\right]\log\left(e+\frac{|u-(u)_{B_r}|}{r}\right)dx.
\end{aligned}$$

Referring to the integral term herein, we obtain the following estimate

$$
\begin{aligned}
&\int_{B_r}\left[\frac{|u|^p}{r^p}+a(x)\frac{|u|^q}{r^q}\right]\log\left(e+\frac{|u|}{r}\right)dx\\
&\le c\int_{B_r}\left[\frac{|u-(u)_{B_r}|^p}{r^p}+a(x)\frac{|u-(u)_{B_r}|^q}{r^q}\right]\log\left(e+\frac{|u-(u)_{B_r}|}{r}\right)dx\\
&\quad+c\int_{B_r}\left[\frac{|(u)_{B_r}|^p}{r^p}+a(x)\frac{|(u)_{B_r}|^q}{r^q}\right]\log\left(e+\frac{|(u)_{B_r}|}{r}\right)dx \text{ (by (10) for some } c=c(q)>0)\\
&\le c\int_{B_r}\left[\frac{|u-(u)_{B_r}|^p}{r^p}+a(x)\frac{|u-(u)_{B_r}|^q}{r^q}\right]\log\left(e+\frac{|u-(u)_{B_r}|}{r}\right)dx\\
&\quad+c\int_{B_r}\left[\frac{|(u)_{B_r}|^p}{r^p}+\|a\|_\infty\frac{|(u)_{B_r}|^q}{r^q}\right]\log\left(e+\frac{|(u)_{B_r}|}{r}\right)dx\\
&\le c\int_{B_r}\left[\frac{|u-(u)_{B_r}|^p}{r^p}+a(x)\frac{|u-(u)_{B_r}|^q}{r^q}\right]\log\left(e+\frac{|u-(u)_{B_r}|}{r}\right)dx\\
&\quad+c\frac{|B_r|}{|A|}\int_{B_r}\left[\frac{|u-(u)_{B_r}|^p}{r^p}+a(x)\frac{|u-(u)_{B_r}|^q}{r^q}\right]\log\left(e+\frac{|u-(u)_{B_r}|}{r}\right)dx.
\end{aligned}
$$

Finally, involving the modular version of the Sobolev-Poincaré inequality, that is we use inequality (26), then we conclude that

$$
\begin{aligned}
&\fint_{B_r}\left[\frac{|u|^p}{r^p}+a(x)\frac{|u|^q}{r^q}\right]\log\left(e+\frac{|u|}{r}\right)dx\\
&\le c\left(1+\frac{1}{c_0}\right)\fint_{B_r}\left[\frac{|u-(u)_{B_r}|^p}{r^p}+a(x)\frac{|u-(u)_{B_r}|^q}{r^q}\right]\log\left(e+\frac{|u-(u)_{B_r}|}{r}\right)dx\\
&\le c\left(1+\frac{1}{c_0}\right)\left[\left(\fint_{B_R}\left(|\nabla u|^p+a(x)|\nabla u|^q\right)^{\frac{1}{s}}\log^{\frac{1}{s}}(e+|\nabla u|)\,dx\right)^s+1\right]\quad\text{(by (26))}.
\end{aligned}
$$

This completes the proof. $\square$

Remark 4. The Caccioppoli inequality (23) together with the Sobolev-Poincaré inequality (4) give us the necessary tools to obtain a reverse Hölder inequality. This fact will be noted in next section.

Following the similar arguments in [21], one can establish the Harnack inequality for $\mathcal{H}_L$ in the following form.

Lemma 2. *Let $\mathcal{H}_L$ be the function defined by* (2) *and* (12)*. Assume that* (13) *holds if $p<N$ (resp.,* (14) *holds if $p\ge N$). If $u\in W^{1,\mathcal{H}_L}_{\mathrm{loc}}(\Omega)$ is a non-negative local K-quasiminimizer of the $\mathcal{H}_L$-energy, then we can find r_0 satisfying*

$$
\operatorname*{ess\,sup}_{x\in B_r} u(x)\le c\left(\operatorname*{ess\,inf}_{x\in B_r} u(x)+r\right) \tag{27}
$$

for all $0<r\le r_0$, where $c=c(N,K,\|u\|_{L^\infty(B_{2r})},\mathcal{H}_L)$, $B_{2r}\Subset\Omega$.

