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On the critical behavior for time-fractional pseudo-parabolic-type equations with combined nonlinearities

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Abstract

We are concerned with the existence and nonexistence of global weak solutions for a certain class of time-fractional inhomogeneous pseudo-parabolic-type equations involving a nonlinearity of the form $|u|^p + \iota|\nabla u|^q$, where $p, q > 1$, and $\iota \geq 0$ is a constant. The cases $\iota = 0$ and $\iota > 0$ are discussed separately. For each case, the critical exponent in the Fujita sense is obtained. We point out two interesting phenomena. First, the obtained critical exponents are independent of the fractional orders of the time derivative. Secondly, in the case $\iota > 0$, we show that the gradient term induces a discontinuity phenomenon of the critical exponent.

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1 Introduction

In this paper, we consider equations driven by time-fractional pseudo-parabolic-type operators of the form $u \mapsto \partial_t^\alpha u - k\Delta \partial_t^\beta u$ for functions $u = u(t, x)$ defined on $(0, \infty) \times \mathbb{R}^N$, where $N \geq 1$, $k > 0$, and $0 < \alpha, \beta < 1$. Here, for $\rho \in \{\alpha, \beta\}$, ∂_t^ρ denotes the time-Caputo fractional derivative of order ρ (see Sect. 2). We study the inhomogeneous Cauchy problem

$$\begin{cases} \partial_t^\alpha u - k\Delta \partial_t^\beta u - \operatorname{div}(|x|^\theta \nabla u) = |u|^p + \iota|\nabla u|^q + w(x), & (t, x) \in (0, \infty) \times \mathbb{R}^N, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \quad (1.1)$$

where $0 \leq \theta < 2$, $p, q > 1$, $\iota \geq 0$, and $u_0, w \in L^1_{\text{loc}}(\mathbb{R}^N)$ with $w \not\equiv 0$. Precisely, we are concerned with the existence and nonexistence of global weak solutions to (1.1) (see Definition 2.1). The cases $\iota = 0$ and $\iota > 0$ are discussed separately. The nonexistence results we establish use a priori estimates for the solutions and employ appropriate contradiction arguments. The idea behind this approach is clearly presented and deeply discussed by Mitidieri and Pokhozhaev [1], who provided useful a priori estimates for solutions of the involved nonlinear equations under initial conditions, then obtained asymptotic properties

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of these estimates, and finally established the nonexistence proof by contradiction. Differently from the classical approach, the Mitidieri–Pokhozhaev-type approach (namely, the rescaled test-function method) does not use the maximum principle (see, for example, Pucci and Serrin [2, Ch. 3]) or other comparison results (see [2, Ch. 2]). This is as the mentioned approach does not impose any parabolicity condition on the operator or restrictive sign-conditions on the solutions. Because we also do not use any parabolicity condition, the similar arguments herein hold for the analogous elliptic problems. About the elliptic problems, we refer to the work of Papageorgiou and Scapellato [3], who prove a bifurcation-type result for a parametric nonlinear boundary value problem driven by the $(p, 2)$ -Laplacian. In [3] the authors look for positive solutions and use a nonlinear maximum principle in their proofs. We finally mention the work of Figueiredo and Silva [4] dealing with elliptic problems defined in a half-space and involving a nonlinearity with critical growth. In [4] the authors establish the existence of positive solutions using variational methods together with Brouwer theory of degree.

Now let us mention some motivations for studying problems of type (1.1). When $\theta = 0$, $\iota = 0$, $w \equiv 0$, and $\alpha, \beta \rightarrow 1$, (1.1) reduces to

$$\begin{cases} \partial_t u - k\Delta \partial_t u - \Delta u = |u|^p, & (t, x) \in (0, \infty) \times \mathbb{R}^N, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.2}$$

When $k = 0$, (1.2) is just the semilinear heat equation with source term $f(u) = |u|^p$. For $k > 0$, (1.2) is said to be pseudo-parabolic (see Showalter and Tin [5]). Pseudo-parabolic equations model a variety of phenomena arising in science and engineering, such as the aggregation of population [6], long waves [7], the seepage of homogeneous fluids through a fissured rock [8], and the nonstationary processes in semiconductors [9]. Cao et al. [10] investigated the large-time behavior of solutions to (1.2). Namely, like for the corresponding Cauchy problem for the semilinear heat equation (see Fujita [11]), it was shown that the Fujita critical exponent for (1.2) is equal to $1 + \frac{2}{N}$, which leads to the following bifurcation-type result:

- (i) If $1 < p \leq 1 + \frac{2}{N}$ and $u_0 \geq 0$, $u_0 \not\equiv 0$, then any solution to (1.2) blows up in finite time;
- (ii) If $p > 1 + \frac{2}{N}$ and $u_0 \geq 0$ is sufficiently small, then the solution to (1.2) exists globally.

Indeed, the work of Fujita [11] originated a wide discussion over the link between the problem of critical exponents and the necessary conditions for the existence of solutions to evolution equations (see also [1, Part II] for more information).

When $\theta = 0$, $\iota = 0$, and $\alpha, \beta \rightarrow 1$, (1.1) reduces to the Cauchy problem

$$\begin{cases} \partial_t u - k\Delta \partial_t u - \Delta u = |u|^p + w(x), & (t, x) \in (0, \infty) \times \mathbb{R}^N, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.3}$$

Recently, Zhou [12] studied (1.3), where $u_0 \geq 0$ and $w \not\equiv 0$. It was shown that the Fujita critical exponent for (1.3) is equal to

$$p^*(N) = \begin{cases} \infty & \text{if } N \in \{1, 2\}, \\ 1 + \frac{2}{N-2} & \text{if } N \geq 3. \end{cases} \tag{1.4}$$

Precisely, we have the following situations:

- (i) If $N \in \{1, 2\}$, then for all $p > 1$, any solution to (1.3) blows up in finite time;
- (ii) If $N \geq 3$, $\int_{\mathbb{R}^N} w(x) dx > 0$, and $1 < p < 1 + \frac{2}{N-2}$, then any solution to (1.3) blows up in finite time;
- (iii) If $N \geq 3$ and $p > 1 + \frac{2}{N-2}$, then (1.3) admits global solutions for some $u_0, w > 0$.

From the above results we observe that the Fujita critical exponent for (1.3) jumps from $1 + \frac{2}{N}$ (the critical exponent for (1.2)) to the larger exponent $p^*(N)$. Notice that a similar blow-up phenomenon was observed by Zhang [13] for the inhomogeneous semilinear heat equation (1.3) with $k = 0$.

On the other hand, due to the importance of fractional calculus in applications (see, e.g., [14–16]), a great attention was paid to the study of fractional evolution equations in many aspects. In particular, the large-time behavior of solutions to fractional space pseudo-parabolic equations was considered by many authors (see, e.g., [17–19]). Motivated by these contributions, in this paper, we investigate (1.1). We first consider the case $\iota = 0$, that is, we deal with the problem

$$\begin{cases} \partial_t^\alpha u - k \Delta \partial_t^\beta u - \operatorname{div}(|x|^\theta \nabla u) = |u|^p + w(x), & (t, x) \in (0, \infty) \times \mathbb{R}^N, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.5}$$

For (1.5), we study the existence and nonexistence of global weak solutions and derive the Fujita critical exponent. Next, we extend our study to (1.1) with $\iota > 0$. Our main motivation for considering this case is to study the effect of the nonlinearity $|\nabla u|^q$ on the critical behavior of (1.5). Namely, we show that this nonlinearity induces an interesting phenomenon of discontinuity of the Fujita critical exponent.

The rest of the paper is organized as follows. In Sect. 2, we mention in which sense solutions to (1.1) are considered and present the main results of this paper. In Sect. 3, we establish some preliminary estimates, which will be used in Sect. 4, where we prove our main results. A short Sect. 5 concludes the paper.

