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Searching for exceptional points and inspecting non-contractivity of trace distance in (anti-) \mathcal{PT} -symmetric systems

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Non-Hermitian systems with parity-time (\mathcal{PT}) symmetry and anti- \mathcal{PT} symmetry give rise to exceptional points (EPs) with intriguing properties related to, e.g., chiral transport and enhanced sensitivity, due to the coalescence of eigenvectors. In this paper, we propose a powerful and easily computable tool, based on the Hilbert-Schmidt speed (HSS), which does not require the diagonalization of the evolved density matrix, to detect exactly the EPs and hence the critical behavior of the (anti-) \mathcal{PT} -symmetric systems, especially high-dimensional ones. Our theoretical predictions, made without the need for modification of the Hilbert space, which is performed by diagonalizing one of the observables, are completely consistent with results extracted from recent experiments studying the criticality in (anti-) \mathcal{PT} -symmetric systems. Nevertheless, not modifying the Hilbert space of the non-Hermitian system, we find that the trace distance, a measure of distinguishability of two arbitrary quantum states, whose dynamics is known as a faithful witness of non-Markovianity in Hermitian systems, may be non-contractive under the non-Hermitian evolution of the system. Therefore, it lacks one of the most important characteristics which must be met by any standard witness of non-Markovianity.

I. INTRODUCTION

One of the fundamental postulates of quantum theory is that the Hamiltonian of an isolated system is Hermitian. This Hermiticity seems to be a compelling postulate because it ensures that the eigenvalues of the Hamiltonian are real. Moreover, a Hermitian Hamiltonian leads to a unitary time evolution and consequently the conservation of probability. However, decoherence effects are ubiquitous in nature because the physical systems are always inevitably affected by the environment. Under this condition, the dynamics of the system is dominated by a Lindbladian generalized master equation. Because accurately solving this master equation is usually a very difficult task, especially in high-dimensional systems, an approximate but fruitful approach [1] to describe the dynamics of open quantum systems is directly handling the Schrödinger equation $i\partial_t |\psi\rangle = H |\psi\rangle$, such that the time evolution operator is determined by an effective time-dependent Hamiltonian H which is not necessarily Hermitian [2, 3]. Non-Hermitian Hamiltonians usually have complex eigenvalue spectra and do not conserve probabilities, and therefore they often only serve as phenomenological descriptions of an open quantum system. Nevertheless, there is a certain class of non-Hermitian Hamiltonians, invariant under a combination of parity (\mathcal{P}) and time-reversal (\mathcal{T}) operations, i.e., $[\mathcal{PT}, H] = 0$, whose spectrum can be entirely real as long as they respect \mathcal{PT} symmetry [4–8]. Some applications and features of \mathcal{PT} symmetry are addressed in Refs. [9–19].

In general, \mathcal{PT} -symmetric systems exhibit two phases: the unbroken phase in which the entire eigenspectrum is real, and the broken phase where some or all of the eigenvalues form complex conjugate pairs. This *phase transition* occurs at a special point where n eigenvalues, as well as their correspond-

ing eigenvectors, coalesce [20–22]. This non-Hermitian degeneracy, also known as exceptional point (EP) of order n (EP $_n$), has recently emerged as a new way to engineer the response of the open physical system.

In optics, the abrupt nature of the phase transitions being encountered around EPs has been demonstrated to lead to many intriguing phenomena, such as unidirectional invisibility [23, 24], loss-induced transparency [25], band merging [26, 27], laser mode selectivity [28, 29], topological chirality [30, 31], new types of thresholdless phonon lasers [32, 33], and even exceptional photon blockade [34]. These important phenomena have been experimentally observed in different platforms based on optomechanics [32], electronics [35], metamaterials [36], acoustics [37, 38], and plasmonics [39].

Moreover, recently, it has been demonstrated that the bifurcation properties of second-order non-Hermitian degeneracies can be used as an efficient tool to improve the sensitivity (frequency shifts) of resonant optical structures to external perturbations [40]. In addition, it is of particular interest [41] to use even higher-order EPs (greater than second order), which in principle could considerably amplify the effect of perturbations, leading to greater sensitivity beyond what is possible in standard arrangements [42].

Recently, anti- \mathcal{PT} -symmetric systems where the Hamiltonian is anti-commutative with the joint \mathcal{PT} operator, i.e., $\{H, \mathcal{PT}\} = 0$, have attracted much research interest. Interesting physical phenomena reported in anti- \mathcal{PT} -symmetric systems include optical systems with constant refraction [43] and optical materials with balanced positive and negative index [44]. Moreover, several relevant experiments have been realized in diffusive systems [45], electrical circuit resonators, and atomic [46–48] or molecular systems [49]. Additionally, quantum processes such as the observation of EPs [50], symmetry-breaking transition [46], and simulation of anti- \mathcal{PT} -symmetric Lorentz dynamics [51], which are important phenomena different from Hermitian quantum mechanics, have been addressed in these experiments.

Decoherence control is a key task for practical implemen-

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tations of nanoscale solid-state quantum information processing, in which the decoherence may be mainly affected by non-Markovian dynamics due to the strong backactions from the environment [52–81]. A fundamental issue is how to accurately define Markovianity in quantum processes [82, 83]. In order to present a definition of a Markov process in the quantum regime, it is essential to understand the concept of Markov process in the classical scenario.

A classical *Markov process* is defined as a family of random variables $\{X(t), t \in I \subset \mathbb{R}\}$, for which the probability that X takes a value x_n at any arbitrary time $t_n \in I$, if it took value x_{n-1} at some previous time $t_{n-1} < t_n$, can be uniquely determined, and is not influenced by the possible values of X at previous times to t_{n-1} . One can formulate it in terms of conditional probabilities as follows: $\mathbb{P}(x_n, t_n | x_{n-1}, t_{n-1}; \dots; x_0, t_0) = \mathbb{P}(x_n, t_n | x_{n-1}, t_{n-1})$ for all $\{t_n \geq t_{n-1} \geq \dots \geq t_0\} \subset I$. Roughly speaking, its concept is informally encapsulated by the statement that "a Markov process has no memory of the history of past values of X ".

In order to obtain a similar formulation in the quantum realm we demand a way to define $\mathbb{P}(x_n, t_n | x_{n-1}, t_{n-1}; \dots; x_0, t_0)$ for quantum systems. In the classical scenario we can sample a random variable without affecting its posterior statistics. However, 'sampling' a quantum system requires measuring process, disturbing the state of the system and affecting the subsequent outcomes. Therefore, $\mathbb{P}(x_n, t_n | x_{n-1}, t_{n-1}; \dots; x_0, t_0)$ depend on both the dynamics and the measurement process. Because in such a case the Markovian character of a quantum dynamical system would depend on the the measurement scheme chosen to achieve $\mathbb{P}(x_n, t_n | x_{n-1}, t_{n-1}; \dots; x_0, t_0)$, a definition of quantum Markovianity in terms of which is not an easy task. In fact, the definition of Markovianity should be independent of what is required to verify it.

The aforementioned problem can be solved by adopting a different approach focusing on the study of one-time probabilities $\mathbb{P}(x, t)$. In *linear* quantum evolutions it may lead to concept of *divisibility* which can be defined without any explicit reference to measurement processes in the quantum realm [83]. *Although these probabilities help us to avoid the difficulties associated with the measurement disturbance, their efficiency in the scenarios involving some measurement with postselection is controversial.*

One of the most well-known approaches to identify the non-Markovian character of the system dynamics has been proposed by Breuer-Laine-Piilo (BLP), namely the distinguishability of two evolving states of the quantum system [84, 85]. For two arbitrary states ρ_1 and ρ_2 , this distinguishability is quantified by the trace distance (TD) $D(\rho_1, \rho_2) = \frac{1}{2} \text{Tr}|\rho_1 - \rho_2|$, where $|A| = \sqrt{A^\dagger A}$ for some operator A .

