Two relaxation times and thermal nonlinear waves along wires with lateral heat exchange

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Abstract

We propose a model for studying several nonlinear waves for heat transport along a cylindrical system with lateral non-linear heat transfer to the environment. We consider relaxational equations, each with its own relaxation time, for longitudinal heat transfer and for lateral heat transfer across the wall. We consider two kinds of nonlinear lateral heat transport: radiative heat transport, and flux-limited heat transport. This work generalizes our previous studies in which the relaxation time for the lateral heat transfer was considered equal to that of the longitudinal heat flux. We explore the influence of both relaxation times on the propagation speed of linear and nonlinear waves, and on the form of nonlinear waves.

Keywords: Heat waves. Thermal solitons. Heat flux saturation. Maxwell-Cattaneo law. Radiative transfer; auxiliary equation method.

AMS Subject Classifications: 35C08 (Soliton solutions), 35C07 (Traveling wave solutions), 74J30 (Nonlinear waves), 35Q51 (Soliton-like equations), 35Q79 (PDEs in connection with classical thermodynamics and heat transfer).
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1 Introduction

The appearance of nonlinear phenomena is an appealing feature which interests many branches of physics and biology. One of the most important nonlinear phenomena is the existence of solitons [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. In recent papers we have considered the propagation of heat solitons along thin wires with a nonlinear heat exchange between the wire and environment given by the Stefan-Boltzmann law [13] or by a flux-limited heat exchange [14]. This idea follows the interest on hyperbolic heat propagation along thin wires or along quantum vortex lines [15, 16, 17, 18, 19, 20]. In particular, solitons have been obtained in the context of heat transfer combined with the Fourier's law and some nonlinear heat producing process (exothermic chemical reactions, and phase transitions with latent heat) [21]. Instead, in [22, 23, 24, 25] the authors have considered a nonlinear radiative heat exchange between the system and the environment. Apart from the existence of the thermal solitons, as seen also in [26], other kinds of nonlinear waves could also be of interest in the heat transport.

In this paper we propose an extension of the model introduced in [13, 14] with the transverse heat exchange being considered as an independent field, namely with a further evolution

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equation for it with its own relaxation time different from the relaxation time for the longitudinal heat transfer. The presence of two different relaxation times is not a mere mathematical ingredient but it follows the physical idea that the heat propagation along the wire involves the collision time of phonons, whereas heat transport across the walls involves the emission of photons at the surface of the wire, or the collisions of particles in the environment against the wall of the wire. The combination of two relaxation times has been considered in other papers but in a linear approximation [27, 28, 29]. There, the authors have studied the propagation of the second sound in terms of linear waves along nanowires and thin layers. Here, we consider a different geometry, thin wires, with a nonlinear approach, which would generalize what found in [13, 14].

For the lateral term, we consider both the Stefan-Boltzmann law and flux-limited heat exchange [30, 31]. This analysis is aimed to explore possible strategies for heat transport and signal transmission along thin wires. Indeed, it may answer to: a) the increasing interest in the transmission and processing of information by means of thermal signals along nanowires, and b) the interest in evaluating the energy cost for the transmission of a signal, in the case the signal is transmitted by a solitonic signal. Both of them have been dealt with by means of the soliton solutions following the recent paper but with a single relaxation time [13] .

The paper is organized as follows. In Section 2 we present the mathematical model; in Section 3 we study the linear waves of the model; in Section 4 we study the nonlinear heat waves with some emphasis to the soliton waves and in Section 5 we apply some nonlinear waves to a Si thin wire. The mathematical approach used in Section 4 and in the Appendix is finding exact solutions of the partial differential equation (PDE), namely functions which exactly solve the PDEs and hence the mathematical problem considered. It allows to prove the existence of the obtained solutions. For the sake of simplicity, we will refer to "solutions" for "exact solutions". The last section is for the conclusions.

2 The mathematical model

Following the main ideas of the papers [13, 14], in this section we consider heat propagation along a heat-conducting wire of radius r (composed of a material of mass density ρ and specific heat per unit mass c) but we assume that the longitudinal heat transfer along the cylinder and the transverse heat exchange per unit area have evolution equations characterized by different relaxation times.

