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The inviscid limit and Prandtl's asymptotic expansion for incompressible flows in the half space

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Abstract

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The validity of the inviscid limit for the incompressible Navier-Stokes equations is one of the most important and challenging problems in the mathematical theory of fluid dynamics: the motion of inviscid fluids is described by the Euler equations, so, when the viscosity goes to zero, one would expect the convergence of NS solutions to the Euler solutions. However, NS equations are a singular perturbation of the Euler equations: the change of order of the equation implies that fewer boundary conditions can be imposed on the inviscid flows. Therefore, the no-slip boundary conditions, imposed on the NS solutions, are not satisfied by the Euler flow, for which a tangential slip is allowed. This mismatch between the behaviour at the boundary of the NS solutions and the same behaviour of their supposed limit creates a boundary layer, with large gradients of velocity in the normal direction, which make the diffusive effects comparable to the inertial ones: this situation is classically described by Prandtl's equation. The ill posedness of Prandtl's equation in Sobolev settings require the use of more regular functional spaces: in a holomorphic setting, Prandtl's equation is well posed, and the inviscid limit holds (Sammartino and Caflisch, 1998a; Sammartino and Caflisch, 1998b). In this thesis, we extend this result for incompatible initial data, which satisfy the no-penetration boundary condition, but allow a tangential slip, a kind of data of both numerical and theoretical interest: this extension is not trivial, since the singularity formed by this kind of initial data forces us to use different function spaces, where some of the properties used in the proof of the compatible case do not hold. In Sobolev settings, we see that, for the linearization around an inviscid flow of the NS equations, the inviscid limit actually holds: in this case, Prandtl's asymptotic expansion is not necessary, and convergence can be proved through energy methods in conormal Sobolev spaces. The linearization can be limited to the tangential part of the flow: indeed, this is enough to avoid the interaction between the diffusive effects on the boundary and strong inertial terms, which is believed to cause boundary layer separation.

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List of Abbreviations

ACK theorem	**A**bstract **C**auchy **K**owalewski theorem
NS	**N**avier **S**tokes
BL	**B**oundary **L**ayer

List of Symbols

γ	trace operator
P	Leray's projector
L^p	when the set and/or the variables are not specified, it must be intended as the Lebesgue space over the spatial domain
w'	the tangential part of a generic vector field **w** defined in the half space
w_n	the normal part of a generic vector field **w** defined in the half space
E_2^p	the solution operator of the inhomogeneous heat equation in the half space with homogeneous Neumann boundary conditions and null initial data
E_2^d	the solution operator of the inhomogeneous heat equation in the half space with homogeneous Dirichlet boundary conditions and null initial data
E_0^p	the solution operator of the homogeneous heat equation in the half space with homogeneous Neumann boundary conditions and non-zero initial data
E_0^d	the solution operator of the homogeneous heat equation in the half space with homogeneous Dirichlet boundary conditions and non-zero initial data
E_1	the solution operator of the homogeneous heat equation in the half space with non homogeneous Dirichlet boundary conditions and null initial data
$A : B$	inner product between the generic second order tensors A and B

Chapter 1

The inviscid limit: classical results and recent progresses

1.1 Introduction

The Navier-Stokes equations are a system of partial differential equations describing the motion of an incompressible viscous fluid

$$\begin{aligned} \partial_t \mathbf{u}^{NS} - \nu\Delta\mathbf{u}^{NS} + \mathbf{u}^{NS}\cdot\nabla\mathbf{u}^{NS} + \nabla p^{NS} &= 0,\\ \nabla\cdot\mathbf{u}^{NS} &= 0,\\ \mathbf{u}^{NS}\big|_{t=0} &= \mathbf{u}_0,\\ \gamma\mathbf{u} &= 0, \end{aligned} \tag{1.1}$$

where $\mathbf{u}^{NS}$ is the velocity field, p^{NS} is the pressure, ν is the viscosity, and γ the trace operator. They were introduced by Claude-Louis Navier as a correction of the Euler equations, which do not take into account the effect of the internal friction in the motion of the fluid

$$\begin{aligned} \partial_t \mathbf{u}^{E} + \mathbf{u}^{E}\cdot\nabla\mathbf{u}^{E} + \nabla p^{E} &= 0,\\ \nabla\cdot\mathbf{u}^{E} &= 0,\\ \mathbf{u}^{E}\big|_{t=0} &= \mathbf{u}_0,\\ \gamma_n\mathbf{u}^E &= 0, \end{aligned} \tag{1.2}$$

where $\gamma_n\mathbf{u}^E$ is the normal part of the trace of γ_n. Formally, the Euler equations are obtained by the Navier-Stokes equations by putting $\nu = 0$: therefore, one would expect that $\mathbf{u}^{NS}$ converges to $\mathbf{u}^E$ when the viscosity goes to zero. Actually, the validity of the inviscid limit, i.e. the convergence of the solution of the Navier-Stokes equations to the solution of the Euler equations, is one of the most challenging problems in mathematical fluid dynamics. The problem has already satisfying answers in the case without boundary, at least for Sobolev regular initial data: in particular, for initial data $\mathbf{u}_0 \in W^{s,2}(\mathbb{R}^d)$, $s > \frac{d}{2}+1$, the convergence holds in any time interval $[0,T]$, with T less than the time of existence T_E of the Euler flow in $W^{s,2}(\mathbb{R}^d)$. The key point is that, without boundaries, for the Navier-Stokes flow, viscosity-independent bounds are available for strong Sobolev norms. Through a standard energy argument, we have

$$\frac{d}{dt}||\mathbf{u}^{NS} - \mathbf{u}^{E}||^2_{W^{s-2,2}} \leq C\left(\left(||\mathbf{u}^{E}||_{W^{s,2}} + ||\mathbf{u}^{NS}||_{W^{s,2}}\right)||\mathbf{u}^{NS} - \mathbf{u}^{E}||_{W^{s-2,2}} + \nu||\Delta\mathbf{u}^{E}||_{W^{s-2,2}}\right)||\mathbf{u}^{NS} - \mathbf{u}^{E}||_{W^{s-2,2}} \tag{1.3}$$

which implies the convergence with a rate ν in $L^\infty_T W^{s-2,2}(\mathbb{R}^d)$. Then, convergence in $W^{s',2}(\mathbb{R}^d)$, $s-2 < s' < s$, follows from interpolation. The case $s' = s$ is less simple: Kato (Kato, 1975) proved the small times convergence using a general argument for quasi-linear equations, while Masmoudi (Masmoudi, 2007) obtained the convergence for any $T < T_E$ using a technique involving the regularization of initial data. In a domain Ω with boundaries, one must impose that the flow does not penetrate the solid wall: this leads to the no-penetration boundary conditions $\gamma_n\mathbf{u} = 0$, where γ_n is the normal component of the trace of the vector field $\mathbf{u}$. Inviscid flows satisfy only this boundary condition, while, since Navier-Stokes equations are second order equations, an additional boundary condition is needed. No-slip boundary conditions are usually the choice: due to the effect of the viscosity, the fluid adheres to the boundary $\partial\Omega$, so the velocity of the fluid on $\partial\Omega$ must be the same of the boundary itself. These conditions where proposed by Stokes (Stokes, 1850), and validated by experimental observations. Under these conditions, viscosity-independent bounds for strong norms are not available: a large production of vorticity can occur in a layer near the boundary, creating complicated structures which can eventually detach from the solid wall and propagate inside the bulk. The fact that the L^2 norm of the velocity is bounded by the L^2 norm of the initial data imply that, up to a subsequence, the Navier-Stokes solutions $\mathbf{u}^{NS}(\nu)$ are weakly convergent in L^2; however, due to the nonlinearity of the Navier-Stokes equations, weak convergence is not enough to prove that the limit satisfies the Euler equations. Indeed, also the validity of the weak inviscid limit is still an open problem.

1.1.1 Prandlt's equation

In order to describe the behaviour of the fluid near the boundary, Prandtl (Prandtl, 1904) proposed that a thin boundary layer, of thickness proportional to $\sqrt{\nu}$, exists in a neighborhood of $\partial\Omega$: in this layer, the tangential velocity sees a rapid adjustment from the nonzero value of the outer flow to the no-slip boundary condition used for the Navier-Stokes equations. Outside the layer, the effects of viscosity are negligible, so the motion of the flow is described by the Euler equations. Experimental observations show that the thickness of the boundary layer (at least for laminar flows) is $O(\sqrt{\nu})$. From this, Prandtl assumed that, near the boundary, the velocity field depends on the normal variable y through the rescaled variable $Y = y/\sqrt{\nu}$, obtaining the following equation

$$\begin{aligned}(\partial_t - \partial_{YY})u^P + u^P\partial_x u^P + v^P\partial_Y u^P - (\partial_t + u^E\big|_{y=0}\partial_x)u^E\big|_{y=0} &= 0,\\ u^P\Big|_{Y=0} = 0, \quad u^P\Big|_{Y\to+\infty} &= u^E(x, y=0, t),\\ u^P\Big|_{t=0} &= u_0(x, y=0).\end{aligned} \tag{1.4}$$

With his work, Prandtl laid the foundations for boundary layer theory, one of the most powerful tools in asymptotic analysis. Ironically, the validity of this technique in fluid mechanics is controversial and not fully understood, while in other fields it has been rigorously justified.
The essential problem with Prandtl's equation is that its wellposedness relies either on structural assumptions on the initial data and/or the flow (in particular, monotonicity assumptions) or on highly restrictive regularity hypothesis (analyticity or Gevrey class regularity).

Monotonicity assumption

Monotonicity allows to transform Prandtl's equation into a degenerate quasilinear parabolic equation, for which maximum and comparison principles hold.
Let us first analyze the bidimensional, stationary case in the half plane domain. Assuming that $u^P(x=0,Y)$ is, for $Y>0$, strictly positively increasing and strictly positive, one can introduce the Von Mises transformation

$$(x,Y)\to(x,\phi), \tag{1.5}$$

where ϕ differs from the streamfunction by a function of the x variable; more precisely,

$$u^P=\partial_Y\phi,\quad v^P=v^P(x,Y=0)-\partial_x\phi,\quad \phi(x,Y=0)=0. \tag{1.6}$$

In this new set of independent variables, the function $w=u^2$ satisfies

$$\partial_x w+v^P(x,Y=0)\partial_\phi w=\sqrt{w}\partial_{\phi\phi}w-\gamma\partial_x p^E, \tag{1.7}$$

which is a parabolic equation where x assumes the role of time. If $u^P(x=0,Y)\in C^{2,\alpha}$, $v^P(x,Y=0),\gamma\partial_x p^E\in C^1$ and

$$\partial_{YY}u^P(x=0,Y)-\gamma\partial_x p^E(x=0)-v^P(x,Y=0)\partial_Y u^P(x=0,Y)=O(Y^2) \tag{1.8}$$

as $Y\to 0$, then Prandtl's equation admits a unique solution in $C^2([0,\bar{x}]\times\mathbb{R}_+)$ (Oleĭnik, 1963). Furthermore, if either $\gamma\partial_x p^E\le 0$ and $v^P(x,Y=0)\le 0$ or $\gamma\partial_x p^E<0$ (favourable pressure gradient), then the solution is global. The fact that, under an adverse gradient of pressure, the solution may be only local, is highly expected from a physical point of view: indeed, under an adverse pressure gradient, the flow near the boundary slows down and, if the gradient is strong enough, assumes a direction opposite to the one of the outer (eulerian) velocity. When this reverse flow occurs, the boundary layer detaches from the boundary, creating vortices and eddies that propagate in the inner region. The point x^* at which boundary layer separation occurs is called point of boundary layer separation, or Goldstein singularity point. The expected behaviour of the flow near the point of separation has been known for decades (Goldstein, 1948; Stewartson, 1958): at this point, the shear stress vanishes, and the trace of u^P approaches this point as $(x^*-x)^{1/2}$. However, these properties were obtained via formal calculations, using asymptotic analysis: they have been rigorously justified only recently in (Dalibard and Masmoudi, 2019), where the authors proved that the stationary Prandtl's equation predicts separation, and with the properties predicted by Goldstein.
As in the stationary case, local well posedness of the unsteady Prandtl equation under monotonicity assumptions was first proved by Oleĭnik (Oleĭnik, 1966). In the unsteady case, Crocco's transformation takes the place of Von Mises transformation:

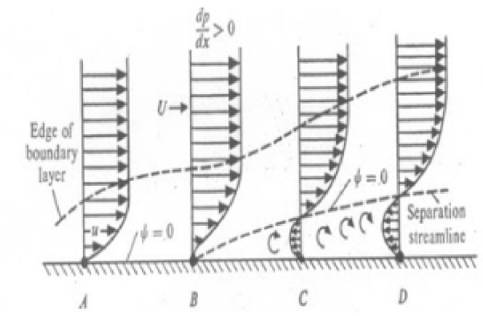

FIGURE 1.1: Boundary layer separation caused by an adverse gradient of pressure

this time, the change of variables is given by

$$(x, Y) \to (x, \eta), \tag{1.9}$$

where

$$\eta = \frac{u^P}{\gamma u^E}, \tag{1.10}$$

while the new independent variable w is given by

$$w = \frac{\partial_y u^P}{\gamma u^E} \tag{1.11}$$

Once again, we have to make assumptions of monotonicity: we require that the initial data satisfy $\partial_y u_0 > 0$, and, in the case of domains $[0, L] \times \mathbb{R}_+$ with finite length in the tangential direction, $\partial_y u^P(x = 0) > 0$. Crocco's change of variables allows to obtain a single quasilinear parabolic equation for the new independent variable w,

$$\partial_t w = A\partial_\eta w + B\partial_x w + Cw + w^2 \partial_{\eta\eta} w. \tag{1.12}$$

where A, B and C depend on η and on the trace of the Euler flow. Unlike Von Mises' transform, Crocco's transform reduces the unbounded domain $[0, L] \times \mathbb{R}_+$ to the bounded domain $[0, L] \times [0, 1[$. Furthermore, although the equation obtained is linear, the boundary conditions are not

$$(w\partial_\eta w)\,\big|_{\eta=0} = \frac{\gamma \partial_x p^E}{\gamma u^E}, \;\; w\big|_{\eta=1} = 0. \tag{1.13}$$

Under suitable regularity assumptions and the aforementioned monotonicity conditions, Oleĭnik proved the local wellposedness of Prandtl's equation: in particular, she proved small time existence for arbitrarily large spatial domains and all time existence for domains $[0, L] \times \mathbb{R}_+$, with L suitably small. Global existence under monotonicity conditions can be proved for weak solutions (Xin and Zhang, 2004), using the viscous splitting method, although uniqueness is not granted for such solutions.

Interestingly, Oleĭnik result for the unsteady Prandtl equation can be obtained via energy methods, without using Crocco's transformation: however, while the independent variables are not changed, we still need a change of dependent variable, so that the most problematic terms in the energy estimates cancel out. See (Xu and

Zhang, 2017) for the linearized case and (Masmoudi and Wong, 2015) for the nonlinear case.
We stress that, compared to the steady case, the relation between the appearance of singularity in Prandtl's solution and the separation of the boundary layer is much more controversial: indeed, for high Reynold's numbers, numerical computations show that the interaction between the outer eulerian flow and the boundary layer begin well before than the appearance of Prandtl's singularity, and have, in any case, different character, since the blow up in the unsteady case is through "shock formation" (Ee, 2000).
Finally, we stress that the previous results are strictly bidimensional: in the three-dimensional case, Prandtl's equations are ill posed even under Oleĭnik's monotonicity condition (Liu, Wang, and Yang, 2016), and additional structural assumptions must be assumed.

Regularity assumptions

The first existence result for Prandtl's equation which does not rely on monotonicity assumptions can be found in (Sammartino and Caflisch, 1998a): the result states that Prandtl's equation is well posed in an analytic setting, and this was part of the proof of the validity of the inviscid limit and Prandtl's ansatz. The proof is obtained through the use of the abstract Cauchy-Kowalewski (ACK) theorem: this is, essentially, a contraction theorem in a scale of Banach spaces. The key of the proof is that, in an holomorphic setting, one can use Cauchy's formulas for derivatives, so, for functions tangentially analytic on a strip, the tangential derivatives can be controlled by the function once one reduces the strip of analyticity; for the normal variable the domain of analyticity is a cone, so the same argument applies to $Y\partial_Y$ instead of ∂_Y.
Those hypothesis were relaxed in (Lombardo, Cannone, and Sammartino, 2003): indeed, the authors found that only tangential analyticity is needed to obtain the well posedness of Prandtl's equation. Once again, the ACK theorem was used to prove local well-posedness in time: the key difference from (Sammartino and Caflisch, 1998a) is the inversion of the unbounded differential operator

$$\partial_t - \partial_{YY} + Y\alpha(t,x)\partial_Y, \tag{1.14}$$

where $\alpha = \gamma\partial_x u^E$, whose fundamental solution can be found through the method of characteristics. The same authors (Cannone, Lombardo, and Sammartino, 2013) proved well posedness under tangential analyticity assumptions for the case of "eulerian" initial data, which satisfy only the non-penetration boundary condition. Well posedness with tangential-only analyticity can be obtained also through energy methods (Kukavica and Vicol, 2013). A tangential regularity stronger than Sobolev regularity is not a technicality, but it is actually necessary in order to avoid strong instability mechanisms typical of Prandtl's equation. It is well known, indeed, that both the linearized and the nonlinear Prandtl equation are ill posed in Sobolev spaces: in particular, the linearization of Prandtl's equation around a non monotonic shear flow admits approximate solutions with growth rate proportional to $\sqrt{\xi'}$, where ξ' is the Fourier variable corresponding to the tangential variable x (Grenier, 2000). Numerical simulations also seem to suggest that the most unstable mode grows like $\sqrt{\xi'}$ (Gérard-Varet and Dormy, 2010). Therefore, one should expect that Prandtl's equation is well posed in the Gevrey class G^2 for the tangential variable, with G^m

defined as the set of functions $f(x)$ such that

$$|D^j f| \leq C\frac{(j!)^m}{\tau^j}, \tag{1.15}$$

for some positive τ and C. Notice that G^1 the set of analytic functions, while, for $m > 1$, G^m contains also non analytic functions: in particular, those spaces contain compactly supported functions. Well posedness for initial data with Gevrey regularity in the tangential variable and Sobolev regularity in the normal variable was first proved in (Gerard-Varet and Masmoudi, 2015), where the authors proved the result for the class $G^{7/4}$, under an additional structural assumption (namely, that the initial vorticity has a single curve of non-degenerate critical points). The proof uses the vorticity formulation and energy estimates performed not directly on the vorticity, but on smarter energy functionals (which resembles the method used to obtain Oleĭnik's results in the monotonic case through energy estimates). Results in better Gevrey classes were obtained in (Chen, Wang, and Zhang, 2018) for the linearized equation in G^m, $m < 2$; the limit case of G^2 has been proved recently (Li and Yang, 2020).
It is worth noting that, even in the most regular setting (the analytic one) the well-posedness is only local in time, and the solutions can blow up in finite time (E and Engquist, 1997): however, existence becomes global if the data are small and analytic at least in the tangential variable (Paicu and Zhang, 2021).

Relation with Prandtl's ansatz and the inviscid limit

Prandtl's equation is obtained assuming that, near the boundary, the velocity field depends on y through the rescaled variable $Y = y/\varepsilon$, where $\varepsilon = \sqrt{\nu}$; continuing this argument, one can formally expand Navier-Stokes' solutions as

$$\mathbf{u}^{NS} = \mathbf{u}^{out}_{(0)} + \mathbf{u}^{inn}_{(0)} + \varepsilon\left(\mathbf{u}^{out}_{(1)} + \mathbf{u}^{inn}_{(1)}\right) + ... + \varepsilon^N\left(\mathbf{u}^{out}_{(N)} + \mathbf{u}^{in}_{(N)}\right) + O(\varepsilon^{N+1}). \tag{1.16}$$

In particular, for $N = 0$, only Euler's equations and Prandtl's equation appear in the expansion, and the remainder is $O(\sqrt{\nu})$ in the L^∞ norm. In general, there is no clear connection between the well posedness of Prandtl's equation, the validity of the asymptotic expansion (1.16) and the validity of the inviscid limit: only in the analytic setting (Sammartino and Caflisch, 1998a; Sammartino and Caflisch, 1998b) the expansion has been rigorously justified, while, in general, its validity is controversial.
Grenier (Grenier, 2000) proved that this expansion is extremely unstable: even when Prandtl's equation is well posed and the asymptotic expansion holds, an arbitrarly small (polynomial in ν) perturbation of the initial data implies that the perturbed solution of the Navier-Stokes equations differs from the original solution of an $O(\nu^{1/4})$ at a time T_ν which goes to zero with ν. Furthermore, Grenier's proof of this instability casts some doubts on the derivation of Prandtl's equations: indeed, the instability found consists of small periodic structures in the x variable, with a period of order $\nu^{1/2}$, which appear and grow, while Prandtl's equation is obtained assuming that the tangential derivatives are bounded. These structures are not possible in analytic settings, for which Prandtl's expansion actually holds (Sammartino and Caflisch, 1998a; Sammartino and Caflisch, 1998b). The result of Grenier was obtained through a perturbation of an unstable shear flow; later, the same author and Nguyen (Grenier and Nguyen, 2017) proved that Prandtl's ansatz is false also near a stable shear flow, and

that a shear flow stable for the Euler equations becomes unstable adding a small viscosity.
In the monotonic setting, where Prandtl's equation is well posed, no convergence result, involving only conditions on the initial data, is known. However, monotonicity assumptions on the solutions imply the validity of the inviscid limit: in particular, assuming the positivity of the traces of u^E and of ω^{NS}, the inviscid limit holds (Constantin, Kukavica, and Vicol, 2015). While the condition on u^E, for small times, can be obtained by assumptions on the initial data (e.g. $\gamma u_0 \geq C > 0$), the positivity of ω^{NS} for a viscosity-independent time must be imposed a priori.
A stark difference between the well posedness of Prandtl's equation and the validity of the asymptotic expansion (1.16) concerns the anisotropic regularity assumptions: while Prandtl's equation is well posed assuming a Sobolev regularity with respect to the normal variable and an analytic regularity in the tangential variable, Euler's equations, in the same setting, is ill posed (Constantin, Kukavica, and Vicol, 2016). This means that an asymptotic expansion in a similar setting is not possible: furthermore, such a setting does not grant the solvability of Prandtl's equation, since the trace of the Euler flow, which appears in Prandtl's equation, is not analytic in x. This also raises the question of which hypotheses, less restrictive than analyticity in both variables, allow us to have the analyticity (or, at least, Gevrey regularity) of the trace of the Euler flow, necessary to use the anisotropic results for Prandtl's equation.

1.1.2 The inviscid limit

The first proof of the validity of the inviscid limit can be found in (Sammartino and Caflisch, 1998a; Sammartino and Caflisch, 1998b): using an analytic setting, the authors proved the validity of the asymptotic expansion (1.16) (at least, up to order 1), which implied the convergence of $\mathbf{u}^{NS}$ to $\mathbf{u}^E$ with a rate $O(\nu^{1/4})$ in the energy norm. The existence of all the terms of the expansion was proved with the abstract Cauchy Kowalewski theorem; the use of an holomorphic setting, in particular, allowed to tame the instabilities of Prandtl's equation. The result was proved for the half space: this particular geometry allowed to use Ukai's representation formula for the solution of the Stokes equations (Ukai, 1987). Later, this result has been generalized to the exterior of a disk (Caflisch and Sammartino, 1997) and a channel (Kukavica, Lombardo, and Sammartino, 2016).
A different proof of the validity of the inviscid limit in an holomorphic setting can be found in (Wang, Wang, and Zhang, 2017): the authors used the vorticity formulation to obtain energy estimates in conormal analytical function spaces, i.e. in spaces such that the normal derivatives are multiplied by functions going to zero linearly with the distance to the boundary. The main advantage is that, unlike the classical normal derivatives, conormal derivatives are bounded in ν.
Maekawa (Maekawa, 2014) proved the validity of the inviscid limit in the bidimensional half plane for initial data with a Sobolev regularity, with vorticity supported away from the boundary: these initial data are actually analytic near the boundary, since the incompressibility condition and the irrotationality form the Cauchy-Riemann equations for the holomorphic function $v + iu$

$$\begin{aligned} \partial_x v &= \partial_y u \\ \partial_y v &= -\partial_x u. \end{aligned} \tag{1.17}$$

Since, in 2D, the eulerian vorticity is transported by the flow, with $||\mathbf{u}^E||_{L^\infty}$ bounded, ω^E is supported away from the boundary for a finite time: in this time span, Maekawa

proves that the boundary layer part decays exponentially. Therefore, strong interactions between the vorticity produced in the boundary layer and the outer flow are avoided, excluding the instability mechanisms found by Grenier (Grenier, 2000).
A similar absence of strong interaction phenomena is expected also for generic data analytic only near the boundary, so the inviscid limit should hold for this kind of data: this intuition was recently proved by Kukavica, Vicol and Wang (Kukavica, Vicol, and Wang, 2020). One of the main technical difficulties is that the persistence of local properties is difficult to verify for the solutions of the Navier-Stokes equations, due to the presence of nonlocal operators (the pressure in the velocity formulation, the vorticity in the vorticity formulation). The proof is obtained through energy estimates: their use is essential in order to avoid the loss of a derivative in the Sobolev region.
The same authors and Nguyen (Kukavica et al., 2021) proved the validity of the asymptotic expansion (1.16): a key ingredient is the proof of the propagation of the local analyticity for the Euler flow. The use of an L^1 based analytic norm allowed the authors to arrest the expansion at the order zero, while, in the L^2-like setting of (Sammartino and Caflisch, 1998b), an expansion up to order one was needed.
Both the cases of initial vorticity supported away from the boundary and initial data analytic near the boundary have been generalized to the three dimensional case in (Fei, Tao, and Zhang, 2018) and (Wang, 2020), respectively.
Finally, the inviscid limit holds in domains with some symmetric geometry, for initial data with a symmetry which is preserved by the flow: the simplest case is a disk (Kelliher, 2009; Bona and Wu, 2002; Lopes Filho, Mazzucato, and Nussenzveig Lopes, 2008; Lopes Filho et al., 2008), for which the Navier-Stokes equations reduce to the heat equation. Other symmetric configurations are the infinite straight, circular pipe (Han et al., 2012) and the infinite parallel channel (Mazzucato, Niu, and Wang, 2011; Mazzucato and Taylor, 2008).

1.2 Organization of the thesis

In chapter two, we analyze some results obtainable in Sobolev settings. First, we treat the case of the linearization of the Navier-Stokes equations around a generic inviscid flow $\mathbf{u}^E$ through an energy method, we see the convergence of the linearized flow $\mathbf{u}^L$ to $\mathbf{u}^E$ in a conormal Sobolev space. The function chosen in order to define the conormal derivative is $\arctan(y)$: the reason is that it is a smooth, bounded function, so that one does not have to assume additional decay properties on the initial data, and it is positive with derivatives of a fixed sign, so, after an integration by parts on the diffusive term, the sign of the various contributions is clear. The use of a conormal derivative implies that the L^2 product with the gradient of the pressure does not disappear: however, we shall see that the contribution of this term to the estimates is small, since the divergence of $\partial_y \mathbf{u}^L \arctan^2(y)$ is $\partial_y v^L \partial_y \arctan^2(y)$, which means that, using integration by parts, we can remove the gradient from the pressure without adding any derivative on $\partial_y \mathbf{u}^L$. In order to prove the convergence, an expression of $\partial_y u^L$ is needed: this is obtained by writing the flow $\mathbf{u}^L$ as the sum of a Navier-Stokes like flow with Navier slip boundary conditions, $\mathbf{u}^{FI} = \mathbf{u}^F + \mathbf{u}^I$, and a Stokes corrector $\mathbf{u}^B$, which cancels out the trace of the first flow. With this decomposition, the trace of $\partial_y u^L$ is the trace of $\partial_y u^B$.
With a similar decomposition, we are able to prove the convergence also when only the equations for the tangential part of the flow is linearized, i.e. the tangential part of the flow $\mathbf{u}^{PL}$ is transported by $\mathbf{u}^E$, while the normal part is transported by itself.

This is an original contribution. The key point is that, in the expression of $\nu\gamma\partial_y u^{PL}$, every term is either a tangential derivative, which can be moved on the trace of $\mathbf{u}^E$ in an integration by parts, or small in ν: without the linearization of the tangential part of the flow, $\gamma\nu\partial_y u^{NS}$ contains a term which is not small in ν, unless one knows a priori either that $v^{NS} - v^E$ is "$O(\nu^{1/2})$ better" than $u^{NS} - u^E$, or that the tangential derivatives are bounded. Without these assumptions, the estimates do not close, requiring the derivatives of the subsequent order: so, energy estimates themselves suggest the use of an analytic setting.
The appearance of a boundary layer is inevitable in a singular perturbation problem like the inviscid limit: however, the strength of the boundary layer can be reduced if one assumes more favorable boundary conditions for the Navier-Stokes equations. We shall see that, if we impose Navier-friction boundary conditions on the viscous flow, convergence is ensured; these conditions, which relate the shear stress at the boundary with the trace of the tangential part of the flow, were first introduced by Navier, and justified through kinetic theory. In particular, in the frictionless case in the half space, the great compatibility of these conditions with the inviscid limit can be deduced by a symmetry argument, which implies the conservation of $\gamma\partial_y u^E$, if initially zero; in bidimensional, curved domains, what remains null for all time if null at the initial time is the trace of the eulerian vorticity. Of course, this persistency property fails for non compatible data: therefore, we see that, unlike the no-slip case, incompatible initial data affect the rate of convergence.
Then, we see how the solution of the Stokes equations can be used to prove the criteria of Kato and Wang: these criteria link the validity of the inviscid limit, respectively, to the behaviour of the total gradient of $\mathbf{u}^{NS}$ in a sublayer of width $O(\nu)$ and to the behaviour of the tangential derivatives of $\mathbf{u}^{NS}$ in any sublayer of width greater than $O(\nu)$. Using a corrector with an artificial viscosity, only two terms in the energy estimates are not automatically small in ν: essentially, the aforementioned criteria are smallness assumptions on one of those two terms, and those assumptions allow to chose an artificial viscosity which makes the other term small.
Finally, in chapter three, we prove the convergence of the solutions to the Navier-Stokes equations to the solution to the Euler equations in an holomorphic setting, for initial data which satisfy only the non penetration boundary condition: this improves the result of Sammartino and Caflish (Sammartino and Caflisch, 1998a; Sammartino and Caflisch, 1998b), which holds only for compatible data. The incompatibility between the initial data and the boundary condition causes the presence of an initial layer: we see how this singularity is propagated in the asymptotic expansion of the Navier-Stokes flow. Due to this incompatibility, our functional setting is less regular to the one used in the compatible case: therefore, the estimates available for the compatible case do not hold, and new estimates must be obtained. Furthermore, algebra properties can no longer be used to deal with nonlinear terms: therefore, unlike the compatible case, we also need estimates in an $L^\infty_Y L^2_x$-like setting.

Chapter 2

Some results in Sobolev settings

2.1 Introduction

The convergence of the solution to the Navier-Stokes equations is a singular perturbation problem: the term which disappears with the viscosity is the term with the derivative of maximum order, so passing to the limit changes the order of the equation, thus changing the number of boundary conditions that can be imposed. The classic boundary conditions imposed to the Navier-Stokes equations are the no-slip conditions $\gamma \mathbf{u}^{NS} = \mathbf{0}$, where γ is the trace operator; these conditions are not satisfied by the Euler flow, which satisfies only the no-penetration condition $\gamma v^E = 0$, while the trace of the tangential part of the flow is generally non-zero. A much more favorable case for convergence occurs for the flow $\mathbf{u}^N$ obtained pairing the Navier-Stokes equations with the so-called Navier-slip conditions $\gamma[\mathbf{n}S(\mathbf{u}^N)]'$, $\gamma v^N = 0$, where S is the stress tensor, $\mathbf{n}$ is the unit normal and $'$ denotes the tangential part. These are a particular case of the Navier-friction boundary conditions $\gamma[\mathbf{n}S(\mathbf{u}^N) + \alpha \mathbf{u}^N]'$, $\gamma v^N = 0$, α is a friction coefficient, which depends on the roughness of the boundary. The idea that, on the boundary, the shear stress should be proportional to the tangential velocity, was first proposed by Navier (Navier, 1827), who justified it through kinetic theory; much later, Jager and Mikelic (Jäger and Mikelić, 2001) obtained Navier-friction conditions as homogenization of the no-slip conditions on a rough boundary. With such boundary conditions, for general domains, convergence to the inviscid flow holds in the energy norm (Lopes Filho, Nussenzveig Lopes, and Planas, 2005; Iftimie and Planas, 2006). In the half space, for an arbitrary k, it is easy to show the convergence of $\mathbf{u}^N$ to $\mathbf{u}^E$ in $W^{2k+1,2}$ assuming that the initial data $\mathbf{u}_0$ and the forcing term $\mathbf{f}$ satisfy some suitable compatibility conditions, namely $\gamma\partial_y^{2j+1}u_0 = 0$ and $\partial_y^{2j+1}P'\mathbf{f} = 0$, where $P'\mathbf{f}$ is the tangential part of the application of the Leray projector P to $\mathbf{f}$: these conditions imply that both $\partial_y^{2j+1}u^N$ and $\partial_y^{2j+1}u^E$ remain null for all times, so one can employ an energy argument similar to the one used for the whole space in (Constantin and Foias, 1988), (Masmoudi, 2007). The norm $||\mathbf{u}^N - \mathbf{u}^E||_{W^{k,2}}$ goes to zero as ν, which is the same rate of convergence obtained without boundaries. This result is largely unsurprising, since if we extend to the lower half space the tangential parts u^N and u^E evenly and the normal parts v^N and v^E oddly, $\mathbf{u}^N$ and $\mathbf{u}^E$ become, respectively, the solutions to the Navier-Stokes equations and the Euler equations in the whole space, and the compatibility conditions imposed on $\mathbf{u}_0$ and $P\mathbf{f}$ imply that their extension to the whole space preserves the same regularity those terms had in the half space. In general domains, if the tangential part and the curl of both the initial data and the forcing term are zero, both $\gamma v^E(t)$ and $\gamma\boldsymbol{\omega}^E(t)$ remain equal to zero for as long as the strong solution to Euler's equations exists. The convergence in $W^{1,2}(\mathbb{R}^n)$ can be proved even when the aforementioned compatibility conditions are not satisfied: this implies that

$||\gamma u^N - \gamma u^E||_{L^2}$ goes to zero with the viscosity, a result which cannot hold for $\mathbf{u}^{NS}$. This affinity between the inviscid limit problem and the Navier-slip conditions suggests to consider the Navier-Stokes solution $\mathbf{u}^{NS}$ as the sum of a flow $\mathbf{u}^{FI}$ which satisfies a Navier-Stokes-like system of equations and the Navier-slip boundary conditions, and a Stokes flow $\mathbf{u}^{B}$ which cancels out the tangential part of $\mathbf{u}^{FI}$ at the boundary: this strategy allows us to prove the validity of the inviscid limit in the energy norm for a linearized flow in section 2.2, and the same result for a flow obtained linearizing only the Navier-Stokes equations for the tangential component in section 2.3. In the linear case, we actually show the convergence in a norm which controls the norm $W^{1,2}_{y\geq c}$ for any $c > 0$, therefore implying that the inviscid limit hols in $W^{1,2}$ away from the boundary. In the fully nonlinear case, the estimate we obtain for the $k - th$ partial derivative of the difference between $\mathbf{u}^{NS}$ and $\mathbf{u}^{E}$ involves the partial derivative of order $k+1$, so the estimates does not close in a Sobolev setting. A Stokes flow with an arbitrary viscosity coefficient can be used to prove that the inviscid limit holds provided that the gradient of the flow near the boundary does not grow too much: in particular, the inviscid limit holds if one of the following holds

$$\lim_{\nu\to 0} \nu^{1/2}||\nabla \mathbf{u}^{NS}||_{L^2_T L^2(y\leq C\nu)} = 0, \tag{2.1}$$

$$\lim_{\nu\to 0} \nu^{1/2}||\partial_x u^{NS}||_{L^2(y\leq h(\nu))} = 0, \tag{2.2}$$

$$\lim_{\nu\to 0} \nu^{1/2}||\partial_x v^{NS}||_{L^2(y\leq h(\nu))} = 0, \tag{2.3}$$

with

$$\lim_{\nu\to 0} \frac{\nu}{h(\nu)} = 0. \tag{2.4}$$

The first condition is the celebrated Kato's criterion (Kato, 1984b); the other two are due to Wang (Wang, 2001). For both these criteria, we just need a nice behavior of the gradient of the velocity in a layer thinner than the boundary layer predicted by the classical Prandtl theory; this could mean that the validity of Prandtl's asymptotic expansion might not be necessary for the convergence of $\mathbf{u}^{NS}$ to $\mathbf{u}^{E}$.