To explain the physical origin of this interpretation we consider two parties, Alice and Bob, and assume that Alice prepares a quantum system in one of two states ρ_1 or ρ_2 , with a probability of 1/2 each. Then, the system is passed into a "black box" where it may be probed by Bob in any way allowed by the laws of quantum mechanics. Bob's task is to determine whether the system is in the state ρ_1 or ρ_2 , by means of a *single* quantum measurement. It has been shown that the maximal success probability which Bob can achieve

through an optimal strategy is directly related to the trace distance [82, 86]: $P_{\max} = 1/2(1 + D(\rho_1, \rho_2))$. Therefore, the trace distance denotes the bias in favor of a correct state discrimination by Bob, and hence it can be interpreted as the distinguishability of the quantum states ρ_1 and ρ_2 .

The TD is contractive under completely positive and trace-preserving (CPTP) maps, i.e. $D(\mathcal{E}_t(\rho_1), \mathcal{E}_t(\rho_2)) \leq D(\rho_1, \rho_2)$, if \mathcal{E}_t is a CPTP map. Therefore, a monotonic decrease in the distinguishability ($\dot{D} < 0$) indicates unidirectional information flow from the system to the environment. However, an increase in the distinguishability ($\dot{D} > 0$) signifies backflow of information from the environment to the system, indicating that the time evolution of the system is affected by the history of system-environment interaction. This is one of the most well-known definitions of quantum non-Markovianity in the literature. It should be noted that there are also other ways to define and detect non-Markovianity or memory effects in quantum mechanics (see [83] for a review). It should be noted that non-Markovian dynamics is always associated with non-unitary evolution of the system. Contrary to Hermitian systems which can evolve unitarily and non-unitarily, (anti-)PT-dynamics is intrinsically nonunitary both in the unbroken and broken phases [87], satisfying one of the necessary conditions to exhibit non-Markovian dynamics.

Similar to classical scenario, appearance of a measurement with postselection in the process may call into question the validity of this non-Markovianity definition, because the time evolution of the system becomes dependent on not only the history of system-environment interaction but also the output of the postselection. *This dependence of the quantum non-Markovianity definition on the measurement scheme, chosen to achieve the time evolution, is bothering even if we distinguish between memory effects and backflow of information from the environment.* Following this idea, we show that the BLP measure does not have a necessary condition to be a figure of merit for characterizing non-Markovianity, if it is definable in dynamics involving postselection. To this aim, we investigate the TD contractivity, a necessary condition which should be satisfied by this measure when it is used for defining non-Markovianity.

Recently, the Hilbert-Schmidt speed (HSS), a measure of quantum statistical speed not requiring diagonalization of the system reduced density matrix, has been introduced as a faithful witness of non-Markovianity in Hermitian systems, completely consistent with the BLP witness [88]. Moreover, the HSS has been introduced as an efficient figure of merit for quantum estimation of phase encoded into the initial state of open n -qubit systems [89]. Possibility to enhance quantum sensing near the EPs [40, 90–92] and application of the TD to characterize the criticality in (anti-)PT-symmetric systems [49, 87, 93] motivate one to investigate the efficiency of the HSS measure to find these singular points in non-Hermitian systems. Moreover, they motivate us to study the relationship among the HSS, TD, and quantum Fisher information (QFI), playing a central role in quantum estimation theory, in (anti-)PT-symmetric systems.

In this paper, we address the aforementioned issue and

also show how the HSS can effectively determine the EPs and reveal the critical behavior of non-Hermitian (anti-) \mathcal{PT} -symmetric systems. Because finding the coalescence of eigenvectors at EPs by numerical full diagonalization, required when computing the TD, may be a tedious and time-consuming chore, this theoretical development for detecting the EPs could be very useful.

This paper is organized as follows. In Sec. II we briefly review the definition of the Hilbert-Schmidt speed. In Sec. III the time evolution of the (anti-) \mathcal{PT} -symmetric systems as well as non-contractivity of the trace distance are discussed. In Sec. IV we propose a protocol based on the HSS to characterize phase transitions in (anti-) \mathcal{PT} -symmetric systems. The efficiency of this protocol for detecting EPs and criticality in single-qubit systems is discussed in Secs. V and VI. Moreover, the sensitivity of this witness is also studied for a high-dimensional \mathcal{PT} -symmetric system in Sec. VII. Finally, Sec. VIII summarizes the main results and prospects.

II. HILBERT-SCHMIDT SPEED

The distance measure, defined as [94]

$$[d(p, q)]^2 = \frac{1}{2} \sum_x |p_x - q_x|^2, \quad (1)$$

where $p = \{p_x\}_x$ and $q = \{q_x\}_x$ are probability distributions, leads to the classical statistical speed

$$s[p(\varphi_0)] = \frac{d}{d\varphi} d(p(\varphi_0 + \varphi), p(\varphi_0)). \quad (2)$$

Thus, one can define a special kind of quantum statistical speed called the HSS by extending these classical notions to the quantum case. To this aim, we may consider a given pair of quantum states ρ and σ , and write $p_x = \text{Tr}\{E_x \rho\}$ and $q_x = \text{Tr}\{E_x \sigma\}$ denoting the measurement probabilities corresponding to the positive-operator-valued measure (POVM) defined by the $\{E_x \geq 0\}$ which satisfies $\sum_x E_x = \mathbb{I}$. Then the associated quantum distance called the Hilbert-Schmidt distance δ_{HS} [95] can be achieved by maximizing the classical distance of Eq. (1) over all possible choices of POVMs [96]

$$\delta_{HS}(\rho, \sigma) \equiv \max_{\{E_x\}} d(\rho, \sigma) = \sqrt{\frac{1}{2} \text{Tr}[(\rho - \sigma)^2]}. \quad (3)$$

Consequently, the HSS, the corresponding quantum statistical speed, is obtained by maximizing the classical statistical speed of Eq. (2) over all possible POVMs [94, 97]

$$\begin{aligned} HSS(\rho(\varphi)) &\equiv HSS_\varphi \equiv S[\rho(\varphi)] = \max_{\{E_x\}} s[p(\varphi)] \\ &= \sqrt{\frac{1}{2} \text{Tr}\left[\left(\frac{d\rho(\varphi)}{d\varphi}\right)^2\right]}, \end{aligned} \quad (4)$$

which can be easily computed without diagonalizing $d\rho(\varphi)/d\varphi$.

III. TIME EVOLUTION OF THE SYSTEM GOVERNED BY A (AN) (ANTI-) \mathcal{PT} -SYMMETRIC HAMILTONIAN LEADING TO NON-CONTRACTIVITY OF TD

We directly apply the conventional quantum mechanics on (anti-) \mathcal{PT} -symmetric systems to obtain the evolved state. Accordingly, the dynamics governed by a (an) (anti-) \mathcal{PT} -symmetric system with non-Hermitian Hamiltonian $H_{\varphi\mathcal{T}}^{(anti)}$ is described by [87, 98]

$$\rho(t) = \frac{e^{-iH_{\varphi\mathcal{T}}^{(anti)}} \rho(0) e^{iH_{\varphi\mathcal{T}}^{(anti)\dagger}}}{\text{Tr}[e^{-iH_{\varphi\mathcal{T}}^{(anti)}} \rho(0) e^{iH_{\varphi\mathcal{T}}^{(anti)\dagger}}]}, \quad (5)$$

where the usual Hilbert-Schmidt inner product is employed. In this situation, the effective dynamics governed by $H_{\varphi\mathcal{T}}^{(anti)}$ is nonunitary and hence it describes the evolution of an open quantum system [49, 87, 99]. It originates from the fact that a (an) (anti-) \mathcal{PT} -symmetric system cannot be implemented by a closed system. In other words, the lack of Hermiticity of the Hamiltonian, is not observable in closed systems, in contrast to open systems [100]. Hence, we do not use a preferentially selected inner product with which the (anti-) \mathcal{PT} -symmetric Hamiltonian $H_{\varphi\mathcal{T}}^{(anti)}$ can be employed to generate the unitary time evolution for the characterization of a closed quantum system. The metric operators used in this approach for modifying the Hilbert space inner products are not physically observable [100], as discussed above. Such physical constraints prohibit experimentalists from modifying the inner product in a laboratory, although it can be used as an effective mathematical tool to nicely formulate the theory of quantum systems whose dynamics is governed by (anti-) \mathcal{PT} -symmetric Hamiltonians (see Ref. [101, 102]).