2 Main results

Let us first recall some basic notions and properties from fractional calculus. For more detail, we refer to the book of Samko et al. [16]. For $f \in C([0, T])$ with $0 < T < \infty$, the Riemann–Liouville fractional integrals of order $\sigma > 0$ are given as

$$(I_0^\sigma f)(t) = \frac{1}{\Gamma(\sigma)} \int_0^t (t-s)^{\sigma-1} f(s) ds, \quad 0 < t \leq T,$$

and

$$(I_T^\sigma f)(t) = \frac{1}{\Gamma(\sigma)} \int_t^T (s-t)^{\sigma-1} f(s) ds, \quad 0 \leq t < T,$$

where Γ denotes the gamma function (see, e.g., [20]). Notice that the limit of $(I_0^\sigma f)(t)$ as t goes to zero from the right is zero. So we can put $(I_0^\sigma f)(0) = 0$ to extend $I_0^\sigma f$ to $[0, T]$ by continuity. Similarly, we can extend by continuity $I_T^\sigma f$ to $[0, T]$ by taking $(I_T^\sigma f)(T) = 0$.

We have the following integration-by-parts rule:

$$\int_0^T (I_0^\sigma f)(t)g(t) dt = \int_0^T (I_T^\sigma g)(t)f(t) dt, \quad \sigma > 0, f, g \in C([0, T]). \tag{2.1}$$

Let $f \in C^1([0, T])$ and $\sigma \in (0, 1)$. The Caputo fractional derivative of order σ of f is defined by

$$\frac{d^\sigma f}{dt^\sigma}(t) = (I_0^{1-\sigma} f')(t), \quad 0 \leq t \leq T,$$

that is,

$$\frac{d^\sigma f}{dt^\sigma}(t) = \frac{1}{\Gamma(1-\sigma)} \int_0^t (t-s)^{-\sigma} f'(s) ds, \quad 0 \leq t \leq T.$$

Now we present the main result obtained for (1.5). Just before, let us mention in which sense solutions to (1.5) are considered. Let $Q = [0, \infty) \times \mathbb{R}^N$. For $0 < T < \infty$, let $Q_T = [0, T] \times \mathbb{R}^N$. Without loss of generality, we will suppose that $k = 1$.

Definition 2.1 Let $N \geq 1, 0 < \alpha, \beta < 1, 0 \leq \theta < 2, p > 1$, and $u_0, w \in L^1_{loc}(\mathbb{R}^N)$. We say that $u \in L^p_{loc}(Q)$ is a global weak solution to (1.5) if

$$\begin{aligned} & \int_{Q_T} |u|^p \varphi \, dx \, dt + \int_{Q_T} w(x) \varphi \, dx \, dt \\ & \quad + \int_{\mathbb{R}^N} u_0(x) (I_T^{1-\alpha} \varphi(0, x) - I_T^{1-\beta} \Delta \varphi(0, x)) \, dx \\ & = \int_{Q_T} u \partial_t (I_T^{1-\beta} \Delta \varphi) \, dx \, dt - \int_{Q_T} u \partial_t (I_T^{1-\alpha} \varphi) \, dx \, dt \\ & \quad - \int_{Q_T} \operatorname{div}(|x|^\theta \nabla \varphi) u \, dx \, dt \end{aligned} \tag{2.2}$$

for all $0 < T < \infty$ and $\varphi \in C^2(Q_T)$ satisfying:

- (a) $\operatorname{supp}_x \varphi \subset \subset \mathbb{R}^N$;
- (b) $\partial_t(I_T^{1-\alpha} \varphi), \partial_t(I_T^{1-\beta} \Delta \varphi) \in L^\infty(Q_T)$.

Remark 2.1 Multiplying the first equation in (1.5) by φ , integrating over Q_T , and using the integration-by-parts rule (2.1), we obtain identity (2.2).

Theorem 2.1 *The following statements hold:*

- (I) Let $N \geq 1, 0 < \alpha, \beta < 1, 0 \leq \theta < 2$, and $u_0, w \in L^1(\mathbb{R}^N)$. Assuming that $\int_{\mathbb{R}^N} w(x) \, dx > 0$, we have:
 - (i) if $\theta \leq 2 - N$, then for all $p > 1$, (1.5) admits no global weak solution;
 - (ii) if $\theta > 2 - N$, then for all $1 < p < \frac{N}{\theta - 2 + N}$, (1.5) admits no global weak solution.
- (II) Let $0 \leq \theta < 2$ and $\theta > 2 - N$. If $p > \frac{N}{\theta - 2 + N}$, then for any $0 < \alpha, \beta < 1$, (1.5) admits global solutions for some $u_0, w > 0$.

Remark 2.2 The emphasis of the paper is on blow-up results. No condition on the sign of u_0 is imposed on part (I) of Theorem 2.1. The existence result (II) is elementary since it is a consequence of elliptic results.

Remark 2.3 Let $0 \leq \theta < 2$. We point out the following facts:

(i) From Theorem 2.1 we deduce that the Fujita critical exponent for (1.5) is given by

$$p^*(N, \theta) = \begin{cases} \infty & \text{if } \theta \leq 2 - N, \\ \frac{N}{\theta - 2 + N} & \text{if } \theta > 2 - N. \end{cases} \tag{2.3}$$

Observe that $p^*(N, \theta)$ is independent of the values of α and β .

(ii) From (2.3) we deduce that

$$p^*(1, \theta) = \begin{cases} \infty & \text{if } 0 \leq \theta \leq 1, \\ \frac{1}{\theta - 1} & \text{if } 1 < \theta < 2; \end{cases}$$

$$p^*(2, \theta) = \begin{cases} \infty & \text{if } \theta = 0, \\ \frac{2}{\theta} & \text{if } 0 < \theta < 2; \end{cases}$$

$$p^*(N, \theta) = \frac{N}{\theta - 2 + N}, \quad N \geq 3.$$

(iii) Observe that

$$p^*(N, 0) = p^*(N),$$

where $p^*(N)$ (defined by (1.4)) is the Fujita critical exponent for (1.3).

Remark 2.4 For $\theta > 2 - N$, the critical case $p = \frac{N}{\theta - 2 + N}$ is left open.

Next, we consider (1.1) for $\iota > 0$. Without loss of generality, we will suppose that $k = \iota = 1$. So (1.1) reduces to

$$\begin{cases} \partial_t^\alpha u - \Delta \partial_t^\beta u - \operatorname{div}(|x|^\theta \nabla u) = |u|^p + |\nabla u|^q + w(x), & (t, x) \in (0, \infty) \times \mathbb{R}^N, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{2.4}$$

Using the integration-by-parts rule (2.1), we define global weak solutions to (2.4) as follows.

Definition 2.2 Let $N \geq 1$, $0 < \alpha, \beta < 1$, $0 \leq \theta < 2$, $p, q > 1$, and $u_0, w \in L^1_{\text{loc}}(\mathbb{R}^N)$. We say that u is a global weak solution to (2.4) if

- (i) $(u, \nabla u) \in L^p_{\text{loc}}(Q) \times L^q_{\text{loc}}(Q)$;
- (ii) For all $0 < T < \infty$ and $\varphi \in C^2(Q_T)$ satisfying conditions (a) and (b) in Definition 2.1, we have

$$\begin{aligned} & \int_{Q_T} (|u|^p + |\nabla u|^q) \varphi \, dx \, dt \\ & + \int_{Q_T} w(x) \varphi \, dx \, dt + \int_{\mathbb{R}^N} u_0(x) (I_T^{1-\alpha} \varphi(0, x) - I_T^{1-\beta} \Delta \varphi(0, x)) \, dx \\ & = \int_{Q_T} u \partial_t (I_T^{1-\beta} \Delta \varphi) \, dx \, dt - \int_{Q_T} u \partial_t (I_T^{1-\alpha} \varphi) \, dx \, dt \\ & - \int_{Q_T} \operatorname{div}(|x|^\theta \nabla \varphi) u \, dx \, dt. \end{aligned} \tag{2.5}$$

For $0 \leq \theta < 1$, let

$$q^*(N, \theta) = \begin{cases} \infty & \text{if } \theta = 1 - N, \\ \frac{N}{\theta + N - 1} & \text{if } \theta > 1 - N. \end{cases} \tag{2.6}$$

Observe that from (2.6) we have

$$q^*(1, \theta) = \begin{cases} \infty & \text{if } \theta = 0, \\ \frac{1}{\theta} & \text{if } 0 < \theta < 1, \end{cases}$$

and

$$q^*(N, \theta) = \frac{N}{\theta + N - 1}, \quad N \geq 2.$$

Our main result for (2.4) is the following theorem.