2.2 Convergence for the linearized problem

The convergence of the viscous solution to the inviscid one cannot be verified in the case of detachment of the boundary layer (Kelliher, 2008), (Kelliher, 2017a); although the formation of a boundary layer can also occur with purely diffusive equations, it is believed that the interaction with a strong convective term causes the detachment of the boundary layer. Therefore, by linearizing the convective term, it is more reasonable to expect that the inviscid limit holds (Maekawa and Mazzucato, 2016). Various results in this sense can be found for flows linearized around a stationary solution: Temam and Wang (Temam and Wang, 1996) performed an asymptotic analysis for Oseen flows, which are obtained by linearizing the Navier-Stokes equations around a constant flow. Lombardo and Sammartino (Lombardo and Sammartino, 2001) considered, in a channel, the linearization of the equation around a velocity field $(U, 0)$ with zero normal component componenent: the particular geometry of the channel and the only-tangential velocity field $(U, 0)$ allowed them to give an explicit solution to the problem in terms of inverted heat operators and projection operators. Gie (Gie, 2014) proved the validity of the asymptotic expansion of the Stokes solutions even for incompatible data, i.e. for initial data such that $\gamma u_0 \neq 0$; Gie, Kelliher and

Mazzucato (Gie, Kelliher, and Mazzucato, 2018) extended the result to equations linearized around any stationary Euler solution.

In this section, we see how energy methods can be used to prove that, in a conormal Sobolev space, the linearization of the Navier-Stokes equations around an arbitrary (even unsteady) Euler flow $\mathbf{u}^E$ converges to $\mathbf{u}^E$. For the simplicity of notation, we first analyze the case where only the most problematic term of the linearization of $\mathbf{u}^{NS} \cdot \nabla \mathbf{u}^{NS}$ is present, i.e. the case of a flow $\mathbf{u}^L$ convected by $\mathbf{u}^E$; then, at the end of the section, we show that the remaining term in the linearization of the bilinear part of the Navier-Stokes equation is actually easier to estimate.

The flow $\mathbf{u}^L$ satisfies the following system

$$\begin{aligned} \partial_t \mathbf{u}^L - \nu \Delta \mathbf{u}^L + \mathbf{u}^E \cdot \nabla \mathbf{u}^L + \nabla p^L &= \mathbf{f}; \\ \nabla \cdot \mathbf{u}^L &= 0; \\ \gamma \mathbf{u}^L &= \mathbf{0}; \\ \mathbf{u}^L(t=0) &= \mathbf{u}^0. \end{aligned} \tag{2.5}$$

We do not provide an asymptotic expansion of the solution: instead, using an energy argument, we directly prove the convergence in the norm $|||\cdot|||$ defined as

$$|||\mathbf{u}|||^2 = ||\mathbf{u}||^2_{L^2} + ||\partial_x \mathbf{u}||^2_{L^2} + ||\arctan(y)\partial_y \mathbf{u}||^2_{L^2}. \tag{2.6}$$

The classical energy argument heavily relies on the fact that the velocity field and its derivatives are divergence free, which of course is not the case for $\arctan^2(y)\partial_y(\mathbf{u}^L - \mathbf{u}^E)$; however, we have $\nabla \cdot (\arctan^2(y)\partial_y(\mathbf{u}^L - \mathbf{u}^E)) = 2\frac{\arctan(y)}{1+y^2}\partial_y(v^L - v^E)$, so in the L^2 product between $\nabla \partial_y(p^L - p^E)$ and $\arctan^2(y)\partial_y(\mathbf{u}^L - \mathbf{u}^E)$ the gradient on the pressure can be moved on the latter term without the appearance of derivatives not involved in the definition of $|||\cdot|||$. In order to handle both the pressure and the boundary integrals deriving from the heat term, an estimate of $\gamma \partial_y u^L$ is needed: this is obtained by writing $\mathbf{u}^L$ as the sum of terms ($\mathbf{u}^F$ and $\mathbf{u}^I$) which satisfy the Navier-slip boundary conditions, and a Stokes flow $\mathbf{u}^B$ that cancels out the boundary value of $\gamma u^I + \gamma u^F$. Using this decomposition, of course, $\gamma \partial_y u^L = \gamma \partial_y u^B$, where the latter can be written in terms of γu^F and γu^I thanks to the solution formula for the Stokes equations in the half space found by Ukai (Ukai, 1987).

We begin by proving that $|||\mathbf{u}^L|||$ remains bounded in time by a constant independent of the viscosity for as long as the strong solution of the Euler equations exists.

Theorem 2.2.1. *Assume $\boldsymbol{u}_0 \in W^{3,2}$, $\boldsymbol{f} \in L^\infty_T W^{2,2}_{xy}$, $\nu \in [0, \nu_0]$. Then*

$$\sup_{t \in [0,t]} |||\boldsymbol{u}^L||| \leq C(||\boldsymbol{u}_0||_{W^{3,2}}, ||\boldsymbol{f}||_{L^\infty_T W^{2,2}_{xy}}, \nu_0, T) \tag{2.7}$$

for any $T < T_E$, where T_E is the time of existence of the strong solution to the Euler equation.

Proof. Multiplying equation (2.5) by $\mathbf{u}^L$ and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}||\mathbf{u}^L||^2_{L^2} + \nu||\nabla \mathbf{u}^L||^2_{L^2} = \int \mathbf{f} \cdot \mathbf{u}^L dxdy \leq ||\mathbf{f}||_{L^2}||\mathbf{u}^L||_{L^2} : \tag{2.8}$$

this means that $||\mathbf{u}^L||_{L^\infty_T L^2_{xy}}$ and $\nu||\nabla \mathbf{u}^L||_{L^2_{Txy}}$ are bounded by a constant depending only on $\mathbf{u}_0$, $\mathbf{f}$ and T. We now take the derivative of equation (2.5) with respect to the tangential variable x, we multiply the equation by $\partial_x \mathbf{u}^L$ and then we integrate. For the diffusive term, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||\partial_x \mathbf{u}^L||_{L^2}^2 + \nu||\partial_x \nabla \mathbf{u}^L||_{L^2}^2 \leq ||\partial_x u^E||_{L^\infty}||\partial_x \mathbf{u}^L||_{L^2}^2 + \\ + ||\frac{\partial_x v^E}{\arctan(y)}||_{L^\infty}||arctg(y)\partial_y \mathbf{u}^L||_{L^2}||\partial_x \mathbf{u}^L||_{L^2} + ||\partial_x \mathbf{f}||_{L^2}||\partial_x \mathbf{u}^L||_{L^2},\end{aligned} \tag{2.9}$$

where

$$||\frac{\partial_x v^E}{\arctan(y)}||_{L^\infty} \leq C||\partial_x v^E||_{W^{1,\infty}}. \tag{2.10}$$

Now we take the derivative of equation (2.5) with respect to the normal variable y, we multiply the equation by $\partial_y \mathbf{u}^L \arctan^2(y)$ and then we integrate. We obtain

$$\begin{aligned}-\nu \int (\arctan(y))^2 \partial_{yyy}\mathbf{u}^L \cdot \partial_y \mathbf{u}^L dxdy = \\ = \nu \int |\partial_{yy}\mathbf{u}^L|^2 (\arctan(y))^2 dxdy - 2\nu \int |\partial_y \mathbf{u}^L|^2 \frac{1-2y\arctan(y)}{(1+y^2)^2} dxdy,\end{aligned} \tag{2.11}$$

where

$$\left|\nu \int |\partial_y \mathbf{u}^L|^2 \frac{1-2y\arctan(y)}{(1+y^2)^2} dxdy\right| \leq C\nu||\partial_y \mathbf{u}^L||_{L^2}^2, \tag{2.12}$$

whose time integral is bounded in terms of the initial datum and the forcing term. For the nonlinear term, we have

$$\begin{aligned}\left|\int \mathbf{u}^E \cdot \nabla \partial_y \mathbf{u}^L \cdot \partial_y \mathbf{u}^L (\arctan(y))^2 dxdy\right| = \left|-\int v^E |\partial_y \mathbf{u}^L|^2 \frac{\arctan(y)}{1+y^2} dxdy\right| \leq \\ ||\frac{v^E}{\arctan(y)}||_{L^\infty}||\partial_y \mathbf{u}^L \arctan(y)||_{L^2}^2,\end{aligned} \tag{2.13}$$

while

$$\begin{aligned}\left|\int \partial_y \mathbf{u}^E \cdot \nabla \mathbf{u}^L \cdot \partial_y \mathbf{u}^L (\arctan(y))^2 dxdy\right| \leq \\ ||\partial_y \mathbf{v}^E||_{L^\infty}|||\partial_y \mathbf{u}^L \arctan(y)||_{L^2}^2 + \frac{\pi}{2}||\partial_y \mathbf{u}^E||_{L^\infty}||\partial_x \mathbf{u}^L||_{L^2}||\partial_y \mathbf{u}^L \arctan(y)||_{L^2}.\end{aligned} \tag{2.14}$$

As for the pressure, we move the gradient on the vector field $\partial_y \mathbf{u}^L(\arctan(y))^2$, so

$$\begin{aligned}\int \partial_y \nabla p^L \cdot \partial_y \mathbf{u}^L (\arctan(y))^2 dxdy = -\int \partial_y p^L \partial_y v^L \frac{\arctan(y)}{1+y^2} dxdy = \\ = \int \partial_y p^L \partial_x u^L \frac{\arctan(y)}{1+y^2} dxdy.\end{aligned} \tag{2.15}$$

In order to estimate $\partial_y p^L$, we decompose the pressure p^L as

$$p^L = p^T + p^N + p^O, \tag{2.16}$$

where p^T solves the system

$$\begin{aligned}\Delta p^T = -\partial_x(\mathbf{u}^E \cdot \nabla u^L), \\ \gamma \partial_y p^T = 0,\end{aligned} \tag{2.17}$$

p^N solves the system

$$\begin{aligned} \Delta p^N &= -\partial_y(\mathbf{u}^E \cdot \nabla v^L), \\ \gamma \partial_y p^N &= 0, \end{aligned} \tag{2.18}$$

while p^O is the solution to

$$\begin{aligned} \Delta p^O &= 0, \\ \gamma \partial_y p^O &= \nu \partial_{yy} v^L. \end{aligned} \tag{2.19}$$

It is easy to see, through integration by parts with the kernel of the Poisson equation, that $p^T = \partial_x \tilde{p}^T$ and $p^N = \partial_y \tilde{p}^N$, where $\tilde{p}^T$ solves the system

$$\begin{aligned} \Delta \tilde{p}^T &= -(\mathbf{u}^E \cdot \nabla u^L), \\ \gamma \partial_y \tilde{p}^T &= 0, \end{aligned} \tag{2.20}$$

and $\tilde{p}^N$ solves the system

$$\begin{aligned} \Delta \tilde{p}^N &= -(\mathbf{u}^E \cdot \nabla v^L), \\ \gamma \tilde{p}^N &= 0. \end{aligned} \tag{2.21}$$

Therefore, we have

$$||\partial_y p^T||_{L^2} = ||\partial_y \partial_x \tilde{p}^T||_{L^2} \leq C \left[||u^E||_{L^\infty} ||\partial_x u^L||_{L^2} + || \frac{v^E}{\arctan(y)} ||_{L^\infty} || \arctan(y) \partial_y u^L||_{L^2} \right], \tag{2.22}$$

$$||\partial_y p^N||_{L^2} \leq ||\mathbf{u}^E||_{L^\infty} ||\partial_x \mathbf{u}^L||_{L^2}. \tag{2.23}$$

The Fourier transform of p^O with respect to the tangential variable is given by

$$\hat{p}^O = \nu e^{-y|\xi'|} \frac{i\xi'}{|\xi'|} \gamma \partial_y \hat{u}^L; \tag{2.24}$$

therefore, using Plancherel's theorem, we have

$$\begin{aligned} &\left| \int \partial_y p^O \partial_x u^L \frac{\arctan(y)}{1+y^2} dx dy \right| = \nu \left| \int e^{-y|\xi'|} \xi' \cdot \gamma \partial_y \hat{u}^L \xi' \cdot \hat{u}^L \frac{\arctan(y)}{1+y^2} d\xi' dy \right| \leq \\ &\nu ||| \xi' |^{3/2} e^{-y|\xi'|} y \gamma \partial_y \hat{u}^L ||_{L^2} ||| \xi' |^{1/2} \hat{u}^L ||_{L^2} \leq C \nu || \gamma \partial_y u^L ||_{L^2_x} || u^L ||_{L^2}^{1/2} || \partial_x u^L ||_{L^2}^{1/2}. \end{aligned} \tag{2.25}$$

In the first inequality, we used the fact that $\arctan(y) \leq y$ and $1/(1+y^2) \leq 1$; the second inequality follows from

$$\int_0^{+\infty} |\xi'|^3 y^2 e^{-2|\xi'|y} dy = \int_0^{+\infty} \sigma^2 e^{-2\sigma} d\sigma = C. \tag{2.26}$$

The estimates of $\gamma \partial_y u^L$ are given by the following proposition, whose proof can be found in appendix A

Proposition 2.2.1. *The following estimates hold for* $\gamma\partial_y u^L$

$$||\gamma\partial_y u^L||_{L^2_x} \leq C\left[\frac{1}{\nu^{1/2}}\left(||\boldsymbol{u}_0||_{W^{2,2}} + ||\boldsymbol{f}||_{L^\infty_T W^{2,2}}\right) + \frac{||\boldsymbol{u}^E||_{L^\infty_T W^{1,\infty}}}{\nu^{3/4}} \sup_{s\in[0,t]} |||\boldsymbol{u}^L|||(s)\right], \tag{2.27}$$

$$||\gamma\partial_y u^L||_{L^2_x} \leq C\left[\frac{1}{\nu^{1/2}}\left(||\boldsymbol{u}_0||_{W^{2,2}} + ||\boldsymbol{f}||_{L^\infty_T W^{2,2}} + ||\boldsymbol{u}^E||^2_{L^\infty_T W^{1,\infty}}\right) + \frac{||\boldsymbol{u}^E||_{L^\infty_T W^{2,2}}}{\nu^{3/4}} \sup_{s\in[0,t]} |||\boldsymbol{u}^L - \boldsymbol{u}^E|||(s)\right]. \tag{2.28}$$

We use equation (2.27) here, while the estimate (2.28) is used later in order to prove the validity of the inviscid limit. Collecting the estimates, we have that, for $t \in [0,T]$ and $\nu \in [0,\nu_0]$

$$\frac{d}{dt}|||\mathbf{u}^L|||^2 \leq C(T,\nu_0)\Bigg(|||\mathbf{u}^L|||^2||\mathbf{u}^E||_{L^\infty W^{1,\infty}} + |||\mathbf{u}^L|||\left(||\mathbf{f}||_{L^\infty_T W^{2,2}} + ||\mathbf{u}_0||_{W^{2,2}}\right) + |||\mathbf{u}^L|||||\mathbf{u}^E||_{L^\infty_T W^{1,\infty}} \sup_{s\in[0,t]} |||\mathbf{u}^L|||(s)\Bigg) \leq A \sup_{s\in[0,t]} |||\mathbf{u}^L|||^2(s) + B, \tag{2.29}$$

where A and B depend on ν_0, T, $\mathbf{f}$, $\mathbf{u}_0$, $\mathbf{u}^E$, but not on the actual viscosity ν. We integrate in time and take the supremum over t, obtaining

$$\sup_{s\in[0,t]} |||\mathbf{u}^L|||^2 \leq |||\mathbf{u}_0||| + BT + A\int_0^t d\tau \sup_{s\in[0,\tau]} |||\mathbf{u}^L|||^2(\tau), \tag{2.30}$$

so, by Gronwall's lemma

$$\sup_{s\in[0,t]} |||\mathbf{u}^L|||^2 \leq (BT + |||\mathbf{u}_0|||)e^{At}. \tag{2.31}$$

□

A similar argument allows to prove the validity of the inviscid limit, with the rate predicted by Prandtl's asymptotic expansion.

Theorem 2.2.2. *Under the assumptions of theorem 2.2.1, we have*

$$|||\boldsymbol{u}^L - \boldsymbol{u}^E||| \leq \nu^{1/4} C(||\boldsymbol{u}_0||_{W^{3,2}}, ||\boldsymbol{f}||_{L^\infty_T W^{2,2}_{xy}}, \nu_0, T). \tag{2.32}$$

Proof. We take the difference between the linearized Navier-Stokes equation and the Euler equation, then we multiply by $\mathbf{u}^L - \mathbf{u}^E$ and we integrate, thus obtaining

$$\frac{1}{2}\frac{d}{dt}||\mathbf{u}^L - \mathbf{u}^E||^2_{L^2} + \nu||\nabla\mathbf{u}^L||^2_{L^2} = \nu\int \mathbf{u}^L \cdot \Delta\mathbf{u}^E dxdy - \nu\int_{y=0} \gamma\partial_y u^L u^E dx \leq \nu||\mathbf{u}_0||_{L^2}||\Delta\mathbf{u}^E||_{L^2} + \nu||\gamma u^E||_{L^2_x}||\gamma\partial_y u^L||_{L^2_x}, \tag{2.33}$$

where, by equation (2.28),

$$\nu||\gamma\partial_y u^L||_{L^2_x} \leq$$
$$\leq C_1\nu^{1/2} + C_2\nu^{1/4}\sup_{s\in[0,t]}|||\mathbf{u}^L-\mathbf{u}^E|||(s) \leq C\left(\nu^{1/2}+\sup_{s\in[0,t]}|||\mathbf{u}^L-\mathbf{u}^E|||^2(s)\right). \tag{2.34}$$

The estimates for $||\partial_x(\mathbf{u}^L-\mathbf{u}^E)||_{L^2}$ follow the same argument used for $||\partial_x\mathbf{u}^L||_{L^2}$; the only difference is that the tangential component of $\gamma\partial_x(\mathbf{u}^L-\mathbf{u}^E)$ is not zero, which implies that

$$\begin{aligned}-\nu\int\partial_x\Delta\mathbf{u}^L\cdot\partial_x\left(\mathbf{u}^L-\mathbf{u}^E\right)dxdy = \nu||\partial_x\nabla\mathbf{u}^L||^2_{L^2} - \nu\int_{y=0}\partial_y u^L\partial_{xx}u^E dx+\\ +\nu\int\partial_x\mathbf{u}^L\cdot\Delta\partial_x\mathbf{u}^E dxdy,\end{aligned} \tag{2.35}$$

where the last two terms can be bounded by

$$\nu||\partial_x\mathbf{u}^L||_{L^2}||\partial_x\Delta\mathbf{u}^E||_{L^2} + \nu||\gamma\partial_y u^L||_{L^2_x}||\gamma\partial_{xx}u^E|| \leq C\left(\nu^{1/2}+\nu^{1/4}\sup_{s\in[0,t]}|||\mathbf{u}^L-\mathbf{u}^E|||(s)\right). \tag{2.36}$$

Therefore, we have

$$\frac{1}{2}\frac{d}{dt}||\partial_x\left(\mathbf{u}^L-\mathbf{u}^E\right)||^2_{L^2} + \nu||\partial_x\nabla\mathbf{u}^L||^2_{L^2} \leq C\left(\nu^{1/2}+\sup_{s\in[0,t]}|||\mathbf{u}^L-\mathbf{u}^E|||^2(s)\right). \tag{2.37}$$

Finally, we estimate $||\arctan(y)\partial_y\left(\mathbf{u}^L-\mathbf{u}^E\right)||_{L^2}$: we have that

$$\begin{aligned}-\nu\int\Delta\partial_y\mathbf{u}^L\cdot\partial_y\left(\mathbf{u}^L-\mathbf{u}^E\right)\arctan^2(y)dxdy = \nu\int|\partial_y\nabla\mathbf{u}^L|^2\arctan^2(y)dxdy+\\ +\nu\int|\partial_y\mathbf{u}^L|^2\frac{2y\arctan(y)}{1+y^2}dxdy - \nu\int\nabla\partial_y\mathbf{u}^L:\nabla\left(\partial_y\mathbf{u}^E\arctan^2(y)\right)dxdy+\\ -\nu\int\frac{|\partial_y\mathbf{u}^L|^2}{(1+y^2)^2}dxdy,\end{aligned} \tag{2.38}$$

with

$$\nu\left|\int\nabla\partial_y\mathbf{u}^L:\nabla\left(\partial_y\mathbf{u}^E\arctan^2(y)\right)dxdy\right| \leq \frac{\nu}{2}||\partial_y\nabla\mathbf{u}^L\arctan(y)|| + C\nu||\mathbf{u}^E||_{W^{k,2}}, \tag{2.39}$$

while $\nu||\partial_y\mathbf{u}^L/(1+y^2)||^2_{L^2}$ is controlled by the positive term $\nu||\partial_y\mathbf{u}^L||^2_{L^2}$ appearing in the balance of $||\mathbf{u}^L-\mathbf{u}^E||^2_{L^2}$. For the pressure, we can write

$$p^L - p^E = p^D + p^O, \tag{2.40}$$

where p^D is the solution to

$$\begin{aligned}\Delta p^D = -\nabla\mathbf{u}^E:\nabla(\mathbf{u}^L-\mathbf{u}^E),\\ \gamma\partial_y p^D = 0,\end{aligned} \tag{2.41}$$

while p^O is defined in (2.19). The same arguments used for the estimates of p^N and p^T can be used in the estimates of p^D, while p^O can be treated as in equation (2.25); this implies that

$$\left|\int \partial_y \nabla p \cdot \partial_y \left(\mathbf{u}^L - \mathbf{u}^E\right) \arctan^2(y) dxdy\right| \leq C \left(\nu^{1/2} + \sup_{s\in[0,t]} |||\mathbf{u}^L - \mathbf{u}^E|||^2(s)\right) \tag{2.42}$$

With no major difference from the proof of theorem 2.2.1, we see that

$$\left|\int \partial_y \left(\mathbf{u}^E \cdot \nabla(\mathbf{u}^L - \mathbf{u}^E)\right) \cdot \partial_y(\mathbf{u}^L - \mathbf{u}^E) dxdy\right| \leq C|||\mathbf{u}^L - \mathbf{u}^E|||^2. \tag{2.43}$$

Therefore, we have

$$\frac{1}{2}\frac{d}{dt}||\partial_y(\mathbf{u}^L - \mathbf{u}^E)\arctan(y)||^2_{L^2} + \frac{\nu}{2}||\partial_y \nabla \mathbf{u}^L \arctan(y)||^2_{L^2} - \nu \int \frac{|\partial_y \mathbf{u}^L|^2}{(1+y^2)^2} dxdy \leq$$
$$\leq C\left(\nu^{1/2} + \sup_{s\in[0,t]} |||\mathbf{u}^L - \mathbf{u}^E|||^2(s)\right); \tag{2.44}$$

combining with (2.33) and (2.37), we obtain

$$\frac{d}{dt}|||\mathbf{u}^L - \mathbf{u}^E|||^2 \leq C\left(\sup_{s\in[0,t]} |||\mathbf{u}^L - \mathbf{u}^E|||^2(s) + \nu^{1/2}\right), \tag{2.45}$$

so, by Gronwall lemma,

$$\sup_{t\in[0,T]} |||\mathbf{u}^L - \mathbf{u}^E|||^2(t) \leq C'\nu^{1/2}. \tag{2.46}$$

□

2.2.1 The full linearized problem

The linearization of the Navier-Stokes equations around the Euler solution satisfies

$$\begin{aligned} \partial_t \mathbf{u}^1 - \nu\Delta\mathbf{u}^1 + \mathbf{u}^E \cdot \nabla\mathbf{u}^E + \left(\mathbf{u}^1 - \mathbf{u}^E\right) \cdot \nabla\mathbf{u}^E + \mathbf{u}^E \cdot \nabla\left(\mathbf{u}^1 - \mathbf{u}^E\right) + \nabla p^1 &= \mathbf{f}, \\ \nabla \cdot \mathbf{u}^1 &= 0, \\ \gamma\mathbf{u}^1 &= 0, \\ \mathbf{u}^1(t=0) &= \mathbf{u}^0; \end{aligned} \tag{2.47}$$

therefore, in the difference between $\mathbf{u}^1$ and $\mathbf{u}^E$, the additional term $\left(\mathbf{u}^1 - \mathbf{u}^E\right) \cdot \nabla\mathbf{u}^E$ must be taken into account, both in the nonlinear term and in the pressure. Anyway, the $|||\cdot|||$ norm of this term is bounded by $|||\mathbf{u}^1 - \mathbf{u}^E|||\,||\mathbf{u}^E||_{W^{2,\infty}}$, so the estimates for the nonlinear term are easy to achieve. As regards the pressure, we can use a decomposition $p^1 - p^E = p^{O_1} + p^{D_1}$ formally identical to the one used for $p^L - p^E$ in (2.40); the laplacian of p^{D_1} is given by

$$(\nabla\mathbf{u}^E)^T : \nabla(\mathbf{u}^1 - \mathbf{u}^E) + \nabla(\mathbf{u}^1 - \mathbf{u}^E)^T : \nabla\mathbf{u}^E = 2(\nabla\mathbf{u}^E)^T : \nabla(\mathbf{u}^1 - \mathbf{u}^E), \tag{2.48}$$

so the same estimates used for p^D can be applied here. As for p^{O_1}, the additional terms are actually easier to estimate, since they depend on $(\mathbf{u}^L - \mathbf{u}^E) \cdot \nabla \mathbf{u}^E$ instead of $\mathbf{u}^E \cdot \nabla(\mathbf{u}^L - \mathbf{u}^E)$.

2.3 The partially nonlinear case

The boundedness in L^2 of the Navier-Stokes flow derives from a property of the nonlinear term which holds component-wise: i.e., we have

$$\int \mathbf{w} \cdot \nabla g g dx dy = 0, \tag{2.49}$$

for any $\mathbf{w}$ divergence free with zero normal component at the boundary and any scalar function g. This allows us to consider a partial linearization of the Navier-Stokes equations, where only the tangential component is convected by the Euler flow,

$$\begin{aligned} \partial_t u^{PL} - \nu \Delta u^{PL} + \mathbf{u}^E \cdot \nabla u^{PL} + \partial_x p^{PL} &= f', \\ \partial_t v^{PL} - \nu \Delta v^{PL} + \mathbf{u}^{PL} \cdot \nabla v^{PL} + \partial_y p^{PL} &= f_n, \\ \nabla \cdot \mathbf{u}^{PL} &= 0, \\ \gamma \mathbf{u}^{PL} &= \mathbf{0}, \\ \mathbf{u}^{PL}(t=0) &= \mathbf{u}_0. \end{aligned} \tag{2.50}$$

We shall prove the convergence of $\mathbf{u}^{PL}$ to $\mathbf{u}^E$ in the energy norm.
As for the linear case, in order to prove the validity of the inviscid limit, we need to estimate the L^2_x product between γu^E and $\gamma \partial_y u^{PL}$. The key ingredients of the proof are the following: the use of Plancherel theorem allows us to give the $L^\infty_{\xi'}$ norm to the Fourier transform of $\partial_y u^{PL}$. By using Young's convolution inequality, the $L^\infty_{\xi'}$ norm of the bilinear terms is bounded by the product of the $L^2_{\xi'}$ norm of the terms. An additional advantage of the $L^\infty_{\xi'}$ setting is that some of the pseudodifferential operators appearing in the definition of $\gamma \partial_y u^{PL}$ are bounded. However, even in this setting, $\nu \gamma \partial_y u^{PL}$ is not bounded: nonetheless, it is the sum of terms which are bounded, and terms which are, essentially, tangential derivatives of bounded quantities. This property is a direct consequence of the linearization of the equation of the tangential part of the flow. Therefore, we can move the tangential derivatives on γu^E, obtaining the desired estimates.
The standard energy argument shows that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||\mathbf{u}^{PL} - \mathbf{u}^E||^2_{L^2} + \nu||\nabla \mathbf{u}^{PL}||^2_{L^2} \leq \nu \frac{||\nabla \mathbf{u}^E||^2_{L^2}}{2} + \nu \frac{||\nabla \mathbf{u}^{PL}||^2_{L^2}}{2} + \\ + ||\nabla v^E||_{L^\infty} ||\mathbf{u}^{PL} - \mathbf{u}^E||^2_{L^2} + \nu \left| \int_{y=0} \partial_y u^{PL} u^E dx \right|. \end{aligned} \tag{2.51}$$

From appendix A, the boundary value of $\partial_y u^{PL}$ is given by

$$\gamma \partial_y u^{PL} = r + \nu \gamma D_1 \gamma \partial_{yy} E^p_2 \left(\partial_x q - \mathbf{u}^E \cdot \nabla \mathbf{u}^{PL} \right) + \partial_x \mathbf{N}' \cdot E^p_2 \left(\partial_x q - \mathbf{u}^E \cdot \nabla \mathbf{u}^{PL} \right), \tag{2.52}$$

where q solves

$$\begin{aligned} \Delta q = \partial_{xx}(u^E u^{PL}) + \partial_{xy}(v^E u^{PL}) + \partial_{xy}(v^{PL} u^{PL}) + \partial_{yy}(v^{PL^2}) = h, \\ \gamma \partial_y q = 0, \end{aligned} \tag{2.53}$$

while r collects terms not too large in ν,

$$||r||_{L^2_x} \leq \frac{C}{\nu^{1/2}}, \tag{2.54}$$

with the constant C depending on $\mathbf{f}$ and $\mathbf{u}_0$. Using the Fourier transform in the tangential variable, the explicit expression of q is

$$\hat{q} = -\int_0^y \left(\frac{e^{-|\xi'|(y-y')} + e^{-|\xi'|(y+y')}}{2|\xi'|} \right) \hat{h}(y')dy' - \int_y^{+\infty} \left(\frac{e^{|\xi'|(y-y')} + e^{-|\xi'|(y+y')}}{2|\xi'|} \right) \hat{h}(y')dy'. \tag{2.55}$$

Using integration by parts, we have

$$\begin{aligned} -\int_0^y \left(\frac{e^{-|\xi'|(y-y')} + e^{-|\xi'|(y+y')}}{2|\xi'|} \right) i\xi' \partial_{y'} \mathcal{F}(v^E u^L + v^L u^L) dy' - \\ \int_y^{+\infty} \left(\frac{e^{|\xi'|(y-y')} + e^{-|\xi'|(y+y')}}{2|\xi'|} \right) i\xi' \partial_{y'} \mathcal{F}(v^E u^L + v^L u^L) dy' = \\ i\xi' \int_0^y \left(\frac{e^{-|\xi'|(y-y')} - e^{-|\xi'|(y+y')}}{2} \right) \mathcal{F}(v^E u^L + v^L u^L) dy' + \\ i\xi' \int_y^{+\infty} \left(\frac{e^{|\xi'|(y-y')} + e^{-|\xi'|(y+y')}}{2} \right) \mathcal{F}(v^E u^L + v^L u^L) dy', \end{aligned} \tag{2.56}$$

while

$$\begin{aligned} -\int_0^y \left(\frac{e^{-|\xi'|(y-y')} + e^{-|\xi'|(y+y')}}{2|\xi'|} \right) \partial_{y'y'} \mathcal{F}(v^{L^2}) dy' - \\ \int_y^{+\infty} \left(\frac{e^{|\xi'|(y-y')} + e^{-|\xi'|(y+y')}}{2|\xi'|} \right) \partial_{y'y'} \mathcal{F}(v^{L^2}) dy' \\ = \mathcal{F}(v^{L^2}) - |\xi'|^2 \Bigg[\int_0^y \left(\frac{e^{-|\xi'|(y-y')} + e^{-|\xi'|(y+y')}}{2|\xi'|} \right) \mathcal{F}(v^{L^2}) dy' - \\ \int_y^{+\infty} \left(\frac{e^{|\xi'|(y-y')} + e^{-|\xi'|(y+y')}}{2|\xi'|} \partial_{y'y'} \right) \mathcal{F}(v^{L^2}) \Bigg]. \end{aligned} \tag{2.57}$$

Therefore, we have

$$\hat{q} = \mathcal{F}(v^2) + |\xi'| \hat{q}^*, \tag{2.58}$$

with

$$||q^*||_{L^\infty_{\xi' y}} \leq C ||\mathbf{u}^{PL}||_{L^2} \left(||\mathbf{u}^{PL}||_{L^2} + ||\mathbf{u}^E||_{L^2} \right) \leq C, \tag{2.59}$$

which leads to

$$\nu \left| \int_{y=0} d\xi' \gamma \hat{u}^E \left(\nu\gamma D_1 \gamma \partial_{yy} E_2^p i\xi' |\xi'| \hat{q}^* + \partial_x N' E_2^p i\xi' \hat{q}^* \right) \right| \leq$$
$$C\nu^{1/2} \int_{y=0} |\xi'|^2 |\hat{u}^E| |d\xi' \leq C\nu^{1/2} ||\gamma u^E||_{W_x^{2+j,2}} \leq C\nu^{1/2} ||\mathbf{u}^E||_{W^{\frac{3}{2}+j,2}}, \tag{2.60}$$

with $j > (d-1)/2$. In the first line of (2.60), we used Holder inequality, giving the $L^\infty_{\xi'} L^1_y$ norm to the kernel of E_2^p; the other steps are straightforward. A similar argument leads to

$$\nu \left| \int_{y=0} dx u^E \left(\nu\gamma D_1 \gamma \partial_{yy} E_2^p \partial_x v^{PL^2} + \partial_x N' E_2^p i \partial_x v^{PL^2} \right) \right| \leq C ||\mathbf{u}^E||_{W^{\frac{1}{2}+j,2}}$$
$$\sup_{s\in[0,t]} \left(||v^{PL} - v^E||^2_{L^2}(s) + \nu^{1/4} ||v^E||_{L^\infty_y L^2_x} ||v^{PL} - v^E||_{L^2}(s) + \nu^{1/2} ||v^E||^2_{L^\infty_y L^2_x} \right). \tag{2.61}$$

This inequality is obtained writing v^{PL^2} as $(v^{PL} - v^E)^2 + 2(v^{PL} - v^E)v^E + v^{E^2}$: then, we give the L^1_y norm to $(v^{PL} - v^E)^2$, the L^2_y norm to $v^E(v^{PL} - v^E)$ and the L^∞_y norm to v^{E^2}. Finally, we write

$$\mathbf{u}^E \cdot \nabla u^{PL} = \partial_x (u^E (u^{PL} - u^E)) + \partial_y (v^E (v^{PL} - v^E)) + \mathbf{u}^E \cdot \nabla u^E, \tag{2.62}$$

with

$$\nu \left| \int_{y=0} dx u^E \left(\nu\gamma D_1 \gamma \partial_{yy} E_2^p \left(\partial_x (u^E (u^{PL} - u^E)) + \mathbf{u}^E \cdot \nabla u^E \right) + \partial_x N' E_2^p i \mathbf{u}^E \cdot \nabla u^{PL} \right) \right| \leq$$
$$\leq C \left(\nu^{1/4} \sup_{s\in[0,t]} ||u^{PL} - u^E||_{L^2}(s) + \nu^{1/2} \right). \tag{2.63}$$