Let's consider the following mathematical model:

$$\begin{cases} \rho c \dfrac{\partial T}{\partial t} = -\dfrac{\partial}{\partial z} q - \dfrac{2}{r} q_t \\ \tau \dfrac{\partial q}{\partial t} + q = -\lambda \dfrac{\partial}{\partial z} T \\ \tau_1 \dfrac{\partial q_t}{\partial t} + q_t = f(T) \end{cases} \tag{2.1}$$

where the first equation is the energy balance equation expressed in terms of the temperature field T, $\mathbf{q} = q(z)\, \hat{\mathbf{z}}$ is the longitudinal heat transfer along the cylinder, described by the so-called Maxwell-Cattaneo equation [26, 28] (second equation), and q_t is the transverse heat exchange per unit area from the cylinder to the environment.
The coefficient τ is the relaxation time of the longitudinal heat flux q and τ_1 is the relaxation time of the transverse heat exchange q_t; λ is the thermal conductivity of the material. In

[13, 14] we have considered the case $\tau_1 = \tau$. This was motivated on purely physical grounds and it may be expected that τ and τ_1, corresponding to different processes, will be different. Indeed, τ corresponds to the characteristic collision time of heat carriers inside the wire, whereas τ_1 corresponds to the characteristic emission time of electromagnetic radiation on the walls of the wire. It is expected that τ_1 is shorter than τ, but on mathematical grounds we aim to analyze the general situation of independent values of τ and τ_1.

The function $f(T)$ is the souce term for the transverse heat exchange q_t, which may be given by the Stefan-Boltzmann law for radiative transfer

$$f(T) = \sigma_{SB}(T^4 - T_0^4), \tag{2.2}$$

with σ_{SB} the Stefann-Boltzmann constant or by a nonlinear flux-limited heat exchange, as for instance

$$f(T) = \frac{\sigma(T - T_0)}{\sqrt{1 + \left(\frac{\sigma'(T-T_0)}{T}\right)^2}} \tag{2.3}$$

with σ the heat exchange coefficient between the medium and the environment. Indeed, if in this equation one ignores the denominator, it reduces to the Newton heat transfer law, with the stated meaning for the coefficient σ. In such equation, q_t increases indefinitely with an increase of $T - T_0$. If in contrast we consider the full expression (2.3), including the denominator, q_t does no longer indefinitely increases with $T - T_0$ but tends to a saturation value given by $\sigma T/\sigma'$. Thus, the coefficient σ' is related to the saturation value of the heat flux; when σ' is zero there is no saturation.

3 Linear heat waves

In this section we investigate the linear waves of the linearized system (2.1) near a reference steady-state, where system (2.1) becomes

$$\begin{cases} \dfrac{\partial}{\partial z}q + \dfrac{2}{r}q_t = 0 \\ q = -\lambda\dfrac{\partial}{\partial z}T \\ q_t = f(T) \end{cases} \tag{3.1}$$

$f(T)$ being given by (2.2) or by (2.3).

3.1 Linear waves from the Stefan-Boltzmann law

By substituting the last two equations into the first equation of the system (3.1) and using the Stefan-Boltzmann law (2.2), corresponding to the case when radiative exchange is dominating, we find

$$\frac{\partial^2}{\partial z^2}T - \frac{2}{\lambda r}\sigma_{SB}(T^4 - T_0^4) = 0 \tag{3.2}$$

which is a second-order nonlinear partial differential equation. Apart from the constant solution $T = T_0$ (equilibrium temperature), which leads to the null solutions for the longitudinal and transverse heat transfer $q^0(z) = -\lambda\frac{\partial}{\partial z}T_0 = 0$ and $q_t^0(z) = f(T_0) = 0$, equation (3.2) has more non trivial solutions. Indeed, as shown in the Appendix A, some solutions of equation

(3.2) can be expressed in terms of the elliptic and hyperelliptic functions, which we have avoid to report here because we will consider the simplest stationary solution: the equilibrium temperature T_0.

Let's set $\Gamma^{(0)}(z) = (T^{(0)}(z), q^{(0)}(z), q_t^{(0)}(z))$ the stationary solution of the system (3.1). The nonlinear term $f(T)$ can be approximated around the stationary solution by

$$f(T) = f(T^{(0)}) + f'(T^{(0)})\left(T - T^{(0)}\right), \tag{3.3}$$

which, for the Stefan-Boltzmann law (2.2), becomes:

$$f(T) = \sigma_{SB}(T^{(0)4} - T_0^4) + 4\sigma_{SB}T^{(0)3}(T - T^{(0)}) \tag{3.4}$$

Now, we consider the propagation of harmonic plane waves of the three fields $\Gamma(z,t) = (T, q, q_t)$ along the z-axis

$$\Gamma(z,t) = \Gamma^{(0)}(z) + \tilde{\Gamma}\mathrm{e}^{i(K(z)-\omega t)} \tag{3.5}$$

where $K(z) = k_r(z) + ik_i(z)$ is the wave number, which here we assume that may depend on z in the case of inhomogeneity, ω is the real frequency, and $\tilde{\Gamma} = (\tilde{T}, \tilde{q}, \tilde{q}_t)$ denotes small amplitudes of the fields, whose product can be neglected.