Theorem 2.2 *The following statements hold:*

(I) *Let $N \geq 1, 0 < \alpha, \beta < 1, u_0, w \in L^1(\mathbb{R}^N)$, and $\int_{\mathbb{R}^N} w(x) dx > 0$. If*

$$0 \leq \theta < 2, \quad 1 < p < p^*(N, \theta) \quad \text{or} \quad 0 \leq \theta < 1, \quad p > 1, 1 < q < q^*(N, \theta),$$

then (2.4) admits no global weak solution.

(II) *Let $0 \leq \theta < 1$ and $\theta > 2 - N$. If*

$$p > p^*(N, \theta) \quad \text{and} \quad q > q^*(N, \theta),$$

then for any $0 < \alpha, \beta < 1$, (2.4) admits global solutions for some $u_0, w > 0$.

Remark 2.5 From Theorem 2.2 and (2.3), and (2.6), for $0 \leq \theta < 1$ and $\theta > 2 - N$, we deduce that the Fujita critical exponent for (2.4) is given by

$$p^*(N, \theta, q) = \begin{cases} \infty & \text{if } 1 < q < q^*(N, \theta) = \frac{N}{\theta + N - 1}, \\ p^*(N, \theta) = \frac{N}{\theta - 2 + N} & \text{if } q > q^*(N, \theta). \end{cases}$$

Observe that the nonlinearity $|\nabla u|^q$ induces an interesting phenomenon of discontinuity of the Fujita critical exponent $p^*(N, \theta, q)$ jumping from $p = \frac{N}{\theta - 2 + N}$ to $p = \infty$, as q reaches the value $\frac{N}{\theta + N - 1}$ from above.

3 Preliminary estimates

We first investigate a priori estimates of certain integral terms, which will be involved in the proofs of Theorems 2.1 and 2.2 in Sect. 4. We stress that we will consider these estimates in the application of a rescaled test-function method. This general approach to the proof of nonexistence of solutions was originally developed by Mitidieri and Pokhozhaev [1] in the case of general forms of nonlinear partial differential equations. We remark that a characteristic feature of this approach is that it requires no any comparison principle (see again [2, Ch. 2]), but uses a contradiction argument employing suitable estimates. Let

us nevertheless mention that the starting point is the definition of weak solution (that is, the above method works well for solution in integral (weak) form). Here, given $0 < T < \infty$ and $\lambda \gg 1$ (λ is large enough), we consider the function

$$F(t) = T^{-\lambda}(T - t)^\lambda, \quad t \in [0, T]. \tag{3.1}$$

Now we gather auxiliary results in the form of lemmas and discuss the validity of these estimates in \mathbb{R}^N . The first lemma gives us a useful representation formula for a Riemann–Liouville fractional integral of order $\rho > 0$ and its Caputo fractional derivative (recall the corresponding notions at the beginning of Sect. 2).

Lemma 3.1 *For all $\rho > 0$,*

$$(I_T^\rho F)(t) = \frac{\Gamma(\lambda + 1)}{\Gamma(\rho + \lambda + 1)} T^{-\lambda}(T - t)^{\lambda + \rho}, \quad t \in [0, T], \tag{3.2}$$

$$(I_T^\rho F)'(t) = -\frac{\Gamma(\lambda + 1)}{\Gamma(\lambda + \rho)} T^{-\lambda}(T - t)^{\lambda + \rho - 1}, \quad t \in [0, T]. \tag{3.3}$$

Proof For $t \in [0, T]$, we have

$$\begin{aligned} (I_T^\rho F)(t) &= \frac{T^{-\lambda}}{\Gamma(\rho)} \int_t^T (s - t)^{\rho - 1} (T - s)^\lambda ds \\ &= \frac{T^{-\lambda}}{\Gamma(\rho)} \int_t^T (s - t)^{\rho - 1} [(T - t) - (s - t)]^\lambda ds \\ &= \frac{T^{-\lambda}(T - t)^\lambda}{\Gamma(\rho)} \int_t^T (s - t)^{\rho - 1} \left(1 - \frac{s - t}{T - t}\right)^\lambda ds. \end{aligned}$$

Hence it is sufficient to make the change of variable $z = \frac{s - t}{T - t}$, and we obtain the representation

$$\begin{aligned} (I_T^\rho F)(t) &= \frac{T^{-\lambda}(T - t)^{\lambda + \rho}}{\Gamma(\rho)} \int_0^1 z^{\rho - 1} (1 - z)^{(\lambda + 1) - 1} dz \\ &= \frac{T^{-\lambda}(T - t)^{\lambda + \rho}}{\Gamma(\rho)} B(\rho, \lambda + 1), \end{aligned}$$

where $B(\rho, \lambda + 1) = \int_0^1 z^{\rho - 1} (1 - z)^{(\lambda + 1) - 1} dz$ is the well-known beta function (whose definition ensures the symmetry $B(\rho, \lambda + 1) = B(\lambda + 1, \rho)$; see [20, Eq. (1.2.10), p. 3]). This function is linked to the gamma function $\Gamma(\rho, \lambda + 1)$ by the property

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)}, \quad a, b > 0,$$

and hence we obtain the final representation (3.2). The second representation, equation (3.3), follows by differentiating (3.2) and using the functional equation

$$\Gamma(a + 1) = a\Gamma(a), \quad a > 0$$

(see also [20, Eq. (1.2.2), p. 3]). This completes the proof. □

Consider now a cut-off argument introducing a function $\psi \in C^\infty([0, \infty))$ satisfying

$$0 \leq \psi \leq 1, \quad \psi \equiv 1 \quad \text{in } [0, 1], \quad \psi \equiv 0 \quad \text{in } [2, \infty). \tag{3.4}$$

Given $0 < T < \infty$, $\xi > 0$, and $L \gg 1$, we use the cut-off function $\psi \in C^\infty([0, \infty))$ to introduce the auxiliary function

$$G(x) = \psi \left(\frac{|x|^2}{T^{2\xi}} \right)^L, \quad x \in \mathbb{R}^N. \tag{3.5}$$

To simplify the notation, throughout this paper we denote by C positive constants independent on T and u with values changing from line to line. We start by estimating the integral of G over \mathbb{R}^N .

Lemma 3.2 *We have the estimate*

$$\int_{\mathbb{R}^N} G(x) \, dx \leq CT^{\xi N}.$$

Proof Using the cut-off argument (3.4) for the function G given in (3.5), we have

$$\begin{aligned} \int_{\mathbb{R}^N} G(x) \, dx &= \int_{|x| < \sqrt{2}T^\xi} \psi \left(\frac{|x|^2}{T^{2\xi}} \right)^L \, dx \\ &\leq \int_{|x| < \sqrt{2}T^\xi} \, dx \\ &\leq CT^{\xi N}, \end{aligned}$$

which trivially gives the desired estimate. □

On this basis, we construct the next estimates.