In more detail, we see that the conventional metric used in Eq. (5) leads to results completely consistent with experimental observations [49, 91, 102–109] provided that it is applied correctly. In fact, in order to physically implement the non-unitary evolution leading to Eq. (5), we can embed the (anti) \mathcal{PT} -symmetric system into a larger Hermitian system, realized by adding ancillary qubits, and perform a measuring process [93, 103]. This idea originates from the Naimark dilation procedure for quantum measurement [87, 110–112]: by including an ancilla and extending the Hilbert space, any nonunitary dynamics can be implemented by a unitary dynamics of the total closed system followed by quantum measurement acting on the ancilla. When a measurement is performed on the ancilla and a special definite state is postselected, the evolved state (5) is realized. Because of this post-selection occurring in the measuring process, the successful implementation of the non-unitary gate is a probabilistic procedure. This experimental limitation [49, 102, 103, 105], is similar to the situation which occurs in Bell inequality tests [113–115]. Therefore, we can solve the paradoxes [7, 101, 116, 117] associated with violation of *no-go theorems* in \mathcal{PT} -symmetric theory [101] using normalized density matrix (5), without the need for any modification in the Hilbert space (for more details, see Refs. [87, 103]).

A significant property of the extended Hamiltonian discussed above is that the original system is non-Hermitian if

and only if the characteristic interaction between the original system and the ancilla is nonzero. On the one hand, the information flowed into the environment is actually stored in the entanglement with the ancilla. On the other hand, because the interaction is global, the quantum correlation between the system and the ancilla may oscillate in time [87]. Moreover, the information exchange between the system and this entangled partner hidden in the environment may be one of the physical origins of the time oscillations of the distance measures for quantum states of the (anti-) \mathcal{PT} -symmetric system. The aforementioned reasons motivates us to define non-Markovianity concept in the (anti-) \mathcal{PT} -symmetric systems, however, as also described in the Introduction, the dependence of this definition on the postselection process, appearing in all current experimental realizations of (anti-) \mathcal{PT} -symmetric dynamics [49, 91, 102–106, 109], may call into question its validity. Demonstrating the failure of the BLP measure in defining possible non-Markovian effects in (anti-) \mathcal{PT} -symmetric systems (see Refs. [105, 118]) can support this reasoning.

Adopting the BLP's definition of non-Markovianity, one finds that the P divisibility of a linear quantum dynamical map is equivalent to Markovianity of the dynamics [119]. In spite of the fact that the (anti-) \mathcal{PT} dynamics given by Eq. (5) is indeed P divisible [98], the divisibility cannot capture the non-Markovianity of the (anti-) \mathcal{PT} dynamics because of the *non-linearity* of quantum operation \mathcal{E} [87]. It is emphasized that even in the linear evolutions the divisibility of map has not been used to generally define Markovianity, because the concept of divisibility is limited to processes for which the inverse of the dynamical map exists [82, 120], a property which cannot be guaranteed e.g., examples on quantum semi-Markov processes [121] or the damped Jaynes-Cummings model on resonance [85], thus making the concept of divisibility sometimes ill defined.

In order to answer the question of whether the BLP's definition of non-Markovianity can be used in non-Hermitian systems [49, 87, 93], we focus on TD contractivity. As known, the TD always characterizes the distinguishability between two quantum states. When this property is associated with contractivity, it can be used for the definition of non-Markovianity interpreted as backflow of information from the environment to the system. However, in the absence of contractivity, its oscillation can only be attributed to oscillation of distinguishability, not existence of information backflow from the environment. Hence, to answer the question we should investigate the contractivity of TD in non-Hermitian systems.

Concerning the mutual relations of the quantum operations and trace distance, as referred briefly in Introduction, the following important result is well known [1, 122–124]: if there is no initial correlation between the system and environment and \mathcal{E} is a trace-preserving quantum operation then $D(\mathcal{E}(\rho), \mathcal{E}(\sigma)) \leq D(\rho, \sigma)$ where ρ and σ denote arbitrary normalized quantum states. This result is usually referred to as contractivity of the trace distance under the linear trace-preserving quantum operations.

However, the quantum operation \mathcal{E} , generating the evolved state (5), is non-trace-preserving [1], since it does not provide a complete description of the processes occurring in the sys-

tem. This nondeterministic feature originates from the fact that other measurement outcomes may take place with some probability. Therefore, the contractivity of the TD under the evolution given in Eq. (5) should be investigated in more detail. Our numerical calculation, presented in the next sections, shows that *the trace distance may exhibit non-contractivity under (anti-) \mathcal{PT} dynamics and hence in such systems it loses one of the necessary conditions which should be satisfied by a faithful witness of non-Markovianity.*

It should be noted that contractivity is not a universal feature but depends on the metric: the dynamics may be contractive with respect to a given metric and may not be contractive with respect to other metric measures [123]. In addition, contractivity of quantum evolution can break down when the system is initially correlated with its environment (for details see [124]).

IV. DETECTING CRITICALITY THROUGH HSS IN NON-HERMITIAN SYSTEMS

In this section we provide the witness based on the Hilbert-Schmidt speed to faithfully identify the EPs and phase transitions in non-Hermitian systems.

It is known that in the \mathcal{PT} -symmetric systems the trace distance oscillates with evolution time when the system symmetry is unbroken while in anti- \mathcal{PT} -symmetric systems, the oscillations of the distinguishability occur if the symmetry is broken [49, 87, 93, 102]. Inspired by this fact and the close relationship between HSS and TD described in Ref. [88], we propose the following easily computable witness to characterize the criticality in (anti-) \mathcal{PT} -symmetric systems:

For a quantum system with an n -dimensional Hilbert space \mathcal{H} , let us consider an initial state given by

$$|\psi_0\rangle = \frac{1}{\sqrt{n}}(e^{i\varphi}|\psi_1\rangle + \dots + |\psi_n\rangle), \quad (6)$$

where φ is an unknown phase shift and $\{|\psi_i\rangle, i = 1, \dots, n\}$ represents the computational orthonormal basis. Then, we find that when the dynamics of the HSS, computed with respect to the initial phase φ , exhibit an oscillating pattern, the (anti-) \mathcal{PT} -symmetric system is in (anti-) \mathcal{PT} -unbroken (broken) phase. At EPs or broken (unbroken) phase of the (anti-) \mathcal{PT} -symmetric system, no oscillation is observed in the HSS dynamics. Therefore, the EPs can be easily detected by investigating the time evolution of the HSS.

The sanity check of this protocol as a faithful witness of EPs and criticality is performed in the following section.

V. \mathcal{PT} -SYMMETRIC TWO-LEVEL SYSTEM

A. Hamiltonian model and computing the witness

As the first example, we consider the paradigmatic model of a two-level system described by the Hamiltonian

$$H_{\mathcal{PT}} = \varepsilon(\sigma_x + ia\sigma_z) = \begin{pmatrix} ia\varepsilon & \varepsilon \\ \varepsilon & -ia\varepsilon \end{pmatrix}, \quad (7)$$

where $\varepsilon \geq 0$ is an energy scale and $a \geq 0$ denotes the degree of non-Hermiticity. This model has been previously realized in both classical [125–127] and quantum [49, 103, 128] experiments. The eigenenergies are $\pm\varepsilon\sqrt{1-a^2}$, and therefore one has an EP2 (exceptional point of order 2) given by $a = 1$.