Substituting (3.5) in the linearized system, the following equations for small amplitude waves are obtained

$$\begin{cases} -i\omega\rho c\tilde{T} + iK'\tilde{q} + \dfrac{2}{r}\tilde{q}_t = 0 \\ i\lambda K'\tilde{T} + (1 - i\omega\tau)\tilde{q} = 0 \\ -4\sigma_{SB}T^{(0)3}\tilde{T} + (1 - i\omega\tau_1)\tilde{q}_t = 0 \end{cases} \tag{3.6}$$

where $K' = dK/dz$ is the derivative of $K(z)$ with respect to z, and $T^{(0)}(z)$ is the stationary solution.

The dispersion relation follows by requiring a nonzero solution for the fields $\tilde{\Gamma} = (\tilde{T}, \tilde{q}, \tilde{q}_t)$, namely by setting zero the determinant of the linear system (3.6). Thus, we find the following dispersion relation

$$ci\rho\tau\tau_1\omega^3 - ci\rho\omega - c\rho\tau\omega^2 - c\rho\tau_1\omega^2 - i\lambda K'^2\tau_1\omega - \frac{8i\sigma_{SB}\tau T^{(0)3}(z)\omega}{r} + \lambda K'^2 + \frac{8\sigma_{SB}T^{(0)3}(z)}{r} = 0 \tag{3.7}$$

which focuses the importance of the nonlinearity through $T^{(0)3}(z)$ [32]

Note that if we assume that $K(z)$ is a linear function of z, namely $K(z) = Kz$ and hence $K' = K$, and that all the parameters in (3.7) are independent of z, then the stationary solution $T^{(0)}(z)$ in (3.7) cannot depend on z, but it has to be constant. From equation (3.2) it follows that $T^{(0)} = T_0$ (with T_0 the temperature at equilibrium). This means that homogeneous perturbations along z, given by (3.5), take part only for homogeneous distribution of the stationary solution of the temperature, which has to coincide with the temperature T_0 at equilibrium, because it is the only constant solution of equation (3.2). The general case $K(z) \neq Kz$ is much more demanding to analyze because of the hyperelliptic functions involved (see the Appendix for more details) and the further integration required to find $K(z)$ from $K'(z)$ in (3.7). Thus, for the sake of simplicity we discard this situation and we study propagation in equilibrium reference states.

Let's assume that $K(z) = K\,z = k_r\,z + ik_s\,z$. Then equation (3.7) can be split into real and imaginary part

$$\begin{cases} -c\rho\tau\omega^2 - c\rho\tau_1\omega^2 + \lambda k_r^2 + 2\lambda k_r k_s \tau_1 \omega - \lambda k_s^2 + \dfrac{8\sigma_{SB}T_0^3}{r} = 0 \\ \omega\left(c\rho\left(\tau\tau_1\omega^2 - 1\right) + \lambda k_s^2 \tau_1 - \dfrac{8\sigma_{SB}\tau T_0^3}{r}\right) - \lambda k_r'^2 \tau_1 \omega + 2\lambda k_r' k_s' = 0 \end{cases} \tag{3.8}$$

If we assume k_s small enough that k_s^2 can be neglected in (3.8), (namely, that the attenuation is relatively small) we find expressions for the phase-velocity $v = \omega/k_r$ and k_s:

$$\begin{cases} v = \dfrac{\omega}{k_r} = \pm\dfrac{U_0}{\sqrt{1 - 8\frac{\sigma_{SB}T_0^3}{c\rho r\tau\omega^2}\frac{(1+\tau\tau_1\omega^2)}{(1+\tau_1^2\omega^2)}}} \\ k_s = \pm\dfrac{1 + 8\frac{\sigma_{SB}T_0^3}{c\rho r}\frac{(\tau-\tau_1)}{1+\omega^2\tau_1^2}}{2U_0\tau r\sqrt{1 - 8\frac{\sigma_{SB}T_0^3(1+\omega^2\tau\tau_1)}{c\rho\tau r\omega^2(1+\tau_1^2\omega^2)}}} \end{cases}. \tag{3.9}$$