Lemma 3.3 *Let $\theta \geq 0$ and $p > 1$. Then*

$$\int_{\mathbb{R}^N} G(x)^{\frac{-1}{p-1}} \left| \operatorname{div}(|x|^\theta \nabla G(x)) \right|^{\frac{p}{p-1}} \, dx \leq CT^{\xi(N + \frac{(\theta-2)p}{p-1})}.$$

Proof Referring to (3.5), the cut-off argument in (3.4) leads to

$$\begin{aligned} &\int_{\mathbb{R}^N} G(x)^{\frac{-1}{p-1}} \left| \operatorname{div}(|x|^\theta \nabla G) \right|^{\frac{p}{p-1}} \, dx \\ &= \int_{T^\xi < |x| < \sqrt{2}T^\xi} \psi \left(\frac{|x|^2}{T^{2\xi}} \right)^{\frac{-L}{p-1}} \left| \operatorname{div} \left(|x|^\theta \nabla \psi \left(\frac{|x|^2}{T^{2\xi}} \right)^L \right) \right|^{\frac{p}{p-1}} \, dx. \end{aligned} \tag{3.6}$$

On the other hand, an elementary calculation shows that

$$|x|^\theta \nabla \psi \left(\frac{|x|^2}{T^{2\xi}} \right)^L = 2LT^{-2\xi} |x|^\theta \psi \left(\frac{|x|^2}{T^{2\xi}} \right)^{L-1} \psi' \left(\frac{|x|^2}{T^{2\xi}} \right) x. \tag{3.7}$$

Let us denote the inner product in \mathbb{R}^N by “ \cdot ”. Thus the divergent term can be written as follows:

$$\begin{aligned}
 & \operatorname{div}\left(|x|^\theta \nabla \psi\left(\frac{|x|^2}{T^{2\xi}}\right)^L\right) \\
 &= 2LT^{-2\xi} \operatorname{div}\left(|x|^\theta \psi\left(\frac{|x|^2}{T^{2\xi}}\right)^{L-1} \psi'\left(\frac{|x|^2}{T^{2\xi}}\right) x\right) \\
 &= 2LT^{-2\xi} \left[|x|^\theta \psi\left(\frac{|x|^2}{T^{2\xi}}\right)^{L-1} \psi'\left(\frac{|x|^2}{T^{2\xi}}\right) \operatorname{div} x \right. \\
 &\quad \left. + \nabla\left(|x|^\theta \psi\left(\frac{|x|^2}{T^{2\xi}}\right)^{L-1} \psi'\left(\frac{|x|^2}{T^{2\xi}}\right)\right) \cdot x \right] \tag{3.8} \\
 &= 2LT^{-2\xi} \left[N|x|^\theta \psi\left(\frac{|x|^2}{T^{2\xi}}\right)^{L-1} \psi'\left(\frac{|x|^2}{T^{2\xi}}\right) \right. \\
 &\quad \left. + \nabla\left(|x|^\theta \psi\left(\frac{|x|^2}{T^{2\xi}}\right)^{L-1} \psi'\left(\frac{|x|^2}{T^{2\xi}}\right)\right) \cdot x \right].
 \end{aligned}$$

Additionally, for the gradient term in the last line of (3.8), we have

$$\nabla\left(|x|^\theta \psi\left(\frac{|x|^2}{T^{2\xi}}\right)^{L-1} \psi'\left(\frac{|x|^2}{T^{2\xi}}\right)\right) = |x|^{\theta-2} \psi\left(\frac{|x|^2}{T^{2\xi}}\right)^{L-2} \nu(x)x, \tag{3.9}$$

where we consider the function ν given as

$$\begin{aligned}
 \nu(x) &= \theta \psi\left(\frac{|x|^2}{T^{2\xi}}\right) \psi'\left(\frac{|x|^2}{T^{2\xi}}\right) + 2(L-1)T^{-2\xi} |x|^2 \psi'\left(\frac{|x|^2}{T^{2\xi}}\right)^2 \\
 &\quad + 2T^{-2\xi} |x|^2 \psi\left(\frac{|x|^2}{T^{2\xi}}\right) \psi''\left(\frac{|x|^2}{T^{2\xi}}\right).
 \end{aligned}$$

Taking the inner product by x on both sides of equality (3.9) and putting together the x terms in the right-hand side, we get

$$\nabla\left(|x|^\theta \psi\left(\frac{|x|^2}{T^{2\xi}}\right)^{L-1} \psi'\left(\frac{|x|^2}{T^{2\xi}}\right)\right) \cdot x = |x|^\theta \psi\left(\frac{|x|^2}{T^{2\xi}}\right)^{L-2} \nu(x), \tag{3.10}$$

and hence referring to (3.8), by (3.10) we obtain the estimate

$$\begin{aligned}
 \left| \operatorname{div}\left(|x|^\theta \nabla \psi\left(\frac{|x|^2}{T^{2\xi}}\right)^L\right) \right| &\leq CT^{-2\xi} |x|^\theta \psi\left(\frac{|x|^2}{T^{2\xi}}\right)^{L-2}, \\
 T^\xi &< |x| < \sqrt{2}T^\xi.
 \end{aligned} \tag{3.11}$$

To obtain the final a priori estimate, we turn to (3.6) and use (3.11) to get

$$\int_{\mathbb{R}^N} G(x)^{\frac{-1}{p-1}} |\operatorname{div}(|x|^\theta \nabla G)|^{\frac{p}{p-1}} dx \leq CT^{\frac{-2\xi p}{p-1}} \int_{T^\xi < |x| < \sqrt{2}T^\xi} |x|^{\frac{\theta p}{p-1}} \psi\left(\frac{|x|^2}{T^{2\xi}}\right)^{L-\frac{2p}{p-1}} dx.$$

By assumptions L is supposed to be large enough (recall that we set $L \gg 1$) and $\psi \leq 1$ (see the properties of the cut-off function in (3.4)), and hence we conclude that

$$\begin{aligned} \int_{\mathbb{R}^N} G(x)^{\frac{-1}{p-1}} |\operatorname{div}(|x|^\theta \nabla G)|^{\frac{p}{p-1}} dx &\leq CT^{\frac{-2\xi p}{p-1}} \int_{T^\xi < |x| < \sqrt{2}T^\xi} |x|^{\frac{\theta p}{p-1}} dx \\ &= CT^{\frac{-2\xi p}{p-1}} \int_{r=T^\xi}^{\sqrt{2}T^\xi} r^{N-1 + \frac{\theta p}{p-1}} dr \\ &\leq CT^{\xi(N + \frac{(\theta-2)p}{p-1})}. \end{aligned}$$

This completes the proof. □

Lemma 3.4 *Let $\theta \geq 0$ and $q > 1$. Then*

$$\int_{\mathbb{R}^N} |x|^{\frac{\theta q}{q-1}} G(x)^{\frac{-1}{q-1}} |\nabla G(x)|^{\frac{q}{q-1}} dx \leq CT^{\xi(\frac{q}{q-1}(\theta-1)+N)}.$$

Proof The proof follows the schema of the proof of Lemma 3.3. We use the properties of the cut-off function in (3.4) and the definition of the auxiliary function in (3.5) to obtain

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^{\frac{\theta q}{q-1}} G(x)^{\frac{-1}{q-1}} |\nabla G(x)|^{\frac{q}{q-1}} dx \\ = \int_{T^\xi < |x| < \sqrt{2}T^\xi} |x|^{\frac{\theta q}{q-1}} \psi \left(\frac{|x|^2}{T^{2\xi}} \right)^{\frac{-L}{q-1}} \left| \nabla \psi \left(\frac{|x|^2}{T^{2\xi}} \right)^L \right|^{\frac{q}{q-1}} dx. \end{aligned} \tag{3.12}$$

On the other hand, using (3.7), we obtain

$$\begin{aligned} |x|^{\frac{\theta q}{q-1}} \psi \left(\frac{|x|^2}{T^{2\xi}} \right)^{\frac{-L}{q-1}} \left| \nabla \psi \left(\frac{|x|^2}{T^{2\xi}} \right)^L \right|^{\frac{q}{q-1}} &\leq CT^{\frac{-2\xi q}{q-1}} |x|^{\frac{q}{q-1}(\theta+1)} \psi \left(\frac{|x|^2}{T^{2\xi}} \right)^{L - \frac{q}{q-1}}, \\ T^\xi < |x| < \sqrt{2}T^\xi. \end{aligned}$$