The only term which still has to be estimated is

$$\nu^2 \gamma D_1 \gamma \partial_{yy} E_2^p \partial_y (v^E (u^{PL} - u^E)) =$$
$$\int_0^t ds' \int_{\mathbb{R}^{d-1}} \frac{e^{-\frac{|\mathbf{x}' - \mathbf{x}'''|^2}{4\nu(t-s')}}}{(4\pi\nu(t-s'))^{(d-1)/2}} \int_0^{+\infty} dy' \frac{\partial_{y'} (v^E (u^{PL} - u^E))}{2\pi\nu(t-s')}$$
$$\int_{s'}^t ds \frac{e^{-\frac{y'^2}{4\nu(s-s')}}}{\sqrt{(s-s')(t-s)}} \frac{y'^2}{2\nu(s-s')} = \int_0^t ds' \int_{\mathbb{R}^{d-1}} \frac{e^{-\frac{|\mathbf{x}' - \mathbf{x}'''|^2}{4\nu(t-s')}}}{(4\pi\nu(t-s'))^{(d-1)/2}} \tag{2.64}$$
$$\int_0^{+\infty} dy' \frac{\partial_{y'} (\frac{v^E}{y} (u^{PL} - u^E))}{2\pi\nu(t-s')} \int_{s'}^t ds \frac{e^{-\frac{y'^2}{4\nu(s-s')}}}{\sqrt{(s-s')(t-s)}} \left(\frac{y'^2}{2\nu(s-s')} - 2 \right) \frac{y'^2}{2\nu(s-s')},$$

so

$$||\nu^2 \gamma D_1 \gamma \partial_{yy} E_2^p \partial_y (v^E (u^{PL} - u^E))||_{L^\infty_{\xi'}} \leq C\nu^{1/4} \sup_{s\in[0,t]} ||u^{PL} - u^E||_{L^2}. \tag{2.65}$$

Therefore, we have

$$\frac{d}{dt}||\mathbf{u}^{PL}-\mathbf{u}^{E}||^2_{L^2} \leq C\left(\sup_{s\in[0,t]}||\mathbf{u}^{PL}-\mathbf{u}^{E}||^2_{L^2}(s)+\nu^{1/2}\right), \tag{2.66}$$

which implies

$$||\mathbf{u}^{PL}-\mathbf{u}^{E}||_{L^\infty_T L^2_{xy}} \leq C\nu^{1/4}. \tag{2.67}$$

2.4 The energy approach for the full nonlinear equation

Without the linearization of the equation of the tangential part of the flow, $\nu\gamma\partial_y u^{NS}$ contains a term which is not a tangential derivative, and that is controlled, in the L^∞_{ζ}, norm, by $||(v^{NS}-v^E)/y||_{L^2}||u^{NS}-u^E||_{L^2}$. Therefore, we need an estimate of the tangential derivative $\partial_x(u^{NS}-u^E$ in order to close the estimates in the energy norm. It would be tempting to use conormal Sobolev spaces, like we did in the linear case. Indeed, a boundary term appears in the integration by parts of the heat term only for the purely tangential derivatives; and those derivatives can be moved on the trace of the euler flow. However, for this approach, the main problem is the nonlinear term $(\mathbf{u}^{NS}-\mathbf{u}^E)\cdot\nabla\mathbf{u}^{NS}+\mathbf{u}^E\cdot\nabla(\mathbf{u}^{NS}-\mathbf{u}^E)$: we shall see, taking two particular quantities appearing in the estimates as examples, which is the strategy we would like to use, and why it fails.
In order to bound the nonlinear terms, we need estimates of the gradient of the velocity field in L^∞. It is possible to prove that

$$||\partial_x\mathbf{u}^{NS}||_{L^\infty} \leq C\left(\frac{|||\mathbf{u}^{NS}-\mathbf{u}^E|||_k^2}{\nu^{1/2}}+\frac{|||\mathbf{u}^{NS}-\mathbf{u}^E|||_k}{\nu^{1/4}}+1\right), \tag{2.68}$$

where $|||\cdot|||_k$ is the norm of the conormal Sobolev space of order k,

$$|||\mathbf{u}|||_k = \sum_{i+j\leq k}||\partial_x^i(\arctan(y)\partial_y)^j\mathbf{u}||_{L^2}, \tag{2.69}$$

and k large enough. Therefore, the following nonlinear quantity can be estimated as

$$\begin{aligned}\left|\int\partial_x^k(u^{NS}-u^E)\partial_x\mathbf{u}^{NS}\cdot\partial_x^k(\mathbf{u}^{NS}-\mathbf{u}^E)dxdy\right| &\leq \\ C\left(\frac{|||\mathbf{u}^{NS}-\mathbf{u}^E|||_k^4}{\nu^{1/2}}+\frac{|||\mathbf{u}^{NS}-\mathbf{u}^E|||_k^3}{\nu^{1/4}}+|||\mathbf{u}^{NS}-\mathbf{u}^E|||_k^2\right) &\leq \\ C\left(\frac{|||\mathbf{u}^{NS}-\mathbf{u}^E|||_k^4}{\nu^{1/2}}+\nu^{1/2}\right)&,\end{aligned} \tag{2.70}$$

where we used Young's inequality in the last step. If all the terms satisfied a similar inequality, then, multiplying the energy estimates by $\nu^{-1/2}$, we would obtain

$$\frac{d}{dt}\frac{|||\mathbf{u}^{NS}-\mathbf{u}^E|||_k^2}{\nu^{1/2}} \leq C\left(\left(\frac{|||\mathbf{u}^{NS}-\mathbf{u}^E|||_k^2}{\nu^{1/2}}\right)^2+1\right), \tag{2.71}$$

which would imply the validity of the inviscid limit in the $|||\cdot|||_k$ norm, with rate of convergence $\nu^{1/4}$, for a time independent of the viscosity. But, according to the classical boundary layer theory, the right hand side of (2.68) is bounded in ν, so a similar bound cannot hold for $\partial_y \mathbf{u}^{NS}$. However, a bound like (2.68) is satisfied by $||\arctan(y)\partial_y u^{NS}||_{L^\infty}$, not by $||\partial_y u^{NS}||_{L^\infty}$; therefore, the following nonlinear term, containing $\partial_y u^{NS}$ can be estimated as

$$\begin{aligned} \left|\int \partial_x^k (v^{NS} - v^E)\partial_y u^{NS} \partial_x^k (u^{NS} - u^E) dxdy\right| \leq \\ C||\frac{\partial_x^k (v^{NS} - v^E)}{\arctan(y)}||_{L^2}||\arctan(y)\partial_y u^{NS}||_{L^\infty}||\partial_x^k (u^{NS} - u^E)||_{L^2} \leq \\ C||\partial_x^{k+1}(u^{NS} - u^E)||_{L^2}||\arctan(y)\partial_y u^{NS}||_{L^\infty}||\partial_x^k (u^{NS} - u^E)||_{L^2} \end{aligned} \tag{2.72}$$

On the right hand side of the equation above, we have a tangential derivative of order $k+1$, for the estimate of the conormal norm of order k; therefore, the energy approach does not close the estimates in a Sobolev setting. The need to use the derivatives of order $k+1$ to estimate the derivative of order k suggests the use of an analytic setting.

2.5 Convergence for Navier boundary conditions

Although the no-slip conditions are in good agreement with most experimental results, there are certain situations where the fluid motion at the boundary cannot be considered zero: gases and non-newtonian fluids like polymers can slip at the interface with a solid object, for example (Lauga, Brenner, and Stone, 2007). For general fluids, roughness can change nature and thickness of the boundary layer, so that on rougher surfaces the fluid can slip and the resistance to the motion decreases: it might might sound counter-intuitive that roughness decreases drag, but this is why nature gave sharks denticles on their skin, and the reason golf balls are no longer smooth (Kadivar, Tormey, and McGranaghan, 2021). In the nineteenth century, the nature of the boundary conditions suitable for hydrodynamics was widely debated: Navier proposed that, at the boundary,

$$\mathbf{u} \cdot \mathbf{n} = 0; \quad [S(\mathbf{u})\mathbf{n} + \alpha \mathbf{u}]' = 0, \tag{2.73}$$

where $\alpha \geq 0$ is a friction coefficient, and S is the stress tensor, which, for a Newtonian fluid, is proportional to the strain rate $\frac{(\nabla \mathbf{u} + \nabla^T \mathbf{u})}{2}$. We call (2.73) Navier-friction boundary conditions (Maekawa and Mazzucato, 2016). The same conditions were later derived by Maxwell in the context of gas dynamics. In the two dimensional case, the problem of convergence to the inviscid flow is often studied in terms of vorticity, and the inviscid limit holds for $\omega_0 \in L^p$, $p > 2$ (Lopes Filho, Nussenzveig Lopes, and Planas, 2005). In the three dimensional case, the convergence can be proved for initial data in $W^{s,2}$, $s > 5/2$ (Iftimie and Planas, 2006), which implies existence of strong solutions to the Euler equation for a finite time T; however, for general domains, viscosity- independent bounds are not available in such regular spaces, so the validity of the inviscid limit can be proved only for weaker norms. Viscosity-independent bounds for higher order derivatives are available if we use conormal Sobolev spaces (Masmoudi and Rousset, 2012), so, essentially, one can prove higher order convergence away from the boundary. The convergence can also be proved for friction coefficients α depending on the viscosity like $\alpha = \alpha' \nu^{-\beta}$ for $\beta \in [0, 1[$ (Paddick, 2014): interestingly, for $\beta = 1/2$, the equivalent of Prandtl's equation for Navier-friction boundary conditions shows the same instability of actual Prandtl's equation, which means that well posedness of the boundary layer equation might not be necessary for the inviscid limit to hold. When $\alpha = 0$, conditions (2.73) are called Navier-slip (Busuioc and Ratiu, 2003), or stress free. In the half space, the shear stress at the boundary is equal to $\gamma\partial_y u^N$, so the solution $\mathbf{u}^N$ satisfies

$$\begin{aligned} \partial_t \mathbf{u}^N - \nu\Delta\mathbf{u}^N + \mathbf{u}^N \cdot \nabla\mathbf{u}^N + \nabla p^N &= \mathbf{f}, \\ \nabla \cdot \mathbf{u}^N &= 0, \\ \gamma\partial_y u^N &= 0, \\ \gamma v^N &= 0, \\ \mathbf{u}^N(t=0) &= \mathbf{0}. \end{aligned} \tag{2.74}$$

It is trivial, in this case, to prove the validity of the inviscid limit: indeed, these boundary conditions imply that

$$-\int \Delta\mathbf{u}^N \cdot \left(\mathbf{u}^N - \mathbf{u}^E\right) dxdy = ||\nabla\mathbf{u}^N||_{L^2}^2 - \int \nabla\mathbf{u}^N : \nabla\mathbf{u}^E dxdy, \tag{2.75}$$

so

$$\frac{1}{2}\frac{d}{dt}||\mathbf{u}^N - \mathbf{u}^E||_{L^2}^2 + \frac{\nu}{2}||\nabla\mathbf{u}^N||_{L^2}^2 \leq \frac{\nu}{2}||\nabla\mathbf{u}^E||_{L^2}^2 + ||\nabla\mathbf{u}^E||_{L^\infty}||\mathbf{u}^N - \mathbf{u}^E||_{L^2}^2. \tag{2.76}$$

Therefore, by Gronwall lemma, the inviscid limit holds with rate $\nu^{1/2}$ in the energy norm. The convergence can be proved in any Sobolev space $W^{2k+1,2}$, and with a better rate ν, as long as we assume

$$\gamma\partial_y^{2j+1} u_0 = 0, \tag{2.77}$$

$$\gamma\partial_y^{2j+1} P'\mathbf{f} = 0, \tag{2.78}$$

for $j = 1, ..., k$. This can be easily proved by a symmetrization argument: indeed, the extensions of $\mathbf{u}^E$ and $\mathbf{u}^N$ to the lower half space as $u(x,y) = u(x,-y)$, $v(x,y) = -v(x,-y)$ become, respectively, the solution to the Euler equations and to the Navier-Stokes equations in the whole space. The conditions imposed on the initial data and the forcing term imply that their extension to the lower half space maintains the same regularity they had in the half space: therefore, we obtain an equivalent problem without boundaries. We can also obtain this result by analyzing the behaviour at the boundary of the normal derivative of the tangential part of the flow: indeed,

$$\partial_t\gamma\partial_y^{2j+1} u^E + \gamma\partial_y^{2j+1} P'(\mathbf{u}^E \cdot \nabla\mathbf{u}^E) = \gamma\partial_y^{2j+1} P'\mathbf{f}, \tag{2.79}$$

where

$$\begin{aligned} \gamma\partial_y^{2j+1} P'(\mathbf{u}^E \cdot \nabla\mathbf{u}^E) &= \gamma\partial_y^{2j+1}(\mathbf{u}^E \cdot \nabla\mathbf{u}^E) - \gamma\partial_x\partial_y^{2j+1} p^E = \\ = \gamma\partial_y^{2j+1}(\mathbf{u}^E \cdot \nabla\mathbf{u}^E) &- \partial_x\gamma\partial_y^{2j-1}\left((\nabla\mathbf{u}^E)^T : \nabla\mathbf{u}^E\right) + \partial_x^3\gamma\partial_y^{2j-1} p^E = \\ = \gamma\partial_y^{2j+1}(\mathbf{u}^E \cdot \nabla u^E) &+ \sum_{i=1}^{j}(-1)^i\partial_x^{2i-1}\gamma\partial_y^{2j+1-2i}\left((\nabla\mathbf{u}^E)^T : \nabla\mathbf{u}^E\right). \end{aligned} \tag{2.80}$$

The expression above consists of a sum of products such that at least one factor is a tangential derivative of either γv^E or of a odd normal derivative of u^E of order

$\leq 2j+1$: therefore, under conditions (2.78) and (2.77), $\gamma\partial_y^{2j+1}u^E$ is zero for all the time the Euler solution exists. As for $\mathbf{u}^N$, the equation of $\gamma\partial_y u^N$, taking into account the condition (2.78) for the forcing term and the boundary condition $\gamma\partial_y u^N = 0$, becomes $\nu\gamma\partial_{yyy}u^N = 0$. Iteratively, we obtain $\partial_y^{2j+1}u^N = 0$, $j = 1, ..., k+1$; the conditions (2.77) imply that the initial data are taken with continuity in $W^{2k+1,2}$.
Generic initial data for the Euler equations only satisfy $\gamma v_0 = 0$: this kind of data is incompatible with the condition $\gamma\partial_y u^N = 0$, so they cause the presence of a boundary layer. For the two dimensional case, we prove the convergence in $W^{1,2}$ for initial data which satisfy only $\gamma v_0 = 0$ and for a generic forcing term: the standard energy argument, in term of vorticity, leads to

$$\frac{d}{dt}||\omega^N - \omega^E||^2_{L^2} + \nu||\nabla\omega^N||^2_{L^2} \leq \nu||\nabla\omega^E||^2_{L^2} + \nu\left|\int_{y=0}\partial_y\omega^N\omega^E dx\right| + \\ +||\mathbf{u}^N - \mathbf{u}^E||_{L^2}||\nabla\omega^E||_{L^\infty}||\omega^N - \omega^E||_{L^2}. \tag{2.81}$$

Following the strategy used in the linear case, we can decompose ω^N as

$$\omega^N = E_2^d\omega_0 + E_2^d\left(\nabla\times\mathbf{f} - \mathbf{u}^N\cdot\omega^N\right) = \omega^I + \omega^F \tag{2.82}$$

$\omega^N = \omega^I + \omega^F$. The incompatibility between the initial data and the boundary condition $\gamma\omega^N = 0$ causes the formation of a boundary layer, whose intensity, for the normal derivative of ω^I, is $O((t\nu)^{-1/2})$:

$$\partial_y\omega^I = \int_{\mathbb{R}^{d-1}} dx' \frac{e^{-\frac{|x-x'|^2}{4\nu t}}}{(4\pi\nu t)^{(d-1)/2}}\frac{e^{-\frac{y^2}{4\nu t}}}{\sqrt{\pi\nu t}}\omega_0(x') + E_0^p\partial_y\omega_0, \tag{2.83}$$

with

$$||\gamma\partial_y\omega^I||_{L^2_x} \leq C\left(\frac{1}{\sqrt{\nu t}}||\gamma\omega_0||_{L^2_x} + ||\partial_y\omega_0||_{L^2}\right). \tag{2.84}$$

As regards ω^F, it can be written as $E_2^d(\nabla\times\mathbf{f}) - \partial_x E_d^2(u^N\omega^N) - E_d^2(\partial_y(v^N\omega^N)$. For the first term

$$||\gamma E_2^d(\nabla\times\mathbf{f})||_{L^2_x} \leq \frac{C}{\nu^{1/2}}||\mathbf{f}||_{W^{1,2}}. \tag{2.85}$$

The second term is a tangential derivative, which can be moved on γu^E; furthermore, we give the L^∞_{tx} norm to $\gamma\partial_x u^E$, so that we just need to estimate

$$||\gamma\partial_y E_2^d u^N\omega^N||_{L^1_{tx}} \leq \frac{C_1}{\nu^{1/2}}||\omega^E||_{L^2_{tx}L^\infty_y}||u^E||_{L^2_{tx}L^\infty_y} + \frac{C_2}{\nu^{3/4}} \\ \left(||u^N - u^E||_{L^2_{tx}L^\infty_y}||\omega^N - \omega^E||_{L^2_{txy}} + ||u^E||_{L^2_{tx}L^\infty_y}||\omega^N - \omega^E||_{L^2_{txy}} + \right. \\ \left. +||u^N - u^E||_{L^2_{txy}}||\omega^E||_{L^2_{tx}L^\infty_y}\right) \leq C\left(\frac{1}{\nu^{1/2}} + \frac{1}{\nu}||\omega^N - \omega^E||^2_{L^2_{txy}}\right). \tag{2.86}$$

In the first inequality, we gave the L^1_y norm to the kernel and the L^∞_y norm to $\omega^E u^E$, while, for all the other terms, we gave the L^2_y norm to the kernel; the second inequality comes from Young's inequality and the interpolation $||\partial_y(u^N - u^E)||_{L^\infty_y} \leq$

$||u^N - u^E||^{1/2}_{L^2_y}||\partial_y(u^N - u^E)||^{1/2}_{L^2_y}$. Finally,

$$\begin{aligned}
||\gamma\partial_y E_2^d \partial_y(v^N\omega^N)||_{L^1_{tx}} \le C\Bigg(\frac{1}{\nu^{1/4}}||\frac{v^E}{y}||_{L^2_{tx}L^\infty_y}||\partial_y\omega^N||_{L^2_{txy}}+\\
+\frac{1}{\nu^{1/2}}\Big(||\partial_y v^E||_{L^2_{tx}L^\infty_y}||\omega^E||_{L^2_{tx}L^\infty_y}+||\frac{v^N-v^E}{y}||_{L^2_{txy}}||\partial_y\omega^N||_{L^2_{txy}}\Big)+\\
+\frac{1}{\nu^{3/4}}\Big(||\partial_y v^E||_{L^2_{tx}L^\infty_y}||\omega^N-\omega^E||_{L^2_{txy}}+||\omega^E||_{L^2_{tx}L^\infty_y}+||\partial_y(v^N-v^E)||_{L^2_{txy}}+\\
+||\partial_y(v^N-v^E)||_{L^2_{txy}}||\omega^N-\omega^E||_{L^2_{tx}L^\infty_y}\Big)\Bigg)\le\\
C\left(\frac{1}{\nu^{1/2}}+||\omega^N-\omega^E||^2_{L^2_{txy}}\right)+\frac{||\partial_y\omega^N||_{L^2_{txy}}}{2}.
\end{aligned} \tag{2.87}$$

In the first inequality of the expression above, the term which multiplies $\nu^{-1/4}$ is obtained by giving to the kernel of $\partial_y E_2^d$, multiplied by y', the L^2_y norm; for the terms which multiply $\nu^{-1/2}$, we either gave the L^1_y norm to the kernel, or the L^∞_y norm to the kernel multiplied by y; finally, the terms multiplying $\nu^{-3/4}$ are obtained giving the L^2_y norm to the kernel. The second inequality follows from the application of Young's inequality and Sobolev's embeddings. Collecting the estimates, we have

$$\nu\left|\int_0^t ds\int_{y=0}\partial_y\omega^N\omega^E dx\right| \le C\left(||\omega^N-\omega^E||^2_{L^2_{txy}}+\nu^{1/2}\right)+\frac{\nu}{2}||\partial_y\omega^N||^2_{L^2_{txy}}; \tag{2.88}$$

substituting in (2.81), we have

$$||\omega^N-\omega^E||^2_{L^2}\le C\left(\nu^{1/2}+\int_0^t||\omega^N-\omega^E||^2_{L^2}(s)ds\right), \tag{2.89}$$

which implies

$$\sup_{t\in[0,T]}||\omega^N-\omega^E||_{L^2}\le C\nu^{1/4}. \tag{2.90}$$

Therefore, convergence still holds, but with a worse rate, due to the incompatibility between initial data and boundary conditions.

2.5.1 Curved domains

For a flat domain, it is equivalent to impose conditions on the vorticity, on the normal derivative of the tangential velocity, and on the shear stress: indeed, at the boundary, since $\partial_x v = 0$, they are all equal to $\partial_y u^N$. The situation changes for curved domains: let us consider, for a point P on the boundary of a bidimensional domain, a reference frame such that the boundary is locally described by a function $y = h(x)$, with $h'(x_P) = 0$. Therefore, the normal $\mathbf{n}$ is given by $\frac{1}{\sqrt{1+h'^2(x)}}(h'(x), -1)$, and in P is parallel to the y axis. Since $\mathbf{n}\cdot\mathbf{u}(x, h(x))$ is zero $\forall x$, then the derivative with respect to x is zero, so

$$\begin{aligned} -\frac{h'(x)h''(x)}{(1+h'^2(x))}(h'(x),-1)\cdot\mathbf{u}+ \\ +\frac{1}{\sqrt{1+h'^2(x)}}\left(h''(x)u+h'(x)\partial_x u-\partial_x v+(h'(x))^2\partial_y u-h'(x)\partial_y v\right)=0. \end{aligned} \tag{2.91}$$

In P, we have $h'(x^P)=0$, so $h''(x^P)$ is the curvature k in P, and the equation above reduces to

$$ku=\partial_x v. \tag{2.92}$$

This follows from the fact that, since the direction of the normal is changing, if we move along the x direction the normal to the boundary will no longer be parallel to the y direction, so v will no longer be the component of the velocity normal to the boundary: therefore, there is no reason for $\partial_x v$ to be zero. With the same argument, one can prove that the equation for the trace of the normal derivative of the tangential part of the velocity can no longer be expressed in local terms, so, calling $\mathbf{t}$ the unit tangent to the boundary, the condition $\gamma\mathbf{n}\cdot\nabla\mathbf{u}^E\cdot\mathbf{t}=0$ is no longer preserved by the Euler flow. However, for bidimensional domains, if the initial vorticity and the curl of the forcing term are zero at the boundary, the vorticity ω^E stays zero at the boundary for all times: this led Bardos (Bardos, 1972) to introduce the boundary conditions $\gamma v^E=0$, $\gamma\omega^E=0$, in contrast to the Navier-slip conditions, which prescribe zero shear stress at the boundary. Indeed, we have

$$\partial_t\gamma\omega^E+(\gamma\mathbf{u}^E\cdot\mathbf{t})\mathbf{t}\cdot\gamma\nabla\omega^E+(\gamma\mathbf{u}^E\cdot\mathbf{n})\mathbf{n}\nabla\omega^E=0, \tag{2.93}$$

with

$$(\gamma\mathbf{u}^E\cdot\mathbf{n})=0. \tag{2.94}$$

If we introduce the curvilinear abscissa s on the boundary, we have

$$\mathbf{t}\cdot\gamma\nabla\omega^E=\partial_s\gamma\omega^E(x(s),y(s)), \tag{2.95}$$

so equation (2.93) becomes

$$\partial_t\gamma\omega^E(t,s)+(\mathbf{t}\cdot\gamma\mathbf{u}^E)\partial_s\omega^E(t,s)=0. \tag{2.96}$$

For $\gamma\mathbf{u}^E$ given by the solution to the Euler equations, we have obtained a partial differential equation in s and t for $\gamma\omega^E$: if $\gamma\omega^E$, then $\partial_s\gamma\omega^E=0$, so $\gamma\omega^E=0$ is the unique solution to the equation, given the initial datum $\gamma\omega_0=0$.

In the three-dimensional case, if the normal component of the velocity and all the components of the vorticity are zero at the boundary at the initial time, they remain null for all the time the Euler solution exists: indeed, the only additional term in the equation is $\boldsymbol{\omega}^E\cdot\nabla\mathbf{u}^E$, which is zero at the boundary as long as $\gamma\boldsymbol{\omega}^E$ is zero. However, these are four scalar conditions, and we can only impose three scalar boundary conditions to the Navier-Stokes system. Since $\gamma\partial_y u^E$ is preserved for flat domains, one could hope that $\mathbf{n}\times\boldsymbol{\omega}$ remains null if initially zero: unfortunately, this is not the case (Veiga and Crispo, 2012). Indeed, assume that $\boldsymbol{\omega}_0\in C^1$, and that $\mathbf{n}\times\boldsymbol{\omega}_0=0$ but $\mathbf{n}\cdot\boldsymbol{\omega}_0\neq 0$ for a point P on the boundary. We adopt a reference frame such that the boundary is locally described by a function $x_3=h(x_1,x_2)$, with the normal to the boundary in P parallel to the x_3 axis. Finally, assume that, in P, the hessian matrix of h (which is the shape operator) has full rank. The directions of the axes x_1 and x_2 are tangent to the boundary in P: the conditions $\partial_{x_i}(\mathbf{n}\cdot\mathbf{u}^E)=0$, $i=1,2$, evaluated in P, become

$$\begin{aligned} \partial_{x_1x_1} h\gamma u_1^E + \partial_{x_1x_2} h\gamma u_2^E &= \gamma \partial_{x_1} v^E, \\ \partial_{x_1x_2} h\gamma u_1^E + \partial_{x_2x_2} h\gamma u_2^E &= \gamma \partial_{x_2} v^E. \end{aligned} \tag{2.97}$$

The conditions $\partial_{x_i}(\mathbf{n} \times \boldsymbol{\omega}_0) = \mathbf{0}$, evaluated in P, become the following four linearly independent conditions

$$\gamma \partial_{x_i} \omega_{j0} = -\partial_{x_i x_j} h \gamma \omega_{30}, \quad i, j \in \{1, 2\}. \tag{2.98}$$

Therefore, we have

$$\begin{aligned} (\partial_t \gamma \omega^E)(t=0) = \gamma \Big(&\omega_{10}\partial_{x_1} u_{10} + \omega_{20}\partial_{x_2} u_{10} + \omega_{30}(\partial_{x_3} u_{10} - \partial_{x_1} u_{30}) + \omega_{30}\partial_{x_1} u_{30} + \\ &- u_{10}\partial_{x_1}\omega_{10} - u_{20}\partial_{x_2}\omega_{10} - u_{30}\partial_{x_3}\omega_{10} \Big) = 2\gamma \Big(\omega_{30} \big(\partial_{x_1x_1} h u_{10} + \partial_{x_1x_2} h u_{20} \big) \Big), \end{aligned} \tag{2.99}$$

and similarly,

$$(\partial_t \gamma \omega_2^E)(t=0) = 2\gamma \Big(\omega_{30} \big(\partial_{x_1x_2} h u_{10} + \partial_{x_3x_2} h u_{20} \big) \Big). \tag{2.100}$$

Since the hessian of h has full rank and $\gamma\omega_{30} \neq 0$ in P, the time derivatives of $\gamma\omega_1$ and $\gamma\omega_2$ are zero at initial time if and only if $\gamma \mathbf{u}_0 = 0$. But since $\boldsymbol{\omega}_0$ is continuous, $\mathbf{n} \cdot \boldsymbol{\omega}_0 \neq 0$ in a neighborhood of P, so, in order to have $\mathbf{n} \times \gamma \partial_t \boldsymbol{\omega} = \mathbf{0}$ at the initial time, we must have $\gamma \mathbf{u} = 0$ in a neighborhood of P. But this implies that $\mathbf{n} \cdot \boldsymbol{\omega}_0 = 0$ in the same neighbourhood, which is against our hypothesis. Therefore, the tangential part of the vorticity in P will become instantaneously non-zero.

2.6 Relation between the inviscid limit and the gradient of the flow near the boundary

Aside from some particular, symmetric domains, there are currently only two types of results concerning the validity of the inviscid limit: the first type relies on regularity assumptions on the initial data, requiring analyticity in both the variables, at least in a neighbourhood of the boundary of width $O(1)$ with respect to the viscosity. The second type consists of a priori assumptions on the family of solutions $\mathbf{u}^{NS}(\nu)$: this line of research was initiated by Kato in 1994. In (Kato, 1984a), he showed that the convergence of the Navier-Stokes solutions to the Euler solution is equivalent to the vanishing of the dissipation in a sublayer of thickness $O(\nu)$, therefore smaller than the Prandtl layer. Temam and Wang, in (Temam and Wang, 1997), see also (Cheng and Wang, 2007; Wang, 2001), obtained a condition leading to convergence, based on the pressure gradient at the boundary, which is interesting because the appearance of high stress at the boundary is the first indicator of the deviation of the Navier-Stokes solutions from the Prandtl solution and the precursor of vortices formation and subsequent separation, see (Gargano, Sammartino, and Sciacca, 2011; Gargano et al., 2014; Obabko and Cassel, 2002). During the last 15 years, the Kato criterion has been improved, interpreted in terms of different quantities of physical interest, e.g., vorticity (Kelliher, 2008; Bardos and Titi, 2013; Kelliher, 2017b) or tangential velocity and velocity gradient at the boundary (Constantin et al., 2017), combined with Oleinik's monotonicity setting (Constantin, Kukavica, and Vicol, 2015). Recently, in (Constantin and Vicol, 2018; Constantin et al., 2019; Drivas and Nguyen, 2019) the authors have derived interesting criteria based on the vorticity in the interior of the domain.

In this section, we see how a Stokes flow that cancels out the tangential part of $\mathbf{u}^E$ at the boundary can be used to prove that, under the assumption that some of the derivatives of $\mathbf{u}^{NS}$ near the boundary do not grow too much with ν, the inviscid limit holds. In particular, we prove the results of Wang (Wang, 2001) and Kato (Kato, 1984a). Call $\mathbf{u}^{B\rho}$ the solution to

$$\begin{aligned} \partial_t \mathbf{u}^{B\rho} - \rho\Delta\mathbf{u}^{B\rho} + \nabla p^{B\rho} &= \mathbf{0}, \\ \nabla\cdot\mathbf{u}^{B\rho} &= \mathbf{0}, \\ \gamma u^{B\rho} &= -u^E, \\ \gamma v^{B\rho}(y=0) &= 0, \\ \mathbf{u}^{B\rho}(t=0) &= \mathbf{0}, \end{aligned} \tag{2.101}$$

where $\rho=\rho(\nu)$ Then, the usual energy argument implies that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||\mathbf{u}^{NS}-\mathbf{u}^E-\mathbf{u}^{B\rho}||_{L^2}^2 + \nu||\nabla\mathbf{u}^{NS}||_{L^2}^2 \le \nu||\mathbf{u}^{NS}||_{L^2}||\Delta\mathbf{u}^E||_{L^2}+ \\ \rho||\Delta\mathbf{u}^{B\rho}||_{L^2}||\mathbf{u}^{NS}-\mathbf{u}^E-\mathbf{u}^{B\rho}||_{L^2}^2 + \nu\left|\int \nabla\mathbf{u}^{NS}:\nabla\mathbf{u}^{B\rho}dxdy\right| + \\ +||\nabla\mathbf{u}^E||_{L^\infty}||\mathbf{u}^{NS}-\mathbf{u}^E-\mathbf{u}^{B\rho}||_{L^2}^2 + ||\mathbf{u}^{B\rho}||_{L^2}||\nabla\mathbf{u}^E||_{L^\infty}||\mathbf{u}^{NS}-\mathbf{u}^E-\mathbf{u}^{B\rho}||_{L^2}+ \\ +\left|\int \mathbf{u}^{NS}\cdot\nabla\mathbf{u}^{B\rho}\cdot\left(\mathbf{u}^{NS}-\mathbf{u}^E\right)dxdy\right|. \end{aligned} \tag{2.102}$$

Since the L^2 norm of $\mathbf{u}^{B\rho}$ vanishes with the viscosity, if we can prove that $||\mathbf{u}^{NS}-\mathbf{u}^E-\mathbf{u}^{B\rho}||_{L^2}$ goes to zero with the viscosity, we prove the validity of the inviscid limit. It is easy to see that, for $p\in]1,+\infty[$, $j=0,1,2$,

$$||\partial_x^i\partial_y^j\mathbf{u}^{B\rho}||_{L^p} \le C(p)\rho^{\frac{1}{2}\left(\frac{1}{p}-j\right)}; \tag{2.103}$$

in order to extend the result to $p=\infty$, one can write, for $k>(d-1)/2$,

$$||\partial_x^i\partial_y^j\mathbf{u}^{B\rho}||_{L^\infty} \le C||\partial_y^j\mathbf{u}^{B\rho}||_{W_x^{k,2}L_y^2}^{1/2}||\partial_y^{j+1}\mathbf{u}^{B\rho}||_{W_x^{k,2}L_y^2}^{1/2} \le C\rho^{-\frac{j}{2}}. \tag{2.104}$$

Therefore, in the right hand side of (2.102), the only terms that can be problematic are the ones involving some normal derivatives of $u^{B\rho}$. We have that

$$\rho||\Delta\mathbf{u}^{B\rho}||_L^2 \le C\rho^{\frac{1}{4}}, \tag{2.105}$$

while

$$\begin{aligned} \int v^{NS}\partial_y u^{B\rho}u^E dxdy = -\int \partial_y v^{NS}u^{B\rho}u^E - \int v^{NS}u^{B\rho}\partial_y u^E = -\int u^{NS}\partial_x(u^B u^E)dxdy+ \\ -\int v^{NS}u^B\partial_y u^E dxdy, \end{aligned} \tag{2.106}$$

so the only terms that might be unbounded are

$$|I_1| = \nu \left| \int \partial_y u^{NS} \partial_y u^{B\rho} dxdy \right| \leq \frac{\nu}{4} ||\partial_y \mathbf{u}^{NS}||_L^2 + C\nu\rho^{-1/2}, \tag{2.107}$$

which is unbounded if $\rho/\nu^2 \to 0$, and

$$\begin{aligned} |I_2| = \left| \int v^{NS} \partial_y u^{B\alpha} u^{NS} dxdy \right| \leq C||y^2 \partial_y u^{B\rho}||_{L^\infty} ||\frac{u^{NS}}{y}||_{L^2} ||\frac{v^{NS}}{y}||_{L^2} \leq \\ \leq C\rho^{\frac{1}{2}} ||\partial_y u^{NS}||_{L^2} ||\partial_y v^{NS}||_{L^2}, \end{aligned} \tag{2.108}$$

which is unbounded if $\nu^2\, \rho \to 0$. For $\rho = \nu^2$, both terms are bounded, but not small in ν. However, they are small in time: the only part of $\partial_y u^{B\rho}$ which behaves like $\rho^{-1/2}$ is

$$w = \int_0^t ds \frac{e^{-\frac{y^2}{4\rho(t-s)}}}{\sqrt{\pi\rho(t-s)}} \int_{\mathbb{R}^{d-1}} dx'' \frac{\gamma \partial_t u^E(x'', s) e^{-\frac{|\mathbf{x}'-\mathbf{x}''|^2}{4\rho(t-s)}}}{(4\pi\rho(t-s))^{(d-1)/2}}. \tag{2.109}$$

If we take $\rho = \nu^2$, then for a small time T^* independent of the viscosity, we have

$$\left| \int v^{NS} \partial_y u^{B\rho} u^{NS} dxdy \right| \leq \frac{\nu}{2} ||\nabla \mathbf{u}^{NS}||^2 + C\nu^{1/2}; \tag{2.110}$$

therefore, we obtain, for $t \in [0, T]$,

$$\frac{d}{dt} ||\mathbf{u}^{NS} - \mathbf{u}^E - \mathbf{u}^{B\rho}||_{L^2}^2 \leq C \left(\nu^{1/2} + ||\mathbf{u}^{NS} - \mathbf{u}^E - \mathbf{u}^{B\rho}||_{L^2}^2 + t^{1/2} \sup_{t\in[0,T*]} ||\gamma \partial_t u^E||_{L^2_x}(t) \right), \tag{2.111}$$

which allows to estimate the eventual failure of the inviscid limit in terms of the evolution of the trace of u^E.