where U_0 is the Maxwell-Cattaneo velocity defined by $U_0 = \sqrt{\dfrac{\lambda}{\rho c\tau}}$. Note that the quantity $\rho cr/(\sigma_{SB}T_0^3)$ has the dimensions of time, and characterizes the relaxation of the heat signal because of lateral radiative exchange. This sets a characteristic time scale that must be compared to the timescales set by the longitudinal heat relaxation time τ and the lateral heat relaxation time τ_1.
From (3.9) the following condition arises for the existence of the root square, namely:

$$8\frac{\sigma_{SB}T_0^3(1+\omega^2\tau\tau_1)}{c\rho\tau r\omega^2\left(1+\tau_1^2\omega^2\right)} < 1. \tag{3.10}$$

This means that for sufficiently high frequencies, making that condition (3.10) is fulfilled, namely for ω higher than $\omega_{min}^2 = 8\dfrac{\sigma_{SB}T_0^3}{c\rho r\tau_1}$, the velocity v will be real and waves will propagate. Instead, for smaller frequencies there will not be propagation of waves. Note that for $\sigma_{SB} = 0$, i.e. in the absence of radiative lateral transfer, $\omega_{min} = 0$. Thus, the presence of the lateral heat transfer sets in this case a cutoff minimal frequency for the propagation of heat waves.

It is interesting to analyze the results (3.9) in four main cases, namely: $\omega\tau \gg 1$ (high-frequency asymptotic limit); and three high-frequency (but not infinite) cases, corresponding to $\tau = \tau_1$; $\tau_1 \gg \tau$ and $\tau \gg \tau_1$ [33, 34].

In the high-frequency asymptotic limit, namely $\omega\tau \gg 1$, the velocity of the waves becomes the Maxwell-Cattaneo velocity and the attenuation coefficient k_s becomes:

$$\begin{cases} v = \pm U_0 \\ k_s = \pm\dfrac{1}{2U_0\tau r} \end{cases} \tag{3.11}$$

In the high-frequency cases instead we consider the following three situations $\tau = \tau_1$, $\tau_1 \gg \tau$ and $\tau \gg \tau_1$, corresponding to ω finite but sufficiently high that the propagation condition (3.10) is satisfied.

For $\tau = \tau_1$, velocity and dissipation (3.9) become

$$
\begin{cases}
v = \dfrac{\omega}{k_r} = \pm \dfrac{U_0}{\sqrt{1 - 8\frac{\sigma_{SB} T_0^3}{c\rho r \tau \omega^2}}} \\
k_s = \pm \dfrac{1}{2U_0 \tau r \sqrt{1 - 8\frac{\sigma_{SB} T_0^3}{c\rho \tau r \omega^2}}}
\end{cases}
\tag{3.12}
$$

For $\tau_1 \gg \tau$ and $\tau \simeq 0$, in such a way that $\tau \tau_1 \omega^2 \ll 1$, (3.9) becomes

$$
\begin{cases}
v = \dfrac{\omega}{k_r} = \pm \dfrac{U_0}{\sqrt{1 - 8\frac{\sigma_{SB} T_0^3}{c\rho r \tau \omega^2} \frac{1}{\left(1+\tau_1^2 \omega^2\right)}}} \\
k_s = \pm \dfrac{1 - 8\frac{\sigma_{SB} T_0^3}{c\rho r} \frac{\tau_1}{1+\omega^2 \tau_1^2}}{2U_0 \tau r \sqrt{1 - 8\frac{\sigma_{SB} T_0^3}{c\rho \tau r \omega^2 \left(1+\tau_1^2 \omega^2\right)}}}
\end{cases}
\tag{3.13}
$$

and finally, for $\tau \gg \tau_1$ and $\tau_1 \simeq 0$, in such a way that $\omega^2 \tau_1^2 \ll 1$, (3.9) becomes

$$
\begin{cases}
v = \dfrac{\omega}{k_r} = \pm \dfrac{U_0}{\sqrt{1 - 8\frac{\sigma_{SB} T_0^3}{c\rho r \tau \omega^2} \left(1 + \tau \tau_1 \omega^2\right)}} \\
k_s = \pm \dfrac{1 + 8\frac{\sigma_{SB} T_0^3}{c\rho r} \tau}{2U_0 \tau r \sqrt{1 - 8\frac{\sigma_{SB} T_0^3 \left(1+\omega^2 \tau \tau_1\right)}{c\rho \tau r \omega^2}}}
\end{cases}
\tag{3.14}
$$