Again, recalling that L is supposed to be large enough and the cut-off function ψ is such that $\psi \leq 1$, we have that

$$\begin{aligned} |x|^{\frac{\theta q}{q-1}} \psi \left(\frac{|x|^2}{T^{2\xi}} \right)^{\frac{-L}{q-1}} \left| \nabla \psi \left(\frac{|x|^2}{T^{2\xi}} \right)^L \right|^{\frac{q}{q-1}} &\leq CT^{\frac{-2\xi q}{q-1}} |x|^{\frac{q}{q-1}(\theta+1)}, \\ T^\xi < |x| < \sqrt{2}T^\xi. \end{aligned}$$

From (3.12) we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} |x|^{\frac{\theta q}{q-1}} G(x)^{\frac{-1}{q-1}} |\nabla G(x)|^{\frac{q}{q-1}} dx &\leq CT^{\frac{-2\xi q}{q-1}} \int_{T^\xi < |x| < \sqrt{2}T^\xi} |x|^{\frac{q}{q-1}(\theta+1)} dx \\ &\leq CT^{\frac{-2\xi q}{q-1}} \int_{r=0}^{\sqrt{2}T^\xi} r^{N + \frac{q}{q-1}(\theta+1)-1} dr \\ &= CT^{\frac{-2\xi q}{q-1} + \xi(N + \frac{q}{q-1}(\theta+1))}, \end{aligned}$$

and hence the lemma is proved. □

The last result of this section is an immediate consequence of the definition of the function F given in (3.1), used together with identity (3.3) (that is, the second representation formula stated in Lemma 3.1). Therefore we omit the details of its proof.

Lemma 3.5 *Let $0 < \rho < 1$ and $p > 1$. We have the following estimates:*

$$\int_0^T F(t)^{\frac{-1}{p-1}} |(I_T^\rho F)'(t)|^{\frac{p}{p-1}} dt \leq CT^{\frac{pp-1}{p-1}}, \tag{3.13}$$

$$\int_0^T F(t) dt = CT. \tag{3.14}$$

4 Proofs of the main results

Proof of Theorem 2.1 As already mentioned, our arguments follow the rescaled test-function argument in [1]. Therefore we implement a proof by contradiction. The idea is to assume that there exists a global weak solution from an appropriate class, and then using a priori asymptotic bounds for this solution, we obtain a contradiction, since there is no nontrivial global weak solution. We combine the relevant identities and inequalities established in the previous section.

(I) Suppose that u is a global weak solution to (1.5). By Definition 2.1 (focusing on (2.2)), for all $0 < T < \infty$ and $\varphi \in C^2(Q_T)$ satisfying conditions (a) and (b) therein, we have the inequality

$$\begin{aligned} & \int_{Q_T} |u|^p \varphi \, dx \, dt + \int_{Q_T} w(x)\varphi \, dx \, dt + \int_{\mathbb{R}^N} u_0(x)(I_T^{1-\alpha} \varphi(0, x) - I_T^{1-\beta} \Delta \varphi(0, x)) \, dx \\ & \leq \int_{Q_T} |u| |\partial_t (I_T^{1-\beta} \Delta \varphi)| \, dx \, dt + \int_{Q_T} |u| |\partial_t (I_T^{1-\alpha} \varphi)| \, dx \, dt \\ & \quad + \int_{Q_T} |\operatorname{div}(|x|^\rho \nabla \varphi)| |u| \, dx \, dt. \end{aligned} \tag{4.1}$$

On the other hand, using Young’s inequality, we obtain three inequalities related to each term in the right-hand side of (4.1). For the first term, we have

$$\begin{aligned} & \int_{Q_T} |u| |\partial_t (I_T^{1-\beta} \Delta \varphi)| \, dx \, dt \\ & \leq \frac{1}{3} \int_{Q_T} |u|^p \varphi \, dx \, dt + C \int_{Q_T} \varphi^{\frac{-1}{p-1}} |\partial_t (I_T^{1-\beta} \Delta \varphi)|^{\frac{p}{p-1}} \, dx \, dt. \end{aligned} \tag{4.2}$$

Similarly, for the second term, we get

$$\begin{aligned} & \int_{Q_T} |u| |\partial_t (I_T^{1-\alpha} \varphi)| \, dx \, dt \\ & \leq \frac{1}{3} \int_{Q_T} |u|^p \varphi \, dx \, dt + C \int_{Q_T} \varphi^{\frac{-1}{p-1}} |\partial_t (I_T^{1-\alpha} \varphi)|^{\frac{p}{p-1}} \, dx \, dt \end{aligned} \tag{4.3}$$

and finally for the third term, we have

$$\begin{aligned} & \int_{Q_T} |\operatorname{div}(|x|^\theta \nabla \varphi)| |u| \, dx \, dt \\ & \leq \frac{1}{3} \int_{Q_T} |u|^p \varphi \, dx \, dt + C \int_{Q_T} \varphi^{\frac{-1}{p-1}} |\operatorname{div}(|x|^\theta \nabla \varphi)|^{\frac{p}{p-1}} \, dx \, dt. \end{aligned} \tag{4.4}$$

Starting from (4.1) and using inequality (4.2) together with the inequalities (4.3) and (4.4), we deduce that

$$\begin{aligned} & \int_{Q_T} w(x) \varphi \, dx \, dt + \int_{\mathbb{R}^N} u_0(x) (I_T^{1-\alpha} \varphi(0, x) - I_T^{1-\beta} \Delta \varphi(0, x)) \, dx \\ & \leq C(I_1(\varphi) + I_2(\varphi) + I_3(\varphi)), \end{aligned} \tag{4.5}$$

where

$$I_1(\varphi) = \int_{Q_T} \varphi^{\frac{-1}{p-1}} |\partial_t(I_T^{1-\beta} \Delta \varphi)|^{\frac{p}{p-1}} \, dx \, dt, \tag{4.6}$$

$$I_2(\varphi) = \int_{Q_T} \varphi^{\frac{-1}{p-1}} |\partial_t(I_T^{1-\alpha} \varphi)|^{\frac{p}{p-1}} \, dx \, dt, \tag{4.7}$$

$$I_3(\varphi) = \int_{Q_T} \varphi^{\frac{-1}{p-1}} |\operatorname{div}(|x|^\theta \nabla \varphi)|^{\frac{p}{p-1}} \, dx \, dt.$$

Without loss of generality, consider now the test function

$$\varphi(t, x) = F(t)G(x), \quad (t, x) \in Q_T, \tag{4.8}$$

where F and G are given, respectively, by (3.1) and (3.5) (with $\lambda, L \gg 1$). We can easily see that $\varphi \in C^2(Q_T)$ and it satisfies both conditions (a) and (b) of Definition 2.1. Hence the bound (4.5) holds for a function φ given by (4.8).