The time evolution operator of this system is obtained as [87]

$$U_{\mathcal{PT}} = e^{-iH_{\mathcal{PT}}t} = \frac{1}{\sqrt{1-a^2}} \begin{pmatrix} \sqrt{1-a^2}\cos\theta + a\sin\theta & -i\sin\theta \\ -i\sin\theta & \sqrt{1-a^2}\cos\theta - a\sin\theta \end{pmatrix}, \quad (8)$$

where $\theta = \sqrt{1-a^2}\varepsilon t$.

In order to check the efficiency of our HSS-based witness, we first compute the normalized evolved state of the system when it is prepared in the initial state $\rho_0 = |\psi_0\rangle\langle\psi_0|$, where $|\psi_0\rangle = (e^{i\varphi}|0\rangle + |1\rangle)/\sqrt{2}$. We can easily calculate the HSS analytically by inserting the evolved state $\rho_t(\varphi) = U_{\mathcal{PT}}\rho_0 U_{\mathcal{PT}}^\dagger \left\{ \text{Tr}[U_{\mathcal{PT}}\rho_0 U_{\mathcal{PT}}^\dagger] \right\}^{-1}$ into Eq. (4); however, its explicit expression has a cumbersome form and is not reported here.

B. Dynamical behavior of the witness of the quantum criticality

The qualitative dynamics of the HSS is displayed in Fig. 1 for the \mathcal{PT} -broken phase ($a > 1$) and \mathcal{PT} -unbroken phase ($0 < a < 1$). In the \mathcal{PT} -broken phase, the HSS, as expected, exhibits no oscillations and monotonically decreases with time. However, in the \mathcal{PT} -unbroken phase, it oscillates and eventually returns to its initial value, namely there is a T

$$\text{such that } HSS(T) = HSS(0), \quad (9)$$

where the period is given by $T = \pi [1-a^2]^{-1/2}$. This period T of the oscillation, called the recurrence time, increases as the system approaches the EP ($a = 1$). This period is exactly similar to one achieved theoretically as well as experimentally for the distinguishability oscillations [87, 93]. Moreover, checking the experimental data presented in [93], we find that the EP predicted by the HSS-based witness is quite accurate.

Behavior at the exceptional point

As discussed in the Introduction, the exceptional points (EPs) in non-Hermitian Hamiltonians constitute a threshold for the system parameters individuating a phase transition of the system. In the present case of a \mathcal{PT} -symmetric system, one has a broken-to-unbroken phase transition by diminishing the value of the non-Hermiticity degree a appearing in the Hamiltonian of Eq. (7). Since this phase transition is associated to intriguing physical phenomena, the characterization of the behavior of the non-Hermitian system around EPs is important. Moreover, because of the importance of the HSS

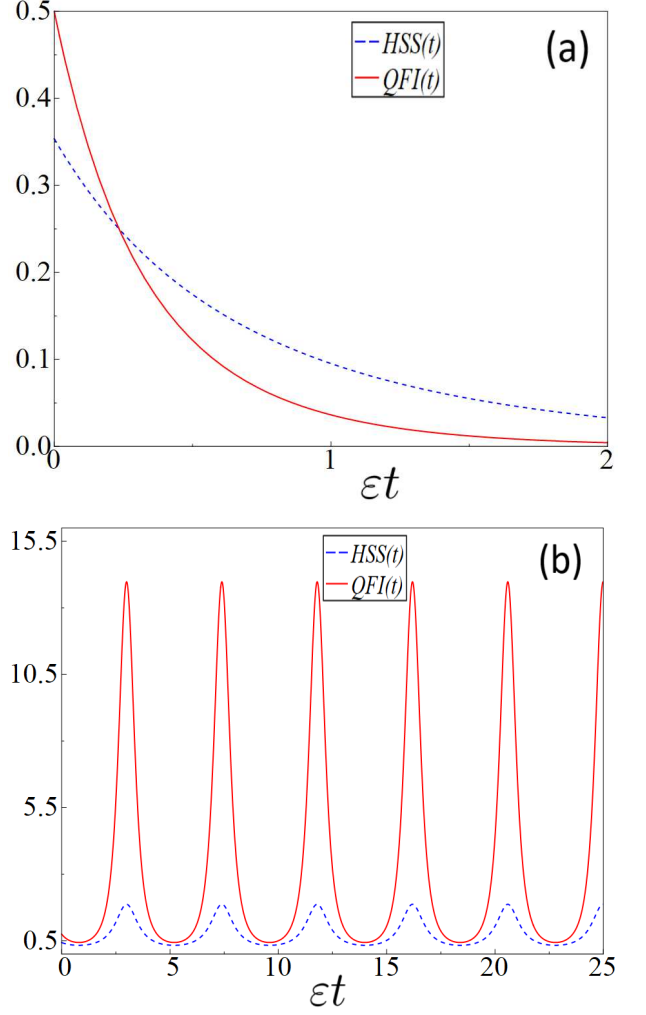


FIG. 1. Dynamics of the Hilbert-Schmidt speed $HSS(\rho_t(\varphi))$ (blue dashed curve) and the Quantum Fisher information $QFI(t)$ (red solid curve) as a function of the dimensionless time εt for a two-level system (a) in the \mathcal{PT} -broken phase ($a > 1$) and (b) in the \mathcal{PT} -unbroken phase ($0 < a < 1$).

in detecting the EPs, it would be interesting to investigate the behavior of this measure when the system approaches the EP, i.e., when $a \rightarrow 1$. In this limit, we find that the HSS is given by

$$HSS_{EP}(t) = \frac{1}{4\varepsilon^2 t^2 - 4\varepsilon^2 t^2 \sin(\varphi) + 2}. \quad (10)$$

It is immediate to see that it decreases monotonically with time according to $HSS_{EP}(t) \propto t^{-2}$; while only for $\varphi = \pi/2$ the HSS gives a constant value equal to $1/2$. These results immediately show that, when the system parameters reach the EP ($a \rightarrow 1$), no oscillation in HSS dynamics is observed which arises from the divergence of period T at this limit.

Overall, we see that the HSS can be employed as an efficient witness to identify phase transitions and detect the EPs in the non-Hermitian system under consideration.

C. Relationship between quantum Fisher information (QFI) and HSS

The fundamental question in the theory of quantum estimation is the following: When performing measurements on the quantum systems affected by some classical parameter φ (which may be the phase encoded into the initial state of the system), how precisely can φ be estimated? The answer is given by the quantum Cramer-Rao bound [129] indicating that the smallest resolvable change in φ is $\delta\varphi = 1/\sqrt{F_\varphi}$ where F_φ denotes the quantum Fisher information (QFI) given by [129, 130]

$$F_\varphi(\rho(t)) = 2 \sum_{i,j} \frac{|\langle \phi_i | \partial_\varphi \rho(\varphi) | \phi_j \rangle|^2}{(\lambda_i + \lambda_j)}, \quad (11)$$

where $|\phi_i\rangle$ and λ_i represent, respectively, the eigenvectors and eigenvalues of the density matrix $\rho(t)$. It should be noted that recently other expressions for the QFI have been proposed in non-Hermitian systems [131], however, for pure states they reduce to the above expression, ignoring a constant coefficient.

Because both QFI and HSS are quantum statistical speeds associated, respectively, with the *Bures* and *Hilbert-Schmidt distances* (for details see [94]), it is reasonable to explore how they can be related to each other. Recently, a strong relationship between the HSS and QFI has been constructed in the process of phase estimation for n -qubit Hermitian systems [89]. It has been found that, when both the HSS and QFI are computed with respect to the phase parameter encoded into the initial state of an n -qubit system, the zeros of the HSS dynamics are actually equal to those of the QFI dynamics. Likewise, the signs of the time-derivatives of both HSS and QFI exactly coincide.