Note that velocity in (3.12), (3.13) and (3.14), as well as in the general expression (3.9), is bigger than the Maxwell-Cattaneo velocity because the denominator is less than the unit. This is not the same for the dissipation coefficient k_s: indeed, it is surely bigger in (3.12) and (3.13) but nothing can be asserted for the expression in (3.14) because the numerator is smaller too. The characteristic attenuation lengths instead (recriprocal of k_s) are shorter than in the Maxwell-Cattaneo case. Note also that if τ_1 is zero, the speed in (3.14) has the same form as in (3.12), corresponding to $\tau = \tau_1$. This is of interest, because it indicates that our previous analysis in [13] where the Stefan-Boltzmann law was generalized to a relaxational transfer equation with relaxation time τ_1 equal to τ, is also valid for the usual (non-relaxational) Stefan-Boltzmann law, corresponding to $\tau_1 = 0$.

In actual physical situations it is expected that τ (related to photon collisions in the wires) is much shorter than τ_1 (related to photon collisions with suspended particles in the environment).

3.2 Linear waves for the flux-limited heat exchange

In this subsection we consider the propagation of linear waves in the case of the flux-limited lateral heat exchange (2.3), which can be approximated to the third-order in the Taylor's series

$$
f(T) = \frac{\sigma(T - T_0)}{\sqrt{1 + \left(\frac{\sigma'(T-T_0)}{T}\right)^2}} \simeq \sigma(T - T_0) - \frac{\sigma \sigma'^2}{2T_0^2}(T - T_0)^3. \tag{3.15}
$$

Thus, the steady-state system (3.1) leads to

$$
\frac{\partial^2}{\partial z^2}T - \frac{2\sigma}{\lambda r}\left[(T-T_0) - \frac{\sigma'^2}{2T_0^2}(T-T_0)^3\right] = 0 \tag{3.16}
$$

for the steady temperature along the wire. In the appendix A we have studied the above equation (3.16) and, apart from the constant solution T_0, we have found some non trivial solutions for the above equation.

Let's consider now the propagation of linear waves and look for their dispersion relation. According to the linearization (3.3) we find

$$
f(T) = \sigma\left(T^{(0)} - T_0\right) - \frac{\sigma\sigma'^2}{2T_0^2}(T^{(0)}-T_0)^3 + \left[\sigma - \frac{3\sigma\sigma'^2}{2T_0^2}(T^{(0)}-T_0)^2\right](T-T_0) \tag{3.17}
$$

Now, we follow the same procedure used in the previous subsection, namely the propagation of harmonic plane waves along the z-axis. For this aim, we consider again $\Gamma(z,t) = (T, q, q_t)$ and (3.5), where $\Gamma^{(0)}(z)$ is the stationary solution which have been found in the Appendix and given in terms of the elliptic function and in the degenerate cases by (A.60), (A.61) and (A.62).

Substituting (3.5) in the linearized system, we find

$$
\begin{cases}
-i\omega\rho c\tilde{T} + iK'\tilde{q} + \dfrac{2}{r}\tilde{q}_t = 0 \\
i\lambda K'\tilde{T} + (1 - i\omega\tau)\tilde{q} = 0 \\
\left[\sigma - \dfrac{3\sigma\sigma'^2}{2T_0^2}\left(T^{(0)} - T_0\right)^2\right]\tilde{T} + (1 - i\omega\tau_1)\tilde{q}_t = 0
\end{cases} \tag{3.18}
$$

where $K' = dK/dz$ is the derivative of K with respect to z.

The dispersion relation is found from the system (3.18):

$$
\begin{aligned}
-T_0\Big(c\rho r\omega\left(-i\tau\tau_1\omega^2 + i + \omega(\tau+\tau_1)\right) + \lambda r K'^2(i\tau_1\omega - 1) + 2\sigma\left(3\sigma'^2T^{(0)} + 1\right)(i\tau\omega - 1)\Big) + \\
+3\sigma\sigma'^2T_0^2(i\tau\omega - 1) + 3\sigma\sigma'^2T^{(0)2}(i\tau\omega - 1) = 0
\end{aligned} \tag{3.19}
$$