Now let us estimate the integrals $I_i(\varphi)$, $i = 1, 2, 3$, always in the case that φ is given by (4.8). We have

$$I_1(\varphi) = \left(\int_0^T F(t)^{\frac{-1}{p-1}} |(I_T^{1-\beta} F)'(t)|^{\frac{p}{p-1}} \, dt \right) \left(\int_{\mathbb{R}^N} G(x)^{\frac{-1}{p-1}} |\Delta G(x)|^{\frac{p}{p-1}} \, dx \right). \tag{4.9}$$

Using (3.13) with $\rho = 1 - \beta$, we obtain

$$\int_0^T F(t)^{\frac{-1}{p-1}} |(I_T^{1-\beta} F)'(t)|^{\frac{p}{p-1}} \, dt \leq CT^{\frac{(1-\beta)p-1}{p-1}}. \tag{4.10}$$

Next, using Lemma 3.3 with $\theta = 0$, we have that

$$\int_{\mathbb{R}^N} G(x)^{\frac{-1}{p-1}} |\Delta G(x)|^{\frac{p}{p-1}} \, dx \leq CT^{\xi(N-\frac{2p}{p-1})}. \tag{4.11}$$

Therefore (4.9), (4.10), and (4.11) yield the estimate

$$I_1(\varphi) \leq CT^{\xi N + \frac{(1-\beta)p-1-2\xi p}{p-1}}. \tag{4.12}$$

Proceeding in a similar way, we now estimate $I_2(\varphi)$. Indeed, considering (4.8), we get the identity

$$I_2(\varphi) = \left(\int_0^T F(t)^{\frac{-1}{p-1}} |(I_T^{1-\alpha} F)'(t)|^{\frac{p}{p-1}} dt \right) \left(\int_{\mathbb{R}^N} G(x) dx \right). \tag{4.13}$$

Using (3.13) with $\rho = 1 - \alpha$, we obtain

$$\int_0^T F(t)^{\frac{-1}{p-1}} |(I_T^{1-\alpha} F)'(t)|^{\frac{p}{p-1}} dt \leq CT^{\frac{(1-\alpha)p-1}{p-1}}, \tag{4.14}$$

and hence using (4.13) and (4.14) together with Lemma 3.2, we deduce that

$$I_2(\varphi) \leq CT^{\xi N + \frac{(1-\alpha)p-1}{p-1}}. \tag{4.15}$$

Employing a similar argument as above, by (4.8) we retrieve the identity

$$I_3(\varphi) = \left(\int_0^T F(t) dt \right) \left(\int_{\mathbb{R}^N} G(x)^{\frac{-1}{p-1}} |\operatorname{div}(|x|^\theta \nabla G)|^{\frac{p}{p-1}} dx \right). \tag{4.16}$$

Then by Lemma 3.3 and identity (3.14), we deduce for (4.16) the estimate

$$I_3(\varphi) \leq CT^{\xi(N + \frac{(\theta-2)p}{p-1}) + 1}. \tag{4.17}$$

Therefore it follows from (4.5), (4.12), (4.15), and (4.17) that

$$\begin{aligned} & \int_{Q_T} w(x)\varphi dx dt + \int_{\mathbb{R}^N} u_0(x)(I_T^{1-\alpha} \varphi(0, x) - I_T^{1-\beta} \Delta \varphi(0, x)) dx \\ & \leq C \left(T^{\xi N + \frac{(1-\beta)p-1-2\xi p}{p-1}} + T^{\xi N + \frac{(1-\alpha)p-1}{p-1}} + T^{\xi(N + \frac{(\theta-2)p}{p-1}) + 1} \right). \end{aligned} \tag{4.18}$$

On the other hand, by (3.5), (4.8), and (3.14) we have

$$\begin{aligned} \int_{Q_T} w(x)\varphi dx dt &= \left(\int_0^T F(t) dt \right) \left(\int_{\mathbb{R}^N} w(x)G(x) dx \right) \\ &= CT \int_{\mathbb{R}^N} w(x)\psi \left(\frac{|x|^2}{T^{2\xi}} \right)^L dx. \end{aligned} \tag{4.19}$$

Notice that since $w \in L^1(\mathbb{R}^N)$, by the dominated convergence theorem and properties of the cut-off function in (3.4), we have the asymptotic behavior

$$\lim_{T \rightarrow +\infty} \int_{\mathbb{R}^N} w(x)\psi \left(\frac{|x|^2}{T^{2\xi}} \right)^L dx = \int_{\mathbb{R}^N} w(x) dx.$$

Since $\int_{\mathbb{R}^N} w(x) dx > 0$, we deduce that for sufficiently large T ,

$$\int_{\mathbb{R}^N} w(x)\psi \left(\frac{|x|^2}{T^{2\xi}} \right)^L dx \geq C \int_{\mathbb{R}^N} w(x) dx. \tag{4.20}$$

Again for sufficiently large T , combining (4.19) and (4.20), it follows that

$$\int_{Q_T} w(x)\varphi \, dx \, dt \geq CT \int_{\mathbb{R}^N} w(x) \, dx. \tag{4.21}$$

On the other hand, using (4.8) and (3.2) with $\rho \in \{1 - \alpha, 1 - \beta\}$, we obtain

$$\begin{aligned} & \int_{\mathbb{R}^N} u_0(x) (I_T^{1-\alpha} \varphi(0, x) - I_T^{1-\beta} \Delta \varphi(0, x)) \, dx \\ &= \int_{\mathbb{R}^N} u_0(x) ((I_T^{1-\alpha} F)(0)G(x) - (I_T^{1-\beta} F)(0)\Delta G(x)) \, dx \\ &= \int_{\mathbb{R}^N} u_0(x) \left(\frac{\Gamma(\lambda + 1)}{\Gamma(2 - \alpha + \lambda)} T^{1-\alpha} G(x) - \frac{\Gamma(\lambda + 1)}{\Gamma(2 - \beta + \lambda)} T^{1-\beta} \Delta G(x) \right) \, dx \\ &= \mu_T(u_0)T, \end{aligned} \tag{4.22}$$

where

$$\mu_T(u_0) = \Gamma(\lambda + 1) \int_{\mathbb{R}^N} u_0(x) \left(\frac{1}{\Gamma(2 - \alpha + \lambda)} T^{-\alpha} G(x) - \frac{1}{\Gamma(2 - \beta + \lambda)} T^{-\beta} \Delta G(x) \right) \, dx.$$

We claim that

$$\lim_{T \rightarrow +\infty} \mu_T(u_0) = 0. \tag{4.23}$$

Indeed, by (3.4) and (3.5), since $u_0 \in L^1(\mathbb{R}^N)$, we have

$$|\mu_T(u_0)| \leq C \left(T^{-\alpha} \int_{\mathbb{R}^N} |u_0(x)| \, dx + T^{-\beta} \int_{T^\xi < |x| < \sqrt{2}T^\xi} |u_0(x)| |\Delta G(x)| \, dx \right).$$

On the other hand, using (3.4) and (3.11) with $\theta = 0$, we obtain

$$|\Delta G(x)| \leq CT^{-2\xi} \psi \left(\frac{|x|^2}{T^{2\xi}} \right)^{L-2} \leq CT^{-2\xi}, \quad T^\xi < |x| < \sqrt{2}T^\xi$$

(recall that $L \gg 1$). Hence we have the estimate

$$|\mu_T(u_0)| \leq C \left(T^{-\alpha} \int_{\mathbb{R}^N} |u_0(x)| \, dx + T^{-\beta-2\xi} \int_{T^\xi < |x| < \sqrt{2}T^\xi} |u_0(x)| \, dx \right).$$

Therefore, passing to the limit as $T \rightarrow +\infty$ in the above inequality, the result in (4.23) is established. If we combine appropriately inequalities (4.18), (4.21), and (4.22), we get

$$\begin{aligned} & T \left(\int_{\mathbb{R}^N} w(x) \, dx + \mu_T(u_0) \right) \\ & \leq C \left(T^{\xi N + \frac{(1-\beta)p-1-2\xi p}{p-1}} + T^{\xi N + \frac{(1-\alpha)p-1}{p-1}} + T^{\xi(N + \frac{(\theta-2)p}{p-1} + 1)} \right), \end{aligned}$$

and hence

$$\int_{\mathbb{R}^N} w(x) \, dx + \mu_T(u_0) \leq C(T^{A_1(\xi)} + T^{A_2(\xi)} + T^{A_3(\xi)}), \tag{4.24}$$

where we set

$$A_1(\xi) = \xi N + \frac{(1 - \beta)p - 1 - 2\xi p}{p - 1} - 1, \tag{4.25}$$

$$A_2(\xi) = \xi N + \frac{(1 - \alpha)p - 1}{p - 1} - 1, \tag{4.26}$$

$$A_3(\xi) = \xi \left(N + \frac{(\theta - 2)p}{p - 1} \right).$$

Now we take $\xi > 0$ such that

$$\xi \left(N - \frac{2p}{p - 1} \right) < \frac{\beta p}{p - 1} \quad \text{and} \quad \xi < \frac{\alpha p}{N(p - 1)}. \tag{4.27}$$

Notice that under the above conditions, we have $A_i(\xi) < 0, i = 1, 2$. Moreover, if $\theta \leq 2 - N$, then $A_3(\xi) < 0$ for all $p > 1$. Hence, passing to the limit as $T \rightarrow +\infty$ in (4.24) and using (4.23), we obtain a contradiction to the positivity condition $\int_{\mathbb{R}^N} w(x) dx > 0$. Therefore we deduce that for all $p > 1$, problem (1.5) admits no global weak solution. This proves part (I)-(i) of Theorem 2.1.