Now computing the QFI with respect to the phase parameter encoded into the initial state of our one-qubit \mathcal{PT} -symmetric system, we obtain the similar result, i.e., both the QFI and HSS exhibit the same qualitative dynamics (see Fig. 1). Moreover, in the \mathcal{PT} -broken phase ($a > 1$) illustrated in Fig. 1(a), both measures are contractive and hence monotonically decrease with time. However, as observed in Fig. 1(b), not only the HSS but also the QFI, contractive under CPTP maps in Hermitian systems, may be non-contractive under the non-trace-preserving evolution in the unbroken phase ($0 < a < 1$). Moreover, this important relationship between the HSS and QFI shows that the HSS may be introduced as an efficient figure of merit for quantum estimation of phase encoded into the initial state of n -qubit \mathcal{PT} -symmetric systems. This will be investigated in detail in future studies.

The non-contractivity of the QFI and HSS confirms the fact that quantum information measures and witnesses may exhibit different behaviors for Hermitian and non-Hermitian systems. It is known that the QFI measures the maximum information about a parameter φ , extractable from a given measurement procedure [129, 130, 132–134]. In our model, because the parameter to be estimated is encoded into the initial state of the system and the system, initially not correlated with the environment, does not sense it later, we reasonably expect that the maximum information, achieved in the estimation process,

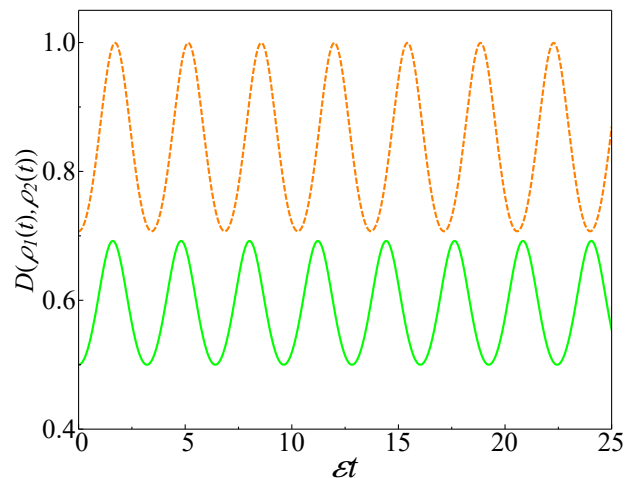


FIG. 2. Dynamics of the distinguishability D between two evolved states $\rho_1(t)$ and $\rho_2(t)$ starting from two special pairs of initial states as a function of the dimensionless time et for a two-level system in the \mathcal{PT} -unbroken phase ($0 < a < 1$).

must be extracted from the initial state itself. Contrary to this intuitive reasoning, it is observed that the non-Hermitian evolution of the system interestingly may enhance the estimation of the initial parameter with time.

We can conclude that the system initially hides some of the encoded information such that it is inaccessible at first and then the non-Hermitian evolution makes this hidden information available for the estimation of the initial phase. In fact, this unusual behavior, leading to non-contractivity of both QFI and HSS, is indistinguishable from the non-Markovian behavior (backflow of information from the environment to the system). This is why we cannot use the standard Hermitian witnesses based on QFI and HSS to detect the non-Markovianity in this context.

D. Non-contractivity of TD in \mathcal{PT} -symmetric one-qubit systems

The non-contractivity of the QFI and HSS motivates us to investigate the TD non-contractivity under non-Hermitian evolution of the system. Our numerical computation shows that the TD may be non-contractive in such systems. Assuming two initial states $|\psi_0^{(1)}\rangle = 1/\sqrt{2}(e^{i\varphi}|0\rangle + |1\rangle)$ and $|\psi_0^{(2)}\rangle = 1/\sqrt{2}(|0\rangle - e^{i\varphi}|1\rangle)$, and computing the TD between the two corresponding evolved states, we can observe the non-contractivity of the TD under \mathcal{PT} dynamics (see Fig. 2, showing the time evolution of the TD, in which the green solid and orange dashed curves are plotted for ($a = 0.2, \varphi = \pi/3$) and ($a = 0.4, \varphi = \pi/4$), respectively).

In our model, initially the system is not correlated with the environment. This preparation alongside the fact that the decoherence effects lead to flow of information from the system to the environment, make us reasonably expect that the distinguishability shows contractivity under the time evolution. However, as discussed in Sec. VC, the system initially

may hide some of the information such that it is inaccessible at first and then the non-Hermitian evolution makes this hidden information available for the system. This counterintuitive behavior may be one of the reasons that the TD exhibits non-contractive behavior. Moreover, this unusual behavior is indistinguishable from the non-Markovian behavior (i.e., backflow of information from the environment to the system). Therefore, we cannot use the BLP's measure as a faithful witness to detect the non-Markovianity in \mathcal{PT} -symmetric systems.

VI. ANTI- \mathcal{PT} -SYMMETRIC TWO-LEVEL SYSTEM

A. Hamiltonian model and witnesses

The generalized form of a single-qubit anti- \mathcal{PT} -symmetric Hamiltonian can be expressed as [4]

$$H_{\mathcal{PT}}^{\text{anti}} = \begin{pmatrix} \lambda\eta e^{i\vartheta} & i\eta \\ i\eta & -\lambda\eta e^{-i\vartheta} \end{pmatrix}, \quad (12)$$

where all of the parameters λ , ϑ and η denote real numbers. It is easy to show that this Hamiltonian satisfies the anti-commutation relation

$$(\mathcal{PT})H_{\mathcal{PT}}^{\text{anti}}(\mathcal{PT})^{-1} = -(H_{\mathcal{PT}}^{\text{anti}})^T = -H_{\mathcal{PT}}^{\text{anti}} \quad (13)$$

where here operator \mathcal{P} denotes Pauli matrix σ_x , \mathcal{T} represents the complex conjugation, and notation A^T means the transpose of matrix A . The eigenvalues of Hamiltonian $H_{\mathcal{PT}}^{\text{anti}}$ are given by $\epsilon_{\pm} = i\lambda\eta \sin\vartheta \pm \sqrt{\lambda^2\eta^2 \cos^2\vartheta - \eta^2}$ and the system is denoted in the regime of unbroken anti- \mathcal{PT} -symmetric phase if $\lambda^2\eta^2 \cos^2\vartheta - \eta^2 < 0$. For simplicity, as well as comparison with the experimental results presented in [49] for this kind of system, we consider the scenario in which $\vartheta = 0$, $\eta \geq 0$ being an energy scale and $\lambda \geq 0$ denoting the degree of Hermiticity. Therefore, the EP2 is located at $\lambda = 1$. We obtain the corresponding time evolution operator of this system as

$$U_{\mathcal{PT}}^{\text{anti}} = \begin{pmatrix} \cos\theta - \frac{i\lambda \sin\theta}{\sqrt{\lambda^2 - 1}} & \frac{\sin\theta}{\sqrt{\lambda^2 - 1}} \\ \frac{\sin\theta}{\sqrt{\lambda^2 - 1}} & \cos\theta + \frac{i\lambda \sin\theta}{\sqrt{\lambda^2 - 1}} \end{pmatrix}, \quad (14)$$

where $\theta = \sqrt{\lambda^2 - 1} \eta t$.

The calculation of the HSS is similar to the approach followed in the previous section. The analytical expressions for this witness is also accessible; however, it does not have compact informative forms, and hence it is not reported here.