Now, we see how assumptions on the behaviour of the tangential derivatives of the viscous flow near the boundary imply the validity if the inviscid limit. Assume that

$$\lim_{\nu\to 0} \nu^{1/2} ||\partial_x u^{NS}||_{L^2(y\leq h(\nu))} = r(\nu) = 0 \tag{2.112}$$

for some function $h(\nu)$ such that

$$\lim_{\nu\to 0} \frac{\nu}{h(\nu)} = 0. \tag{2.113}$$

Then, if

$$\lim_{\nu\to 0} r(\nu) \sqrt{\frac{h(\nu)}{\nu}} = 0, \tag{2.114}$$

using a corrector $\mathbf{u}^{B\rho}$ with "viscosity" $\rho(\nu) = \nu h(\nu)$, we have that

$$||y^2 \partial_y \mathbf{u}^{B\rho}||_{L^\infty(y\geq h(\nu))} \leq C e^{-\frac{h(\nu)^2}{4\nu h(\nu) T}} \sqrt{\nu h(\nu)}, \tag{2.115}$$

so, for I_2, outside the layer

$$\left| \int_0^T dt \int_{y\geq h(\nu)} v^{NS} \partial_y u^B u^{NS} dxdy \right| \leq C\nu ||\nabla \mathbf{u}^{NS}||_{L^2_{Txy}}^2 e^{-\frac{h(\nu)^2}{4\nu h(\nu) T}} \sqrt{\frac{h(\nu)}{\nu}} \to 0, \tag{2.116}$$

while, inside the layer,

$$\left| \int_0^T dt \int_{y \le h(\nu)} v^{NS} \partial_y u^B u^{NS} dxdy \right| \le C \sqrt{\frac{h(\nu)}{\nu)}} r(\nu) \to 0. \tag{2.117}$$

If (2.114) is not satisfied, we can take $\rho(\nu) = \nu^2/r^\beta$ as "viscosity" for $\mathbf{u}^{B\rho}$, with $\beta \in]1/2, 1[$: in this case,

$$||y^2 \partial_y \mathbf{u}^{B\rho}||_{L^\infty(y \ge h(\nu))} \le Ce^{-\frac{h(\nu)^2 r(\nu)^{2\beta}}{4\nu^2 T}} \frac{\nu}{r(\nu)^{\frac{\beta}{2}}} \le Ce^{-cr(\nu)^{2\beta-1}} \frac{\nu}{r(\nu)^{\frac{\beta}{2}}}, \tag{2.118}$$

so, outside the layer,

$$\left| \int_0^T dt \int_{y \ge h(\nu)} v^{NS} \partial_y u^{B\rho} u^{NS} dxdy \right| \le Ce^{-cr(\nu)^{2\beta-1}} \frac{1}{r(\nu)^{\frac{\beta}{2}}} \to 0, \tag{2.119}$$

while, inside the layer,

$$\left| \int_0^T dt \int_{y \le h(\nu)} v^{NS} \partial_y u^{B\rho} u^{NS} dxdy \right| \le C \frac{r(\nu)}{\nu} \frac{\nu}{r(\nu)^{\frac{\beta}{2}}} \to 0. \tag{2.120}$$

In both cases, we have $\nu^2/\rho(\nu) \to 0$ as $\nu \to 0$, so I_1 goes to zero with the viscosity; therefore, the inviscid limit holds. The inviscid limit also holds under the assumption that

$$\lim_{\nu \to 0} \nu^{1/2} ||\partial_x v^{NS}||_{L^2_T L^2(y \le h(\nu))} = 0 : \tag{2.121}$$

indeed,

$$|I_2| \le C||\partial_y u^{NS}||_{L^2} ||v^{NS} y \partial_y u^{B\rho}||_{L^2}, \tag{2.122}$$

with

$$
\begin{aligned}
||v^{NS}\partial_y u^{B\rho}||^2_{L^2} = \left|\int v^{NS^2} y^2 \partial_y u^{B\rho^2} dxdy\right| = \left|\int dxdy v^{NS^2}\partial_y\left(\int_y^{+\infty} y'^2\partial_y u^{B\rho})^2 dy'\right)\right| = \\
2\left|\int dxdy v^{NS}\partial_x u^{NS}\left(\int_y^{+\infty} y'^2\partial_y u^{B\rho^2} dy'\right)\right| \leq \\
2\left|\int dxdy \partial_x v^{NS} u^{NS}\left(\int_y^{+\infty} y'^2\partial_y u^{B\rho^2}\right) dy'\right| + \\
+\left|4\int dxdy v^{NS} u^{NS}\left(\int_y^{+\infty} y'^2\partial_y u^{B\rho}\partial_x\partial_y u^{B\rho} dy'\right)\right| \leq \\
2\int dxdy\left|\partial_x v^{NS}\frac{u^{NS}}{y}\left(\int_y^{+\infty} y'^3 u^{B\rho^2}\right) dy'\right| + \\
4\int dxdy\left|\frac{v^{NS}}{y}\frac{u^{NS}}{y}\left(\int_y^{+\infty} y'^4\partial_y u^{B\rho}\partial_x\partial_y u^{B\rho} dy'\right)\right| \leq \\
2\int dxdy\left|\partial_x v^{NS}\frac{u^{NS}}{y}\left(\int_y^{+\infty} y'^3\partial_y u^{B\rho^2}\right) dy'\right| + C\rho(\nu)^{3/2}||\partial_y \mathbf{u}^{NS}||^2_{L^2}.
\end{aligned}
\tag{2.123}
$$

The second inequality follows from $1 \leq y'/y$, while the last inequality derives from estimates similar to (2.103). In the last line, the second term goes to zero with ν as long as $\rho(\nu)$ goes to zero faster than $\nu^{4/3}$, while for the first term we can use the same argument used for $\partial_x u^{NS}$ and $\partial_y u^{NS}$ to prove the smallness in ν, under the assumption (2.121). These results were already obtained by Wang (Wang, 2001), using a different, compactly supported corrector, defined starting from its streamfunction. It resembles the Kato's criterion (Kato, 1984b), which ensures the validity of the inviscid limit as long as

$$\lim_{\nu\to 0}\nu^{1/2}||\nabla \mathbf{u}^{NS}||_{L^2_T L^2(y\leq c\nu)} = 0. \tag{2.124}$$

Wang's criterion requires a thicker layer, but involves only the tangential derivatives of the velocity field, which are bounded for the traditional boundary layer theory. Actually, in a layer of the thickness $c\nu$ proposed by Kato, only the norm of $\nu^{1/2}\partial_y u^N$ needs to disappear. Furthermore, the assumptions of $\partial_y u^N$ are essentially assumptions on I_1, which is an L^2 product between $\partial_y u^N$ and $\partial_y u^{B\rho}$, with the behaviour of $\partial_y u^{B\rho}$, in terms of ν, which does not depend on the choice of the norm in the variables t and x. Therefore, we can formulate Kato's criterion in terms of an $L^p_T L^2_y L^q_x$ norm, for any $p, q \in [1, 2]$, or an $L^p_T L^r_{\xi'} L^2_y$ norm, for any $p \in [1, 2]$, $r \in [2, +\infty]$. Finally, instead of the L^1_T norm of $\partial_y u^{NS}$, the criterion can be formulated in terms of $\int_0^t u^{NS} d\tau$: the difference is that time oscillations are canceled out by the time integral, but not by the norm.

So, assume that

$$\lim_{\nu\to 0} \nu^{1/2}||\partial_y u^{NS}||_{L^p_T L^2(y\le h(\nu))L^q_x} = r(\nu) = 0, \tag{2.125}$$

and consider a corrector with viscosity $\rho = \nu^2 r(\nu)$. Then, for I_1, inside the layer

$$\left|\int_0^t ds \int_{y\le C\nu} \partial_y u^{NS}\partial_y u^{B\rho} dxdy\right| \le \nu||\partial_y u^{NS}||_{L^p_T L^2(y\le c\nu)L^q_x}||\partial_y u^{B\rho}||_{L^{p'}_T L^2_y L^{q'}_x} \le \\ Cr(\nu)\nu^{1/2}(\nu^2 r(\nu))^{-1/4} \to 0, \tag{2.126}$$

while, outside the layer,

$$\left|\int_0^t ds \int_{y\ge C\nu} \partial_y u^{NS}\partial_y u^{B\rho} dxdy\right| \le C\nu||\partial_y u^{NS}||_{L^2_{Txy}}||\partial_y u^{B\rho}||_{L^2_T L^2(y\le C\nu)}, \tag{2.127}$$

with

$$\nu^{1/2}||\partial_y u^{B\rho}||_{L^2(y\ge c\nu)} \le C\nu^{1/2} e^{-\frac{c\nu^2}{\nu^2 r(\nu)T}}||\partial_y u^{B2\rho}||_{L^2} \le \frac{e^{-\frac{c\nu^2}{\nu^2 r(\nu)T}}}{(r(\nu))^{1/4}} \to 0. \tag{2.128}$$

Finally, since $\rho/\nu^2 \to 0$, $|I_2|$ goes to zero with the viscosity: therefore, the inviscid limit holds. If the assumption (2.125) is done in terms of $\int_0^t \partial_y u^{NS} d\tau$, we can use the same argument once we express, through by integration by parts, the time integral of I_1 in terms of $\int_0^t \partial_y u^{NS} d\tau$

$$\nu\int_0^T dt \int \partial_y u^{NS}\partial_y u^{B\rho} dxdy = \\ = \nu\int dxdy \left(\int_0^T \partial_y u^{NS} dt\right)\partial_y u^{B\rho}(T) - \nu\int_0^T dt\int dxdy\left(\int_0^t \partial_y u^{NS} d\tau\right)\partial_t\partial_y u^{B\rho}. \tag{2.129}$$

Chapter 3

The inviscid limit in the half space with non compatible data

3.1 Introduction

In this chapter, we shall study the solutions of the 3D Navier-Stokes(NS) equations in the half-space when the initial datum and the boundary datum are incompatible. At the boundary, we impose the no-slip boundary condition and, therefore, the incompatibility means that we consider initial data having non-zero tangential component at the boundary; we shall call these data *Euler type* or *not well prepared* initial data. We shall establish the existence and uniqueness of the NS solutions for a time short but independent of the viscosity. The central hypothesis we use is the analyticity of the initial datum.

The study of the NS equations in the half-space (or half-plane in 2D) is a central subject in the mathematical theory of fluid dynamics because it is a prototypical case where one can study the interaction between a fluid and a wall. The introduction by S. Ukai of an exact formula for solutions to the Stokes equations in the half-space (Ukai, 1987), had a significant impact on the mathematical theory of the NS equations; since Ukai's result, several papers appeared concerning the existence and uniqueness of solutions for the Navier-Stokes equations in the half-space with general initial data, see for example (Cannone, Planchon, and Schonbek, 2000; Maremonti, 2008) and references therein. However, these results, by one side, do not allow Euler type initial data, i.e., data with a non zero tangential component at the boundary; on the other hand, they rely on estimates that, in the zero viscosity limit, would degrade.

Euler type initial data are important classical configurations; among them, we mention the impulsively started disk or plate (Batchelor, 2000) and the flow generated by the interaction between a wall and a point vortex or a core vortex (Sir Horace Lamb, 1975). Incompatible data are an interesting subject also from a numerical point of view; in (Boyd and Flyer, 1999; Chen, Qin, and Temam, 2010; Chen, Qin, and Temam, 2011; Hamouda, Temam, and Zhang, 2017; Temam, 2006), one can find in-depth studies of how the lack of compatibility between initial and boundary data can lead to loss of numerical accuracy; and on the appropriate compatibility conditions, one should impose to ensure the required accuracy. In incompressible fluid dynamics Euler type data typically arise when, to the Navier-Stokes equations, one imposes initial data that are stationary solutions of the Euler equations and are therefore of great interest for the numerical and theoretical study of the boundary layer theory; for example, they have been used to test the Prandtl equations' effectiveness to reproduce the separation phenomena occurring at the boundary, see (Gargano, Sammartino, and Sciacca, 2011; Gargano et al., 2014; Obabko and Cassel, 2002).

As we mentioned before, we want to construct the Navier-Stokes equation solution starting from an initial datum that has a non-zero tangential slip. Moreover, we want that the time of existence of the solution does not shrink to zero when the viscosity goes to zero; this is not trivial because it is well known that, in 3D, the time of existence of the NS solutions is short and that, in the presence of boundaries, the existence time, in general, would go to zero with the viscosity. Therefore, we shall have to handle both the initial layer, due to the presence of the initial discontinuity, and the boundary layer, due to the mismatch between the no-slip boundary condition and the Euler solution. Following (Sammartino and Caflisch, 1998b), we shall decompose the solution to the Navier-Stokes equations as the sum of the solution to the Euler equations, the solution to the Prandtl equations with non-compatible data, and a remainder, which needs a further decomposition. The singularity due to the initial layer in Prandtl's equation is passed to the other terms of the asymptotic expansion, although in a less severe form. In particular, this singularity forces us to solve the equation of the overall error in a functional setting more singular than the one found in (Sammartino and Caflisch, 1998b), which requires different strategies: one of the main differences is that we can no longer use algebra properties to deal with the nonlinear terms. In our procedure we shall rely on the well-posedness result for Prandtl's equations with non-compatible data (Cannone, Lombardo, and Sammartino, 2013).

We also point out the recent work that has tackled similar problems, although considering a linearized version of the NS equations (Gie, 2013; Gie, Kelliher, and Mazzucato, 2018).

The results of this chapter can be found in (Argenziano, Cannone, and Sammartino, 2022).

3.1.1 Asymptotic decomposition of the NS solution

Consider the Navier-Stokes equations in the half-plane or in the half-space $\Pi^+ = \{(x,y) : y \geq 0\}$ where, in 3D, $x = (x_1, x_2)$:

$$\begin{aligned} \partial_t \mathbf{u} - \nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = 0, \\ \nabla\cdot\mathbf{u} = 0, \\ \mathbf{u}\big|_{t=0} = \mathbf{u}_0, \\ \gamma\mathbf{u} = 0. \end{aligned} \tag{3.1}$$

In the above equation γ is the trace operator, i.e., for regular functions,

$$\gamma\mathbf{v} = \mathbf{v}(y=0). \tag{3.2}$$

When the viscosity ν is small, its effects are mainly concentrated in a layer near the boundary, whose thickness is proportional to $\varepsilon = \sqrt{\nu}$. For this kind of singularly perturbed problems, one looks for solutions that, at the formal level, can be written as (Dyke, 1964):

$$\mathbf{u} = \mathbf{u}^{out}_{(0)} + \mathbf{u}^{inn}_{(0)} + \varepsilon\left(\mathbf{u}^{out}_{(1)} + \mathbf{u}^{inn}_{(1)}\right) + ... + \varepsilon^N\left(\mathbf{u}^{out}_{(N)} + \mathbf{u}^{in}_{(N)}\right) + O(\varepsilon^{N+1}), \tag{3.3}$$

having carried out the expansion up to order N. In the above expansion $\mathbf{u}^{out}_{(i)}(x,y,t)$ and $\mathbf{u}^{in}_{(i)}(x,Y,t)$ for $i = 0, ...N$ describe the outer flow (away from the boundary) and the inner flow (close to the boundary) respectively, being $Y = y/\varepsilon$ the rescaled variable.

The rigorous proof that an expansion like (3.3) holds in the compatible case, was given in (Sammartino and Caflisch, 1998b), where the authors proved that the solution of the NS equations can be written as:

$$\mathbf{u} = \mathbf{u}^E + \bar{\mathbf{u}}^P + \varepsilon \left(\mathbf{u}_1^E + \bar{\mathbf{u}}_1^P \right) + \varepsilon \mathbf{e}.$$

In the above expansion the outer leading order term, $\mathbf{u}^E$, is the solution of the Euler equations:

$$\begin{aligned} \partial_t \mathbf{u}^E + \mathbf{u}^E \cdot \nabla \mathbf{u}^E + \nabla p^E &= 0, \\ \nabla \cdot \mathbf{u}^E &= 0, \\ \mathbf{u}^E\big|_{t=0} &= \mathbf{u}_0, \\ \gamma_n \mathbf{u}^E &= 0, \end{aligned} \tag{3.4}$$

where $\gamma_n \mathbf{v}$ is the normal component of the trace of $\mathbf{v}$.

The leading order inner solution $\bar{\mathbf{u}}^P$, is linked to the solution of the Prandtl equations as follows: let u^P the solution of Prandtl's equations

$$\begin{aligned} (\partial_t - \partial_{YY}) u^P + u^P \partial_x u^P + v^P \partial_Y u^P - (\partial_t + u^E|_{y=0} \partial_x) u^E|_{y=0} &= 0, \\ u^P\Big|_{Y=0} = 0, \quad u^P\Big|_{Y \to +\infty} &= u^E(x, y=0, t), \\ u^P\Big|_{t=0} &= u_0(x, y=0), \end{aligned} \tag{3.5}$$

where

$$v^P = - \int_0^Y \partial_x u^P dY'. \tag{3.6}$$

We define the inner solution as:

$$\bar{\mathbf{u}}^P = (\tilde{u}^P, \varepsilon \bar{v}^P), \tag{3.7}$$

where

$$\tilde{u}^P = u^P - u^E\big|_{y=0}, \tag{3.8}$$

$$\bar{v}^P = \int_Y^{+\infty} \partial_x \tilde{u}^P dY'. \tag{3.9}$$

The $\bar{\mathbf{u}}^P$ defined above decays away from the boundary, so that it does not interfere strongly with the outer solution $\mathbf{u}^E$. Moreover $\bar{\mathbf{u}}^P$ cancels the tangential flow generated by $\mathbf{u}^E$. However $\bar{\mathbf{u}}^P$ generates a normal inflow at the boundary $Y = 0$, see (3.9). This inflow is canceled by the first order outer solution $\varepsilon \mathbf{u}_1^E$. This outer solution generates tangential flow which is canceled by first order inner solution $\varepsilon \bar{\mathbf{u}}_1^P$.

The overall remainder $\varepsilon \mathbf{e}$ closes the procedure. Notice, however, that the remainder, which at the formal level should be $O(\varepsilon^2)$, due to the nonlinear interactions between the boundary layer terms and the outer Euler terms, can be proven to be $O(\varepsilon)$ only.

3.1.2 Statement of the main result

We give an informal statement of the main result of this chapter.

Theorem 3.1.1 (Informal statement)**.** *Suppose that $\boldsymbol{u}_0$ has analytic regularity, with $\gamma_n \boldsymbol{u}_0 = 0$. Then an analytic solution to the Navier-Stokes equations exists for a time independent of the viscosity; this solution can be written as*

$$\boldsymbol{u} = \boldsymbol{u}^{out}_{(0)} + \boldsymbol{u}^{inn}_{(0)} + \varepsilon \left(\boldsymbol{u}^{out}_{(1)} + \boldsymbol{u}^{inn}_{(1)} + \boldsymbol{e} \right), \tag{3.10}$$

where the outer solutions $\boldsymbol{u}^{out}_{(0)}$ and $\boldsymbol{u}^{out}_{(1)}$ depend on (x, y, t), while the inner solutions $\boldsymbol{u}^{inn}_{(0)}$ and $\boldsymbol{u}^{inn}_{(1)}$ depend on (x, Y, t), and are exponentially decaying outside the boundary layer.

Moreover $\boldsymbol{u}^{inn}_{(0)}$, $\boldsymbol{u}^{out}_{(1)}$ and $\boldsymbol{u}^{inn}_{(1)}$ have singular time derivatives at the initial time.

The rest of this subsection is devoted to illustrating the physical meaning of the terms appearing in the asymptotic expansion of the NS solution, equation (3.10).

The leading-order outer solution

The term $\mathbf{u}^{out}_{(0)}(x, y, t)$ is the solutions $\mathbf{u}^E$ of the Euler equations (3.4). The existence of an analytic solution for the Euler equation has been established in the literature, see among others (Bardos and Benachour, 1977; Levermore and Oliver, 1997; Sammartino and Caflisch, 1998a); therefore, in constructing this part of the NS solution, we shall refer to these results.

The leading-order inner solution

The term $\mathbf{u}^{inn}_{(0)} = \bar{\mathbf{u}}^P(x, Y, t)$ correlates to the solution of the Prandtl equations (3.5) through subtraction of a constant (in Y) function that makes it decaying outside the BL. The incompatibility, in equations (3.5), between the initial data and the boundary data leads to the occurrence of a singular part $\mathbf{u}^S$, so that the $\bar{\mathbf{u}}^P$ must be decomposed in a regular and a singular part:

$$\bar{\mathbf{u}}^P = \bar{\mathbf{u}}^R + \mathbf{u}^S. \tag{3.11}$$

The singular part $\mathbf{u}^S = (u^S, \varepsilon v^S)$ is in the form of an initial layer; u^S solves the following system:

$$\begin{aligned} (\partial_t - \partial_{YY}) u^S &= 0, \\ u^S|_{Y=0} &= -u_{00}(x), \\ u^S|_{t=0} &= 0, \end{aligned} \tag{3.12}$$

where u_{00} is the trace at the boundary of the tangential part of the initial NS datum, i.e.,

$$u_{00} = \gamma u_0; \tag{3.13}$$

clearly, were the initial and boundary data compatible, $u_{00} = 0$ and, consequently, $u^S = 0$. One can write the explicit expression of u^S,

$$u^S = -\gamma u_0 \int_Y^{+\infty} \frac{e^{-\frac{\sigma^2}{4t}}}{\sqrt{\pi t}} d\sigma, \tag{3.14}$$

while v^S derives from the incompressibility condition,

$$v^S = \int_Y^{+\infty} \partial_x u^S dY'. \tag{3.15}$$

Notice how the tangential part u^S has Y- and t- derivatives that, at the initial time, are singular at the boundary. For the rest of the chapter, we shall say that a function has a "gaussian" singularity if it is bounded away from the origin $(Y,t) = (0,0)$ and behaves near the origin like

$$t^{-\beta} p(Y/\sqrt{t}) e^{-Y^2/ct} \tag{3.16}$$

for some $\beta > 0$, $c > 0$ and some polynomial p. This kind of singularity is concentrated at the boundary at the initial time, and that can be tamed by multiplying the function by suitable powers of Y and/or t. The regular part $\bar{\mathbf{u}}^R$ can be proven to be more regular. The fact that the Prandtl equation with incompatible data admits the solution given in (3.11) was proven in (Cannone, Lombardo, and Sammartino, 2013).

In the sequel it will be useful to introduce the following notation for the value at the boundary of the influx generated by the Prandtl solution,

$$\begin{gathered} g \equiv \gamma \bar{v}^P = -\int_0^{+\infty} \partial_x \tilde{u}^P dY' = g^R + g^S, \\ g^R = -\int_0^{+\infty} \partial_x \tilde{u}^R dY' \qquad g^S = -\int_0^{+\infty} \partial_x u^S dY'. = -\sqrt{\frac{4}{\pi}}\sqrt{t}\partial_x u_0(x, y=0) \end{gathered} \tag{3.17}$$

The first-order correction to the outer solution

The first order outer flow $\mathbf{u}^{out}_{(1)} = \mathbf{u}^E_{(1)}(x,y,t)$ is the solution of a linearized Euler's system:

$$\begin{aligned} \partial_t \mathbf{u}^E_{(1)} + \mathbf{u}^E_{(1)} \cdot \nabla \mathbf{u}^E + \mathbf{u}^E \cdot \nabla \mathbf{u}^E_{(1)} + \nabla p^E_{(1)} &= \mathbf{0}, \\ \nabla \cdot \mathbf{u}^E_{(1)} &= 0, \\ \gamma_n \mathbf{u}^E_{(1)} &= -\gamma \bar{v}^P, \\ \mathbf{u}^E_{(1)}\big|_{t=0} &= \mathbf{0}. \end{aligned} \tag{3.18}$$

The role of the correction to the Euler flow is to cancel the normal inflow generated at the boundary by the boundary layer corrector $\bar{\mathbf{u}}^P$. We shall decompose $\mathbf{u}^E_{(1)}$ in a regular and a singular part:

$$\mathbf{u}^E_{(1)} = \boldsymbol{w}^R + \boldsymbol{w}^S. \tag{3.19}$$

We shall see that the singular part can be written as $\sqrt{t}\boldsymbol{w}^S_b(x,y)$, where $\boldsymbol{w}^S_b$ does not depend on t. This time behavior means that the singularity, formed at $t = 0$ in the inner flow $\mathbf{u}^{inn}_{(0)}$, does not remain confined in the boundary layer, but instantaneously propagates in the whole space with an $O(\varepsilon)$ intensity. This is consistent with the parabolic nature of the Navier-Stokes equations, leading to an infinite speed of propagating disturbances.

While the time derivative of $\mathbf{u}^E_{(1)}$ is singular everywhere at the initial time, its growth as t goes to zero is less pronounced than the one of $\partial_t u^S$: the reason is that the singularity has been passed at $O(\varepsilon)$ terms through the incompressibility condition that gives g^S in (3.17), and the integration in Y has a regularizing effect on "gaussian" singularities. For the same reason, we shall see that the overall error $\mathbf{e}$ is regular, with a bounded time derivative.

The first-order correction to the inner solution

The first order inner flow $\mathbf{u}^{inn}_{(1)} = \bar{\mathbf{u}}^P_{(1)}(x, Y, t) = \left(\bar{u}^P_{(1)}, \varepsilon\bar{v}^P_{(1)}\right)$ is the solution of a linearized Prandtl's system,

$$\begin{aligned} (\partial_t - \partial_{YY})\bar{u}^P_{(1)} &= 0, \\ \gamma\bar{u}^P_{(1)} &= -\gamma u^E_{(1)}, \\ \bar{u}^P_{(1)}(t=0) &= 0, \end{aligned} \tag{3.20}$$

with the normal component given by the incompressibility condition,

$$\bar{v}^P_{(1)} = \int_Y^{+\infty} \partial_x \bar{u}^P_{(1)} dY'. \tag{3.21}$$

The role of $\bar{\mathbf{u}}^P_{(1)}$ is to correct the tangential component at the boundary generated by $\mathbf{u}^E_{(1)}$. The term $\bar{\mathbf{u}}^P_{(1)}$ admits a decomposition in a regular part and a singular part with a "gaussian" singularity less severe then the one in $\mathbf{u}^{inn}_{(0)}$.

The error $\mathbf{e}(x, Y, t)$

The term $\mathbf{e}(x, Y, t)$ is an overall error that closes the asymptotic expansion. It satisfies a NS type equation with a source term and with a boundary condition that cancels the inflow generated by $\bar{\mathbf{u}}^P_{(1)}$, without generating tangential flow:

$$\begin{aligned} \partial_t \mathbf{e} + \left(\mathbf{u}^{NS}_{(0)} + \varepsilon\mathbf{u}^{NS}_{(1)}\right)\cdot\nabla\mathbf{e} + \mathbf{e}\cdot\nabla\left(\mathbf{u}^{NS}_{(0)} + \varepsilon\mathbf{u}^{NS}_{(1)}\right) + \varepsilon\mathbf{e}\cdot\nabla\mathbf{e} + \nabla p^e &= \Xi, \\ \nabla\cdot\mathbf{e} &= 0, \\ \gamma\mathbf{e} &= \left(0, -\gamma\bar{v}^P_{(1)}\right), \\ \mathbf{e}(t=0) &= 0, \end{aligned} \tag{3.22}$$

where we have defined the zero-th and first order approximation of the NS solution:

$$\mathbf{u}^{NS}_{(0)} = \mathbf{u}^E + \bar{\mathbf{u}}^P, \qquad \mathbf{u}^{NS}_{(1)} = \mathbf{u}^E_{(1)} + \bar{\mathbf{u}}^P_{(1)}.$$

The source term Ξ is generated by the discrepancies between the NS equation and the equations satisfied by the approximating terms in the asymptotic expansion. The explicit expression can be found in appendix B.

A key point in the proof of the regularity of $\mathbf{e}$ is that, in its equations, the singular terms multiply terms which go to zero with t or Y in a sufficiently fast way.

All the singular terms appearing in (3.10) have explicit expressions and finite L^2 norms in appropriate spaces of holomorphic functions. The expansion (3.10) implies in particular the validity of the inviscid limit in the energy norm with an $O(\sqrt{\varepsilon})$ rate, since $||\bar{\mathbf{u}}^P||_{L^2_{xy}} = \sqrt{\varepsilon}||\bar{\mathbf{u}}^P||_{L^2_{xY}}$.

Plan of the chapter

The organization of the chapter is the following: in 3.2 we introduce the function spaces needed to prove the validity of the inviscid limit, and we analyze some of their properties. In 3.3 we introduce the abstract Cauchy-Kowalewski theorem; this theorem is essentially a fixed point method used to prove the existence of solutions to differential equations in a scale of Banach spaces. In 3.4 we present the main result of the chapter. The other sections deal with the terms of order one and the remainder deriving from the asymptotic expansion of $\mathbf{u}^{NS}$.

3.2 Function spaces

Define the strip $D(\rho)$, the angular sector $\Sigma(\theta)$ and the conoid $\Sigma(\theta, a)$ as follows

$$D(\rho) = \{x \in \mathbb{C} : |Im(x)| < \rho\}, \tag{3.23}$$

$$\Sigma(\theta) = \{y \in \mathbb{C} : Re(y) > 0,\ |Im(y)| < Re(y)\tan(\theta)\}, \tag{3.24}$$

$$\begin{aligned}\Sigma(\theta, a) = \{y \in \mathbb{C} : 0 < Re(y) \le a,\ |Im(y)| < Re(y)\tan(\theta)\} \\ \cup\{y \in \mathbb{C} : Re(y) > a,\ |Im(y)| < a\tan(\theta)\}.\end{aligned} \tag{3.25}$$

In what follows, we shall deal with spaces of functions analytic in some of the above domains; in those spaces, the chosen paths of integration in the x variable are lines parallel to the real axis, while we shall adopt piecewise linear paths for the y and Y variables.

$$\Gamma(b) = \{x \in \mathbb{C} : Im(x) = b\}, \tag{3.26}$$

$$\begin{aligned}\Gamma(\theta', a) = \{y \in \mathbb{C} : 0 < Re(y) \le a,\ Im(y) = Re(y)\tan(\theta')\}\cup \\ \{y \in \mathbb{C} : Re(y) > a,\ Im(y) = a\tan(\theta')\}.\end{aligned} \tag{3.27}$$

Definition 3.2.1. *The space $H^{l,\rho}$ is the set of all complex functions $f(x)$ such that*

1. *f is analytic in $D(\rho)$;*
2. $|f|_{l,\rho} = \sum_{|\alpha|\le l} \sup_{|\lambda|<\rho} |\partial_x^\alpha f(\cdot + i\lambda)|_{L^2(\mathbb{R})} < \infty.$

When dealing with the 3D Navier Stokes system, x is a 2D vector, and α is a multi-index, while in the two-dimensional case, x is a scalar. The use of an L^2 norm instead of a generic L^p norm allows an important characterization for Hardy spaces, see (Paley and Wiener, 1934) for a proof:

Theorem 3.2.1 (Paley-Wiener Theorem for the strip)**.** *Let $f \in L^2(\mathbb{R})$, $\rho > 0$, denote with $\hat{f}$ the Fourier transform of f. The following are equivalent:*

1. *f is the restriction to the real line of a function holomorphic on the strip $D(\rho)$, with* $\sup_{|\lambda|<\rho} |f(\cdot + i\lambda)|_{L^2(\mathbb{R})} < \infty$;
2. $e^{\rho|\xi|}\hat{f} \in L^2(\mathbb{R})$.

We shall use extensively this result, and we shall often work in the Fourier variable ξ' corresponding to the physical variable x: to simplify the notation, we denote with the same symbol a function f and its Fourier transform with respect to the x variable, and similarly we use the same notation for a pseudodifferential operator and its symbol.

3.2.1 Function spaces for the outer flow: zero-th and first-order Euler equations

In this section, we shall define the appropriate function spaces for studying Euler equations and the first order correction to the Euler equations. First, we introduce the space of functions, depending on x and y. To construct the solution of the Euler equations, the main tool is the half-plane Leray projector, which allows as many derivatives in x and y. This reflects in the following definition.

Definition 3.2.2. *$H^{l,\rho,\theta}$ is the set of all functions $f(x,y)$ such that:*

1. *f is analytic in $D(\rho) \times \Sigma(\theta, a)$;*
2. $|f|_{l,\rho,\theta} = \sum_{|\alpha_1|+\alpha_2 \le l} \sup_{|\theta'|<\theta} ||\partial_y^{\alpha_2}\partial_x^{\alpha_1} f|_{0,\rho}|_{L^2(\Gamma(\theta',a))} < \infty.$

The initial value for the Euler equations is given in $H^{l,\rho,\theta}$, with $l \ge 6 > 1 + d/2$ and $\theta < \pi/4$.

We now pass to introduce function spaces with time dependence. In all the spaces defined below, the width of the analyticity domain diminishes linearly with the time t.

For a given Banach scale $\{X_\rho\}_{0<\rho\le\rho_0}$, with $X_{\rho''} \subset X_{\rho'}$ and $|\cdot|_{\rho'} \le |\cdot|_{\rho''}$ when $\rho' \le \rho'' \le \rho_0$, denote with $B^j_\beta([0,T], X_{\rho_0})$ the set of all functions f such that, for $k = 0, ..., j$, $\partial_t^k f$ is continuous from $[0,\tau]$ to $X_{\rho_0-\beta\tau}$ $\forall 0 < \tau \le T \le \rho_0/\beta$, with norm

$$|f|_{k,\rho,\beta} = \sum_{j=0}^{k} \sup_{t\in[0,T]} |\partial_t^j f(t)|_{\rho_0-\beta t}. \tag{3.28}$$

The following spaces are where one can prove the existence of the outer solutions.

Definition 3.2.3. *$H^{l,\rho,\theta}_{\beta,T}$, with $T \le \min\{\rho/\beta, \theta/\beta\}$ is the space of all functions $f(x,y,t)$ such that $f \in \bigcap_{i=0}^{l} B^i_\beta([0,T], H^{l-i,\rho,\theta})$, with norm*

$$|f|_{l,\rho,\theta,\beta,T} = \sum_{i=0}^{l} \sup_{t\in[0,T]} |\partial_t^i f(\cdot,\cdot,t)|_{l-i,\rho-\beta t,\theta-\beta t}. \tag{3.29}$$

The above space is the natural space for analytic solutions of the Euler equations; see Theorem 4.1 in (Sammartino and Caflisch, 1998a).