We leave the proof of this result to the reader (we refer also to [21]). We only note that to establish Lemma 2, the function $\mathcal{H}_L$ needs to satisfy all the properties ($A0$), ($A1$), ($A1$-N), ($aDec$) and ($aInc$) (recall Remark 3 and Proposition 2 (for ($A1$-N))) given in Section 2. Clearly, the constant c in (27) depends also on the parameters involved in these properties. As a consequence of Lemma 2, we have the following result about Hölder continuity of the local K-quasiminimizers.

Corollary 1. *Let $\mathcal{H}_L$ be the function defined by* (2) *and* (12). *Assume that* (13) *holds if $p < N$ (resp.* (14) *holds if $p \geq N$). If u is a local K-quasiminimizer of the $\mathcal{H}_L$-energy, then $u \in C^{0,\beta}_{\mathrm{loc}}(\Omega)$ for some suitable $\beta > 0$.*

4. Higher integrability

In this section we collect the proofs of main Theorems 2 and 3. We recall that the first one is a result about the local higher integrability for the gradient of the integrand in (1), imposing sufficient regularities (as discussed in the Introduction). The proof follows the classical arguments and is established by using the Caccioppoli type inequality, the modular version of the Sobolev-Poincaré inequality and the reverse Hölder inequality, however for completeness and convenience of the reader, we give a short proof.

Proof of Theorem 2. We design the setting required to apply the Sobolev-Poincaré inequality (26). Thus, we assume that $1 < s < \min\{p, \frac{N}{N-1}\}$ and choose a ball B_r with $B_{2r} \Subset \Omega$ satisfying the constraint $\|\nabla u\|_{L^{\mathcal{H}_L}(B_{2r})} < 1$ (here, with a slight abuse of notation with respect to our convention in (21), we have denoted by $\|\nabla u\|_{L^{\mathcal{H}_L}(B_{2r})}$ the Luxemburg norm). The Caccioppoli inequality for a local K-quasiminimizer u and $B_\gamma \Subset B_r$, gives us that

$$\fint_{B_\gamma} \mathcal{H}_L(x, |\nabla u|)dx \leq c \fint_{B_{2\gamma}} \mathcal{H}_L\left(x, \frac{|u-(u)_{B_{2\gamma}}|}{\gamma}\right) dx.$$

Thus, we can use the modular version of the Sobolev-Poincaré inequality (recall (26)) to obtain that

$$\fint_{B_\gamma} \mathcal{H}_L(x, |\nabla u|)dx \leq c \left(\fint_{B_{2\gamma}} \mathcal{H}_L(x, |\nabla u|)^{\frac{1}{s}} dx\right)^s + c.$$

If we choose $g = c\chi_\Omega$ in Lemma 1, then we conclude that

$$\fint_{B_r} \mathcal{H}_L(x, |\nabla u|)^t dx \leq c \left[\left(\fint_{B_{2r}} \mathcal{H}_L(x, |\nabla u|)dx\right)^t + \fint_{B_{2r}} g^t dx\right] < +\infty$$

for some $t > 1$. So, $\mathcal{H}_L(\cdot, |\nabla u|)$ has higher integrability in the ball B_r. We choose $\Omega' \Subset \Omega$ and consider the covering of $\overline{\Omega'}$ by balls B_r, whose properties are depicted at the beginning of the proof. We note that $\overline{\Omega'}$ is compact, and hence there is a finite subcover $\{B_{r_i}\}_{i=1}^k$ of Ω' and some $\varepsilon > 0$ such that

$$\fint_{\Omega'} \mathcal{H}_L(x, |\nabla u|)^{1+\varepsilon} dx \leq c \sum_{i=1}^k \left[\left(\fint_{B_{2r_i}} \mathcal{H}_L(x, |\nabla u|)dx\right)^{1+\varepsilon} + \fint_{B_{2r_i}} c^{1+\varepsilon} dx\right] < +\infty.$$

It follows that $\mathcal{H}_L(\cdot, |\nabla u|) \in L^{1+\epsilon}_{\mathrm{loc}}(\Omega)$ for some $\varepsilon > 0$. □

Next, we prove Theorem 3, that is the higher integrability result up to the boundary for the gradient of a local K-quasiminimizer to the functional

$$u \mapsto \int_{B_r} \mathcal{H}_L(x, |\nabla u|)dx \quad \text{(recall (1))},$$

by the synergic use of the auxiliary results given in previous sections.