B. Dynamical behavior of the witness of quantum criticality

We again see that the HSS works well in detecting the critical behavior of the system. In the anti- \mathcal{PT} -unbroken phase ($\lambda < 1$), as shown in Fig. 3(a) plotted for $\lambda = 0.2$, the HSS does not oscillate over time. However, its oscillatory behavior for $\lambda > 1$ exactly predicts that the system is in anti- \mathcal{PT} -broken phase (see Fig. 3(b) plotted for $\lambda = 1.4$). Moreover, we find that the intensity of the HSS oscillations is related to the parameter λ , denoting the degree of Hermiticity,

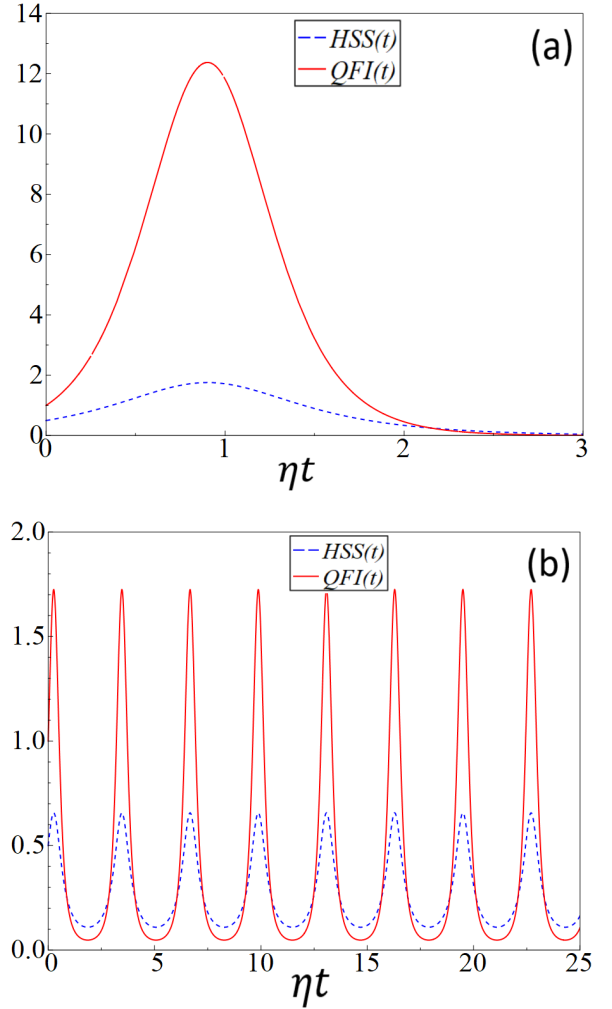


FIG. 3. Dynamics of the Hilbert-Schmidt speed $HSS(\rho_t(\varphi))$ (blue dashed curve) and QFI $D(t)$ (red solid curve) as a function of the dimensionless time ηt for a two-level system (a) in the anti- \mathcal{PT} -unbroken phase ($\lambda < 1$) and (b) in the anti- \mathcal{PT} -broken phase ($\lambda > 1$).

such that with an increase in this parameter, its oscillation gradually weakens.

In Ref. [49], the authors proposed an algorithm for the implementation of the above generalized anti- \mathcal{PT} -symmetric evolution with a circuit-based quantum computing system applying a three-qubit scheme including two ancillary qubits and one working qubit. The implementation scheme is based on decomposing the non-Hermitian Hamiltonian evolution into a linear combination of unitary operators and realizing the scheme in an enlarged Hilbert space with ancillary qubits. The experimental results clearly show that the distinguishability oscillates with period

$$T = \frac{\pi}{\eta \sqrt{\lambda^2 - 1}}, \quad (15)$$

when the system symmetry is broken. Moreover, the intensity of the distinguishability oscillation is connected to parameter

λ such that the transition between the broken and unbroken phases is completely determined by this parameter. Comparing our findings to these experimental results, we see that they are completely consistent. In particular we find that the periods of the HSS and distinguishability oscillations are exactly the same. This fact hence proves the efficiency of our proposed witness in faithfully detecting the critical behavior of the anti- \mathcal{PT} -symmetric systems.

Behavior at the exceptional point

In this case of an anti- \mathcal{PT} -symmetric system, the exceptional point (EP) individuates a phase transition of the system from the unbroken to the broken phase when increasing the parameter λ of the Hamiltonian of Eq. (12). In order to complete the analysis, we investigate the behavior of the HSS at the EP $\lambda = 1$, where it is given by

$$HSS_{EP}^{\text{anti}}(t) = \frac{1}{2 |2\eta^2 t^2 + 2\eta t [\eta t \sin \varphi + \cos \varphi] + 1|}. \quad (16)$$

It is easily seen that the time behavior of the HSS is similar to that we have shown in Fig. 3(a) depicting the HSS dynamics in the unbroken phase. This result also confirms that the HSS can be used to detect phase transitions at the EPs in this type of non-Hermitian systems.

C. Relationship between QFI and HSS

Calculating the QFI with respect to phase parameter φ , we again find that both the QFI and HSS exhibit the same qualitative dynamics in this one-qubit anti- \mathcal{PT} -symmetric system (see Fig. 3).

Interestingly, in both unbroken and broken phases, depicted in Fig. 3, both measures does not necessarily exhibit contractivity, usually assumed as a necessary property for any faithful witness of non-Markovianity.

D. Non-contractivity of TD in anti- \mathcal{PT} -symmetric one-qubit systems

Numerically investigating the TD contractivity under the anti- \mathcal{PT} -symmetric evolution of the system, we find that it is not necessarily contractive in these systems. Starting from two initial states $|\psi_0^{(1)}\rangle = 1/\sqrt{2}(e^{i\varphi}|0\rangle + |1\rangle)$ and $|\psi_0^{(2)}\rangle = 1/\sqrt{2}(|0\rangle + e^{i\varphi}|1\rangle)$, and computing the TD between the two corresponding evolved states, we may observe the non-contractivity of the TD under anti- \mathcal{PT} symmetric dynamics in both unbroken and broken phases (see Fig. 4(a) (4(b)), showing the time evolution of the TD in the unbroken (broken) phase, where the green solid and orange dashed curves are plotted for $(\lambda = 0.5 (\lambda = 1.5), \varphi = 3.1)$ and $(\lambda = 0.6 (\lambda = 1.3), \varphi = 2.9)$, respectively).

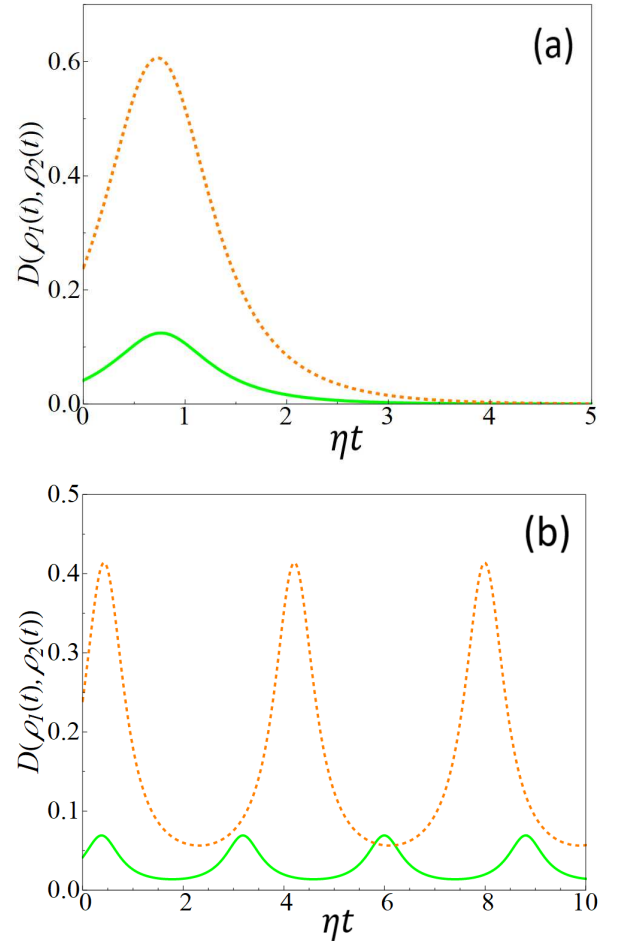


FIG. 4. Dynamics of the distinguishability D between two evolved states $\rho_1(t)$ and $\rho_2(t)$ starting from two special pairs of initial states as a function of the dimensionless time ηt for a two-level system (a) in the anti- \mathcal{PT} -unbroken phase ($\lambda < 1$) and (b) in the anti- \mathcal{PT} -broken phase ($\lambda > 1$).