Definition 3.2.4. *$\tilde{H}^{l,\rho,\theta}_{\beta,T,1}$ is the set of all functions $f(t,x,y)$ such that*

1. $f \in B^0_\beta([0,T], H^{l,\rho,\theta}) \cap B^1_\beta([0,T], H^{l-1,\rho,\theta})$;
2. $|f|_{l,\rho,\theta,\beta,T,1} = \sup_{t\in[0,T]} |f|_{l,\rho-\beta t,\theta-\beta t} + \sup_{t\in[0,T]} |\partial_t f|_{l-1,\rho-\beta t,\theta-\beta t} < \infty.$

The above space is the natural space where to prove the existence of the correction of Euler flow, $\mathbf{u}^E_{(1)}$: the singularity deriving from Prandtl equations implies that only one time-derivative is allowed; on the other hand, the regularity in y depends only on operators (like the Leray projector) with some symmetry between the behavior in x and y, so that we can use as many derivatives with respect to y as for x.

3.2.2 Function spaces for the inner flow: zero-th and first order boundary layer equations

We now introduce the space of functions analytic in x and Y, in the strip $D(\rho)$ and in the cone $\Sigma(\theta)$, respectively. Moreover, we impose exponential decay in the Y-variable so that for large Y (away from the boundary, in the outer flow), the boundary layer-type solutions do not influence the Euler solutions.

Definition 3.2.5. *The space $K^{l,\rho,\theta,\mu}$ with $\mu > 0$ is the set of all functions $f(x,Y)$ such that*

1. *f is analytic in $D(\rho) \times \Sigma(\theta)$;*
2. *$\partial_Y^{\alpha_1}\partial_x^{\alpha_2} f \in C^0(\Sigma(\theta), H^{0,\rho})$ for any $\alpha_1 \le 2$ and $|\alpha_2| \le l - \alpha_1$;*
3. *$|f|_{l,\rho,\theta,\mu} = \sum_{\alpha_1 \le 2} \sum_{|\alpha_2| \le l-\alpha_1} \sup_{Y\in\Sigma(\theta)} e^{\mu Re(Y)} |\partial_Y^{\alpha_1}\partial_x^{\alpha_2} f(\cdot, Y)|_{0,\rho} < \infty.$*

The asymmetry on the regularity requirements in x and Y is due to the diffusion in Y that allows regularity only up to second-order Y-derivatives (unless one imposes stronger compatibility conditions).

We now introduce the functions depending on x and t. The presence of the diffusion allows regularity in t only up to first-order derivative.

Definition 3.2.6. *$K^{l,\rho}_{\beta,T}$ is the set of all functions $f(x,t) \in B^0_\beta([0,T], H^{l,\rho}) \cap B^1_\beta([0,T], H^{l-1,\rho})$, with norm*

$$|f|_{l,\rho,\beta,T} = \sum_{i=0}^{1} \sum_{|\alpha|\le l} \sup_{0\le t\le T} |\partial_t^i \partial_x^\alpha f(\cdot,t)|_{0,\rho-\beta t}. \tag{3.30}$$

The next space is where we shall prove the existence of the regular part of the solutions of the zeroth and first-order BL equations. The diffusion in Y allows regularity only up to second-order Y-derivatives and first-order t-derivatives.

Definition 3.2.7. *The space $K^{l,\rho,\theta,\mu}_{\beta,T}$ is the set of all functions $f(x,Y,t)$ such that:*

1. *$f \in C^0([0,t], K^{l,\rho-\beta t,\theta-\beta t,\mu-\beta t})$ and $\partial_t\partial_x^\alpha f \in C^0([0,t], K^{0,\rho-\beta t,\theta-\beta t,\mu-\beta t})$, with $t \le T \le \min\{\rho/\beta, \theta/\beta, \mu/\beta\}$ and $|\alpha| \le l-2$;*
2. *$|f|_{l,\rho,\theta,\mu,\beta,T} = \sum_{\alpha_1\le 2} \sum_{\alpha_1+|\alpha_2|\le l} \sup_{0\le t\le T} |\partial_Y^{\alpha_1}\partial_x^{\alpha_2} f(\cdot,\cdot,t)|_{0,\rho-\beta t,\theta-\beta t,\mu-\beta t} + \sum_{|\alpha|\le l-2} \sup_{t\in[0,T]} |\partial_t \partial_x^\alpha f(\cdot,\cdot,t)|_{0,\rho-\beta t,\theta-\beta t,\mu-\beta t} < \infty.$*

3.2.3 Function spaces for the overall error equation

Definition 3.2.8. *$S^{l,\rho,m,\theta}$ is the set of all functions $f(x,Y)$ such that:*

1. *f is analytic inside $D(\rho) \times \Sigma(\theta, \frac{a}{\varepsilon})$ and in $L^2\left(\Gamma(\rho') \times \Gamma(\theta', \frac{a}{\varepsilon}\right) \forall |\rho'| < \rho, \forall |\theta'| < \theta$;*
2. *$|f|_{l,\rho,m,\theta} = \sum_{k\le m} |\partial_Y^k f|_{l-k,\rho,\theta} < \infty.$*

Definition 3.2.9. *$S^{l,\rho,m,\theta}_{\beta,T}$ the set of all functions $f(t,x,Y)$ such that:*

1. *$f \in B^0_\beta\left([0,T], S^{l,\rho,m,\theta}\right)$;*

2. $|f|_{l,\rho,m,\theta,\beta,T} = \sup_{t\in[0,T]} |f|_{l,\rho-\beta t,m,\theta-\beta t} < \infty.$

In the equation (3.71) of section 3.7, we further decompose the error **e**: the space $S^{l,\rho,1,\theta}_{\beta,T}$ is the functional setting in which the ACK theorem can be applied for the term $\mathbf{e}^*$, and the image of $\mathbf{e}^*$ under the operator $\mathcal{N}^*$ defined in 3.7.3 is in the space $L^{l,\rho,\theta}_{\beta,T}$, defined below.

Definition 3.2.10. *$L^{l,\rho,\theta}_{\beta,T}$ is the set of functions $f(x,Y,t)$ such that*
$f \in C^0([0,t], S^{l,\rho-\beta t,0,\theta-\beta t})$, $\partial_Y f$, $\partial_{YY} f$, $\partial_t f \in C^0([0,t], S^{l-2,\rho-\beta t,0,\theta-\beta t})$ $\forall t \leq T \leq \min\{\rho/\beta, 0, \theta\}$, with norm

$$|f|_{l,\rho,\theta,\beta,T} = \sum_{j=0}^{1} \sum_{\alpha\leq l-2j} \sup_{t\in[0,T]} |\partial_t^j \partial_x^\alpha f(\cdot,\cdot,t)|_{0,\rho-\beta t,\theta-\beta t} + \sum_{1\leq\alpha_1\leq 2} \sum_{|\alpha_2|\leq l-2} \sup_{0\leq t\leq T} |\partial_Y^{\alpha_1}\partial_x^{\alpha_2} f(\cdot,\cdot,t)|_{0,\rho-\beta t,\theta-\beta t}. \tag{3.31}$$

3.2.4 Algebra properties and Cauchy estimates

Let d' be the dimension of x, i.e. $d' = 1$ or $d' = 2$ when solving NS equations in the half-plane or in the half-space respectively; moreover, call $d = d' + 1$

Lemma 3.2.1. *Let $u(x)$, $v(x)$ be in $H^{l,\rho}$, with $l > d'/2$. Then $uv \in H^{l,\rho}$, with*

$$|uv|_{l,\rho} \leq C|u|_{l,\rho}|v|_{l,\rho}. \tag{3.32}$$

Lemma 3.2.2. *Let $u(x,y)$, $v(x,y)$ be in $H^{l,\rho,\theta}$, with $l > d/2$. Then $uv \in H^{l,\rho,\theta}$, with*

$$|uv|_{l,\rho,\theta} \leq C|u|_{l,\rho,\theta}|v|_{l,\rho,\theta}. \tag{3.33}$$

One can easily verify both lemmas using the Paley-Wiener theorem and the usual argument used to prove the algebra properties of Sobolev spaces.

The use of complex variables allows to have simple estimates for the norms of the derivatives: using Cauchy formula for derivatives, we immediately have that

$$|\partial_x^k u|_{0,\rho'} \leq \frac{k!}{(\rho-\rho')^k}|u|_{0,\rho}. \tag{3.34}$$

For functions holomorphic in a cone or in a conoid, we cannot bound the norm of the derivatives with the norm of the function in a larger cone with the same vertex, because we cannot use the Cauchy formula for derivatives with a fixed radius in all the path of integration. If we use a radius linearly growing with $|y|$ for $Re(y) < a$, constant for $Re(y) \geq a$, we easily obtain for a function holomorphic in $\Sigma(\theta, a)$ that

$$\sup_{|\theta'|<\theta} |\min\{a,|y|\}\partial_y f(y)|_{L^2(\Gamma(\theta',a))} \leq \frac{C}{\bar\theta-\theta} \sup_{|\theta'|<\bar\theta} |f|_{L^p(\Gamma(\theta',a))}. \tag{3.35}$$

From equation (3.35), we obtain

Lemma 3.2.3. *Assume that u,v $\in H^{l,\rho,\theta}$, with $u(y=0) = 0$ and $l > d/2+1$. Then for any $\theta' < \theta$ we have*

$$|u\partial_y v|_{l,\rho,\theta'} \leq C|u|_{l,\rho,\theta'}\frac{|v|_{l,\rho,\theta}}{\theta-\theta'}. \tag{3.36}$$

This lemma is crucial for the estimate of the nonlinear term in the Euler equations. The Prandtl equations with initial data exponentially decaying in Y need the following estimates.

Lemma 3.2.4. *Let $f \in K^{l,\rho',\theta'',\mu''}$. Then*

$$|min\{1,|Y|\}\partial_Y f|_{l,\rho',\theta',\mu'} \leq \frac{|f|_{l,\rho',\theta'',\mu'}}{\theta''-\theta'} + \mu'|f|_{l,\rho',\theta',\mu'} \tag{3.37}$$

$$|Y\partial_Y f|_{l,\rho',\theta',\mu'} \leq \frac{|f|_{l,\rho',\theta'',\mu'}}{\theta''-\theta'} + \mu'\frac{|f|_{l,\rho',\theta',\mu''}}{\mu''-\mu'} + |f|_{l,\rho',\theta',\mu'} \tag{3.38}$$

Both the lemmas above can be found in (Sammartino and Caflisch, 1998a). Notice that a different exponential decay is needed in order to estimate the product of $\partial_Y f$ with a linearly increasing function. The Prandtl equations are still well posed when the initial data decay with Y with a polynomial rate, see (Kukavica and Vicol, 2013), and also (Cannone, Lombardo, and Sammartino, 2013); in this case, using a radius linearly growing with $|Y|$ in the Cauchy formula for derivatives, it is easy to obtain

$$\sup_{Y\in\Sigma(\theta)} (1+|Y|)^\alpha |Y\partial_Y f(Y)| \leq \frac{C}{\bar{\theta}-\theta} \sup_{Y\in\Sigma(\bar{\theta})} (1+|Y|)^\alpha |f(Y)|, \tag{3.39}$$

which means that, unlike the exponential case, there is no need to change the polynomial rate of decay.

3.2.5 Paths of integration

In this chapter, we shall solve equations involving heat operators in several instances: for example, boundary layer equations, where diffusion in the normal Y-variable appears, or the error equation, where diffusion is both in the x- and Y-variable is present. The method we shall use is based on representing the solutions utilizing the convolution with the appropriate gaussian. As long as the integrand is holomorphic, one can deform the integration path, choosing the most convenient one. However, when estimating the solution and passing the modulus inside the integral, the integrand is no longer holomorphic, and the integral depends on the particular path of integration; therefore, in many estimates, a careful a priori choice of the path of integration is helpful. Taking the convolution with respect to x, the best choice for the path is the one that makes the argument of the gaussian real, i.e., the path $x' \in \Gamma(Im(x))$. On the other hand, for the convolution in the Y variable between a gaussian and a function analytic in a conoid, given that the domain of analyticity shrinks near the origin, one cannot take $Im(Y') = Im(Y)$; therefore, we choose the path consisting of the segment connecting the origin and Y, and the half-line from Y to $+\infty + iIm(Y)$ parallel to the real axis. Said differently, one has:

$$Y' = \begin{cases} \frac{Y}{|Y|}r & r \in [0,|Y|] \\ r + iY_{im} & r \in [Y_r, +\infty[\end{cases} . \tag{3.40}$$

In the first part of the above path of integration one has

$$\left| e^{-\frac{(Y-Y')^2}{4(t-s)}} \right| = e^{-\frac{(|Y|-r)^2}{4(t-s)} \frac{Y_r^2 - Y_{im}^2}{|Y|^2}} \leq e^{-\frac{(|Y|-r)^2}{4(t-s)} \cos(2\theta)}; \tag{3.41}$$

to get the last inequality we have used

$$\frac{Y_r^2 - Y_{im}^2}{Y_r^2 + Y_{im}^2} = \frac{Y_r^2(1-\tan^2(\theta_Y))}{Y_r^2(1+\tan^2(\theta_Y))} = \cos(2\theta_Y) \geq \cos(2\theta), \tag{3.42}$$

where θ_Y is the argument of the complex number Y. Analogously, one can show that

$$\left| e^{-\frac{(Y+Y')^2}{4(t-s)}} \right| \leq e^{-\frac{(|Y|+r)^2}{4(t-s)}\cos(2\theta)}, \tag{3.43}$$

$$\left| e^{-\frac{(Y')^2}{4s}} \right| \leq e^{-\frac{r^2}{4s}\cos(2\theta)}, \tag{3.44}$$

where the latter inequality will be used for estimating the singular term u^S.

In the second part of the path (3.40), we have that

$$\left| e^{-\frac{(Y-Y')^2}{4(t-s)}} \right| = e^{-\frac{(Y_r-r)^2}{4(t-s)}}, \tag{3.45}$$

while

$$\left| e^{-\frac{(Y+Y')^2}{4(t-s)}} \right| = e^{\frac{-(Y_r+r)^2+4Y_{im}^2}{4(t-s)}} \leq e^{-(1-\tan^2(\theta))\frac{(Y_r+r)^2}{4(t-s)}}, \tag{3.46}$$

where we have used

$$(2Y_{im})^2 \leq \tan^2(\theta_Y)(2Y_r)^2 \leq \tan^2(\theta)(Y_r+r)^2.$$

Finally, since $(Y_{im})^2 \leq \tan^2(\theta)r^2$, we have

$$\left| e^{-\frac{(Y')^2}{4s}} \right| \leq e^{-(1-\tan^2(\theta))\frac{r^2}{4s}}. \tag{3.47}$$

Therefore, we conclude that after a change of variable, one can deduce the estimates involving the complex gaussian on the paths (3.40) by estimates involving the real gaussian; these estimates are up constants that blow up when $\theta \to \pi/4$.

For a function analytic in a cone, we take as path the half line which starts at the origin and passes through Y: therefore, the estimates (3.41), (3.43) and (3.44) are valid along all the path.

3.3 The abstract Cauchy Kowalewski theorem

Consider the equation

$$u + F(t,u) = 0 \quad t \in [0,T]. \tag{3.48}$$

Let $\{X_\rho\}_{\rho\in]0,\rho_0]}$ a scale of Banach spaces, with $X_{\rho'} \subseteq X_{\rho''}$ and $|\cdot|_{\rho'} \leq |\cdot|_{\rho''}$ when $\rho'' \leq \rho' \leq \rho$

Theorem 3.3.1. *(ACK theorem) Suppose that $\exists R > 0$, $\rho_0 > 0$, $\beta_0 > 0$ such that if $0 < \tau \leq T \leq \rho_0/\beta_0$ the following properties hold:*

1. *$\forall 0 < \rho' < \rho \leq \rho_0 - \beta_0\tau$ and $\forall u$ such that $\{u \in X_\rho : \sup_{t\in[0,\tau]} |u(t)|_\rho \leq R\}$ the map $F(u,t) : [0,\tau] \mapsto X_{\rho'}$ is continuous;*

2. $\forall 0 < \rho \le \rho_0 - \beta_0\tau$ *the function* $F(t,0) : [0,\tau] \mapsto \{u \in X_\rho : \sup_{t\in[0,\tau]} |u(t)|_\rho \le R\}$ *is continuous and*
$$|F(t,0)|_\rho \le R_0 < R; \tag{3.49}$$

3. *for any* $\beta \le \beta_0$, $0 < \rho' < \rho(s) \le \rho_0 - \beta_0 s$ *and* u^1 *and* $u^2 \in \{u : u(t) \in X_{\rho_0-\beta t} \le R\}$
$$|F(t,u^1) - F(t,u^2)|_{\rho'} \le$$
$$C\int_0^t \frac{|u^1-u^2|_{\rho(s)}}{\rho(s)-\rho'} + \frac{|u^1-u^2|_{\rho'}}{(t-s)^\alpha} ds, \tag{3.50}$$
with $\alpha < 1$ *and* C *independent of* $t, \tau, u^1, u^2, \beta, \rho', \rho(s)$.

Then $\exists\beta > \beta_0$ *such that equation* (3.48) *has a unique solution* u *such that* $\forall\rho \in]0,\rho_0[$ $u(t) \in X_\rho$ $\forall t \in [0,(\rho_0-\rho)/\beta]$; *moreover* $\sup_{\rho+\beta t<\rho_0} |u(t)|_\rho \le R$.

The spaces introduced in 3.2 are Banach scales with respect to the parameters defining the complex domains and the exponential decay. The use of an analytic setting is mainly due to Prandtl's equations, since the well posedness results available for these equations use either an analytic setting (Kukavica and Vicol, 2013) and (Lombardo, Cannone, and Sammartino, 2003) or some monotonicity assumption (Oleinik and Samokhin, 1997).

3.4 The main result

We now state the main result of the chapter. An informal statement was given by Theorem 3.10, in section 3.1.2, where the reader can find a detailed explanation of the meaning of the different terms appearing in the asymptotic expansion (3.51) below.

Theorem 3.4.1. *Assume that* $\boldsymbol{u}_0 \in H^{l,\rho,\theta}$, *with* $\gamma_n \boldsymbol{u}_0 = 0$, $l \ge 6$. *Then, for any* $\mu > 0$, *there exist* $\bar\rho < \rho$, $\bar\theta < \theta$, $\bar\beta > 0$, *all independent of* ν, *such that the solution of the Navier-Stokes equation* (3.1) *can be written as*

$$\boldsymbol{u} = \boldsymbol{u}^E + \bar{\boldsymbol{u}}^P + \varepsilon[\boldsymbol{u}^E_{(1)} + \bar{\boldsymbol{u}}^P_{(1)} + \boldsymbol{e}], \tag{3.51}$$

where:

1. *The term* $\boldsymbol{u}^E \in H^{l,\bar\rho,\bar\theta}_{\bar\beta,T}$ *is the solution of the Euler equations* (3.4).

2. *The term* $\bar{\boldsymbol{u}}^P$ *is the modified Prandtl solution given by* (3.8) *and* (3.9). *The following decomposition in regular and singular part holds:*
$$\bar{\boldsymbol{u}}^P = \bar{\boldsymbol{u}}^R + \bar{\boldsymbol{u}}^S,$$
where the regular term $\bar{\boldsymbol{u}}^R \in K^{l,\bar\rho,\bar\theta,\mu}_{\bar\beta,T}$, *and the singular term* $\boldsymbol{u}^S$ *is given by* (3.14) *and* (3.15).

3. *The term* $\boldsymbol{u}^E_{(1)}$ *is the first order correction to the inviscid flow solving the system* (3.18). *The following decomposition in regular and singular part holds:*
$$\boldsymbol{u}^E_{(1)} = \boldsymbol{u}^{ER}_{(1)} + \sqrt{t}\boldsymbol{w}^S_b(x,y),$$

where $\boldsymbol{u}^{ER}_{(1)} \in \tilde{H}^{l,\bar{\rho},\bar{\theta}}_{\bar{\beta},T,1}$ *and* $\boldsymbol{w}^S_b \in H^{l,\bar{\rho},\bar{\theta}}$ *is given by* (3.56).

4. $\bar{\boldsymbol{u}}^P_{(1)} = \left(\bar{u}^P_{(1)}, \varepsilon\bar{v}^P_{(1)}\right)$ *is the first order correction to the boundary layer flow;* $\bar{u}^P_{(1)}$ *solves the linear heat equation* (3.20)*, while* $\bar{v}^P_{(1)}$ *is given by the incompressibility condition* (3.21)*. The following decomposition in regular and singular part holds:*

$$\bar{\boldsymbol{u}}^P_{(1)} = \bar{\boldsymbol{u}}^R_{(1)} + \bar{\boldsymbol{u}}^S_{(1)} \tag{3.52}$$

where the regular part $\bar{\boldsymbol{u}}^R_{(1)} \in K^{l,\bar{\rho},\bar{\theta},\mu}_{\bar{\beta},T}$ *and the singular part* $\bar{\boldsymbol{u}}^S_{(1)}$ *is given by* (3.64).

5. *The term* $\boldsymbol{e}$ *is an overall error that closes the asymptotic procedure with* $\boldsymbol{e} \in L^{l,\bar{\rho},\bar{\theta}}_{\bar{\beta},T}$.

The result concerning the Euler flow can be found in (Sammartino and Caflisch, 1998a) as Theorem 4.1. : the key point is to use Leray's projector P for the half space, which is the projection operator of L^2 on the space of divergence-free L^2 functions with zero normal component at the boundary. Using P, the Euler equation can be written as

$$\mathbf{u}^E = \mathbf{u}_0 + \int_0^t P\left(\mathbf{u}^E \cdot \nabla \mathbf{u}^E\right) d\tau, \tag{3.53}$$

which is a suitable form for the application of the ACK theorem.

The result for $\bar{\mathbf{u}}^P$ can be easily obtained with a slight modification of the argument used in (Cannone, Lombardo, and Sammartino, 2013), where the authors supposed a polynomial decay of the initial data, rather than an exponential one: they decomposed the solution as the sum of a singular term u^S given by (3.14), the solution u^D of Prandtl's equations with compatible data ($\gamma u^D = u_{00}$) and an initial datum given by the initial value of the Euler flow at the boundary, and an interaction term.

The other terms of the asymptotic expansions are analyzed in the following sections.

Remark 1. *To prove theorem 3.4.1, we use the fact that when we reduce the strip of x-analyticity, we can get as much x-regularity as as needed. However, while we can obtain arbitrary regularity in the tangential variable, this does not hold for the normal variable since the reduction of the cone of analyticity does not provide any additional regularity: the regularity in the normal variable does not exceed the regularity of the initial datum.*

3.5 Correction to the Euler flow

The first-order correction to the Euler flow is produced by the inflow at the boundary generated by the Prandtl solution; given that the Prandtl solution has a regular and a singular part, in (3.17) we have split the inflow g in singular and regular part, g^S and g^R, respectively. The first order Euler equations (3.18) are linear; therefore, by the superposition principle, we can decompose the first order Euler correction $\bar{\mathbf{u}}^E_{(1)}$

as in (3.19), i.e., as singular and regular part. The singular part $\boldsymbol{w}^S$ solves

$$\begin{aligned} \partial_t \boldsymbol{w}^S + \boldsymbol{w}^S \cdot \nabla \mathbf{u}^E + \mathbf{u}^E \cdot \nabla \boldsymbol{w}^S + \nabla p^{w^S} &= \mathbf{0}, \\ \nabla \cdot \boldsymbol{w}^S &= 0, \\ \gamma_n \boldsymbol{w}^S = g^S = -\sqrt{\frac{4}{\pi}}\sqrt{t}\partial_x u_{00}(x)&, \\ \boldsymbol{w}^S|_{t=0} &= \mathbf{0}, \end{aligned} \tag{3.54}$$

while the regular part $\boldsymbol{w}^R$ solves the system obtained by (3.54) replacing g^S with g^R. The regularity of g^R is the same that the boundary value of the normal part of the Prandtl's correction would have in the compatible case; therefore, we have $\boldsymbol{w}^R \in \tilde{H}^{l-1,\bar{\rho},\bar{\theta}}_{\bar{\beta},T,1}$.

We now pass to the analysis of the singular part $\boldsymbol{w}^S$. If one denotes by ξ' the Fourier variable corresponding to x, it is useful to write the solution as

$$\boldsymbol{w}^S(t,\xi',y) = \sqrt{t}\boldsymbol{w}^S_b + I_{\sqrt{t}}\boldsymbol{w}^{S^*}, \tag{3.55}$$

where

$$\boldsymbol{w}^S_b(\xi',y) = C_1(-i\frac{\xi'}{|\xi'|},1)e^{-|\xi'|y}i\xi' u_{00}(\xi'), \tag{3.56}$$

is divergence-free and takes into account the normal inflow. The operator $I_{\sqrt{t}}$ is a weighted integration in time:

$$I_{\sqrt{t}}\boldsymbol{w}^{S^*} = \int_0^t \sqrt{\tau}\boldsymbol{w}^{S^*}(\tau,\xi',y)d\tau. \tag{3.57}$$

The main difference with the compatible case, treated in (Sammartino and Caflisch, 1998b), is the use of $I_{\sqrt{t}}$ instead of a simple integration in time. The point is that the use of a weighted integration allows to eliminate $\sqrt{t}$ from the forcing term of the equation of w^{S^*}, which is essential in order to take a time derivative in equation (3.58) below.

The rest of this section is devoted to the construction $\boldsymbol{w}^{S^*}$. Substituting equation (3.55) into (3.54), and then applying the Leray projector, one derives the following equation for $\boldsymbol{w}^{S^*}$,

$$\boldsymbol{w}^{S^*} + P\left[\frac{I_{\sqrt{t}}\boldsymbol{w}^{S^*}}{\sqrt{t}} \cdot \nabla \mathbf{u}^E + \mathbf{u}^E \cdot \nabla \frac{I_{\sqrt{t}}\boldsymbol{w}^{S^*}}{\sqrt{t}} + \boldsymbol{w}^S_b \cdot \nabla \mathbf{u}^E + \mathbf{u}^E \cdot \nabla \boldsymbol{w}^S_b\right] = \mathbf{0}, \tag{3.58}$$

which is a suitable form for the application of the abstract Cauchy-Kowalewski theorem. Since P is a bounded operator in $H^{l,\rho,\theta}$ (see (Sammartino and Caflisch, 1998a)) and commutes with time derivatives, we can estimate $\mathbf{v}$ instead of $P\mathbf{v}$ for any $\mathbf{v}$. We shall show that, assuming that $u_{00} \in H^{l+1,\rho}$ and $\mathbf{u}^E \in H^{l,\rho,\theta}_{\beta,T}$, then $\boldsymbol{w}^{S*} \in \tilde{H}^{l-1,\rho,\theta}_{\bar{\beta},T,1}$ for some $\bar{\beta} > \beta$: such a regularity for u_{00} can always be obtained from the trace of a function $u_0 \in H^{l,\rho,\theta}$ by reducing the strip of analyticity.

3.5.1 The forcing term

For any $\rho' < \rho - \beta t$, $\theta' \le \theta - \beta t$, we use the algebra property (3.33) to obtain

$$|w_b^S \cdot \nabla \mathbf{u}^E|_{l-1,\rho',\theta'} \le C|w_b^S|_{l-1,\rho',\theta'}|\mathbf{u}^E|_{l,\rho',\theta'} \le C|u_{00}|_{l,\rho}|\mathbf{u}^E|_{l,\rho,\theta,\beta,T} \tag{3.59}$$

and similarly

$$|\mathbf{u}^E \cdot \nabla w_b^S|_{l-1,\rho',\theta'} \le C|u_{00}|_{l+1,\rho}|\mathbf{u}^E|_{l-1,\rho,\theta,\beta,T}. \tag{3.60}$$

3.5.2 Quasi contractiveness

For the term $\mathbf{u}^E \cdot \nabla \frac{I_{\sqrt{t}} w^{S*}}{\sqrt{t}}$, we have

$$\begin{aligned} \left| u^E \partial_x \frac{I_{\sqrt{t}} w^{S*}}{\sqrt{t}} \right|_{l-1,\rho',\theta'} &\le C|u^E|_{l-1,\rho',\theta'} \int_0^t \frac{\sqrt{\tau}}{\sqrt{t}} |\partial_x w^{S^*}|_{l-1,\rho',\theta'} d\tau \le \\ &\le C|u^E|_{l-1,\rho,\theta,\beta,T} \int_0^t \frac{|w^{S*}|_{l-1,\rho(s),\theta'}}{\rho(s)-\rho'} ds, \end{aligned} \tag{3.61}$$

while, with an argument similar to the one used in the proof of lemma 3.2.3, since $\partial_y \frac{I_{\sqrt{t}} w^{S*}}{\sqrt{t}}$ multiplies v^E, which goes to zero linearly as y approaches zero, we have

$$\left| v^E \partial_y \frac{I_{\sqrt{t}} w^{S^*}}{\sqrt{t}} \right|_{l-1,\rho',\theta'} \le C|v^E|_{l-1,\rho,\theta,\beta,T} \int_0^t \frac{|w^{S^*}|_{l-1,\rho',\theta(s)}}{\theta(s)-\theta'} ds. \tag{3.62}$$

The term $\frac{I_{\sqrt{t}} w^{S^*}}{\sqrt{t}} \cdot \nabla \mathbf{u}^E$ is easier to estimate, since no Cauchy estimate is needed

$$\left|\frac{I_{\sqrt{t}} w^{S^*}}{\sqrt{t}} \cdot \nabla \mathbf{u}^E\right| \le C|\mathbf{u}^E|_{l,\rho,\theta,\beta,T} \int_0^t |w^{S*}|_{l-1,\rho',\theta'} ds. \tag{3.63}$$

By the ACK theorem, we obtain the existence of a unique solution of equation (3.58) in $\tilde{H}^{l-1,\rho,\theta}_{\tilde{\beta},T,0}$; taking the time derivative of that equation, it is easy to see that $w^{S*} \in \tilde{H}^{l-1,\rho,\theta}_{\tilde{\beta},T,1}$.

3.6 Boundary layer corrector

The first order correction $\bar{\mathbf{u}}^P_{(1)} = (u^P_{(1)}, \varepsilon \bar{v}^P_{(1)})$ is the sum (see (3.52)) of the solution $\bar{\mathbf{u}}^S_{(1)}$ of the system

$$\begin{aligned} (\partial_t - \partial_{YY}) u^S_{(1)} &= u^S, \\ \gamma u^S_{(1)} &= -\gamma w^S, \\ u^S_{(1)}(t=0) &= 0, \\ \bar{v}^S_{(1)} &= \varepsilon \int_Y^{+\infty} \partial_x u^S dY', \end{aligned} \tag{3.64}$$

and the solution $\bar{\mathbf{u}}^R_{(1)}$ of the system obtained by (3.64) replacing $\boldsymbol{w}^S$ with $\boldsymbol{w}^R$ We have $\bar{\mathbf{u}}^R_{(1)} \in K^{l,\bar{\rho},\bar{\theta},\mu}_{\bar{\beta},T}$: this is obtained through estimates of the heat operators in analytic settings, see (Sammartino and Caflisch, 1998b) for proof. In Fourier terms, $u^S_{(1)}$ is given by

$$u^S_{(1)} = -C_1|\xi'|u_{00}(\xi')F_{1/2}(t,Y) \tag{3.65}$$

where

$$F_{1/2} = \int_0^t \frac{Y}{2(t-s)} \frac{e^{-\frac{Y^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}}\sqrt{s}ds = \frac{1}{\sqrt{\pi}} \int_{\frac{Y}{2\sqrt{t}}}^{+\infty} e^{-\sigma^2}\sqrt{t-\frac{Y^2}{4\sigma^2}}d\sigma, \tag{3.66}$$

We estimate $u^S_{(1)}$ in terms of F: we have

$$|F_{1/2}| \le Ce^{-\frac{Y^2}{8t}}\sqrt{t}. \tag{3.67}$$

The derivative with respect to Y is

$$\partial_Y F_{1/2} = -\frac{1}{\sqrt{\pi}} \int_{\frac{Y}{2\sqrt{t}}}^{+\infty} e^{-\sigma^2}\frac{Y}{4\sigma^2}\frac{d\sigma}{\sqrt{t-\frac{Y^2}{4\sigma^2}}} = -\frac{2}{\sqrt{\pi}}\int_0^t e^{-\frac{Y^2}{4(t-s)}}\frac{ds}{\sqrt{s}\sqrt{(t-s)}}, \tag{3.68}$$

so

$$|\partial_Y F_{1/2}| \le Ce^{-\frac{Y^2}{4t}}. \tag{3.69}$$

The term $\partial_{YY}F_{1/2}$ is singular near the boundary (although $Y\partial_{YY}F_{1/2}$ is not).
The normal component $\bar{v}^S_{(1)}$ has a better regularity with respect to t and Y; we have

$$|\partial^j_Y \bar{v}^S_{(1)}|_{k,\rho} \le C|u_{00}|_{k+2,\rho}e^{-c\frac{|Y|^2}{t}}t^{1-\frac{j}{2}} \tag{3.70}$$

3.7 The remainder e

The remainder **e** satisfies equations (3.22), which are Navier-Stokes type equations with source term and non-homogeneous boundary conditions. It is important to notice that the source term contains $\partial_{xx}u^S$ which is the most singular term. Therefore, we decompose the error **e** as

$$\mathbf{e} = \mathcal{N}^*\mathbf{e}^* + \boldsymbol{\sigma} + \mathbf{h}. \tag{3.71}$$

In the above decomposition, the tangential part of $\mathbf{h} = (h', \varepsilon h_n)$ takes care of the most singular term $\partial_{xx}u^S$; in fact, h' solves the heat equation in the half space with forcing term $\varepsilon\partial_{xx}u^S$, and homogeneous boundary and initial conditions, see the system (3.72); the normal component h_n, is obtained imposing the incompressibility condition and the decay at infinity in Y, see (3.76). Therefore h_n has a non zero trace at the boundary. The construction of **h** will be accomplished in subsection 3.7.1, while the fact that $\mathbf{h} \in L^{l,\bar{\rho},\theta}_{\beta,T}$ is stated in Proposition 3.7.2.

The term $\boldsymbol{\sigma}$ solves the Stokes equations with boundary conditions, see the system (3.78) below. We have introduced the term $\boldsymbol{\sigma}$ to take into account the boundary conditions of $\mathbf{e}$, deriving from the boundary layer corrector, and the boundary conditions generated by $\mathbf{h}$. The construction of σ will be accomplished in subsection 3.7.2, while the fact that $\sigma \in L^{l,\bar{\rho},\theta}_{\beta,T}$ is stated in Proposition 3.7.3.

Finally, $\mathcal{N}^*\mathbf{e}^*$ solves the Stokes equations with homogeneous boundary conditions and with forcing term $\mathbf{e}^*$, see the system (3.85). The Navier-Stokes operator $\mathcal{N}^*$ can be defined explicitly in terms of the Leray projector and and heat operator, see the formula (3.86). The construction of the operator $\mathcal{N}^*$, and the necessary estimates, will be presented in subsection 3.7.3

To construct $\mathbf{e}^*$ we shall use the ACK theorem in the $S^{l,\rho,1,\theta}$ setting: given that $u^S \notin S^{l,\rho,1,\theta}$, this has led us to isolate the effect of $\varepsilon\partial_{xx}u^S$ introducing the term $\mathbf{h}$. The construction of $\mathbf{e}^*$ will be accomplished in subsection 3.7.4.

For the compatible case, in (Sammartino and Caflisch, 1998b), the authors showed that $\mathbf{e} \in L^{l,\rho,\theta}_{\beta,T}$ by proving that both σ and $\mathbf{e}^*$ were in that space. In our case, σ is still in $L^{l,\rho,\theta}_{\beta,T}$; however, we can say that $\mathbf{e}^* \in S^{l,\rho,1,\theta}$, only; nevertheless, we shall prove that the image under $\mathcal{N}^*$ is still in $L^{l,\rho,\theta}_{\beta,T}$.