Next, if $\theta > 2 - N$, then we observe that $A_3(\xi) < 0$ for all $1 < p < \frac{N}{\theta - 2 + N}$. So, proceeding as in the previous case, we arrive again at contradiction to $\int_{\mathbb{R}^N} w(x) dx > 0$. Thus part (I)-(ii) of Theorem 2.1 is also established.

Now we focus on the second part of the theorem.

(II) We assume the restrictions

$$\frac{1}{p - 1} < \delta < \frac{N - 2 + \theta}{2 - \theta} \tag{4.28}$$

and

$$0 < \varepsilon < [(2 - \theta)\tau]^{\frac{1}{p-1}}, \tag{4.29}$$

where

$$\tau = N - (2 - \theta)(\delta + 1). \tag{4.30}$$

Notice that since $0 \leq \theta < 2, \theta > 2 - N$, and $p > \frac{N}{\theta - 2 + N}$, the set of values δ satisfying (4.28) and the set of values ε satisfying (4.29) are nonempty. Without loss of generality, consider the function

$$u(x) = \varepsilon(1 + r^{2-\theta})^{-\delta}, \quad x \in \mathbb{R}^N, |x| = r. \tag{4.31}$$

An elementary calculation shows that

$$\begin{aligned} & -\operatorname{div}(|x|^\theta \nabla u) \\ &= -r^{1-N} \partial_r (r^{N-1+\theta} \partial_r u) \\ &= \varepsilon(2 - \theta) [N(1 + r^{2-\theta})^{-\delta-1} - (2 - \theta)(\delta + 1)r^{2-\theta} (1 + r^{2-\theta})^{-\delta-2}]. \end{aligned} \tag{4.32}$$

Let

$$w(x) = -\operatorname{div}(|x|^\theta \nabla u) - |u(x)|^p, \quad x \in \mathbb{R}^N, \tag{4.33}$$

so that by (4.31) and (4.32) we obtain

$$\begin{aligned} w(x) &= \varepsilon(2 - \theta) [N(1 + r^{2-\theta})^{-\delta-1} - (2 - \theta)(\delta + 1)r^{2-\theta}(1 + r^{2-\theta})^{-\delta-2}] \\ &\quad - \varepsilon^p(1 + r^{2-\theta})^{-\delta p} \\ &\geq \varepsilon(2 - \theta) [N(1 + r^{2-\theta})^{-\delta-1} - (2 - \theta)(\delta + 1)(1 + r^{2-\theta})^{-\delta-1}] \\ &\quad - \varepsilon^p(1 + r^{2-\theta})^{-\delta p} \\ &= \varepsilon(2 - \theta)\tau(1 + r^{2-\theta})^{-\delta-1} - \varepsilon^p(1 + r^{2-\theta})^{-\delta p}. \end{aligned}$$

Next, using (4.28) and (4.29), we get the positivity condition

$$\begin{aligned} \varepsilon^{-1}w(x) &\geq (2 - \theta)\tau(1 + r^{2-\theta})^{-\delta-1} - \varepsilon^{p-1}(1 + r^{2-\theta})^{-\delta p} \\ &\geq (2 - \theta)\tau[(1 + r^{2-\theta})^{-\delta-1} - (1 + r^{2-\theta})^{-\delta p}] \\ &> 0. \end{aligned}$$

This shows that for all δ and ε satisfying, respectively, (4.28) and (4.29), the function u defined by (4.31) is a stationary solution (and hence a global solution) to (1.5), where $u_0(x) = \varepsilon(1 + r^{2-\theta})^{-\delta} > 0$, and $w > 0$ is given by (4.33). This proves part (II) of Theorem 2.1. \square

Next, we give the proof of the second main result of this manuscript (i.e., we consider (1.1) when $\iota > 0$).

Proof of Theorem 2.2 We construct the proof following a similar strategy to the previous proof, and hence we aim to obtain a contradiction to the assumption that there exists a global weak solution to problem (2.4). We provide the precise details as follows.

(I) Suppose that u is a global weak solution to (2.4). We first consider the case

$$0 \leq \theta < 2 \quad \text{and} \quad 1 < p < p^*(N, \theta).$$

From (2.5) we deduce that u solves (4.1) for all $0 < T < \infty$ and $\varphi \in C^2(Q_T)$, $\varphi \geq 0$, satisfying conditions (a) and (b) of Definition 2.1. Hence in this case the proof is the same as that of part (I) of Theorem 2.1. Consider now the case

$$0 \leq \theta < 1 \quad \text{and} \quad p > 1, \quad 1 < q < q^*(N, \theta). \tag{4.34}$$

From (2.5), for all $0 < T < \infty$, we have

$$\begin{aligned} &\int_{Q_T} (|u|^p + |\nabla u|^q)\varphi \, dx \, dt + \int_{Q_T} w(x)\varphi \, dx \, dt \\ &\quad + \int_{\mathbb{R}^N} u_0(x)(I_T^{1-\alpha}\varphi(0, x) - I_T^{1-\beta}\Delta\varphi(0, x)) \, dx \end{aligned} \tag{4.35}$$

$$\begin{aligned}
 &= \int_{Q_T} |u| |\partial_t(I_T^{1-\beta} \Delta \varphi)| \, dx \, dt + \int_{Q_T} |u| |\partial_t(I_T^{1-\alpha} \varphi)| \, dx \, dt \\
 &\quad - \int_{Q_T} \operatorname{div}(|x|^\theta \nabla \varphi) u \, dx \, dt,
 \end{aligned}$$

where φ is the test function given by (4.8). On the other hand, integrating by parts, we obtain

$$- \int_{Q_T} \operatorname{div}(|x|^\theta \nabla \varphi) u \, dx \, dt = \int_{Q_T} |x|^\theta \nabla \varphi \cdot \nabla u \, dx \, dt,$$

which yields (by the Cauchy–Schwarz inequality)

$$- \int_{Q_T} \operatorname{div}(|x|^\theta \nabla \varphi) u \, dx \, dt \leq \int_{Q_T} |x|^\theta |\nabla \varphi| |\nabla u| \, dx \, dt.$$

Hence by (4.35) we obtain the identity

$$\begin{aligned}
 &\int_{Q_T} (|u|^p + |\nabla u|^q) \varphi \, dx \, dt + \int_{Q_T} w(x) \varphi \, dx \, dt \\
 &\quad + \int_{\mathbb{R}^N} u_0(x) (I_T^{1-\alpha} \varphi(0, x) - I_T^{1-\beta} \Delta \varphi(0, x)) \, dx \\
 &= \int_{Q_T} |u| |\partial_t(I_T^{1-\beta} \Delta \varphi)| \, dx \, dt + \int_{Q_T} |u| |\partial_t(I_T^{1-\alpha} \varphi)| \, dx \, dt \\
 &\quad + \int_{Q_T} |x|^\theta |\nabla \varphi| |\nabla u| \, dx \, dt.
 \end{aligned} \tag{4.36}$$