VII. HIGH-DIMENSIONAL \mathcal{PT} -SYMMETRIC SYSTEM

A. Hamiltonian model and witnesses

Now we consider an open, high-dimensional system described by a 4×4 Hamiltonian [105]

$$H_{\mathcal{PT}} = -JS_x + i\gamma S_z \quad (17)$$

in which S_x and S_z denote spin-3/2 representations of the SU(2) group. In the orthonormal computational basis $\{|1\rangle, |2\rangle, |3\rangle, |4\rangle\}$, the Hamiltonian can be written in the following matrix form

$$H_{\mathcal{PT}} = \frac{1}{2} \begin{pmatrix} 3i\gamma & -\sqrt{3}J & 0 & 0 \\ -\sqrt{3}J & i\gamma & -2J & 0 \\ 0 & -2J & -i\gamma & -\sqrt{3}J \\ 0 & 0 & -\sqrt{3}J & -3i\gamma \end{pmatrix}, \quad (18)$$

representing a \mathcal{PT} -symmetric qudit with $d = 4$. The eigenvalues of $H_{\mathcal{PT}}$ are simply given by $\lambda_k =$

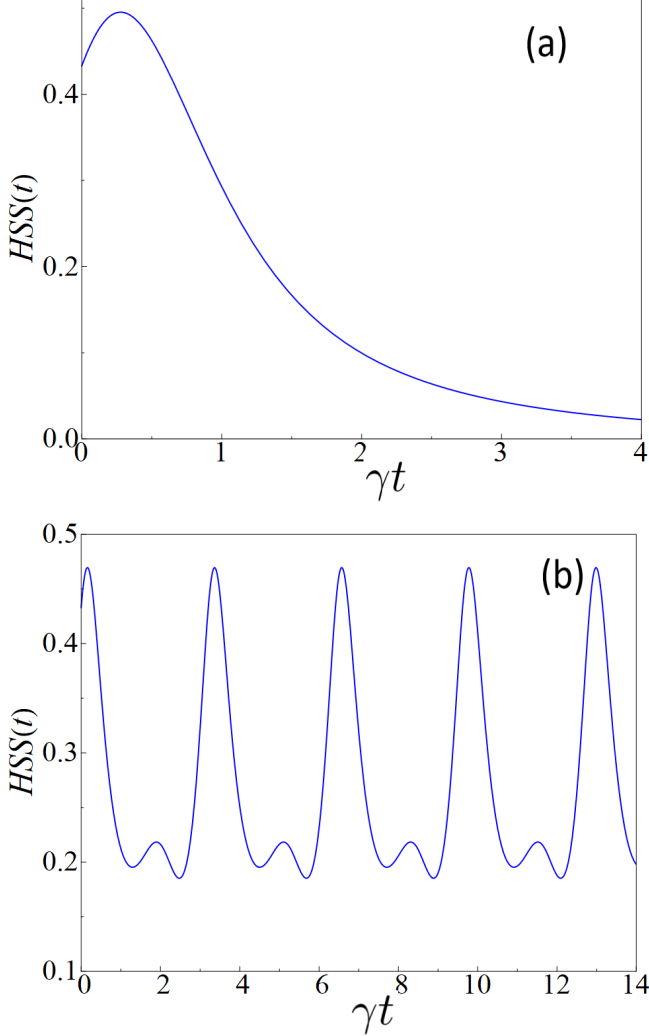


FIG. 5. Dynamics of the Hilbert-Schmidt speed $HSS(\rho_i(\varphi))$ as a function of the dimensionless time γt for the qudit system (a) in the \mathcal{PT} -broken phase ($\gamma > J$) and (b) in the \mathcal{PT} -unbroken phase ($\gamma < J$).

$\{-3/2, -1/2, 1/2, 3/2\} \sqrt{J^2 - \gamma^2}$ ($k = 1, 2, 3, 4$), leading to an EP4 at the \mathcal{PT} -breaking threshold $\gamma = J$. This Hamiltonian can be easily generalized to an arbitrary dimensional system and has an EP with the order equal to the dimension of the system [41, 135, 136].

Let us now assume that the qudit system is 4-dimensional ($d = 4$) and prepared in the pure initial state of Eq. (6): $|\psi_0\rangle = (e^{i\varphi}|1\rangle + |2\rangle + |3\rangle + |4\rangle) / \sqrt{4}$. To obtain the time evolution operator $U = e^{-iH_{\mathcal{PT}}t}$, we can expand the initial state in terms of the non-orthogonal eigenvectors $|\zeta_k\rangle$ of the non-Hermitian Hamiltonian $H_{\mathcal{PT}}$ as $|\psi_0\rangle = \sum_k \beta_k |\zeta_k\rangle$, where the coefficients β_k 's should be determined. To this aim, we first define a square $d \times d$ matrix Φ such that the normalized eigenvectors $|\zeta_k\rangle$ are concatenated as its columns. Therefore, the initial state can be represented as $|\psi_0\rangle := \Phi \beta$ where β denotes a column matrix with elements β_i . Because the columns of Φ are linearly independent, it is invertible, and hence we can

write $|\psi_0\rangle := \Phi \Phi^{-1} |\psi_0\rangle$, leading to the relation $\beta = \Phi^{-1} |\psi_0\rangle$. After computing the coefficients β_k 's by the above method, we can easily obtain the evolved state of the system. Although the computation of the HSS (associated with the evolved density matrix $\rho_i(\varphi) = |\psi(t)\rangle \langle \psi(t)| / \text{Tr}[|\psi(t)\rangle \langle \psi(t)|]$ where $|\psi(t)\rangle = e^{-iH_{\mathcal{PT}}t} |\psi_0\rangle = \sum_k e^{-i\lambda_k t} \beta_k |\zeta_k\rangle$, with respect to initial phase φ) is straightforward, the explicit analytic expression is extremely complex and is not reported in this paper, whereas the results are described below.

B. Dynamical behavior of the witness and EP role

Analyzing the HSS dynamics reveals that the EP4 with $\gamma = J$ determines the border between \mathcal{PT} -symmetric unbroken and broken phases of the qudit (see Fig. 5). In the \mathcal{PT} -broken phase ($\gamma > J$), the HSS first may increase with time and show a peak. Nevertheless, no oscillation in its dynamics is observed and then it monotonously falls with time (see Fig. 5(a) plotted for $J/\gamma = 0.9$). However, in the \mathcal{PT} -symmetry unbroken region ($\gamma < J$), as expected and shown in Fig. 5(b) plotted for $J/\gamma = 2.2$, the HSS dynamics exhibits periodic oscillations (see relation (9)).

C. Non-contractivity of TD in high dimensional \mathcal{PT} -symmetric systems

The computation of the trace distance requires diagonalization of $\rho_1(t) - \rho_2(t)$ for the pair of optimal initial states $\rho_1(0)$ and $\rho_2(0)$. The optimization process with complexity of computing the eigenvectors of the high dimensional complicated density matrices $\rho_i(t)$ ($i = 1, 2$), makes the trace distance too difficult to compute.

We check numerically the contractivity of TD under the \mathcal{PT} -symmetric evolution of the qudit for a large number of random pairs of initial states. We explicitly show that the TD may exhibit non-contractivity in both unbroken and broken phases. For example, starting from two initial states $|\psi_0^{(1)}\rangle = (e^{i\varphi}|1\rangle + |2\rangle + |3\rangle + |4\rangle) / \sqrt{4}$ and $|\psi_0^{(2)}\rangle = (e^{i\theta}|1\rangle + e^{i\varphi}|2\rangle + |3\rangle + |4\rangle) / \sqrt{4}$, and computing the TD between the two corresponding evolved states, one can observe the non-contractivity of the TD under \mathcal{PT} symmetric dynamics of the qudit in both phases (see Fig. 6(a) (6(b)), representing the dynamics of the distinguishability in the broken (unbroken) phase, in which the green solid and orange dashed curves are plotted for $(\theta = \pi/4, \varphi = \pi/2)$ and $(\theta = 1.1 (\theta = 1.5), \varphi = 2.1 (\varphi = 0.1))$, respectively).