3.7.1 Heat term

The tangential part h' of $\mathbf{h}$ satisfies

$$\begin{aligned} (\partial_t - \partial_{YY})h' &= \varepsilon\partial_{xx}u^S, \\ \gamma h' &= 0, \\ h'(t=0) &= 0. \end{aligned} \tag{3.72}$$

One can write the explicit expression of h':

$$h' = \varepsilon\partial_{xx}u_{00}F(t,Y), \tag{3.73}$$

where

$$F = \frac{2}{\sqrt{\pi}}\int_0^t ds \int_0^{+\infty} (E_0^- - E_0^+) \int_{\frac{Y'}{2\sqrt{s}}}^{+\infty} e^{-\sigma^2} d\sigma dY', \tag{3.74}$$

and E_0^- and E_0^+ are:

$$E_0^- = \frac{e^{-\frac{(Y-Y')^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}};\; E_0^+ = \frac{e^{-\frac{(Y+Y')^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}}. \tag{3.75}$$

The normal part of $\mathbf{h}$ is obtained through the incompressibility condition expressed in the Y variable; thus, it is a $O(\varepsilon)$ with respect to h'. We shall denote the normal part of $\mathbf{h}$ as εh_n to stress this fact. We have

$$h_n = \int_Y^{+\infty} \partial_x h' dY'. \tag{3.76}$$

One can give the following estimate:

Proposition 3.7.1. *For $j = 0, 1, 2$, $\forall \mu > 0$, we have*

$$\sup_{Y \in \Sigma(\theta)} e^{\mu Re(Y)} |\partial_Y^j h'|_{l-2-j,\rho} \leq C \varepsilon t^{\frac{2-j}{2}} |u_{00}|_{l-j,\rho},$$
$$\sup_{Y \in \Sigma(\theta)} e^{\mu Re(Y)} |\partial_Y^j h_n|_{l-3-j,\rho} \leq C \varepsilon t^{\frac{3-j}{2}} |u_{00}|_{l-j,\rho}. \tag{3.77}$$

The proof of the proposition is in appendix C. The exponential decay in the Y variable stated in proposition 3.7.1 implies the boundedness in L^2_Y; moreover every $u_0 \in H^{l,\rho,\theta}$ has trace $u_{00} \in H^{l-1,\rho} \subset H^{k,\bar{\rho}}$ $\forall k \geq l$, $\forall \bar{\rho} < \rho$. Therefore, we obtain the following result:

Proposition 3.7.2. *Assume $u_0 \in H^{l,\rho,\theta}$: then $\forall \bar{\rho} < \rho$, $\forall T$, $\forall \beta$, we have $\boldsymbol{h} \in L^{l,\bar{\rho},\theta}_{\beta,T}$.*

The estimates given in Proposition 3.7.1 imply an L^∞_Y boundedness, which will be useful in the treatment of the nonlinear terms in the equation of $\mathbf{e}^*$.

3.7.2 Boundary value of the error

The term $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$ is needed to cancel the boundary value of h_n and $\bar{v}^P_{(1)}$; therefore, it satisfies

$$\begin{aligned} (\partial_t - \partial_{YY})\boldsymbol{\sigma} + \nabla \phi &= 0, \\ \nabla \cdot \boldsymbol{\sigma} &= 0,, \\ \gamma \boldsymbol{\sigma} &= (0, \varepsilon G), \\ \boldsymbol{\sigma}(t=0) &= \mathbf{0}, \end{aligned} \tag{3.78}$$

where

$$G = -\gamma \bar{v}^R_{(1)} - \gamma \bar{v}^P_{(1)} - \gamma h_n = \partial_x \int_0^{+\infty} \left(u^R_{(1)} + u^P_{(1)} + h' \right) dY = \partial_x \tilde{G}. \tag{3.79}$$

The above system is a Stokes problem with boundary datum and homogeneous initial datum. In (Sammartino and Caflisch, 1998b) one can find the procedure to solve such a problem; the solution writes as

$$\sigma_1 = \varepsilon N' E_1 G - N' \varepsilon \int_0^Y \varepsilon |\xi'| e^{-\varepsilon |\xi'| (Y - Y')} E_1 G dY' - N' e^{-\varepsilon |\xi'| Y} G,$$
$$\sigma_2 = \varepsilon e^{-|\xi'| \varepsilon Y} G + \varepsilon \int_0^Y \varepsilon |\xi'| e^{-\varepsilon |\xi'| (Y - Y')} E_1 G, \tag{3.80}$$

where

$$N' = \frac{i \xi'}{|\xi'|}, \tag{3.81}$$

while $E_1 f(t)$ gives the solution of the heat equation with boundary datum $f(t)$ and homogeneous initial datum; the explicit expression is

$$E_1 f = \int_0^t \frac{Y}{2(t-s)} \frac{e^{-\frac{Y^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}} f(s) ds. \tag{3.82}$$

Proposition 3.7.3. *Let $\sigma = (\sigma_1, \sigma_2)$ with σ_i given in (3.80). Then $\sigma \in L^{l,\bar\rho,\bar\theta}_{\bar\beta,\bar T}$ and, $\forall|\theta'| < \theta$, $\forall\theta \in]0, \pi/4[$, $\forall a$ the following estimates, hold:*

$$\begin{aligned} ||\sigma|_{k,\rho}|_{L^2(\Gamma(\theta',a))} &\le C|\tilde G|_{k+1,\rho}, \\ ||\partial_Y\sigma|_{k,\rho}|_{L^2(\Gamma(\theta',a))} &\le C\varepsilon\left[t^{3/4}|\partial_t G|_{k,\rho} + |G|_{\rho,k+1}\right], \\ ||\partial_{YY}\sigma|_{k,\rho}|_{L^2(\Gamma(\theta',a))} &\le C\varepsilon\left[|G|_{k+2,\rho} + t^{1/4}|\partial_t G|_{k+1,\rho}\right], \\ ||\partial_t\sigma|_{k,\rho}|_{L^2(\Gamma(\theta',a))} &\le C\left|\partial_t\tilde G\right|_{k+1,\rho}, \end{aligned} \tag{3.83}$$

$$\begin{aligned} \sup_Y |\sigma|_{k,\rho} &\le C|G|_{k,\rho}, \\ \sup_Y |\partial_Y\sigma|_{k,\rho} &\le C\varepsilon\left[|G|_{k+1,\rho} + t^{1/2}|\partial_t G|_{k,\rho}\right], \\ \sup_Y |\partial_{YY}\sigma|_{k,\rho} &\le C\varepsilon\left[|G|_{k+2,\rho}|\partial_t G|_{k+1,\rho}\right]. \end{aligned} \tag{3.84}$$

The proof of the proposition is straightforward, and can be obtained by applying Young's convolution inequality to the expressions (3.80).

Remark 2. *In the expression of σ, the term $N'e^{-\varepsilon|\xi'|Y}G$ is the only one which is not $O(\varepsilon)$; however, $\partial_Y\sigma$ is $O(\epsilon)$, which implies that $\partial_y\sigma$ is $O(1)$.*

Remark 3. *The estimates for σ and $\partial_Y\sigma$ show that, near $t = 0$, they are small in t; in fact, $G(t = 0) = 0$ and continuosly differentiable in t, so that G goes linearly to zero with t. The regularity of G with respect to time is due to the regularizing effect that the integration in Y, appearing in (3.79), has on "gaussian"-type singularities.*

These remarks and the estimates in the L^∞_Y norm in (3.84) will be useful in the estimates for the nonlinear terms in the equation of $\mathbf{e}^*$.

3.7.3 The Navier-Stokes operator

In the present subsection we shall introduce the operator $\mathcal{N}^*$ and give the estimate Proposition 3.7.7, which is the main result of the subsection. The Navier-Stokes operator $\mathcal{N}^*$ solves the time-dependent Stokes equation with a forcing term: $\boldsymbol{w} = \mathcal{N}^*\boldsymbol{w}^*$ is the solution of

$$\begin{aligned} (\partial_t - \varepsilon^2\partial_{xx} - \partial_{YY})\boldsymbol{w} + \nabla p^w &= \boldsymbol{w}^* \\ \nabla\cdot\boldsymbol{w} &= 0, \\ \gamma\boldsymbol{w} &= \mathbf{0}, \\ \boldsymbol{w}(t=0) &= \mathbf{0}. \end{aligned} \tag{3.85}$$

One can write the explicit expression of $\mathcal{N}^*$ in terms of a projection operator $\bar P^\infty$, of the heat operator E_2^d, and of the operator $\mathcal{S}$ which solves the Stokes equations with boundary data:

$$\mathcal{N}^* = \bar P^\infty E_2^d - \mathcal{S}\gamma\bar P^\infty E_2^d. \tag{3.86}$$

In the rest of the section we shall give the explicit expression of $\bar P^\infty$, E_2^d, and $\mathcal{S}$, and state the necessary estimates for $\mathcal{N}^*$.

If $\mathbf{v}$ is a vector field defined on the upper plane, $\bar P^\infty\mathbf{v}$ is obtained in the following way: extend $\mathbf{v}$ oddly for $Y < 0$, then apply the Leray projector for functions defined on the whole space, and finally restrict the result to $Y \ge 0$. The explicit expression of the normal component is given by

$$\bar{P}_n^{\infty}\mathbf{v} = \frac{1}{2}\varepsilon|\xi'|\Bigg[\int_0^Y dY'\left(e^{-\varepsilon|\xi'|(Y-Y')} - e^{-\varepsilon|\xi'|(Y+Y')}\right)(-N'v_1+v_2)+ \\ +\int_Y^{+\infty} dY'\left(e^{\varepsilon|\xi'|(Y-Y')}(N'v_1+v_2) - e^{-\varepsilon|\xi'|(Y+Y')}(-N'v_1+v_2)\right)\Bigg], \tag{3.87}$$

where the Riesz-type operator N' is defined in (3.81). The tangential component is given by

$$\bar{P}^{\infty\prime}\mathbf{v} = v_1 - \frac{\varepsilon|\xi'|}{2}\Bigg[\int_0^Y dY'\left(e^{-\varepsilon|\xi'|(Y-Y')} - e^{-\varepsilon|\xi'|(Y+Y')}\right)(v_1+N'v_2)+ \\ +\int_Y^{+\infty} dY'\left(e^{\varepsilon|\xi'|(Y-Y')}(v_1-N'v_2) - e^{-\varepsilon|\xi'|(Y+Y')}(v_1+N'v_2)\right)\Bigg]. \tag{3.88}$$

Notice that, if we had extended the tangential part evenly and the normal part oddly, we would have obtained the Leray projector for the half space.

The operator E_2^d is such that $E_2^d f$ solves the heat equation in the half space with source f and homogeneous data:

$$\begin{aligned} (\partial_t - \varepsilon^2\partial_{xx} - \partial_{YY})u &= f \\ \gamma u &= 0 \\ u(t=0) &= 0 \end{aligned} \tag{3.89}$$

The explicit expression can be given in terms of convolutions with gaussians:

$$E_2^d f(x,Y,t) = \int_0^t ds \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-x')^2}{4(t-s)\varepsilon^2}}}{\sqrt{4\pi\varepsilon^2(t-s)}}dx' \\ \int_0^{+\infty}\left(\frac{e^{-\frac{(Y-Y')^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}} - \frac{e^{-\frac{(Y+Y')^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}}\right) f(x',Y',s)dY'. \tag{3.90}$$

The Stokes operator $\mathcal{S}$ is the operator such that $\mathcal{S}\mathbf{g}$ solves the Stokes equations with boundary datum $\mathbf{g}$:

$$\begin{aligned} (\partial_y - \varepsilon^2\Delta)\mathbf{u}^S + \nabla p^S &= 0, \\ \nabla\cdot\mathbf{u}^S &= 0, \\ \gamma\mathbf{u}^S &= \mathbf{g}, \\ \mathbf{u}^S(t=0) &= \mathbf{0}. \end{aligned} \tag{3.91}$$

We now give some bounds in a time integrated form that will be useful for the application of the ACK theorem.

Proposition 3.7.4. *Assume that $u \in S^{l,\rho,1,\theta}_{\beta,T}$. Then we have that, for $\rho' < \rho - \beta t$, $\theta' < \theta - \beta t$,*

$$|\mathcal{N}^* u|_{l,\rho',0,\theta'} \leq C \int_0^t ds |u(\cdot,\cdot,s)|_{l,\rho',0,\theta'}, \tag{3.92}$$

$$|\partial_Y \mathcal{N}^* u|_{l-1,\rho',0,\theta'} \leq C \int_0^t ds \frac{|u(\cdot,\cdot,s)|_{l,\rho',1,\theta'}}{(t-s)^{1/4}}. \tag{3.93}$$

Contrarily to the compatible case, the mild singularity in time cannot be completely eliminated, even when $\gamma \mathbf{u} = \mathbf{0}$: this is due to the fact that the functional setting $S^{l,\rho,1,\theta}_{\beta,T}$ is more singular than the one used in the compatible case. In particular, this setting does not allow a time derivative. The presence of this singularity implies that, in order to estimate a derivative of order $l+1$, like $\partial_Y \partial_x^l$, we cannot use the Cauchy estimates, otherwise we would have, at the denominator, $(t-s)^{1/4}(\rho(s) - \rho')$, which is not allowed by the ACK theorem. Therefore, we need to estimate the derivatives of order $l+1$ in a better way. A similar problem would appear if, in order to deal with the bilinear terms appearing in the error equation, one tries to use the algebra properties of $S^{l,\rho,1,\theta}_{\beta,T}$; therefore, we need some estimates in an $L^\infty_Y L^2_x$-like setting, so that the algebra properties are used only in the tangential variable.

Proposition 3.7.5. *For any $u \in S^{l,\rho,1,\theta}_{\beta,T}$, we have for $\rho' < \rho - \beta t$, $\theta' < \theta - \beta t$*

$$|\mathcal{N}^* u|_{l+1,\rho',0,\theta'} \leq C \int_0^t \frac{|u|_{l,\rho(s),0,\theta'}}{\rho(s) - \rho'} ds, \tag{3.94}$$

$$|\mathcal{N}^* u|_{l+1,\rho',0,\theta'} \leq \frac{C}{\varepsilon} \int_0^t \frac{|u|_{l,\rho',0,\theta'}}{(t-s)^{1/2}} ds, \tag{3.95}$$

$$|\partial_Y \mathcal{N}^* u|_{l,\rho',0,\theta'} \leq C \int_0^t \frac{|u|_{l,\rho',1,\theta'}}{(t-s)^{1/2}} + \frac{|u|_{l,\rho(s),1,\theta'}}{\rho(s) - \rho'} ds, \tag{3.96}$$

$$|\partial_{YY} \mathcal{N}^* u|_{l-1,\rho',0,\theta'} \leq C \int_0^t \frac{|u|_{l,\rho',1,\theta'}}{(t-s)^{1/2}} + \varepsilon^{1/2} \frac{|u|_{l-1/2,\rho',0,\theta'}}{(t-s)^{3/4}} ds. \tag{3.97}$$

Furthermore, if $\gamma \mathbf{u} = \mathbf{0}$, then

$$|\partial_Y \mathcal{N}^* u|_{l,\rho',0,\theta'} \leq C \int_0^t \frac{|u|_{l,\rho',1,\theta'}}{(t-s)^{1/2}} ds. \tag{3.98}$$

The case $\gamma \mathbf{u} = \mathbf{0}$ is used to verify the quasi-contractiveness hypothesis of the ACK theorem: in this case, if we don't use Cauchy estimates, the only derivative of order $l+1$ which can be an $O(1/\varepsilon)$ is the purely tangential one, $\partial_x^{l+1} \mathcal{N}^* \mathbf{u}$. In the error equation, nonlinear terms in $\mathcal{N}^* \mathbf{e}^*$ appear: for those terms, it is useful to have some properties, in the L^∞_Y norm.

Proposition 3.7.6. *For any $\boldsymbol{u} \in S^{l,\rho,1,\theta}_{\beta,T}$ we have that, $\forall \rho' < \rho - \beta t$, $\forall t \in [0,T]$*

$$\sup_Y |\mathcal{N}^* \boldsymbol{u}|_{l,\rho'} \leq C \int_0^t \frac{|\boldsymbol{u}|_{l,\rho',0,\theta'}}{(t-s)^{1/4}}, \tag{3.99}$$

$$\sup_Y |\partial_Y \mathcal{N}^* \boldsymbol{u}|_{l-1,\rho'} \leq C \int_0^t \frac{|\boldsymbol{u}|_{l,\rho',0,\theta'}}{(t-s)^{3/4}}. \tag{3.100}$$

By estimates (3.92), (3.93), (3.97), and considering that

$$\partial_t \mathcal{N}^* \mathbf{u} = \bar{P^{\infty}} \mathbf{u} + \varepsilon^2 \partial_{xx} \mathcal{N}^* \mathbf{u} + \partial_{YY} \mathcal{N}^* \mathbf{u}, \tag{3.101}$$

the following proposition holds.

Proposition 3.7.7. *Suppose that $\boldsymbol{u} \in S^{l,\rho,1,\theta}_{\beta,T}$, then $\mathcal{N}^* \boldsymbol{u} \in L^{l,\rho,\theta}_{\beta,T}$ and*

$$|\mathcal{N}^* \boldsymbol{u}|_{l,\rho,\theta,\beta,T} \leq C |\boldsymbol{u}|_{l,\rho,1,\theta,\beta,T}. \tag{3.102}$$

3.7.4 The error equation

Given the expression (3.71) for $\mathbf{e}$, the system (3.22) is now an equation for $\mathbf{e}^*$ that can be cast in the following form:

$$\mathbf{e}^* = \mathbf{F}(\mathbf{e}^*, t), \tag{3.103}$$

where

$$\begin{aligned} \mathbf{F}(\mathbf{e}^*, t) = \mathbf{k} - \{ [\mathbf{u}^{NS0} + \varepsilon(\mathbf{u}^E_{(1)} + \bar{\mathbf{u}}^P_{(1)} + \boldsymbol{\sigma})] \cdot \nabla \mathcal{N}^* \mathbf{e}^* + \mathcal{N}^* \mathbf{e}^* \cdot \nabla [\mathbf{u}^{NS0} + \\ + \varepsilon(\mathbf{u}^E_{(1)} + \bar{\mathbf{u}}^P_{(1)} + \boldsymbol{\sigma})] + \varepsilon \mathcal{N}^* \mathbf{e}^* \cdot \nabla \mathcal{N}^* \mathbf{e}^* \}. \end{aligned} \tag{3.104}$$

The forcing term $\mathbf{k}$ is given by

$$\mathbf{k} = \Xi_r + \boldsymbol{\Psi}. \tag{3.105}$$

In the above expression, Ξ_r is the regular part of the source term appearing in the equation of the remainder $\mathbf{e}$, see the system (3.22). The expression for Ξ_r is reported in (B.5). The term $\boldsymbol{\Psi}$ derives from the introduction of $\mathbf{h}$ and $\boldsymbol{\sigma}$ in the decomposition (3.71); the explicit expression of $\boldsymbol{\Psi}$ is given in (B.7). The following proposition states that the forcing term is bounded.

Proposition 3.7.8. *There exists a constant R_0, independent of ε, such that*

$$|\boldsymbol{k}|_{l,\bar{\rho},1,\bar{\theta},\bar{\beta},\bar{T}} \leq R_0 \tag{3.106}$$

The key of the proof is to rearrange $\mathbf{k}$ in a way such that terms that are $O(\varepsilon^{-1})$ multiply terms that are $O(\varepsilon)$, and terms that have singular derivative with respect to Y multiply terms that go to zero with Y or t. The details are reported in the appendix B.

Quasi contractiveness hypothesis

In this subsection we shall prove that right hand side of equation (3.103), explicitly defined in (3.104), satisfies the quasi-contractiveness hypothesis of the ACK Theorem. It is useful to distinguish between the nonlinear and the linear terms present in (3.104).

As for the nonlinear term, a special attention is needed for $(\mathcal{N}^*(\mathbf{e}^{*1}-\mathbf{e}^{*2}))_n \partial_Y(\mathcal{N}^*\mathbf{e}^{*2})'$: indeed, since $\gamma\mathbf{e}^{*2}\neq\mathbf{0}$, the $|\,|_{l,\rho',0,\theta'}$ norm of $\partial_Y(\mathcal{N}^*\mathbf{e}^{*2})'$ cannot be bounded by a constant, due to a term which behaves like $\partial_Y\tilde{E}_1\gamma\bar{P}^\infty E_2^d\mathbf{e}^{2*}$. However, for the properties of $\tilde{E}_1$, $Y\partial_Y\tilde{E}_1\gamma\bar{P}^\infty E_2^d\mathbf{e}^{2*}$ behaves like $\tilde{E}_1\gamma\bar{P}^\infty E_2^d\mathbf{e}^{2*}$, while $(\mathcal{N}^*(\mathbf{e}^{*1}-\mathbf{e}^{*2}))_n/Y$ can be estimated in terms of $\partial_Y(\mathcal{N}^*(\mathbf{e}^{*1}-\mathbf{e}^{*2}))_n$. Therefore, multiplying and dividing by Y, we essentially move the normal derivative on the term with a more favourable estimate.
The other nonlinear terms are easily estimated using Propositions 3.7.4, 3.7.5 and 3.7.6, so we obtain the following result.

Proposition 3.7.9. *Suppose $\boldsymbol{e}^{*i}\in S^{l,\rho',1,\theta'}$ for $i=1,2$. Then the following estimate holds:*

$$\left|\varepsilon\mathcal{N}^*\boldsymbol{e}^{*1}\cdot\nabla\mathcal{N}^*\boldsymbol{e}^{*1}-\varepsilon\mathcal{N}^*\boldsymbol{e}^{*2}\cdot\nabla\mathcal{N}^*\boldsymbol{e}^{*2}\right|_{l,\rho',1,\theta'}\leq C\int_0^t\frac{|\boldsymbol{e}^{*1}-\boldsymbol{e}^{*2}|_{l,\rho',1,\theta'}}{(t-s)^{3/4}}ds. \tag{3.107}$$

For the linear terms, since $\gamma\mathbf{e}^{*1}=\gamma\mathbf{e}^{*2}$, we can apply the results of proposition 3.7.5 regarding the case with zero trace. The linear terms are problematic essentially for the presence of u^S and its Y-derivatives, as well for the presence of the $O(\varepsilon^{-1})$ term $\partial_y\tilde{u}^P$. In Appendix E we show how it is possible to estimate these terms.

Proposition 3.7.8, the estimate of the nonlinear term (3.107), and the estimates on the linear terms given in Appendix E allow us to use the ACK theorem in the functional setting $S^{l,\bar{\rho},1,\bar{\theta}}_{\bar{\beta},T}$. This leads to the following result:

Proposition 3.7.10. *Assume that $\boldsymbol{u}_0\in H^{l,\rho,\theta}$, with $\gamma_n\boldsymbol{u}_0=0$; then $\boldsymbol{e}^*\in S^{l,\bar{\rho},1,\bar{\theta}}_{\bar{\beta},T}$ for some $T>0$, $\bar{\rho}<\rho$, $\bar{\theta}<\theta$, $\bar{\beta}>\beta$, with all those parameters independent of the viscosity.*

By proposition 3.7.7, we therefore have $\mathcal{N}^*\mathbf{e}^*\in L^{l,\bar{\rho},\bar{\theta}}_{\bar{\beta},T}$; combining with propositions 3.7.2 and 3.7.3, we have that the overall error $\mathbf{e}$ is in $L^{l,\bar{\rho},\bar{\theta}}_{\bar{\beta},T}$, and this concludes the proof of theorem 3.4.1.

3.7.5 More general initial conditions

The estimates performed for the forcing term in the equation of $\mathbf{e}^*$ heavily relied on the assumption that the initial condition are purely eulerian, which means that $\mathbf{e}(t=0)=\mathbf{u}^E_{(1)}(t=0)=\bar{\mathbf{u}}^P_{(1)}(t=0)=\bar{\mathbf{u}}^P(t=0)=\mathbf{0}$. The zero viscosity limit holds also for more general initial conditions like

$$\mathbf{u}_0=\mathbf{u}^E_0+\bar{\mathbf{u}}^P_0+\varepsilon\left(\mathbf{u}^E_{(1),0}+\bar{\mathbf{u}}^P_{(1),0}+\mathbf{e}_0\right) \tag{3.108}$$

with

$$\gamma v^E_{(1),0}=-\gamma\bar{v}^P_0,\ \gamma u^P_{(1),0}=-\gamma u^E_{(1),0},\ \gamma\mathbf{e}_0=(0,-\gamma\bar{v}^P_{(1),0}); \tag{3.109}$$

this kind of initial data allows a zero order incompatibility with no-slip boundary condition, since we are not assuming that $\gamma\tilde{u}^P_0=-\gamma u^E_0$. The most challenging variation needed is to prove that the forcing term is still in $S^{l,\bar{\rho},1,\bar{\theta}}_{\bar{\beta},\bar{T}}$ and $O(1)$: we need to regroup some terms carefully in order to show the derivative $\partial_Y \mathbf{k}$ has the desired regularity. This is done in appendix B .

Chapter 4

Conclusions and future improvements

The convergence of the NS solutions to the Euler flow, for Sobolev-regular initial data and under the no-slip boundary condition, is still an open problem: in such a functional setting, we need to linearize at least the equation of the tangential part of the flow, so that strong interactions between the inertial terms and the diffusive effects near the boundary are avoided. Although they are both singular perturbations problems, the case with Navier-type boundary conditions is much simpler than the no-slip one: the boundary layer produced is weaker, and, for homogeneous Dirichlet conditions imposed on the vorticity, reflects a persistence property of the Euler flow.
For holomorphic initial data, we proved the existence of solutions of the Navier-Stokes equations in two and three dimensions in the half space when the initial data are not compatible with the boundary data, for a small time which is independent of the viscosity. When the viscosity approaches zero, the Navier-Stokes solution approaches the Euler solution away from the boundary and the Prandtl solution inside a boundary layer whose thickness is proportional to the square root of the viscosity: the fact that the incompatibility did not change the rate of convergence is not trivial, since this is not the case for Navier-slip boundary conditions. We used analytic initial data for two reasons: the first is that, in order to prove well posedness of the Prandtl equation without a monotonicity assumption, the use of data analytic in at least the tangential variable is "almost necessary"; the second reason is that we managed to obtain estimates independent of the viscosity by exploiting the restriction of the strip of analyticity in the tangential variable. Furthermore, while Prandtl's equation requires only tangential analyticity, the Euler equations are not well posed in a tangential-only analytic setting, so we imposed a similar regularity also for the normal variable. The incompatibility between the initial data and the no-slip condition cause the presence of singular terms which are $O(1)$ near the boundary and $O(\varepsilon)$ away from the boundary: the fact that the singularity is propagated in all the half space immediately through the term $\varepsilon\sqrt{t}w_b^S$ is coherent with the parabolic nature of the Navier-Stokes equations, which leads to an infinite speed of propagation of the singularity. Of course, when the viscosity goes to zero, the parabolic nature of the equations is lost, and indeed we have that $\varepsilon\sqrt{t}w_b^S$ goes to zero, leaving a singularity confined at the boundary.
Our proof relies on the solution, found by Ukai, of the Stokes problem in the half space: it would be interesting to extend our proof to different geometries. Furthermore, analyticity is not needed in the whole space for the inviscid limit to hold, so a similar asymptotic expansion, in the incompatible case, could be performed for initial data with analytic regularity only near the boundary. Of course, this would require some severe changes in the proof: in particular, an energy method would be

necessary, in order to avoid the loss of one derivative in the Sobolev region.

Appendix A

The estimates of $\gamma\partial_y \mathbf{u}_L$

In order to estimate $||\gamma\partial_y u^L||_{L^2_x}$, we decompose $\mathbf{u}^L$ as $\mathbf{u}^L = \mathbf{u}^F + \mathbf{u}^B + \mathbf{u}^I$, where $\mathbf{u}^F$ solves the system

$$\begin{aligned} \partial_t \mathbf{u}^F - \nu\Delta\mathbf{u}^F + \nabla p^F &= \mathbf{f} - \mathbf{u}^L \cdot \nabla \mathbf{u}^L, \\ \nabla \cdot \mathbf{u}^F &= \mathbf{0}, \\ \gamma\partial_y u^F(y=0) &= 0, \\ \gamma v^F &= 0, \\ \mathbf{u}^F(t=0) &= \mathbf{0}, \end{aligned} \tag{A.1}$$

$\mathbf{u}^I$ solves the system

$$\begin{aligned} \partial_t \mathbf{u}^I - \nu\Delta\mathbf{u}^I + \nabla p^I &= \mathbf{0}, \\ \nabla \cdot \mathbf{u}^I &= \mathbf{0}, \\ \gamma\partial_y u^I &= 0, \\ \gamma v^I &= 0, \\ \mathbf{u}^I(t=0) &= \mathbf{u}_0, \end{aligned} \tag{A.2}$$

and $\mathbf{u}^B$ solves the system

$$\begin{aligned} \partial_t \mathbf{u}^B - \nu\Delta\mathbf{u}^B + \nabla p^B &= \mathbf{0}, \\ \nabla \cdot \mathbf{u}^B &= \mathbf{0}, \\ \gamma u^B &= -\gamma u^F - \gamma u^I, \\ \gamma v^B &= 0, \\ \mathbf{u}^B(t=0) &= \mathbf{0}. \end{aligned} \tag{A.3}$$

We have then

$$\mathbf{u}^F = E_2^{pd} P\mathbf{f} - E_2^{pd} P(\mathbf{u} \cdot \nabla \mathbf{u}), \tag{A.4}$$

where P is the Leray projector on the half space and the operator E_2^{pd}, applied to a generic vector $\mathbf{h}$, has tangential component given by

$$\begin{aligned} (E_2^{pd}\mathbf{h})'(x,y,t) = E_2^p h'(x,y,t) = \int_0^t ds \int_{\mathbb{R}^{d-1}} dx' \frac{e^{-\frac{|x-x'|^2}{4\nu(t-s)}}}{(4\pi\nu(t-s))^{(d-1)/2}} \\ \int_0^{+\infty} dy' h'(s,x',y') \left(\frac{e^{-\frac{|y-y'|^2}{4\nu(t-s)}}}{\sqrt{4\pi\nu(t-s)}} + \frac{e^{-\frac{|y+y'|^2}{4\nu(t-s)}}}{\sqrt{4\pi\nu(t-s)}} \right) \end{aligned} \tag{A.5}$$

and normal component given by

$$(E_2^{pd}\mathbf{h})_n(x,y,t) = E_2^d h_n(\mathbf{x}',y,t) = \int_0^t ds \int_{\mathbb{R}^{d-1}} dx' \frac{e^{-\frac{|x-x'|^2}{4\nu(t-s)}}}{(4\pi\nu(t-s))^{(d-1)/2}} \int_0^{+\infty} dy' h_n(s,x',y') \left(\frac{e^{-\frac{|y-y'|^2}{4\nu(t-s)}}}{\sqrt{4\pi\nu(t-s)}} - \frac{e^{-\frac{|y+y'|^2}{4\nu(t-s)}}}{\sqrt{4\pi\nu(t-s)}} \right). \tag{A.6}$$

Applying Leray's projector to system (A.2), we see that $\mathbf{u}^I$ satisfies the same system without the pressure: thus, we have $\nabla p^I = 0$ and $\mathbf{u}^I$ is given by $\mathbf{E}_0^{pd}\mathbf{u}_0$, whose tangential component is given by

$$(E_0^p d\mathbf{u}_0)'(x,y,t) = E_0^p u_0(x,y,t) = \int_{\mathbb{R}^{d-1}} dx' \frac{e^{-\frac{|x-x'|^2}{4\nu(t)}}}{(4\pi\nu t)^{(d-1)/2}} \int_0^{+\infty} dy' u_0(x',y') \left(\frac{e^{-\frac{|y-y'|^2}{4\nu t}}}{\sqrt{4\pi\nu t}} + \frac{e^{-\frac{|y+y'|^2}{4\nu t}}}{\sqrt{4\pi\nu t}} \right), \tag{A.7}$$

while the normal component is given by

$$(E_0^p d\mathbf{u}_0)_n(x,y,t) = (E_0^d v_0)(x,y,t) = \int_{\mathbb{R}^{d-1}} dx' \frac{e^{-\frac{|x-x'|^2}{4\nu(t)}}}{(4\pi\nu t)^{(d-1)/2}} \int_0^{+\infty} dy\, v_0(x',y') \left(\frac{e^{-\frac{|y-y'|^2}{4\nu t}}}{\sqrt{4\pi\nu t}} - \frac{e^{-\frac{|y+y'|^2}{4\nu t}}}{\sqrt{4\pi\nu t}} \right). \tag{A.8}$$

Finally, $\mathbf{u}^B$ has tangential component given by (Ukai, 1987)

$$u^B = -E_1(\gamma u^F + \gamma u^I) - UN'E_1N' \cdot (\gamma u^F + \gamma u^I) \tag{A.9}$$

and normal component given by

$$v^B = UE_1N' \cdot (\gamma u^F + \gamma u^I) \tag{A.10}$$

where

$$(\hat{U h}) = |\xi'| \int_0^y e^{-|\xi'|(y-y')} \hat{h}(\xi',y')dy', \tag{A.11}$$

$$N' = \frac{i\xi'}{|\xi'|} \tag{A.12}$$

and E_1 is the operator such that

$$\begin{aligned} (\partial_t - \nu\Delta)E_1 h &= 0, \\ \gamma E_1 h &= h, \\ E_1 h(t=0) &= 0. \end{aligned} \tag{A.13}$$

The explicit expression of E_1 is

$$E_1 h(x,y,t) = \int_0^t ds \frac{y e^{-\frac{y^2}{4\nu(t-s)}}}{2\sqrt{\pi\nu}((t-s))^{3/2}} \int_{\mathbb{R}^{d-1}} \frac{e^{-\frac{|x-x'|^2}{4\nu(t-s)}}}{(4\pi\nu(t-s))^{(d-1)/2}} h(x',s)dx'. \quad \text{(A.14)}$$

When $h(t=0)=0$, we have

$$\partial_y E_1 h = \int_0^t ds \frac{e^{-\frac{y^2}{4\nu(t-s)}}}{\sqrt{\pi\nu(t-s)}} \int_{\mathbb{R}^{d-1}} dx'' \frac{(\nu\partial_{xx}h - \partial_t h) e^{-\frac{|x-x'|^2}{4\nu(t-s)}}}{(4\pi\nu(t-s))^{(d-1)/2}} = D_1(\nu\partial_{xx}h - \partial_t h). \quad \text{(A.15)}$$

With this decomposition, we have $\gamma\partial_y u^L = \gamma\partial_y u^B$, where

$$\begin{aligned} \gamma\partial_y u^B &= -\gamma\partial_y E_1(\gamma u^F + \gamma u^I) - \partial_x \gamma E_1 N' \cdot (\gamma u^F + \gamma u^I) = \\ &= -\partial_x N' \cdot \gamma u^I - \gamma\partial_y E_1 \gamma u^I - \partial_x N' \cdot \gamma u^F - \gamma\partial_y E_1 \gamma u^F = A_1 + A_2 + A_3 + A_4. \end{aligned} \quad \text{(A.16)}$$