Proof of Theorem 3. We involve the function u defined by

$$u(x) = v(x) \text{ if } x \in B_r \text{ and } u(x) = w(x) \text{ if } x \in B_{2r} \setminus B_r.$$

We choose $z \in B_r$ with $B_{2\gamma}(z) \subset B_r$ and satisfying the constraint $\|\nabla u\|_{L^{\mathcal{H}_L}(B_{2\gamma}(z))} < 1$. By Theorem 4, we deduce that

$$\begin{aligned}
&\int_{B_\gamma(z)} [|\nabla v|^p + a(x)|\nabla v|^q] \log(e + |\nabla v|)dx \\
&\le c \int_{B_{2\gamma}(z)} \left[\frac{|v-(v)_{B_{2\gamma}(z)}|^p}{\gamma^p} + a(x)\frac{|v-(v)_{B_{2\gamma}(z)}|^q}{\gamma^q}\right] \log\left(e + \frac{|v-(v)_{B_{2\gamma}(z)}|}{\gamma}\right) dx.
\end{aligned}$$

Using the modular version of Sobolev-Poincaré inequality, we derive that

$$\begin{aligned}
&\fint_{B_\gamma(z)} [|\nabla v|^p + a(x)|\nabla v|^q] \log(e + |\nabla v|)dx \\
&\le c \left(\fint_{B_{2\gamma}(z)} [|\nabla v|^p + a(x)|\nabla v|^q]^{\frac{1}{s}} \log^{\frac{1}{s}}(e + |\nabla v|)dx\right)^s + c
\end{aligned}$$

for some $1 < s < \min\{p, \frac{N}{N-1}\}$. So, the higher integrability is immediately obtained as byproduct of Lemma 1.

On the other hand, we assume that $B_{2\gamma}(z) \subset B_{2r}$ with $z \in \partial B_r$ and fix $\gamma \le t < s \le 2\gamma$. Then, we introduce a cut-off function $\eta \in C_0^\infty(B_{2\gamma}(z))$ such that

$$0 \le \eta \le 1, \text{ supp}(\eta) \subset B_s(z), \ \eta = 1 \text{ on } B_t(z) \text{ and } |\nabla \eta| \le \frac{2}{s-t}.$$

For a test function $h = v - \eta(v-w) \in W^{1,\mathcal{H}_L}(B_r)$ with $\operatorname{supp}(v-h) \subset B_s(z)$, using the properties of $\eta(\cdot)$ and the estimate

$$|\nabla h| = |(1-\eta)\nabla v + \eta \nabla w + (w-v)\nabla \eta| \le (1-\eta)|\nabla v| + |\nabla w| + 2\frac{|w-v|}{s-t},$$

together with (10), we obtain

$$\begin{aligned}
&\int_{B_t(z)\cap B_r} [|\nabla v|^p + a(x)|\nabla v|^q] \log(e + |\nabla v|)dx \\
&\le \int_{B_s(z)\cap B_r} [|\nabla v|^p + a(x)|\nabla v|^q] \log(e + |\nabla v|)dx \\
&\le K \int_{B_s(z)\cap B_r} [|\nabla h|^p + a(x)|\nabla h|^q] \log(e + |\nabla h|)dx \\
&\le c \int_{(B_s(z)\setminus B_t(z))\cap B_r} [|\nabla v|^p + a(x)|\nabla v|^q] \log(e + |\nabla v|)dx \quad (\text{as } \eta = 1 \text{ on } B_t(z)) \\
&\quad + c \int_{B_s(z)\cap B_r} \left[\frac{|w-v|^p}{(s-t)^p} + a(x)\frac{|w-v|^q}{(s-t)^q}\right] \log\left(e + \frac{|w-v|}{s-t}\right) dx \\
&\quad + c \int_{B_s(z)\cap B_r} [|\nabla w|^p + a(x)|\nabla w|^q] \log(e + |\nabla w|)dx. \qquad (28)
\end{aligned}$$