Recently, this interesting four-dimensional system, has been implemented experimentally with single photons and a cascaded interferometric setup [105]. In that work, 4×4 nonunitary evolution operations were realized by six beam displacers and another one is used for state preparation. Moreover, two different measurements, i.e., the projective measurement and the quantum state tomography of a four-level system, are carried out at the output. We find that our theoretical predictions are completely consistent with the experimental

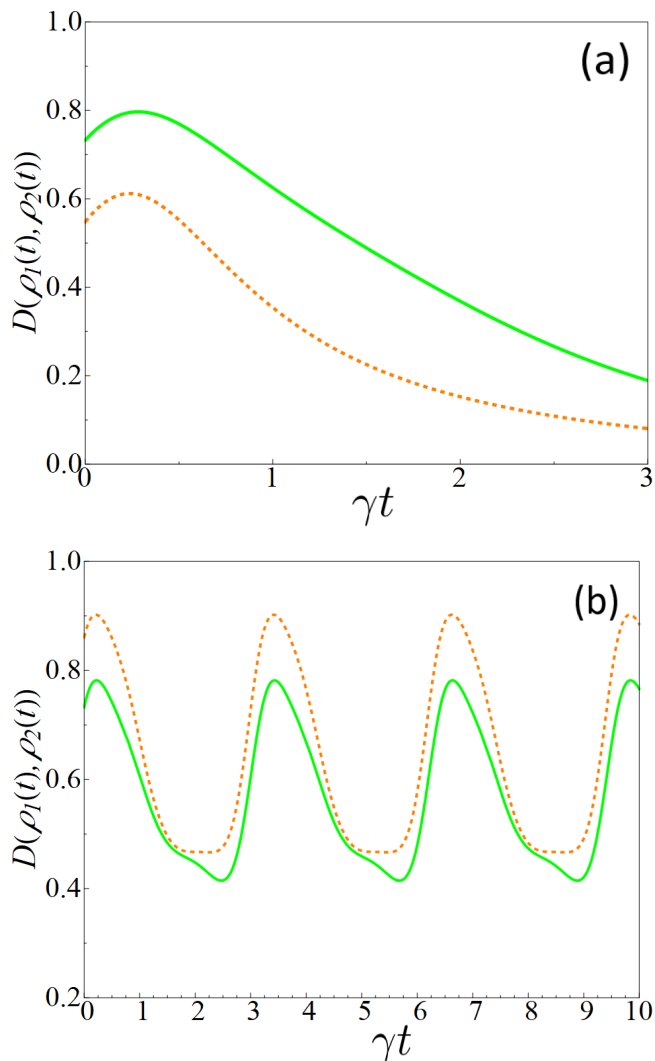


FIG. 6. Dynamics of the distinguishability D between two evolved states $\rho_1(t)$ and $\rho_2(t)$ starting from two special pairs of initial states of the qudit system as a function of the dimensionless time γt (a) in the \mathcal{PT} -broken phase ($J/\gamma = 0.9$) and (b) in the \mathcal{PT} -unbroken phase ($J/\gamma = 2.2$).

results discussed in [105] in which explicitly reported that the EP4 ($\gamma = J$) plays the main role in determining the critical behavior of the qudit. Moreover, our theoretical analysis shows that the HSS oscillates with period $T = 2\pi/(\sqrt{J^2 - \gamma^2})$ in the unbroken phase. This is exactly the period measured experimentally in [105] for oscillations of the dynamics of the quantum information (as quantified by entropy). Because the TD is not contractive in this model, these oscillations cannot be interpreted as evidence of information backflow from the environment or the signature of non-Markovianity. They can be attributed to the non-Hermitian evolution of the system making the initial hidden information available, as discussed in Sec. VC.

VIII. CONCLUSIONS

We have proposed a powerful and easily computable witness, based on the Hilbert-Schmidt speed (HSS), which is a special case of quantum statistical speed, to detect the quantum criticality in systems governed by non-Hermitian (anti-) \mathcal{PT} -symmetric Hamiltonians. Surprisingly, our theoretical predictions can exactly predict the experimental results.

In addition to its conceptual interest, we remark that the HSS-based witness of criticality does not require the diagonalization of the reduced density matrix of the system. Hence, as discussed in the paper for a four-dimensional qudit, it can be introduced as a faithful witness for characterizing criticality in high-dimensional (anti-) \mathcal{PT} -symmetric systems in which computation of other measures leads to serious challenges. The role of the HSS-based measure in detecting the exceptional points (EPs) at which breaking of (anti-) \mathcal{PT} -symmetry occurs has been also analyzed. We have especially illustrated that the system critical behavior appearing at EPs is similar to that of (anti-) \mathcal{PT} -symmetric system in its broken (unbroken) phase. We stress that our theoretical findings all are in complete agreement with experimental observations. These results thus indicate that the HSS-based witness can be adopted to exactly identify the parameter values where phase transitions in the physical behavior of non-Hermitian systems occur.

As an interesting outlook, the introduced HSS-based witness can be employed to characterize controlled speedup of quantum processes in non-Hermitian systems. Such a line of investigation is suggested by recent advances in the context of shortcuts to adiabaticity (STA), which enable us to control a quantum system evolution with no need of slow driving [137–140]. Strategies for STA are typically engineered by means of non-Hermitian control Hamiltonians [140–142]. On the basis of this argument and seeing the results presented here, one may thus expect that the HSS measure plays a role in optimizing STA via non-Hermitian Hamiltonians. This study will be carried on elsewhere.

Another important feature which should be addressed is the non-contractivity of both trace distance (TD) and HSS in (anti-) \mathcal{PT} -symmetric non-Hermitian systems. Because of this non-contractive behavior, the change in the TD of two arbitrary states can no longer be interpreted as a flow of information between the system and the environment. Therefore, contrary to what happens in Hermitian system [84, 88], both TD and HSS can no longer be used as general measures or definitions of non-Markovianity in (anti-) \mathcal{PT} -symmetric systems. Our results show that the non-contractivity of the Hilbert-Schmidt speed in a given model may be a signature of the trace distance non-contractivity and its failure in detecting non-Markovianity.

We explain that the definition of non-Markovianity remains as an important open question in the theory of non-Hermitian quantum systems. In fact, the distinguishability and HSS oscillations in (anti-) \mathcal{PT} -symmetric dynamics may be attributed to the result of the dynamical overlap between the skew eigenstates in the (anti-) \mathcal{PT} -symmetric, an interesting characteristic absent in conventional open quantum systems.

In the unbroken (broken) phase, this overlap is nontrivial and leads to a beat from the two skew dynamical eigenstates, such that the beat period equals the oscillation period. However, in the broken (unbroken) phase, the aforementioned overlap becomes trivial, and in addition, the amplitudes of the eigenstates monotonically shrink or grow under quantum evolution, leading to a monotonic decrease in the distinguishability or the HSS dynamics (see [106] for more details).

In Figs. 1 and 3, we have compared the behaviors of HSS and quantum Fisher information (QFI) associated with the initial phase φ for one-qubit (anti-)PT-symmetric systems. This investigation shows that the HSS and QFI exhibit the same qualitative dynamics in one-qubit systems. According to the discussion presented in the previous paragraph, we find that by inspecting the QFI dynamics one cannot exactly detect the non-Markovian evolution of the (anti-)PT-symmetric systems. However, as known, the QFI is a faithful witness of

non-Markovianity in Hermitian systems [143, 144]. Therefore, we conclude that the witnesses proposed for Hermitian systems should be reexamined to check their efficiency in non-Hermitian systems.

Our work thus motivates deeper analyses to clarify the applicability of other witnesses of the non-Markovianity to detect the memory effects in Hermitian and non-Hermitian systems. Our results also pave the way to further studies on HSS applications in detecting the criticality in high-dimensional non-Hermitian multi-qudit systems.

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