For A_1, we have

$$||\partial_x N' \cdot \gamma u^I||_{L^2_x} \leq C||\partial_x u_0||_{L^\infty_Y L^2_x} \leq C||u_0||_{W^{2,2}}, \quad \text{(A.17)}$$

while, for A_2,

$$\nu\partial_{xx}u^I - \partial_t u^I = -\nu\partial_{yy}u^I = -\nu E_0^p \partial_{yy} u^0 + \nu \int_{\mathbb{R}^{d-1}} dx' \frac{e^{-\frac{|\mathbf{x}-\mathbf{x}'|^2}{4\nu(t)}}}{(4\pi\nu t)^{(d-1)/2}} \frac{e^{-\frac{y^2}{4\nu t}}}{\sqrt{\pi\nu t}} \gamma\partial_y u_0, \quad \text{(A.18)}$$

so

$$||\gamma D_1(\nu\partial_{xx}u^I - \partial_t u^I)||_{L^2_x} \leq C\left[\nu^{1/2}||\partial_{yy}u_0||_{L^2} + ||\gamma\partial_y u_0||_{L^2_x}\right] \leq C||u_0||_{W^{2,2}}. \quad \text{(A.19)}$$

When we evaluate $A_3 = \partial_x N' \cdot \gamma u^F$, for the part depending on $\mathbf{u}^E \cdot \nabla \mathbf{u}^L$, we let ∂_x hit the kernel, and we use Young's convolution inequality in the x and Holder inequality in the y, giving the $||\cdot||_{L^1_x L^2_y}$ norm to the kernel; for the part depending on $\mathbf{f}$, we can let the tangential derivative hit $\mathbf{f}$. Therefore, we have that

$$||\partial_x N' \cdot \gamma u^F||_{L^2_x} \leq C \int_0^t ds \frac{||\mathbf{u}^E||_{L^\infty_{Txy}} ||\partial_x \mathbf{u}^L||_{L^2}(s) + ||\frac{v^E}{\arctan(y)}||_{L^\infty_{Txy}} ||\arctan(y)\partial_y u^L||_{L^2}(s)}{(\nu(t-s))^{3/4}} + C||\mathbf{f}||_{L^\infty_{Ty} W^{1,2}_x}. \quad \text{(A.20)}$$

We can obtain an estimate in terms of $\mathbf{u}^L - \mathbf{u}^E$ by writing $\mathbf{u}^E \cdot \nabla \mathbf{u}^L$ as $\mathbf{u}^E \cdot \nabla \mathbf{u}^E + \mathbf{u}^E \cdot \nabla\left(\mathbf{u}^L - \mathbf{u}^E\right)$, so that we can give the $L^2_x L^\infty_y$ norm to $\mathbf{u}^E \cdot \nabla \mathbf{u}^E$ and the $L^2_x L^1_y$ norm to the kernel

$$||\nabla'\mathbf{N}'\cdot\gamma u^F||_{L^2_x} \leq \frac{C}{\nu^{1/2}}||\mathbf{u}^E||^2_{L^\infty_T W^{1,\infty}} + C||\mathbf{f}||_{L^\infty_{Ty}W^{1,2}_x} +$$
$$C\int_0^t ds \frac{||\mathbf{u}^E||_{L^\infty_{Txy}}||\partial_x\left(\mathbf{u}^L-\mathbf{u}^E\right)||_{L^2}(s) + ||\frac{v^E}{\arctan(y)}||_{L^\infty_{Txy}}||\arctan(y)\partial_y\left(u^L-u^E\right)||_{L^2}(s)}{(\nu(t-s))^{3/4}}. \tag{A.21}$$

The last term that we have to evaluate, A_4, is the most challenging one: for a generic g, we have

$$\nu\gamma\partial_{xx}E^p_2 g - \gamma\partial_t E^p_2 g = -\nu\gamma\partial_{yy}E^p_2 g - \gamma g. \tag{A.22}$$

The problem is that, in the estimate of $\gamma\partial_{yy}E^p_2 g$, we would like to give the kernel the L^2_y norm and both the derivatives with respect to y; this would cause the appearance of $1/(t-s)^{5/4}$ inside the time integral, which of course is not integrable. However, since $\partial_{yy}E_2$ appears inside D_1, we can use the following strategy: we give one derivative to the kernel and one to g, we switch the order of integration in time of the operators D_1 and E_2, and then we move the derivative from g to the kernel. The change of the order of integration puts the gaussian of the kernel of E_2 under a time integral, which enhances the regularity in time and allows to take the second derivative of the kernel. For a generic g, we have that

$$\gamma D_1\nu\gamma\partial_{yy}E^p_2 g =$$
$$= \nu\int_0^t \frac{ds}{\sqrt{\pi\nu(t-s)}}\int_{\mathbb{R}^{d-1}} dx' \frac{e^{-\frac{|x-x'|^2}{4\nu(t-s)}}}{(4\pi\nu(t-s))^{(d-1)/2}}\int_0^s ds' \int_{\mathbb{R}^{d-1}} dx'' \frac{e^{-\frac{|x'-x''|^2}{4\nu(s-s')}}}{(4\pi\nu(s-s'))^{(d-1)/2}}$$
$$\int_0^{+\infty} dy' \frac{y' e^{-\frac{y'^2}{4\nu(s-s')}}}{\sqrt{4\pi}(\nu(s-s'))^{3/2}}\partial_{y'}g(x'',y',s') = \nu\int_0^t ds' \int_{\mathbb{R}^{d-1}} \frac{e^{-\frac{|x'-x'''|^2}{4\nu(t-s')}}}{(4\pi\nu(t-s'))^{(d-1)/2}}$$
$$\int_0^{+\infty} dy'\partial_{y'}g\int_{s'}^t ds \frac{y' e^{-\frac{y'^2}{4\nu(s-s')}}}{\sqrt{4\pi}(\nu(s-s'))^{3/2}\sqrt{\pi\nu(t-s)}}. \tag{A.23}$$

We changed the order of integration for the time and we used the well-known fact that the convolution between two gaussians is a gaussian: the resulting gaussian in the tangential variable x depends on time through $(s-s')+(t-s) = t-s'$, so it is independent of s. The gaussian in the y' variable is now under the regularizing effect of the time integration, so now it can bear an additional derivatives with respect to y'. Call $\sigma = y'/\sqrt{\nu(s-s')}$; the last integral in (A.23) in given by

$$\int_{\frac{y'}{\sqrt{\nu(t-s')}}}^{+\infty} \frac{e^{-\sigma^2/4}}{\pi\nu^{3/2}\sqrt{t-s'-\frac{y'^2}{\nu\sigma^2}}}d\sigma, \tag{A.24}$$

so its value for $y'=0$, (A.24) is given by

$$\frac{1}{\pi\nu^{3/2}\sqrt{t-s'}}\int_0^{+\infty} e^{-\frac{\sigma^2}{4}}d\sigma = \frac{1}{\nu^{3/2}\sqrt{\pi(t-s')}}, \tag{A.25}$$

which is the boundary term which appears in the integration by parts. In order to take the derivative with respect to y of (A.24), we use the following manipulation: from

$$\frac{1}{\sqrt{(t-s)}(s-s')^{3/2}} = \frac{1}{\sqrt{t-s}(t-s')\sqrt{s-s'}} + \frac{\sqrt{t-s}}{(t-s')(s-s')^{3/2}} \tag{A.26}$$

we have that

$$\int_{s'}^{t} ds \frac{y' e^{-\frac{y'^2}{4\nu(s-s')}}}{\sqrt{4\pi}(\nu(s-s'))^{3/2}\sqrt{\pi\nu(t-s)}} = \frac{1}{2\pi\nu^2(t-s')}\int_{s'}^{t} ds \frac{y' e^{-\frac{y'^2}{4\nu(s-s')}}}{\sqrt{(s-s')(t-s)}} + \frac{1}{\pi\nu^{3/2}(t-s')}\int_{\frac{y'}{\sqrt{\nu(t-s')}}}^{+\infty} d\sigma e^{-\frac{\sigma^2}{4}}\sqrt{t-s'-\frac{y'^2}{\nu\sigma^2}}. \tag{A.27}$$

In the right hand side of (A.27), every term has non singular derivatives with respect to y': the derivative is given by

$$\frac{1}{2\pi\nu^2(t-s')}\int_{s'}^{t} ds \frac{e^{-\frac{y'^2}{4\nu(s-s')}}}{\sqrt{(s-s')(t-s)}}\left(1-\frac{y'^2}{2\nu(s-s')}\right) + -\frac{1}{2\pi\nu^2(t-s')}\int_{s'}^{t} ds \frac{e^{-\frac{y'^2}{4\nu(s-s')}}}{\sqrt{(s-s')(t-s)}} = -\frac{1}{2\pi\nu^3(t-s')}\int_{s'}^{t} ds \frac{e^{-\frac{y'^2}{4\nu(s-s')}}}{\sqrt{(s-s')(t-s)}}\frac{y'^2}{2(s-s')}. \tag{A.28}$$

Therefore, we have

$$\int_0^{+\infty} dy' \partial_{y'} g \int_{s'}^{t} ds \frac{y' e^{-\frac{y'^2}{4\nu(s-s')}}}{\sqrt{4\pi}(\nu(s-s'))^{3/2}\sqrt{\pi\nu(t-s)}} = -\frac{\gamma g}{\nu\sqrt{\pi\nu(t-s')}} + + \int_0^{+\infty} dy' \frac{g(y',s')}{2\pi\nu^2(t-s')}\int_{s'}^{t} ds \frac{e^{-\frac{y'^2}{4\nu(s-s')}}}{\sqrt{(s-s')(t-s)}}\frac{y'^2}{2\nu(s-s')}, \tag{A.29}$$

which means that

$$\gamma D_1 \nu\gamma\partial_{yy} E_2^p g = -D_1\gamma g + \int_0^t ds' \int_{\mathbb{R}^{d-1}} \frac{e^{-\frac{|x-x''|^2}{4\nu(t-s')}} dx''}{(4\pi\nu(t-s'))^{(d-1)/2}} \int_0^{+\infty} dy' \frac{g(y',s')}{2\pi\nu(t-s')}\int_{s'}^{t} ds \frac{e^{-\frac{y'^2}{4\nu(s-s')}}}{\sqrt{(s-s')(t-s)}}\frac{y'^2}{2\nu(s-s')}, \tag{A.30}$$

so

$$\gamma\partial_y E_1 \gamma E_2 g = \int_0^t ds' \int_{\mathbb{R}^{d-1}} \frac{e^{-\frac{|x-x''|^2}{4\nu(t-s')}} dx''}{(4\pi\nu(t-s'))^{(d-1)/2}} \int_0^{+\infty} dy' \frac{g(y',s')}{2\pi\nu(t-s')}\int_{s'}^{t} ds \frac{e^{-\frac{y'^2}{4\nu(s-s')}}}{\sqrt{(s-s')(t-s)}}\frac{y'^2}{2\nu(s-s')}. \tag{A.31}$$

When $g = P'(\mathbf{u}^E \cdot \nabla \mathbf{u}^L)$, we have, giving the L^2_y norm to the kernel of E^p_2,

$$||\gamma\partial_y E_1 \gamma E_2 P'(\mathbf{u}^E \cdot \nabla \mathbf{u}^L)||_{L^2_x} \leq \frac{C}{\nu^{3/4}} \int_0^t ds' \frac{||(\mathbf{u}^E \cdot \nabla \mathbf{u}^L)||_{L^2}}{(t-s')^{3/4}} \leq \frac{C}{\nu^{3/4}} \int_0^t ds' \frac{||\mathbf{u}^E||_{W^{1,\infty}} \left(||\partial_x \mathbf{u}^L||_{L^2}(s') + ||\arctan(y)\partial_y u^L||^2_L(s')\right)}{(t-s')^{3/4}}, \tag{A.32}$$

while, for $g = P'(\mathbf{u}^E \cdot \nabla \mathbf{u}^L)$ we can give the L^1_y norm to the kernel, so

$$||\gamma\partial_y E_1 \gamma E_2 P' \mathbf{f}||_{L^2_x} \leq \frac{C}{\nu^{1/2}} ||\mathbf{f}||_{L^\infty_T W^{1,2}}. \tag{A.33}$$

With the same argument used for equation (A.21), an estimate in terms of $\mathbf{u}^L - \mathbf{u}^E$ is given by

$$||\gamma\partial_y E_1 \gamma E_2 P'(\mathbf{u}^E \cdot \nabla \mathbf{u}^L)||_{L^2_x} \leq \frac{C}{\nu^{1/2}} t^{1/2} ||\mathbf{u}^E||^2_{L^\infty_T W^{1,\infty}} + \frac{C}{\nu^{3/4}} \int_0^t ds' \frac{||\mathbf{u}^E||_{W^{1,\infty}} \left(||\partial_x \left(\mathbf{u}^L - \mathbf{u}^E\right)||_{L^2}(s') + ||\arctan(y)\partial_y \left(u^L - u^E\right)||^2_L(s')\right)}{(t-s')^{3/4}}. \tag{A.34}$$

Remark 4. *The cancellation of $D_1\gamma g$ in the expression of $\gamma\partial_y E_1\gamma E_2 g$ is not important for the estimates to hold: indeed, for $g = P'\boldsymbol{f}$, we can give one derivative with respect to y to $P'\boldsymbol{f}$, as well as the L^∞_y norm, with $||g||_{L^\infty_y L^2_x} \leq ||g||^{1/2}_{L^2_{xy}} ||\partial_y g||^{1/2}_{L^2_{xy}}$, so*

$$||D_1 \nu \partial_{yy} E^p_2 P' \boldsymbol{f}||_{L^2_x} \leq C||\boldsymbol{f}||_{L^\infty_T W^{2,2}}, \tag{A.35}$$

while

$$||D_1 \gamma P' \boldsymbol{f}||_{L^2_x} \leq \frac{C}{\nu^{1/2}} ||\boldsymbol{f}||_{L^\infty_T W^{1,2}}. \tag{A.36}$$

For $g = P'(\boldsymbol{u}^E \cdot \nabla \boldsymbol{u}^L)$, instead, we have

$$\gamma g = \gamma(\boldsymbol{u}^E \cdot \nabla u^L) - \gamma\partial_x q = \gamma \partial_x q, \tag{A.37}$$

where q solves

$$\Delta q = \nabla \cdot (\boldsymbol{u}^E \cdot \nabla \boldsymbol{u}^L) = (\nabla \boldsymbol{u}^E)^T : \nabla \boldsymbol{u}^L, \quad \gamma \partial_y q = 0, \tag{A.38}$$

with

$$A : B = \sum_{i,j} A_{i,j} B_{i,j}, \tag{A.39}$$

so

$$||\gamma\partial_x q||_{L^2_x} \leq ||\partial_x q||^{1/2}_{L^2} ||\partial_{xy} q||^{1/2}_{L^2} \leq C||\boldsymbol{u}^E||_{W^{2,\infty}} \left(||\partial_x \boldsymbol{u}^L||_{L^2} + ||\arctan(y)\partial_y u^L||\right). \tag{A.40}$$

Remark 5. *If the initial data satisfy only the non penetration boundary conditions $\gamma v_0 = 0$, then equation (A.15) no longer holds, and an additional term appears in the expression of*

$\partial_y \boldsymbol{u}^B$*, given by*

$$-\frac{e^{-\frac{y^2}{\nu t}}}{\sqrt{\pi\nu t}}\int_{\mathbb{R}^{d-1}} dx' \frac{e^{-\frac{|x-x'|^2}{4\nu t}}}{(4\pi\nu t)^{(d-1)/2}}\gamma u_0(x'). \tag{A.41}$$

Therefore, the estimates given in proposition 2.2.1 must be modified adding, in the right hand side, $C\frac{||\gamma u_0||_{L^2_x}}{\sqrt{\nu t}}$. Since the singularity in time is integrable, the inviscid limit still holds in the $|||\cdot|||$ norm.

Appendix B

The source terms

B.1 The source term for the error equation 3.22

The source term Ξ has the following expression:

$$\begin{aligned}\Xi = \mathbf{f} + \left(g\partial_y \tilde{u}^P, 0\right) + \\ - \left[\mathbf{u}^{NS}_{(0)} \cdot \nabla \bar{\mathbf{u}}^P_{(1)} + \bar{\mathbf{u}}^P_{(1)} \cdot \nabla \mathbf{u}^{NS}_{(0)} + \left(\bar{\mathbf{u}}^P + \varepsilon \mathbf{u}^{NS}_{(1)}\right) \cdot \nabla \mathbf{u}^E_{(1)} + \mathbf{u}^E_{(1)} \cdot \nabla \left(\bar{\mathbf{u}}^P + \varepsilon \bar{\mathbf{u}}^P_{(1)}\right) + \right. \\ \left. \varepsilon \bar{\mathbf{u}}^P_{(1)} \cdot \nabla \bar{\mathbf{u}}^P_{(1)}\right] \\ + \varepsilon^2 \left[\Delta \mathbf{u}^E_{(1)} + \left(\partial_{xx} \bar{u}^P_{(1)}, 0\right)\right] - \left(0, \left(\partial_t - \varepsilon^2 \Delta\right) \bar{v}^P_{(1)}\right)\end{aligned} \tag{B.1}$$

where $\mathbf{f}$ is given by:

$$\mathbf{f} = \mathbf{f}_r + (\varepsilon \partial_x^2 u^S, 0), \tag{B.2}$$

$$\begin{aligned} f_r^1 &= -\varepsilon^{-1}\{\tilde{u}^P(\partial_x u^E - \partial_x u^E|_{y=0}) + \partial_x \tilde{u}^P (u^E - u^E|_{y=0}) + \\ &\quad \partial_y \tilde{u}^P (v^E + y \partial_x u^E|_{y=0})\} - \bar{v}^P \partial_y u^E + \varepsilon \Delta u^E + \varepsilon \partial_x^2 \tilde{u}^R, \\ f_r^2 &= -[\partial_t \bar{v}^R + u^{NS0} \partial_x \bar{v}^P + v^{NS0} \partial_y \bar{v}^P + \bar{v}^P \partial_y v^E] - \varepsilon^{-1} \tilde{u}^P \partial_x v^E + \\ &\quad + \varepsilon \Delta v^E + \varepsilon^2 \Delta v^R + \varepsilon^2 \partial_x^2 \bar{v}^S. \end{aligned} \tag{B.3}$$

Writing down the expression of f^2, we used the fact that $\partial_t \bar{v}^S - \partial_Y^2 \bar{v}^S = 0$ to get rid of those singular terms. We can write $g = g^R + g^S$, with

$$g^S = -\partial_x u_0(x, y=0) \int_0^{+\infty} dY \int_Y^{+\infty} \frac{e^{\frac{-Y'^2}{4t}}}{\sqrt{t\pi}} dY' = C_1 \sqrt{t} \partial_x u_0(x, y=0). \tag{B.4}$$

In equation (B.2), we isolated the most singular term of $\mathbf{f}$; in a similar way, we write

$$\Xi_r = \Xi - \varepsilon(\partial_x^2 u^S, 0), \tag{B.5}$$

where Ξ is given in (B.1). Notice that $\mathbf{f} + \left(g\partial_y \tilde{u}^P, 0\right)$ is the forcing term in the equation of the remainder $\mathbf{R}$ in the decomposition

$$\mathbf{u}^{NS} = \mathbf{u}^E + \bar{\mathbf{u}}^P + \varepsilon \mathbf{R} \tag{B.6}$$

The term $\mathbf{f}$ is $O(1)$ with respect to ε, while $g\partial_y \tilde{u}^P$ is $O(\varepsilon^{-1})$: so an additional decomposition of the remainder $\mathbf{R}$ is needed to prove the validity of the inviscid limit.

B.2 Estimates for the forcing term of $\mathbf{e}^*$

The source term $\mathbf{k}$ in (3.105) is decomposed as $\mathbf{k} = \Xi_r + \mathbf{\Psi}$ where

$$\begin{aligned}\mathbf{\Psi} = -\Big(&(\mathbf{u}^{NS0} + \varepsilon(\mathbf{u}^E_{(1)} + \bar{\mathbf{u}}^P_{(1)})) \cdot \nabla(\mathbf{h}+\sigma) + (\mathbf{h}+\sigma)\cdot\nabla(\mathbf{u}^{NS0} + \varepsilon(\mathbf{u}^E_{(1)} + \bar{\mathbf{u}}^P_{(1)})) \\ &+(\mathbf{h}+\sigma)\cdot\nabla(\mathbf{h}+\sigma)\Big) + \varepsilon^2\partial_{xx}(\mathbf{h}+\sigma) - \varepsilon(0,(\partial_t - \partial_{YY})h_n) = \\ &-\{[(\mathbf{u}^{NS0} + \varepsilon(\mathbf{u}^E_{(1)} + \bar{\mathbf{u}}^P_{(1)} + \mathbf{h} + \sigma))\cdot\nabla(\bar{\mathbf{u}}^P_{(1)} + \mathbf{h}) + (\mathbf{u}^E_{(1)}\cdot\nabla\bar{\mathbf{u}}^P - (g\partial_y\tilde{u}^P,0)) + \\ &(\bar{\mathbf{u}}^P_{(1)} + \mathbf{h}+\sigma)\cdot\nabla\bar{\mathbf{u}}^P] + [(\mathbf{u}^E_{(1)} + \bar{\mathbf{u}}^P_{(1)} + \mathbf{h})\cdot\nabla\mathbf{u}^E + (\bar{u}^P + \varepsilon(\mathbf{u}^E_{(1)} + \bar{\mathbf{u}}^P_{(1)} + \mathbf{h}+\sigma))\cdot \\ &\nabla\mathbf{u}^E_{(1)} + (\mathbf{u}^{NS0} + \varepsilon(\mathbf{u}^E_{(1)} + \bar{\mathbf{u}}^P_{(1)} + \mathbf{h}+\sigma))\cdot\nabla\sigma]\} + \mathbf{f}_r - (0,(\partial_t - \varepsilon^2\Delta)\varepsilon(\bar{v}^P_{(1)} + h_n)) \\ &+\varepsilon^2[\Delta\mathbf{u}^E_{(1)} + \partial_{xx}(u^P_{(1)} + h',0) + \partial_{xx}\sigma].\end{aligned} \tag{B.7}$$

We want to prove that the forcing term $\mathbf{k}$ is in $S^{l,\bar{\rho},1,\bar{\theta}}_{\bar{\beta},\bar{T}}$ and that it is $O(1)$ with respect to ε. In our analysis we shall focus on the most challenging terms. We begin with:

$$[\mathbf{u}^{NS0} + \varepsilon(\mathbf{u}^E_{(1)} + \bar{\mathbf{u}}^P_{(1)} + \mathbf{h}+\sigma))]\cdot\nabla(\bar{\mathbf{u}}^P_{(1)} + \mathbf{h}). \tag{B.8}$$

This term is $O(1)$ because $\partial_y(\bar{\mathbf{u}}^P_{(1)} + \mathbf{h})$, which has an $O(1/\varepsilon)$ L^2_Y norm, multiplies a term with an $O(\varepsilon)$ L^∞_Y norm. When we take a partial derivative with respect to Y, some singular terms appear, $\partial_Y\tilde{u}^S\partial_x(\bar{\mathbf{u}}^P_{(1)} + \mathbf{h})$ and $[\mathbf{u}^{NS0}/\varepsilon + (\mathbf{u}^E_{(1)} + \bar{\mathbf{u}}^P_{(1)} + \mathbf{h} + \sigma)]_2\partial_{YY}u^S_1$: for the first term, since $v^E = \varepsilon Y(\partial_y v^E)(\lambda(y)y)$ for some $\lambda(y)\in[0,1]$, we have that $\bar{\mathbf{u}}^S_{(1)}$ goes to zero as $\sqrt{t}$ in the L^∞_Y norm, $\bar{\mathbf{u}}^R_{(1)}$ and $\mathbf{h}$ go to zero linearly with t, and this is enough to balance the singularity of $\partial_Y\tilde{u}^S$; for the second term, $v^E/\varepsilon + \bar{v}^S + (\mathbf{u}^E_{(1)} + \bar{\mathbf{u}}^P_{(1)} + \mathbf{h}+\sigma))_2$ goes to zero linearly with Y, and this allows to balance the singularity of $\partial_{YY}u^S_{(1)}$. So all the singular terms in $[\mathbf{u}^{NS0} + \varepsilon(w + w + \sigma)]\cdot\nabla(\bar{\mathbf{u}}^P_{(1)} + \mathbf{h})$ are balanced, and its norm is controlled through the usual arguments involving algebra properties.

We can now pass to estimate the terms:

$$\mathbf{u}^E_{(1)}\cdot\nabla\bar{\mathbf{u}}^P - (g\partial_y\tilde{u}^P,0) \tag{B.9}$$

The most problematic term is $(v^E_{(1)} - g)\partial_y\tilde{u}^P$, which is $O(1)$ since $v^E_{(1)} - g$ goes to zero linearly in $y = \varepsilon Y$; since $\partial_y u^S_{(1)}$ is multiplied by εY times a bounded function, the product has finite $|\cdot|_{l,\rho',0,\theta'}$ norm. When we take the partial derivative with respect to Y, $\partial_Y v^E_{(1)} = \varepsilon\partial_y v^E_{(1)}$, so $\partial_Y v^E_{(1)}\partial_y\tilde{u}^P$ is still $O(1)$, and since $\partial_Y v^E_{(1)}$ has a regular part which goes to zero linearly with t and a singular part which goes to zero with t like $\sqrt{t}$, this product has finite $|\cdot|_{l,\rho',0,\theta'}$ norm. Finally, $(v^E_{(1)} - g)\partial_{YY}\tilde{u}^P = (\partial_Y v^E_{(1)})(\lambda y)Y\partial_{YY}\tilde{u}^P$, with the regular part of $\partial_Y v^E_{(1)}$ going to zero linearly with t, while the singular part goes to zero like $\sqrt{t}$.

We now estimate the terms:

$$(\bar{\mathbf{u}}^P_{(1)} + \mathbf{h} + \sigma)\cdot\nabla\bar{\mathbf{u}}^P. \tag{B.10}$$

Both $(u^P_{(1)} + h' + \sigma_1)\partial_Y\partial_x\tilde{u}^P$ and $\left(\varepsilon\partial_Y\bar{v}^P_{(1)} + \varepsilon\partial_Y h_n + \partial_Y\sigma_2/\right)\partial_y\bar{\mathbf{u}}^P = -(\partial_x u^P_{(1)} + \partial_x h' +$

$\partial_x\sigma_1)\partial_Y\bar{\mathbf{u}}^P$ are products of terms whose L^∞ norm is $O(1)$ and goes to zero with t at least as $\sqrt{t}$ and a function whose L^2_Y behaves like $t^{-1/4}$; in $(\bar{v}^P + h_n + \sigma_2/\varepsilon)\partial_{YY}\bar{\mathbf{u}}^P$, the first factor has an $O(1)$ L^∞_Y norm which goes to zero linearly with t, and this balances the singularity of the second factor.

The term

$$\varepsilon^{-1}\tilde{u}^P(\partial_x u^E - \partial_x \gamma u^E) \tag{B.11}$$

is $O(1)$ because $\partial_x u^E - \partial_x\gamma u^E$ goes to zero linearly with $y = \varepsilon Y$, and for the same reason $\varepsilon^{-1}\partial_Y\tilde{u}^P(\partial_x u^E - \partial_x\gamma u^E)$ is $O(1)$ and with finite $|\cdot|_{l,\rho',0,\theta'}$ norm, while $\varepsilon^{-1}\tilde{u}^P\partial_Y\partial_x u^E = \tilde{u}^P\partial_y\partial_x u^E$ is of course $O(1)$ and bounded. The term $\varepsilon^{-1}\partial_x\tilde{u}^P(u^E - \gamma u^E)$ can be treated in a similar way.

Finally, the term

$$\frac{1}{\varepsilon}\partial_y\tilde{u}^P(v^E + y\partial_x\gamma u^E) = \frac{1}{\varepsilon^2}\partial_Y\tilde{u}^P(v^E + y\partial_x\gamma u^E) \tag{B.12}$$

is $O(1)$ because $(v^E + y\partial_x\gamma u^E)$ goes to zero quadratically in $y^2 = \varepsilon^2 Y^2$ due to the incompressibility condition, and for the same reason $\frac{1}{\varepsilon^2}\partial_{YY}\tilde{u}^P(v^E + y\partial_x\gamma u^E)$ is $O(1)$ and with finite $|\cdot|_{l,\rho',0,\theta'}$ norm, while $\frac{1}{\varepsilon}\partial_Y\tilde{u}^P(\partial_y v^E + \partial_x\gamma u^E)$ has $\partial_y v^E + \partial_x\gamma u^E$ which goes to zero linearly with εY

All the other terms in **k** are easier to estimate.

B.3 Forcing term k with general initial conditions

When the initial conditions are not purely eurelian, we arrange the terms so that their sum is zero at the boundary. This is done in a different way for the tangential and the normal components.

The sum of $\partial_Y u^S\partial_x w_1^R$, deriving from the derivative of $u^S\partial_x w_1^R$, and $\partial_Y u^S\partial_x u^R_{(1)}$, deriving from the derivative of $u^S\partial_x u^R_{(1)}$, has finite $|\cdot|_{l-1,\rho',0,\theta'}$ norm because $\partial_x w_1^R + \partial_x u^R_{(1)}$ goes to zero linearly in Y (as well in t, since $\gamma w^{RR}_{1,0} = -\gamma w^{RR}_{1,0}$).

The sum of $\partial_Y u^S\varepsilon\partial_x\bar{v}^R_{(1)}$, deriving from the derivative of $u^S\partial_x\bar{v}^R_{(1)}$, and $\partial_Y\tilde{u}^S\partial_x\sigma_2$, deriving from the derivative of $\tilde{u}^S\partial_x\sigma_2$, has finite $|\cdot|_{l-1,\rho',0,\theta'}$ norm because $\varepsilon\partial_x\bar{v}^R_{(1)} + \partial_x\sigma_2$ goes to zero linearly in Y (as well in t).

The sum of $w_1^R\partial_x\partial_Y u^S$, deriving from the derivative of $w_1^R\partial_x u^S$, and $u^R_{(1)}\partial_x\partial_Y u^S$, deriving from the derivative of $w_1^{RR}\partial_x u^S$, has finite $|\cdot|_{l-1,\rho',0,\theta'}$ norm because $w_1^R + u^R_{(1)}$ goes to zero linearly in Y (and also in t).

The sum of $\partial_y w_2^R\partial_Y u^S = -\partial_x w_1^R\partial_Y u^S$, deriving from the derivative of $(w_2^R - g^R)\partial_y u^S$, and $(\partial_Y\bar{v}^R_{(1)} + \varepsilon^{-1}\partial_Y\sigma_2)\partial_Y u^S = -(\partial_x u^R_{(1)} + \partial_x\sigma_1)\partial_Y u^S$, deriving from the derivative of $(\bar{v}^R_{(1)} + \varepsilon^{-1}\sigma_2)\partial_Y u^S$, has finite $|\cdot|_{l-1,\rho',0,\theta'}$ norm because $\partial_x w_1^R + \partial_x u^R_{(1)}$ goes to zero linearly in Y (and in t) and $\partial_x\sigma_1$ goes to zero linearly with t.

The sum of $-w_2^R\partial_{YY}\bar{v}^S$, deriving from the derivative of $-w_2^{RR}\partial_Y\bar{v}^S$, and $\varepsilon^{-1}v^{NS0}$ $\partial_{YY}\bar{v}^S$, deriving from the derivative of $\varepsilon v^{NS0}\partial_Y\bar{v}^S$, has finite $|\cdot|_{l-1,\rho',0,\theta'}$ norm: indeed, $v^{NS0} = v^E + \varepsilon\bar{v}^R + \varepsilon\bar{v}^S$, where v^E goes to zero linearly with $y = \varepsilon Y$, $\bar{v}^S$ goes to zero like $\sqrt{t}$ and $-w_2^R + \bar{v}^R$ goes to zero linearly in Y (and in t).

For all the other terms there is no need to change the argument used in the case $\mathbf{u}_0 = \mathbf{u}_0^E$.

Appendix C

Proof of Proposition 3.7.1

The estimate on h' is a direct consequence of the expression (3.73), and of the following estimates on F and on its Y-derivatives. We begin with F.

$$\sup_{Y\in\Sigma(\theta)} e^{\mu Y}|F| = \sup_{Y}\left|\frac{2}{\sqrt{\pi}}\int_0^t ds \int_0^{+\infty} e^{\mu Re(Y-Y')}(E_0^- - E_0^+)e^{\mu Re(Y')}\int_{\frac{Y'}{2\sqrt{t}}}^{+\infty} e^{-\sigma^2}d\sigma dY'\right| \leq$$

$$\sup_{Y}\left|\frac{2}{\sqrt{\pi}}\int_0^t ds \int_0^{+\infty} e^{\mu Re(Y-Y')}(E_0^- - E_0^+)\right| \leq ct \leq C$$

To pass from the first to the second line we have taken into account the argument of subsection 3.2.5 about the complex gaussian and the fact that

$$\int_{\frac{Y'}{2\sqrt{s}}}^{+\infty} e^{-\sigma^2}d\sigma \leq e^{-\frac{Y'^2}{8s}}\int_0^{+\infty} e^{-\frac{\sigma^2}{2}}d\sigma,$$

while, to get the last inequality, we have used that for $z \in \mathbb{R}$

$$e^{\mu z}e^{-c\frac{z^2}{t}} \leq e^{\mu z}e^{-c\frac{z^2}{2T}}e^{-c\frac{z^2}{2t}} \leq C(\mu, T)e^{-c\frac{z^2}{2t}}.$$

A similar argument shows that

$$\sup_{Y\in\Sigma(\theta)} e^{\mu Y}|\partial_Y F| \leq c\sqrt{t}.$$

To estimate $\partial_{YY}F$, we first compute ∂_Y. Integrating by parts, we have

$$\partial_Y F = -\int_0^t \frac{e^{-\frac{Y^2}{4(t-s)}}}{\sqrt{\pi(t-s)}}ds + \int_0^t ds \int_0^{+\infty}(E_0^- + E_0^+)\frac{e^{-\frac{Y'^2}{4s}}}{\sqrt{\pi s}}dY'. \tag{C.1}$$

We can now estimate $\partial_{YY}F$ as follows:

$$e^{\mu Re(Y)}|\partial_{YY}F| =$$

$$e^{\mu Re(Y)}\left|\int_0^t \frac{Ye^{-\frac{Y^2}{4(t-s)}}}{2\sqrt{\pi}(t-s)^{3/2}}ds + \int_0^t ds \int_0^{+\infty} \partial_Y(E_0^- + E_0^+)\frac{e^{-\frac{Y'^2}{4s}}}{\sqrt{\pi s}}dY'\right| \leq$$

$$e^{\mu Re(Y)} \int_{\frac{Y}{2\sqrt{t}}}^{+\infty} \frac{e^{-\sigma^2}}{\sqrt{\pi}} d\sigma + e^{\mu Re(Y-Y')} +$$

$$\int_0^t ds \int_0^{+\infty} e^{\mu Re(Y-Y')}\partial_Y(E_0^- + E_0^+)e^{\mu Re(Y')}\frac{e^{-\frac{Y'^2}{4s}}}{\sqrt{\pi s}}dY' \leq$$

$$C + C\int_0^t \frac{ds}{\sqrt{s(t-s)}} \leq C$$

In the first of the inequalities above, in the first integral we have simply used a standard change of variable. To pass from the third line to the fourth, in the first integral we have used the exponential decay of the integral while, in the second integral, the fact that the gaussian in Y' dominates the exponential $e^{\mu Re(Y')}$, that the gaussian in $Y - Y'$ dominates the exponential $e^{\mu Re(Y-Y')}$, and a standard change of variable.