Now, we apply the Widman's hole filling method, that is, we add the quantity

$$c \int_{B_t(z)\cap B_r} [|\nabla v|^p + a(x)|\nabla v|^q] \log(e + |\nabla v|)dx$$

to both sides of (28) and dividing the obtained inequality by $c+1$, we have

$$\begin{aligned}
&\int_{B_t(z)\cap B_r} [|\nabla v|^p + a(x)|\nabla v|^q]\log(e+|\nabla v|)dx \\
&\le \vartheta \int_{B_s(z)\cap B_r} [|\nabla v|^p + a(x)|\nabla v|^q]\log(e+|\nabla v|)dx \\
&\quad + c\int_{B_s(z)\cap B_r} \left[\frac{|w-v|^p}{(s-t)^p} + a(x)\frac{|w-v|^q}{(s-t)^q}\right]\log\left(e+\frac{|w-v|}{s-t}\right)dx \\
&\quad + c\int_{B_s(z)\cap B_r} [|\nabla w|^p + a(x)|\nabla w|^q]\log(e+|\nabla w|)dx
\end{aligned} \tag{29}$$

for some $0<\vartheta<1$. Let us consider the sequence $\{t_i\}$, defined as follows

$$t_0=\gamma \quad \text{and} \quad t_{i+1}-t_i=(1-\lambda)\lambda^i\gamma$$

for some $0<\lambda<1$. Using (29) and a standard iteration (see the proof of Theorem 4), we get

$$\begin{aligned}
&\int_{B_\gamma(z)\cap B_r} [|\nabla v|^p + a(x)|\nabla v|^q]\log(e+|\nabla v|)dx \\
&\le c\int_{B_{2\gamma}(z)\cap B_r} \left[\frac{|w-v|^p}{\gamma^p} + a(x)\frac{|w-v|^q}{\gamma^q}\right]\log\left(e+\frac{|w-v|}{\gamma}\right)dx \\
&\quad + c\int_{B_{2\gamma}(z)\cap B_r} [|\nabla w|^p + a(x)|\nabla w|^q]\log(e+|\nabla w|)dx \\
&= c\int_{B_{2\gamma}(z)} \left[\frac{|w-u|^p}{\gamma^p} + a(x)\frac{|w-u|^q}{\gamma^q}\right]\log\left(e+\frac{|w-u|}{\gamma}\right)dx \\
&\quad + c\int_{B_{2\gamma}(z)\cap B_r} [|\nabla w|^p + a(x)|\nabla w|^q]\log(e+|\nabla w|)dx,
\end{aligned}$$

where we have used the fact that $v=u$ on B_r. It follows that

$$\begin{aligned}
&\int_{B_\gamma(z)} [|\nabla u|^p + a(x)|\nabla u|^q]\log(e+|\nabla u|)dx \\
&= \int_{B_\gamma(z)\cap B_r} [|\nabla v|^p + a(x)|\nabla v|^q]\log(e+|\nabla v|)dx \\
&\quad + \int_{B_\gamma(z)\setminus B_r} [|\nabla w|^p + a(x)|\nabla w|^q]\log(e+|\nabla w|)dx \\
&\le c\int_{B_{2\gamma}(z)} \left[\frac{|w-u|^p}{\gamma^p} + a(x)\frac{|w-u|^q}{\gamma^q}\right]\log\left(e+\frac{|w-u|}{\gamma}\right)dx \\
&\quad + c\int_{B_{2\gamma}(z)} [|\nabla w|^p + a(x)|\nabla w|^q]\log(e+|\nabla w|)dx.
\end{aligned}$$

By the definition of u, we have that $u-w=0$ on $B_{2r}\setminus B_r$ and therefore on $B_{2\gamma}(z)\setminus B_r$. From $|B_{2\gamma}(z)\setminus B_r|\ge c_0|B_{2\gamma}(z)|$ for some $0<c_0<1$, we can use the Sobolev-Poincaré type inequality (4), and hence we get

$$\begin{aligned}
&\fint_{B_\gamma(z)} [|\nabla u|^p + a(x)|\nabla u|^q] \log(e + |\nabla u|)dx \\
&\leq c \left(\fint_{B_{2\gamma}(z)} [|\nabla (w-u)|^p + a(x)|\nabla (w-u)|^q]^{\frac{1}{s}} \log^{\frac{1}{s}}(e + |\nabla (w-u)|)dx \right)^s \\
&\quad + c \fint_{B_{2\gamma}(z)} [|\nabla w|^p + a(x)|\nabla w|^q] \log(e + |\nabla w|)dx + c,
\end{aligned}$$

for some $1 < s < \min\{p, \frac{N}{N-1}\}$. Further, we get

$$\begin{aligned}
&\fint_{B_\gamma(z)} [|\nabla u|^p + a(x)|\nabla u|^q] \log(e + |\nabla u|)dx \\
&\leq c \left(\fint_{B_{2\gamma}(z)} [|\nabla u|^p + a(x)|\nabla u|^q]^{\frac{1}{s}} \log^{\frac{1}{s}}(e + |\nabla u|)dx \right)^s \\
&\quad + c \fint_{B_{2\gamma}(z)} [|\nabla w|^p + a(x)|\nabla w|^q] \log(e + |\nabla w|)dx + c.
\end{aligned}$$