On the other hand, by Young’s inequality we have

$$\begin{aligned}
 &\int_{Q_T} |x|^\theta |\nabla \varphi| |\nabla u| \, dx \, dt \\
 &\leq \varepsilon \int_{Q_T} |\nabla u|^q \varphi \, dx \, dt + C \int_{Q_T} |x|^{\frac{\theta q}{q-1}} \varphi^{\frac{-1}{q-1}} |\nabla \varphi|^{\frac{q}{q-1}} \, dx \, dt,
 \end{aligned} \tag{4.37}$$

where $0 < \varepsilon < 1$. Therefore, using appropriately (4.36), (4.37), (3.10), (4.3), (4.21), and (4.22), we deduce that

$$T \left(\int_{\mathbb{R}^N} w(x) \, dx + \mu_T(u_0) \right) \leq C(I_1(\varphi) + I_2(\varphi) + J(\varphi)), \tag{4.38}$$

where $I_1(\varphi)$ and $I_2(\varphi)$ are given, respectively, by (4.6) and (4.7), and

$$J(\varphi) = \int_{Q_T} |x|^{\frac{\theta q}{q-1}} \varphi^{\frac{-1}{q-1}} |\nabla \varphi|^{\frac{q}{q-1}} \, dx \, dt.$$

Referring to (3.14) and (4.8), by Lemma 3.4 we deduce that

$$\begin{aligned}
 J(\varphi) &= \left(\int_0^T F(t) \, dt \right) \left(\int_{\mathbb{R}^N} |x|^{\frac{\theta q}{q-1}} G(x)^{\frac{-1}{q-1}} |\nabla G(x)|^{\frac{q}{q-1}} \, dx \right) \\
 &\leq CT^{1+\xi(\frac{q}{q-1}(\theta-1)+N)},
 \end{aligned} \tag{4.39}$$

Therefore it follows from (4.38), (4.39), (4.12), and (4.15) that

$$T \left(\int_{\mathbb{R}^N} w(x) dx + \mu_T(u_0) \right) \leq C \left(T^{\xi N + \frac{(1-\beta)p-1-2\xi p}{p-1}} + T^{\xi N + \frac{(1-\alpha)p-1}{p-1}} + T^{1+\xi \left(\frac{q}{q-1}(\theta-1)+N \right)} \right),$$

and hence

$$\int_{\mathbb{R}^N} w(x) dx + \mu_T(u_0) \leq C \left(T^{A_1(\xi)} + T^{A_2(\xi)} + T^{B(\xi)} \right), \tag{4.40}$$

where $A_1(\xi)$ and $A_2(\xi)$ are given respectively by (4.25) and (4.26), and

$$B(\xi) = \xi \left(\frac{q}{q-1}(\theta-1) + N \right).$$

Now we take $\xi > 0$ such that (4.27) is satisfied, which guarantees that $A_i(\xi) < 0, i = 1, 2$. On the other hand, (4.34) guarantees that $B(\xi) < 0$. Hence, passing to the limit as $T \rightarrow +\infty$ in (4.40) and using (4.23), we obtain a contradiction to the positivity condition $\int_{\mathbb{R}^N} w(x) dx > 0$. This proves part (I) of Theorem 2.2.

(II) We consider the ranges

$$\max \left\{ \frac{1}{p-1}, \frac{(1-\theta)q}{(2-\theta)(q-1)} - 1 \right\} < \delta < \frac{N+\theta-2}{2-\theta} \tag{4.41}$$

and

$$0 < \varepsilon < \min \left\{ \left[\frac{(2-\theta)\tau}{2} \right]^{\frac{1}{p-1}}, \left[\frac{(2-\theta)^{1-q}\delta^{-q}\tau}{2} \right]^{\frac{1}{q-1}} \right\}, \tag{4.42}$$

where τ is given by (4.30). Notice that since $0 \leq \theta < 1, \theta > 2 - N, p > \frac{N}{\theta-2+N}$, and $q > \frac{N}{\theta+N-1}$, the set of values δ satisfying (4.41) and the set of values ε satisfying (4.42) are nonempty. Let u be the function defined by (4.31) and set

$$w(x) = -\operatorname{div}(|x|^\theta \nabla u) - |u(x)|^p - |\nabla u|^q, \quad x \in \mathbb{R}^N. \tag{4.43}$$

An elementary calculation shows that

$$|\nabla u|^q = \varepsilon^q \delta^q (2-\theta)^q r^{(1-\theta)q} (1+r^{2-\theta})^{-(\delta+1)q}, \quad r = |x|.$$

Hence, using (4.31) and (4.32), we obtain

$$\begin{aligned} w(x) &= \varepsilon(2-\theta) \left[N(1+r^{2-\theta})^{-\delta-1} - (2-\theta)(\delta+1)r^{2-\theta}(1+r^{2-\theta})^{-\delta-2} \right] \\ &\quad - \varepsilon^p (1+r^{2-\theta})^{-\delta p} - \varepsilon^q \delta^q (2-\theta)^q r^{(1-\theta)q} (1+r^{2-\theta})^{-(\delta+1)q} \\ &\geq \varepsilon(2-\theta) \left[N(1+r^{2-\theta})^{-\delta-1} - (2-\theta)(\delta+1)(1+r^{2-\theta})^{-\delta-1} \right] \\ &\quad - \varepsilon^p (1+r^{2-\theta})^{-\delta p} - \varepsilon^q \delta^q (2-\theta)^q (1+r^{2-\theta})^{-\left(\frac{1}{2-\theta}+\delta\right)q} \end{aligned}$$

$$\begin{aligned}
&= \varepsilon(2-\theta)\tau(1+r^{2-\theta})^{-\delta-1} - \varepsilon^p(1+r^{2-\theta})^{-\delta p} \\
&\quad - \varepsilon^q \delta^q (2-\theta)^q (1+r^{2-\theta})^{-\left(\frac{1}{2-\theta}+\delta\right)q}.
\end{aligned}$$

Next, using (4.41) and (4.42), we deduce that

$$\begin{aligned}
\varepsilon^{-1}w(x) &\geq (2-\theta)\tau(1+r^{2-\theta})^{-\delta-1} - \varepsilon^{p-1}(1+r^{2-\theta})^{-\delta p} \\
&\quad - \varepsilon^{q-1}\delta^q(2-\theta)^q(1+r^{2-\theta})^{-\left(\frac{1}{2-\theta}+\delta\right)q} \\
&> (2-\theta)\tau(1+r^{2-\theta})^{-\delta-1} - \frac{(2-\theta)\tau}{2}(1+r^{2-\theta})^{-\delta-1} \\
&\quad - \frac{(2-\theta)\tau}{2}(1+r^{2-\theta})^{-\delta-1} \\
&= 0.
\end{aligned}$$

This shows that for all δ and ε satisfying, respectively, (4.41) and (4.42), the function u defined by (4.31) is a stationary solution (and hence a global solution) to (2.4), where $u_0(x) = \varepsilon(1+r^{2-\theta})^{-\delta} > 0$, and $w > 0$ is given by (4.43). This proves part (II) of Theorem 2.2. \square

5 Conclusions

We considered a qualitative study of sufficient and necessary conditions ensuring the existence of global weak solutions to certain classes of inhomogeneous Cauchy problems. It is noted that the presence of the parametric nonlinearity $u \rightarrow \iota|\nabla u|^q$ (namely, the case $\iota > 0$) induces a phenomenon of discontinuity of the Fujita critical exponent in comparison with the “unperturbed” problem (namely, the case $\iota = 0$). For other recent blow-up results and different methodologies of proofs, we refer to the works of Mohammed et al. [21] (where a fully nonlinear uniformly elliptic equation is considered using the Alexandroff–Bakelman–Pucci maximum principle) and Pan and Zhang [22] (where the mass-critical variable coefficient nonlinear Schrödinger equation is approached by the variational characterization of the ground state solutions). Finally, we cite the work of Elhindi et al. [23] dealing with a Bresse–Timoshenko model with various competing effects. The authors study both the global existence and uniqueness of solutions, employing some numerical methods (i.e., the Faedo–Galerkin approximations and multiplier techniques).

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Authors' contributions

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