The above estimate concludes the bound on h'. The estimate on the normal component h_n is a direct consequence of the expression (3.76), and of the above estimates on F. One could in fact say more:

$$\sup_{Y\in\Sigma(\theta)} |\partial_Y^j h_n|_{l-3-j,\rho} \leq C\varepsilon t^{\frac{3-j}{2}} |u_{00}|_{l-j,\rho}. \tag{C.2}$$

Appendix D

Proof of propositions 3.7.4, 3.7.5, 3.7.6

D.1 Properties of $\bar{P}^\infty \tilde{E}_2$

It's easy to see that

$$|\bar{P}^\infty \mathbf{v}|_{l,\rho,j,\theta} \leq C|\mathbf{v}|_{l,\rho,j,\theta}. \tag{D.1}$$

Indeed, the convolution part of $\bar{P}^\infty \mathbf{u}$ can be treated with Young's inequality for convolutions, with the L^1 norm of $\varepsilon|\xi'|e^{-|\xi'|\varepsilon Y}\chi_{Y\geq 0}$ equal to 1; furthermore, taking derivatives with respect to Y, when the derivatives hit the extreme of integration the exponentials multiplying u_1 and u_2 are bounded and the $\varepsilon|\xi'|$ outside the integral is balanced by the fact that we are taking fewer derivatives with respect to x; on the other side, when the derivatives hit the exponentials inside the integral, this simply causes the appearance of an $\varepsilon|\xi'|$. and

The same argument shows that

$$|\sup_Y |\partial_Y^j \partial_x^i \bar{P}^\infty \mathbf{v}||_{0,\rho} \leq C \sum_{h+k\leq i+j, h\leq i} |\sup_Y |\partial_Y^h \partial_x^k \mathbf{v}||_{0,\rho}. \tag{D.2}$$

Moreover, since only the non convolution part of $\bar{P}^{\infty\prime}$ actually sees all the partial derivatives with respect to Y, we can write

$$\sup_Y |\partial_Y^j \partial_x^i \bar{P}^\infty \mathbf{v}|_{0,\rho} \leq \sup_Y |\partial_Y^j \partial_x^i v_1|_{0,\rho} + C \sum_{h+k\leq i+j, h<j} |\sup_Y |\partial_Y^h \partial_x^k \mathbf{v}||_{0,\rho}. \tag{D.3}$$

The above two estimates imply that, instead of $|\bar{P}^\infty \tilde{E}_2 \mathbf{u}|_{l,\rho,2,\theta}$, we can estimate $|\tilde{E}_2 \mathbf{u}|_{l,\rho,2,\theta}$, and that, instead of $\sup_Y |\partial_Y^i \partial_x^j \bar{P}^\infty \tilde{E}_2 \mathbf{u}|_{0,\rho}$, we can estimate the right hand side of (D.3).

It is also useful to notice that

$$\sup_{Y\in\Gamma(\theta')} |f|_{0,\rho} \leq |\sup_{Y\in\Gamma(\theta')} |f||_{0,\rho} \tag{D.4}$$

that can be proven using Minkowski's integral inequality.

We now pass to the estimates involving the $\tilde{E}_2$ operator. Using Minkowski integral inequality to pass under integral sign, then Young inequality for convolutions in both x and Y, we have that

$$|\partial_x^i \tilde{E}_2 f|_{0,\rho',\theta'}(t) \leq C \int_0^t ds |\partial_x^i f|_{0,\rho',\theta'}(s) \leq C|f|_{l,\rho,0,\theta,\beta,T} \tag{D.5}$$

$\forall i = 0, ..., l, \quad \forall t, \quad \forall \rho' \leq \rho - \beta t, \quad \forall \theta' \leq \theta - \beta t$. If we want to take an extra partial derivative, we can either use Cauchy estimates or we can let the derivative hit the kernel, the cost being the appearance of an unbalanced $\frac{1}{\varepsilon\sqrt{t-s}}$

$$|\tilde{E}_2 f|_{l+1,\rho',0,\theta'} \leq C \int_0^t \frac{|f|_{l,\rho(s),0,\theta'}}{\rho(s) - \rho'} ds, \tag{D.6}$$

$$|\tilde{E}_2 f|_{l+1,\rho',0,\theta'} \leq \frac{C}{\varepsilon} \int_0^t \frac{|f|_{l,\rho',0,\theta'}}{\sqrt{t-s}} ds. \tag{D.7}$$

When we use an L^∞ norm for $Y \in \Gamma(\theta')$, we still use Young's inequality for convolution, but this time we use the L^2 norm (with respect to Y) on the kernel, so an additional $\frac{1}{(t-s)^{1/4}}$ appears: we have that

$$\begin{aligned} |\sup_{Y \in \Gamma(\theta')} |\partial_x^i \tilde{E}_2 f||_{0,\rho} \leq C \int_0^t \frac{||\partial_x^i f|_{L^2(\gamma(\theta'))}|_{0,\rho}}{(t-s)^{1/4}} ds = C \int_0^t \frac{||\partial_x^i f|_{0,\rho}|_{L^2(\gamma(\theta'))}}{(t-s)^{1/4}} ds \\ \leq C|f|_{l,\rho,0,\theta,\beta,T}. \end{aligned} \tag{D.8}$$

The key point here is that the L^2 norm in the Y variable and the Hardy norm in the x variable are interchangeable, essentially because the Hardy norm is an L^2 norm on the boundary lines. We also obtain that

$$|\sup_{Y \in \Gamma(\theta')} |\partial_x^{l+1} \tilde{E}_2 f||_{l+1,\rho} \leq \frac{C}{\varepsilon} \int_0^t \frac{||f|_{l,\rho}|_{L^2(\gamma(\theta'))}}{(t-s)^{3/4}} ds. \tag{D.9}$$

The same argument works for the first partial derivative with respect to Y: while we make the derivatives with respect to x act on f, we can leave the derivative with respect to Y on the kernel, so we obtain

$$|\partial_x^i \partial_Y \tilde{E}_2 f|_{0,\rho',\theta'} \leq C \int_0^t ds \frac{|\partial_x^i f|_{0,\rho',\theta'}(s)}{\sqrt{t-s}} \leq C|f|_{l,\rho,0,\theta,\beta,T} \tag{D.10}$$

$\forall i = 0, ..., l-1, \quad \forall t, \quad \forall \rho' \leq \rho - \beta t, \quad \forall \theta' \leq \theta - \beta t$. If $\gamma f = 0$, then we can use integration by parts to move $\partial_Y = -\partial_{Y'}$ on f, so we obtain

$$|\partial_x^i \partial_Y \tilde{E}_2 f|_{0,\rho',\theta'} \leq C \int_0^t |\partial_x^i \partial_Y f|_{0,\rho',\theta'}(s) ds. \tag{D.11}$$

If we want to take l derivatives with respect to x, we still have

$$|\partial_Y \tilde{E}_2 f|_{l,\rho',0,\theta'} \leq C \int_0^t ds \frac{|f|_{l,\rho',0,\theta'}(s)}{\sqrt{t-s}}. \tag{D.12}$$

Using an L^∞ norm on Y, we have that

$$| \sup_{Y \in \Gamma(\theta')} |\partial_x^i \partial_Y \tilde{E}_2||_{0,\rho'} \leq C \int_0^t \frac{|f|_{l,\rho',0,\theta'}}{(t-s)^{3/4}} ds. \tag{D.13}$$

$i = 0, ..., l$ Before we take a second partial derivative with respect to Y, we first perform an integration by parts; if $\gamma f = 0$, then

$$|\partial_{YY} \tilde{E}_2 f|_{i,\rho',0,\theta'} \leq C \int_0^t ds \frac{|\partial_Y f|_{i,\rho',0,\theta'}}{(t-s)^{1/2}} \tag{D.14}$$

$i = 0, ..., l-1$, while if $\gamma f \neq 0$ we have an additional term given by

$$\begin{aligned} C \left| \left| \int_0^t ds \frac{Y e^{-\frac{Y^2}{4(t-s)}}}{2(t-s)^{3/2}} \gamma \partial_x^i \tilde{f} ds \right|_{L^2(\Gamma(\theta'))} \right|_{0,\rho'} &\leq C \left| \int_0^t \frac{|\partial_x^i \tilde{f}|_{W^{1,2}(\Gamma(\theta'))}}{(t-s)^{3/4}} ds \right|_{0,\rho'} \\ &\leq C \int_0^t \frac{|f|_{l,\rho',0,\theta'}(s)}{(t-s)^{3/4}} ds, \end{aligned} \tag{D.15}$$

where

$$\tilde{f} = \int_{-\infty}^{+\infty} \frac{e^{-\frac{(x-x')^2}{4\varepsilon^2(t-s)}}}{\sqrt{4\pi(t-s)}} f(s,x) dx'. \tag{D.16}$$

In the L^∞ norm, if $\gamma f = 0$ then

$$||\partial_x^i \partial_{YY} \tilde{E}_2 f|_{L^\infty(\Gamma(\theta'))}|_{0,\rho'} \leq C \int_0^t \frac{|\partial_Y f|_{i,\rho',0,\theta'}}{(t-s)^{3/4}} ds, \tag{D.17}$$

$i = 0, ..., l-1$; otherwise, since we have an additional term given by

$$C \int_0^t \frac{Y e^{-\frac{Y^2}{4(t-s)}}}{2(t-s)^{3/2}} \gamma \partial_x^i \tilde{f} ds \tag{D.18}$$

this term has finite $|| \cdot |_{l,\rho}|_{L^\infty(\Sigma(\theta'))}$ norm, because

$$\begin{aligned} C \left| \left| \int_0^t \frac{Y e^{-\frac{Y^2}{4(t-s)}}}{2(t-s)^{3/2}} \gamma \partial_x^i \tilde{f} ds \right|_{0,\rho'} \right|_{L_Y^\infty} &\leq C \left| \int_0^t \frac{Y e^{-\frac{Y^2}{4(t-s)}}}{2(t-s)^{3/2}} \gamma |\partial_x^i f|_{0,\rho'} ds \right|_{L_Y^\infty} \\ \leq C \left| \int_0^t \frac{Y e^{-\frac{Y^2}{4(t-s)}}}{2(t-s)^{3/2}} |f|_{l,\rho',1,\theta'} ds \right|_{L_Y^\infty} &\leq C |f|_{l,\rho,1,\theta,\beta,T} \sup_Y | \int_{\frac{Y}{2\sqrt{t}}}^{+\infty} e^{-\sigma^2} d\sigma | \\ &\leq C |f|_{l,\rho,1,\theta,\beta,T}. \end{aligned} \tag{D.19}$$

Notice that, in this case, we are not able to use the $|| \cdot |_{L^\infty(\Sigma(\theta'))}|_{l,\rho}$ norm, because either we have to take the supremum with respect to s before we take the $| \cdot |_{0,\rho'}$ norm (in order to take f outside the time integral), or we have to bring the L_Y^∞ norm inside the time integral, with $\sup_Y |\frac{Ye^{-\frac{Y^2}{4(t-s)}}}{2(t-s)^{3/2}}| = \frac{C}{t-s}$, which is not integrable in time.

D.2 Properties of $\mathcal{S}\gamma\bar{P}^{\infty}\tilde{E}_2$

In this appendix, we shall obtain some estimates of the operator $\mathcal{S}\gamma\bar{P}^{\infty}\tilde{E}_2$; combining these estimates with those obtained in appendix D.1 for $\bar{P}^{\infty}\tilde{E}_2$, we shall obtain the proof of propositions 3.7.4, 3.7.5, 3.7.6.

More specifically: the bounds in an L^2_{xY}-like setting for derivatives of order up to l provide the proof of proposition 3.7.4; the estimates of the derivatives of order $l+1$ (i.e. with an "excessive" derivative), in an L^2_{xY}-like setting, are used to prove proposition 3.7.5; finally, the estimates in $L^{\infty}_Y L^2_x$ are used for proposition 3.7.6.

For notational simplicity we shall introduce the notation

$$\mathbf{g} = \gamma\bar{P}^{\infty}\tilde{E}_2\mathbf{u}. \tag{D.20}$$

The Stokes operator, see Sammartino and Caflisch, 1998b, can be written explicitly as

$$\mathcal{S}\mathbf{g} = \begin{pmatrix} -N'e^{-\varepsilon Y|\xi'|}g_n + N'(1-\bar{U})\tilde{E}_1 V_1\mathbf{g} \\ e^{-\varepsilon|\xi'|Y}g_n + \bar{U}\tilde{E}_1 V_1\mathbf{g} \end{pmatrix} \tag{D.21}$$

where N' and V_1 are defined as

$$N' = \frac{i\xi'}{|\xi'|}, \qquad V_1\mathbf{g} = g_n - N'g',$$

$\bar{U}$ is the Ukai operator

$$\bar{U}f = \varepsilon|\xi'|\int_0^Y e^{-\varepsilon|\xi'|(Y-Y')}f(\xi',Y')dY',$$

and $E_1 f$ solves the heat equation on the semi-space with boundary condition f and homogeneous initial datum, and writes explicitly as:

$$\tilde{E}_1 f = \int_0^t ds \frac{Y}{t-s}\frac{e^{-\frac{Y^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}}\int_{-\infty}^{+\infty} f(x',s)\frac{e^{-\frac{(x-x')^2}{4\varepsilon^2(t-s)}}}{\sqrt{4\pi(t-s)\varepsilon^2}}dx'.$$

It is useful introduce the decomposition $\mathcal{S}\mathbf{g} = \mathcal{S}^E\mathbf{g} + \bar{\mathcal{S}}\mathbf{g} + \mathcal{S}^*\mathbf{g}$, where

$$\mathcal{S}^E\mathbf{g} = \begin{pmatrix} -N'e^{-\varepsilon Y|\xi'|}g_n \\ e^{-\varepsilon|\xi'|Y}g_n \end{pmatrix}, \quad \bar{\mathcal{S}}\mathbf{g} = \begin{pmatrix} -N'\bar{U}\tilde{E}_1 V_1\mathbf{g} \\ \bar{U}\tilde{E}_1 V_1\mathbf{g} \end{pmatrix}, \quad \mathcal{S}^*\mathbf{g} = \begin{pmatrix} N'\tilde{E}_1 V_1\mathbf{g} \\ 0 \end{pmatrix}.$$

In what follows we shall estimate $\mathcal{S}^E\mathbf{g}$, $\bar{\mathcal{S}}\mathbf{g}$ and $\mathcal{S}^*\mathbf{g}$ separately. The general ideas to bound these terms are the following. The estimates of $\mathcal{S}^E\mathbf{g}$ are easy to achieve, since every normal derivative is essentially transformed into ε times a tangential derivative. Concerning $\bar{\mathcal{S}}\mathbf{g}$, since all terms appear inside the operator $\bar{U}$, the first normal derivative operates essentially as ε times a tangential derivative: therefore, the estimates of $\partial_Y\partial_x^j\bar{\mathcal{S}}\mathbf{g}$ are essentially the estimates of $\varepsilon\partial_x^{j+1}\mathcal{S}^*\mathbf{g}$, while $\partial_{YY}\partial_x^j\bar{\mathcal{S}}\mathbf{g}$ has a part that behaves like $\varepsilon\partial_Y\partial_x^{j+1}\mathcal{S}^*\mathbf{g}$ and a part that behaves like $\varepsilon^2\partial_x^{j+2}\mathcal{S}^*\mathbf{g}$. This means that the estimates of $\bar{\mathcal{S}}\mathbf{g}$ follow from the estimates of $\mathcal{S}^*\mathbf{g}$.

D.2.1 Proof of Proposition 3.7.4

We begin with the estimates of $\mathcal{S}^E\mathbf{g}$

$$\begin{aligned} &\left|\partial_x^i\partial_Y^j\mathcal{S}^E\mathbf{g}\right|^2_{0,\rho',\theta'} \le \\ \varepsilon^{2j}\int_{\Gamma(\theta')} dy \int d\xi' e^{2\rho|\xi'|}&\left[\int_0^{+\infty} dy'\varepsilon|\xi'|e^{-\varepsilon|\xi'|(Y+Y')}|\xi'|^{i+j}|\tilde{E}_2u_1|\right]^2 dy' \le \\ \varepsilon^{2j}\int_{\Gamma(\theta')} dy \int d\xi' e^{2\rho|\xi'|}&\int_0^{+\infty} dy'\varepsilon|\xi'|e^{-\varepsilon|\xi'|(2Y+Y')}|\xi'|^{2(i+j)}|\tilde{E}_2u_1|^2 dy' \le \\ C\varepsilon^{2j}\int d\xi' e^{2\rho|\xi'|}&\int_0^{+\infty} dy' e^{-\varepsilon|\xi'|Y'}|\xi'|^{2(i+j)}|\tilde{E}_2u_1|^2 dy' \le C\varepsilon^{2j}|\tilde{E}_2u_1|^2_{i+j,\rho',0,\theta'}. \end{aligned} \tag{D.22}$$

In the second inequality we used Cauchy-Schwartz inequality; in the third inequality we integrated over Y, and in the last inequality we used $|e^{-\varepsilon|\xi'|Y'}| \le 1$.

Now we pass to the estimates of $\mathcal{S}^*\mathbf{g}$: when only tangential derivatives are performed, for $\tilde{\mathcal{S}}\mathbf{g}$, $i \le l$ we have

$$\begin{aligned} |\partial_x^i N'\tilde{E}_1V_1\mathbf{g}|_{0,\rho',\theta'} &\le C\int_0^t ds\frac{|\gamma\bar{P}^\infty\tilde{E}_2\mathbf{u}|_{i,\rho'}}{(t-s)^{3/4}} \le C\int_0^t \frac{ds}{(t-s)^{3/4}}\int_0^s \frac{|u|_{i,\rho',0,\theta'}(s')}{(s-s')^{1/4}}ds' \\ &= C\int_0^t |u|_{i,\rho',0,\theta'}(s')ds'\int_{s'}^t \frac{ds}{(t-s)^{3/4}(s-s')^{1/4}} \le C\int_0^t |u|_{i,\rho',0,\theta'}(s')ds', \end{aligned} \tag{D.23}$$

where we used the fact that

$$\int_{s'}^t \frac{ds}{(t-s)^{3/4}(s-s')^{1/4}} = \int_0^{t-s'} \frac{ds''}{(t-s-s'')^{3/4}(s-s')^{1/4}} = \int_0^1 \frac{d\tau}{(1-\tau)^{3/4}\tau^{1/4}}. \tag{D.24}$$

Now we begin to estimate the normal derivatives of $\mathcal{S}^*\mathbf{g}$: call

$$f = N'V_1\gamma\bar{P}^\infty E_2^d\mathbf{u}. \tag{D.25}$$

We have that

$$\partial_Y\tilde{E}_1 f = D_1(\varepsilon^2\partial_{xx}f - \partial_t f), \tag{D.26}$$

with D_1 defined in (A.15). In our functional setting, we are not allowed to take a time derivative: therefore, we need to use the properties of the operator E_2^d in order to express the time derivative as a function of the normal and tangential derivatives. We have

$$\partial_t f = N'V_1\gamma\bar{P}^\infty\partial_t E_2^d\mathbf{u}, \tag{D.27}$$

with

$$\gamma\bar{P}^{\infty}\partial_t E_2^d\mathbf{u} = \gamma\bar{P}^{\infty}\mathbf{u} + \gamma\bar{P}^{\infty}\varepsilon^2\partial_{xx}E_2^d\mathbf{u} + \gamma\bar{P}^{\infty}\partial_{YY}E_2^d\mathbf{u} = \mathbf{h}_1 + \mathbf{h}_2 + \mathbf{h}_3 \tag{D.28}$$

and

$$\varepsilon^2\partial_{xx}f = N'V_1\varepsilon^2\partial_{xx}\gamma\bar{P}^{\infty}E_2^d\mathbf{u}, \tag{D.29}$$

with

$$\varepsilon^2\gamma\partial_{xx}\bar{P}^{\infty}E_2^d\mathbf{u} = \mathbf{h}_2. \tag{D.30}$$

Although the contribution of $\mathbf{h}_2$ is canceled, it's still useful to perform an estimate for this term, since it is used in the estimate of $\mathbf{h}_3$.
As regards $\mathbf{h}_1$, we have that, for $i \le l-1$

$$||\sup Y|\partial_x^i\mathbf{u}|||_{\rho'} \le C||\mathbf{u}||_{l,\rho',1,\theta'}, \tag{D.31}$$

with the trace which satisfies a similar bound. Therefore

$$|D_1N'\gamma u_1|_{i,\rho',0,\theta'} \le C\int_0^t \frac{|\mathbf{u}|_{l,\rho',0,\theta'}}{(t-s)^{1/4}}ds. \tag{D.32}$$

In order to bound $\mathbf{h}_2 = \gamma\bar{P}^{\infty}\varepsilon^2\partial_{xx}E_2^d\mathbf{u}$, we let one of the tangential derivatives of $\partial_{xx}E_2^d\mathbf{u}$ hit the kernel E_2^d and one hit $\mathbf{u}$, so that, for $i \le l-1$

$$\begin{aligned}|D_1V_1\mathbf{h}_2|_{i,\rho',0,\theta'} \le C\varepsilon\int_0^t \frac{ds}{(t-s)^{1/4}}\int_0^s \frac{|u|_{i+1,\rho',0,\theta'}(s')}{(s-s')^{3/4}}ds' = \\ C\varepsilon\int_0^t |u|_{l,\rho',0,\theta'}(s')ds'.\end{aligned} \tag{D.33}$$

We first estimate $\mathbf{h}_3$, assuming that $\gamma\mathbf{u} = \mathbf{0}$: under this assumption, in $\partial_{YY}E_2^d\mathbf{u}$, we can move one of the normal derivatives on $\mathbf{u}$, without boundary terms. Therefore, for $i \le l-1$ we have, in the L^2_Y setting of the proposition 3.7.4

$$|D_1V_1\mathbf{h}_3|_{i,\rho',0,\theta'} \le C\int_0^t \frac{ds}{(t-s)^{1/4}}\int_0^s \frac{\partial_Y\mathbf{u}|(s')_{i,\rho',0,\theta'}}{(s-s')^{3/4}}|ds \le C\int_0^t |\mathbf{u}|_{l,\rho',1,\theta'}(s)ds, \tag{D.34}$$

If $\gamma\mathbf{u} \neq \mathbf{0}$, an additional term appears in the expression of $\bar{P}^{\infty}\partial_{YY}E_2^d\mathbf{u}$, which roughly behaves like $\tilde{E}_1\gamma u_1$; the loss of half derivative due to the trace operator is not a problem for the tangential derivatives of order up to $l-1$, so inequality (D.34) still holds. This concludes the estimates of the normal derivative of $\bar{\mathcal{S}}$.
As we already said at the beginning of this section, the estimates of $\bar{\mathcal{S}}\mathbf{g}$ can be derived from the ones of $\mathcal{S}^*\mathbf{g}$: therefore, the proof of the proposition is complete.

D.2.2 Proof of Proposition 3.7.5

The estimate (D.22) obtained for $\mathcal{S}^E\mathbf{g}$ still holds for derivatives of order $l+1$; the "excessive" derivative can be treated either using the Cauchy estimate, or giving a tangential derivative to the kernel of E_2^d.

The same applies for $\partial_x^{l+1}\mathcal{S}^*\mathbf{g}$: the only difference is that we change the order of integration in time. This is done, when we use the Cauchy estimate, in order to eliminate the mild singularity in time deriving from the L^2_Y norm of the operato $\tilde{E}_1$, while, when we let one tangential derivative hit the kernel of E^d_2, changing the order of integration allows to reduce the singularity in time. We have

$$|N'\tilde{E}_1V_1\mathbf{g}|_{l+1,\rho',0,\theta'} \leq \frac{C}{\varepsilon}\int_0^t |u|_{l,\rho',0,\theta'}ds'\int_{s'}^t \frac{ds}{(t-s)^{3/4}(s-s')^{3/4}}, \tag{D.35}$$

with

$$\begin{aligned}\int_{s'}^t \frac{ds}{(t-s)^{3/4}(s-s')^{3/4}} &= \int_0^{t-s'} \frac{ds''}{(t-s-s'')^{3/4}(s-s')^{3/4}} \\ &= \frac{1}{(t-s')^{1/2}}\int_0^1 \frac{d\tau}{(1-\tau)^{3/4}\tau^{3/4}},\end{aligned} \tag{D.36}$$

so

$$|N'\tilde{E}_1V_1\mathbf{g}|_{l+1,\rho',0,\theta'} \leq \frac{C}{\varepsilon}\int_0^t \frac{|u|_{l,\rho',0,\theta'}(s')}{(t-s')^{1/2}}ds'. \tag{D.37}$$

Now, we begin to estimate $\partial_Y\mathcal{S} * \mathbf{g}$. First, we estimate the term $\mathbf{h}_1$, introduced in (D.28), under the assumption that $\gamma\mathbf{u} = 0$: in this case,

$$|\mathbf{h}_1| \leq C\sqrt{\varepsilon|\xi'|}|\mathbf{u}|_{L^2_Y}. \tag{D.38}$$

Using

$$(\varepsilon|\xi'|)^{1/2}e^{-\varepsilon^2(t-s)|\xi'|^2} \leq \frac{C}{(t-s)^{1/4}} \tag{D.39}$$

we have that

$$|D_1V_1\mathbf{h}_1|_{l,\rho',0,\theta'} \leq C\int_0^t \frac{|\mathbf{u}|_{l,\rho',0,\theta'}}{(t-s)^{1/2}}ds. \tag{D.40}$$

If $\gamma\mathbf{u} \neq 0$, then $V_1\mathbf{h}_1$ contains $N'\gamma u_1$ as additional term, with

$$|\gamma u_1| \leq |u_1|^{1/2}_{L^2_Y}|\partial_Y u_1|^{1/2}_{L^2_Y}. \tag{D.41}$$

We use the interpolation inequality (D.41), together with the Cauchy estimates, to obtain

$$|D_1N'\gamma u_1|_{l,\rho',0,\theta'} \leq C\int_0^t \frac{|\mathbf{u}|^{1/2}_{l,\rho',0,\theta'}|\mathbf{u}|^{1/2}_{l,\rho(s),1,\theta'}}{(t-s)^{1/4}(\rho(s)-\rho')^{1/2}}ds \leq C\int_0^t \frac{|\mathbf{u}|_{l,\rho',0,\theta'}|}{(t-s)^{1/2}} + \frac{|\mathbf{u}|_{l,\rho(s),1,\theta'}}{(\rho(s)-\rho')}ds. \tag{D.42}$$

This concludes the estimates of $\mathbf{h}_1$.
The bounds of $\mathbf{h}_2 = \gamma\bar{P}^{\infty}\varepsilon^2\partial_{xx}E^d_2\mathbf{u}$ are obtained moving one tangential derivative on the kernel of D_1 and one tangential derivative on the kernel of E^d_2, so

$$|D_1V_1\mathbf{h}_2|_{l,\rho',0,\theta'} \leq C\int_0^t \frac{ds}{(t-s)^{3/4}}\int_0^s \frac{|u|_{l,\rho',0,\theta'}(s')}{(s-s')^{3/4}}ds' = C\int_0^t \frac{|u|_{l,\rho',0,\theta'}(s')}{(t-s')^{1/2}}ds'. \tag{D.43}$$

We begin the estimates of $\mathbf{h}_3$, starting from the case $\gamma\mathbf{u} = \mathbf{0}$: in this case, $\gamma\partial_Y Y E_2^d \mathbf{u} = \mathbf{0}$, so only the integral part of $\gamma\bar{P}^\infty$ is nonzero, when we apply this operator to $\partial_Y Y E_2^d \mathbf{u}$. We can integrate by parts two times, moving the normal derivatives on the kernel of $\bar{P}^\infty$: the result is a term which behaves like $\varepsilon^2\partial_{xx}\bar{P}^\infty E_2^d\mathbf{u}$ (and therefore can be bounded exactly like $\mathbf{h}_2$) plus an additional term, given by

$$Ci\xi'\varepsilon \int_0^s ds' e^{-\varepsilon^2(s-s')\xi'^2} \int_0^{+\infty} \frac{Y'}{(s-s')^{3/2}} e^{-\frac{Y'^2}{4(s-s')}} (u_2, u_1)(\xi', Y, s) dY'. \tag{D.44}$$

This term is inside D_1: we can move $i\xi'$ on the kernel of D_1, so that the singularities in time are better distributed. Therefore, we obtain

$$|D_1 V_1 \mathbf{h}_3|_{l,\rho',0,\theta'} \leq C \int_0^t \frac{|u|_{l,\rho',0,\theta'}(s')}{(t-s')^{1/2}} ds'. \tag{D.45}$$

If $\gamma\mathbf{u} \neq \mathbf{0}$, an additional term appears in the expression of $\bar{P}^\infty \partial_{YY} E_2^d \mathbf{u}$, which roughly behaves like $\tilde{E}_1 \gamma u_1$; as for $\mathbf{h}_1$, in this case we use the interpolation inequality (D.41), the Cauchy estimate and Young's inequality, obtaining

$$|D_1 V_1 \mathbf{h}_3|_{l,\rho',0,\theta'} \leq C \int_0^t \frac{|u|_{l,\rho',0,\theta'}}{(t-s')^{1/2}} + \frac{|u|_{l,\rho(s'),1,\theta'}(s')}{\rho(s') - \rho'} ds'. \tag{D.46}$$

Finally, when we take another derivative with respect to Y, we obtain

$$\begin{aligned} \partial_{YY}\tilde{E}_1 f &= \\ \int_0^t \frac{Y e^{-\frac{Y^2}{4(t-s)}}}{4\sqrt{\pi}(t-s)^{3/2}} ds \int_{-\infty}^{+\infty} \left((\partial_t f)(x', s) - \varepsilon^2(\partial_{xx} f)(x', s)\right) &\frac{e^{-\frac{(x-x')^2}{4\varepsilon^2(t-s)}}}{\sqrt{4\pi\varepsilon^2(t-s)}} dx' \\ &= \tilde{E}_1(\partial_t f - \varepsilon^2 \partial_{xx} f). \end{aligned} \tag{D.47}$$

Therefore, we have to evaluate $\tilde{E}_1\gamma V_1 \mathbf{h}_1$ and $\tilde{E}_1 \gamma V_1 \mathbf{h}_3$: this time, since two normal derivatives have been taken, only up to $l-1$ tangential derivatives can be taken. Therefore, this time, the loss of half derivative due to the trace operator is not a problem, and the case $\gamma\mathbf{u} \neq \mathbf{0}$ does not need to be treated separately. For $\mathbf{h}_1$, we have

$$|\tilde{E}_1 V_1 \mathbf{h}_1|_{i,\rho',0,\theta'} \leq C \int_0^t \frac{|\mathbf{u}|_{i+i,\rho',0,\theta'}}{(t-s)^{3/4}} ds. \tag{D.48}$$

For $\tilde{E}_1 \gamma V_1 \mathbf{h}_3$, a similar estimate holds: therefore, the estimates of $\mathcal{S}^*$ are complete. The estimates of $\bar{\mathcal{S}}\mathbf{g}$ can be derived from the ones of $\mathcal{S}^*\mathbf{g}$.

D.2.3 Proof of proposition 3.7.6

In the $L^\infty_Y L^2_x$-like setting of proposition 3.7.6, the estimates differ from the ones of 3.7.5 an additional $(t-s)^{-1/4}$, which implies a stronger (but still integrable) singularity in time. Indeed, for the estimates of $\mathcal{S}^*\mathbf{g}$ (and therefore of $\bar{\mathcal{S}}\mathbf{g}$, we give the L^∞_Y norm to the kernel of $\mathbf{E}_1$, which is "$(t-s)^{-1/4}$ times worse" of its L^2_Y norm. For the estimates of $\mathcal{S}^E\mathbf{g}$, instead, we give $e^{-\varepsilon|\xi'|Y}$ the L^∞_Y norm, which is 1, while its L^2_Y norm is $C(\varepsilon|\xi'|)^{-1/2}$: moving the additional $(\varepsilon|\xi'|)^{1/2}$ on the kernel of E_2^d, we can use the bound given by equation (D.39), and an additional $(t-s)^{-1/4}$ appears.

Appendix E

Estimate of the linear terms in (3.104)

In $[\mathbf{u}^{NS0} + \varepsilon(w + w + \sigma)] \cdot \nabla \mathcal{N}^* \mathbf{e}^*$ the most problematic term is $u^S \partial_x \mathcal{N}^* \mathbf{e}^*$: using propositions 3.7.4 and 3.7.5, we have that

$$|u^S \partial_x \mathcal{N}^* \mathbf{e}^*|_{l,\rho',0,\theta'} \leq C \int_0^t \frac{|\mathbf{e}^*|_{l,\rho(s),0,\theta}}{\rho(s) - \rho'} ds, \tag{E.1}$$

$$|\partial_Y u^S \partial_x \mathcal{N}^* \mathbf{e}^*|_{l-1,\rho',0,\theta'} \leq \frac{C}{\sqrt{t}} \int_0^t |\mathbf{e}^*|_{l,\rho',0,\theta} ds, \tag{E.2}$$

$$|u^S \partial_Y \partial_x \mathcal{N}^* \mathbf{e}^*|_{l-1,\rho',0,\theta'} \leq C \int_0^t \frac{|\mathbf{e}^*|_{l,\rho',1,\theta}}{(t-s)^{1/2}} ds. \tag{E.3}$$

In $\mathcal{N}^* \mathbf{e}^* \cdot \nabla [\mathbf{u}^{NS0} + \varepsilon(\sigma + w + \sigma)]$, the most problematic term is $\frac{1}{\varepsilon}(\mathcal{N}^* \mathbf{e}^*)_2 \partial_Y \tilde{u}^P$ (the problem with $\tilde{u}^R$ is to prove that the term is $O(1)$). Since $\mathcal{N}^* \mathbf{e}^*$ is divergence free and zero at the boundary, we have

$$(\mathcal{N}^* \mathbf{e}^*)_2 = -\varepsilon \int_0^Y \partial_x \left(\mathcal{N}^* \mathbf{e}^*\right)_1 dY', \tag{E.4}$$

so

$$\begin{aligned} \left| \frac{1}{\varepsilon} (\mathcal{N}^* \mathbf{e}^*)_2 \partial_Y \tilde{u}^P \right|_{l,\rho',0,\theta'} &\leq C \left| \frac{1}{Y} \int_0^Y \partial_x \left(\mathcal{N}^* \mathbf{e}^*\right)_1 dY' \right|_{l,\rho',0,\theta'} \sup_Y \left| Y \partial_Y \tilde{u}^P \right|_{l,\rho'} \leq \\ & C \left| \partial_x \left(\mathcal{N}^* \mathbf{e}^*\right)_1 \right|_{l,\rho',0,\theta'} \leq C \int_0^t \frac{|\mathbf{e}^*|_{l,\rho(s),0,\theta'}}{\rho(s) - \rho'} ds. \end{aligned} \tag{E.5}$$

In the first inequality we used algebra property 3.2.1, in the second inequality we used the boundedness of $\sup_Y \left| Y \partial_Y \tilde{u}^P \right|_{l,\rho'}$ (both for the regular and for the singular part) and the fact that the average operator is bounded from L^2 to L^2 (is bounded by the maximal function, so its trivial), and in the last inequality we used proposition

3.7.6. With a similar argument we have that

$$\left|\frac{1}{\varepsilon}(\mathcal{N}^*\mathbf{e}^*)_2\partial_{YY}\tilde{u}^P\right|_{l-1,\rho',0,\theta'} \leq C\left(1+\frac{1}{t^{1/2}}\right)\int_0^t |\mathbf{e}^*|_{l,\rho',0,\theta'}ds \tag{E.6}$$

and

$$\begin{aligned}\left|\frac{1}{\varepsilon}\partial_Y(\mathcal{N}^*\mathbf{e}^*)_2\partial_Y\tilde{u}^P\right|_{l-1,\rho',0,\theta'} = \left|\partial_x(\mathcal{N}^*\mathbf{e}^*)_1\partial_Y\tilde{u}^P\right|_{l-1,\rho',0,\theta'} \leq \\ C\left(1+\frac{1}{t^{1/2}}\right)\int_0^t |\mathbf{e}^*|_{l,\rho',0,\theta'}ds\end{aligned} . \tag{E.7}$$

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