Referring to Lemma 1, we deduce that there exists $0 < \varepsilon < \delta$ such that the following estimate holds

$$\begin{aligned}
&\fint_{B_\gamma(z)} [|\nabla u|^p + a(x)|\nabla u|^q]^{1+\varepsilon} \log^{1+\varepsilon}(e + |\nabla u|)dx \\
&\leq c \left(\fint_{B_{2\gamma}(z)} [|\nabla u|^p + a(x)|\nabla u|^q] \log(e + |\nabla u|)dx \right)^{1+\varepsilon} \\
&\quad + c \fint_{B_{2\gamma}(z)} [|\nabla w|^p + a(x)|\nabla w|^q]^{1+\varepsilon} \log^{1+\varepsilon}(e + |\nabla w|)dx + c,
\end{aligned}$$

for a positive constant $c = c(p, q, N, K, \mathcal{H}_L)$. This implies that

$$[|\nabla u|^p + a(x)|\nabla u|^q]^{1+\varepsilon} \log^{1+\varepsilon}(e + |\nabla u|) \in L^1(B_\gamma(z)),$$

or equivalently

$$\mathcal{H}_L(x, |\nabla u|)^{1+\varepsilon} \in L^1(B_\gamma(z))$$

and so we have $\mathcal{H}_L(x, |\nabla v|)^{1+\varepsilon} \in L^1(B_\gamma(z) \cap B_r)$, for some $0 < \varepsilon < \delta$. The conclusion follows by a covering argument. □

5. Dirichlet problem

According to the classical theory and methods described in [14, 18], in this section we link the weak solution of the following Dirichlet problem

$$-\Delta_{\mathcal{H}_L} u = f \text{ in } \Omega, \quad u\Big|_{\partial\Omega} = 0, \tag{30}$$

where $f \in L^{\mathcal{H}_L^*}(\Omega)$, to the minimizers of the functional defined by

$$I(u) = \int_\Omega \mathcal{H}_L(x, |\nabla u|)\, dx - \int_\Omega f u\, dx \quad \text{for all } u \in W_0^{1,\mathcal{H}_L}(\Omega). \tag{31}$$

Equation (30) is governed by the following perturbed double phase operator

$$\Delta_{\mathcal{H}_L} u = \operatorname{div}\left(\frac{\mathcal{H}'_L(x,|\nabla u|)}{|\nabla u|}\nabla u\right) \quad \text{for all } x \in \Omega, \text{ all } u \in W_0^{1,\mathcal{H}_L}(\Omega).$$

We recall that by a weak solution to problem (30), we mean a function $u \in W_0^{1,\mathcal{H}_L}(\Omega)$ satisfying

$$\int_\Omega \frac{\mathcal{H}'_L(x,|\nabla u|)}{|\nabla u|}\nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \text{for all } v \in W_0^{1,\mathcal{H}_L}(\Omega). \tag{32}$$

Referring to (22) (the Poincaré inequality) and subsequent discussion, here we consider the norm $\|u\| = \|\nabla u\|_{\mathcal{H}_L}$ for all $u \in W_0^{1,\mathcal{H}_L}(\Omega)$. Further, we remark that $(W_0^{1,\mathcal{H}_L}(\Omega), \|\cdot\|)$ is a reflexive Banach space. Firstly, we get some properties of the functional $I(\cdot)$ given by (31), that is, the energy functional for problem (30). We set

$$\rho_{\mathcal{H}_L}(|\nabla u|) = \int_\Omega \mathcal{H}_L(x,|\nabla u|)\, dx \quad \text{and} \quad F(u) = \int_\Omega f u \, dx \quad \text{for all } u \in W_0^{1,\mathcal{H}_L}(\Omega).$$

Now, we show that $I(\cdot)$ is convex. Indeed, we have

$$\begin{aligned} I(\lambda u + (1-\lambda)v) &= \int_\Omega \mathcal{H}_L(x,|\nabla(\lambda u + (1-\lambda)v)|)\, dx - \int_\Omega f(\lambda u + (1-\lambda)v)\, dx \\ &\le \int_\Omega \mathcal{H}_L(x,\lambda|\nabla u| + (1-\lambda)|\nabla v|)\, dx - \lambda F(u) - (1-\lambda)F(v) \\ &\le \lambda \rho_{\mathcal{H}_L}(|\nabla u|) + (1-\lambda)\rho_{\mathcal{H}_L}(|\nabla v|) - \lambda F(u) - (1-\lambda)F(v) \\ &= \lambda I(u) + (1-\lambda) I(v). \end{aligned}$$

Moreover we note that $I(\cdot)$ is lower semicontinuous, since $\rho_{\mathcal{H}_L}(\cdot)$ is lower semicontinuous and $F(\cdot)$ is continuous. Next, we show that $I(\cdot)$ is coercive, that is, $I(u) \to +\infty$ as $\|u\| = \|\nabla u\|_{\mathcal{H}_L} \to +\infty$. In fact, we note that if $\|u\| > 1$, then by Proposition 3 (iv) we get

$$\begin{aligned} I(u) &\ge \|\nabla u\|^p_{\mathcal{H}_L} - 2\|f\|_{\mathcal{H}^*_L}\|u\|_{\mathcal{H}_L} \\ &\ge \|u\|^p - 2c\|u\| \to +\infty, \end{aligned}$$

as $\|u\| \to +\infty$.

For reader convenience, we recall the following result about the existence of minimizers of sufficiently regular functionals, as can be found in Kinderlehrer and Stampacchia [24].

Theorem 6 ([24], Theorem 2.1)**.** *Let V be a reflexive Banach space. Let $I : V \to \mathbb{R}$ be a convex, lower semicontinuous and coercive operator, then there is an element in V that minimizes $I(\cdot)$.*

Based on Theorem 6 and the involved properties of the functional $I(\cdot)$ defined by (31), we have the following result.

Corollary 2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $f \in L^{\mathcal{H}^*_L}(\Omega)$. Then there exists a function $u \in W_0^{1,\mathcal{H}_L}(\Omega)$ such that*

$$I(u) = \inf_{v \in W_0^{1,\mathcal{H}_L}(\Omega)} I(v),$$

where $I(\cdot)$ is defined by (31).

Finally we provide the correct link between weak solutions to (30) and minimizers to (31).

Lemma 3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $f \in L^{\mathcal{H}_L^*}(\Omega)$. Then each weak solution $u \in W_0^{1,\mathcal{H}_L}(\Omega)$ of problem (30) is a minimizer of the functional $I(\cdot)$ defined by (31).*

Proof. Using the convexity of $I(\cdot)$, we get the inequality

$$I(u+\lambda(v-u)) \leq (1-\lambda)I(u) + \lambda I(v) \quad \text{for all } u,v \in W_0^{1,\mathcal{H}_L}(\Omega).$$

Consequently, we deduce that

$$I(u+\lambda(v-u)) - I(u) \leq \lambda(-I(u)+I(v)),$$

and hence we have

$$\frac{I(u+\lambda w)-I(u)}{\lambda} \leq I(u+w)-I(u),$$

with $w = v-u \in W_0^{1,\mathcal{H}_L}(\Omega)$. Since $I \in C^1(W_0^{1,\mathcal{H}_L}(\Omega))$, passing to the limit as $\lambda \to 0^+$ we get

$$\int_\Omega \frac{\mathcal{H}_L'(x,|\nabla u|)}{|\nabla u|}\nabla u \cdot \nabla w \, dx - \int_\Omega f w \, dx \leq I(u+w)-I(u) \quad \text{for all } w \in W_0^{1,\mathcal{H}_L}(\Omega).$$

The hypothesis that u is a weak solution of problem (30) implies that

$$0 \leq I(u+w)-I(u),$$

that is,

$$I(u) = \inf_{v \in W_0^{1,\mathcal{H}_L}(\Omega)} I(v).$$

This concludes the proof. $\square$

Remark 5. As $I \in C^1(W_0^{1,\mathcal{H}_L}(\Omega))$, we deduce that each minimizer of $I(\cdot)$ is a weak solution to (30).

Keeping in mind that the second derivative of $\mathcal{H}_L$ with respect to its second variable is positive for all $t>0$, then we deduce that the solution to problem (30) is unique. In fact, let $u,w \in W_0^{1,\mathcal{H}_L}(\Omega)$ be two weak solutions to (30) with $u \neq w$, then by (32) with $v = u-w$ we get that

$$\int_\Omega \left[\frac{\mathcal{H}_L'(x,|\nabla u|)}{|\nabla u|}\nabla u - \frac{\mathcal{H}_L'(x,|\nabla w|)}{|\nabla w|}\nabla w\right] \cdot (\nabla u - \nabla w)\, dx = 0,$$

and by $u \neq w$ we know that

$$|\Omega_0| = |\{x \in \Omega : \nabla u \neq \nabla w\}| > 0.$$

Next, by using the inequality

$$\begin{aligned}
&[\mathcal{H}_L'(x,|\nabla u|) - \mathcal{H}_L'(x,|\nabla w|)](|\nabla u| - |\nabla w|)\\
&\leq \left[\frac{\mathcal{H}_L'(x,|\nabla u|)}{|\nabla u|}\nabla u - \frac{\mathcal{H}_L'(x,|\nabla w|)}{|\nabla w|}\nabla w\right] \cdot (\nabla u - \nabla w),
\end{aligned}$$

we get

$$\begin{aligned}
0 < &\int_{\{x \in \Omega_0 : |\nabla u| \neq |\nabla w|\}} [\mathcal{H}_L'(x,|\nabla u|) - \mathcal{H}_L'(x,|\nabla w|)](|\nabla u| - |\nabla w|)\, dx\\
&+ \int_{\{x \in \Omega_0 : |\nabla u| = |\nabla w|\}} \left[\frac{\mathcal{H}_L'(x,|\nabla u|)}{|\nabla u|}\nabla u - \frac{\mathcal{H}_L'(x,|\nabla w|)}{|\nabla w|}\nabla w\right] \cdot (\nabla u - \nabla w)\, dx
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\Omega_0}\left[\frac{\mathcal{H}_L'(x,|\nabla u|)}{|\nabla u|}\nabla u-\frac{\mathcal{H}_L'(x,|\nabla w|)}{|\nabla w|}\nabla w\right]\cdot(\nabla u-\nabla w)\,dx\\
&=\int_{\Omega}\left[\frac{\mathcal{H}_L'(x,|\nabla u|)}{|\nabla u|}\nabla u-\frac{\mathcal{H}_L'(x,|\nabla w|)}{|\nabla w|}\nabla w\right]\cdot(\nabla u-\nabla w)\,dx=0.
\end{aligned}
$$

This is a contradiction and hence we conclude that $u=w$, namely the solution to problem (30) is unique. By Remark 5, we deduce that the minimizer $u\in W_0^{1,\mathcal{H}_L}(\Omega)$ of Theorem 2 is also unique. Consequently, if $f=0\in L^{\mathcal{H}_L^*}(\Omega)$ then we conclude that the unique solution to problem (30) possesses all regularities of local K-quasiminimizers of the $\mathcal{H}_L$-energy.

Acknowledgements

The second author is member of "Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA)" of the Istituto Nazionale di Alta Matematica (INdAM), and he is partially supported by Research project of MIUR (Italian Ministry of Education, University and Research) Print 2022 "Nonlinear differential problems with applications to real phenomena" (Grant No: 2022ZXZTN2).

Statements & Declarations

Funding. This project has received funding from the Natural Science Foundation of Guangxi Grant Nos. 2021GXNSFFA196004 and GKAD23026237, the NNSF of China Grant Nos. 12001478 and 12371312, the China Postdoctoral Science Foundation funded project No. 2022M721560, and the European Union's Horizon 2020 Research and Innovation Programme under the Marie Sklodowska-Curie grant agreement No. 823731 CONMECH. It is also supported by the project cooperation between Guangxi Normal University and Yulin Normal University and Startup Project of Doctor Scientific Research of Yulin Normal University No. G2023ZK13.
Competing interests. The authors read and approved the final manuscript. The authors have no relevant financial or non-financial interests to disclose.
Availability of data and material. This paper has no associated data and material.
Authors' contributions. The authors contributed equally to this paper.

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