Here we can make the same comments we have written below the dispersion relation (3.7) and assume that $K(z) = k_r(z) + ik_s(z)$. When we split the dispersion relation into real and imaginary part we need to take into account whether $T^{(0)}(z)$ is a complex function. As seen in the Appendix, this is the case and what we have seen is that (3.19) leads to a very hard solution for k_r and k_s. For this reason, we consider the constant solution T_0 (the temperature at equilibrium) and we assume that $k_s'^2$ is small enough to be neglected in (3.19). The real and imaginary part are

$$
\begin{cases}
-c\rho\omega^2(\tau+\tau_1) - \lambda k_s^2 + \dfrac{2\sigma}{r} + \lambda k_r^2 + 2\lambda\tau_1\omega k_r k_s = 0 \\
\omega\left(c\rho r\left(\tau\tau_1\omega^2 - 1\right) + \lambda r\tau_1 k_s^2 - 2\sigma\tau\right) + 2\lambda r k_r k_s - \lambda r\tau_1\omega k_r^2 = 0
\end{cases} \tag{3.20}
$$

Thus, we find two expressions for the phase-velocity $v = \omega/k_r$ and the spatial attenuation coefficient k_s:

$$
\begin{cases}
v = \dfrac{\omega}{k_r} = \pm \dfrac{U_0}{\sqrt{1 - \frac{2\sigma}{\omega^2 c\rho r\tau} \frac{\left(\tau\tau_1\omega^2+1\right)}{\left(\tau_1^2\omega^2+1\right)}}} \\
k_s = \pm \dfrac{1 + 2\sigma \frac{(\tau-\tau_1)}{c\rho r\left(\tau_1^2\omega^2+1\right)}}{2U_0\tau\sqrt{1 - \frac{2\sigma}{\omega^2 c\rho r\tau} \frac{\left(\tau\tau_1\omega^2+1\right)}{\left(\tau_1^2\omega^2+1\right)}}}
\end{cases}
\tag{3.21}
$$

where $U_0 = \sqrt{\dfrac{\lambda}{\rho c\tau}}$ is the Maxwell-Cattaneo velocity. The existence of the above solutions is subordinated to the positive value of the argument of the root-square

$$
\frac{2\sigma}{\omega^2 c\rho r\tau} \frac{\left(\tau\tau_1\omega^2+1\right)}{\left(\tau_1^2\omega^2+1\right)} \le 1
\tag{3.22}
$$

In the case of high frequencies ω for the propagating waves, this condition leads to $\omega^2 \ge \omega_{min}^2 = \dfrac{2\sigma}{c\rho r\tau_1}$.

For $\omega \gg 1$ (high frequency limit) the velocity of the waves becomes

$$
\begin{cases}
v = \dfrac{\omega}{k_r} = \pm U_0 \\
k_s = \pm \dfrac{1}{2\tau U_0}
\end{cases}
\tag{3.23}
$$

For $\tau = \tau_1$, we have

$$
\begin{cases}
v = \dfrac{\omega}{k_r} = \pm \dfrac{U_0}{\sqrt{1 - \frac{2\sigma}{\omega^2 c\rho r\tau}}} \\
k_s = \pm \dfrac{1}{2U_0\tau\sqrt{1 - \frac{2\sigma}{\omega^2 c\rho r\tau}}}
\end{cases}
\tag{3.24}
$$

For $\tau_1 \gg \tau$ and $\tau \simeq 0$, in such a way that $\tau\tau_1\omega^2 \ll 1$, the system becomes

$$
\begin{cases}
v = \dfrac{\omega}{k_r} = \pm \dfrac{U_0}{\sqrt{1 - \frac{2\sigma}{\omega^2 c\rho r\tau} \frac{1}{\left(\tau_1^2\omega^2+1\right)}}} \\
k_s = \pm \dfrac{1 - 2\sigma \frac{\tau_1}{c\rho r\left(\tau_1^2\omega^2+1\right)}}{2U_0\tau\sqrt{1 - \frac{2\sigma}{\omega^2 c\rho r\tau} \frac{1}{\left(\tau_1^2\omega^2+1\right)}}}
\end{cases}
\tag{3.25}
$$

and finally, for $\tau \gg \tau_1$ and $\tau_1 \simeq 0$ in such a way that $\omega^2\tau_1^2 \ll 1$, they become

$$
\begin{cases}
v = \dfrac{\omega}{k_r} = \pm \dfrac{U_0}{\sqrt{1 - \frac{2\sigma\left(\tau\tau_1\omega^2+1\right)}{\omega^2 c\rho r\tau}}} \\
k_s = \pm \dfrac{1 + 2\sigma \frac{\tau}{c\rho r\left(\tau_1^2\omega^2+1\right)}}{2U_0\tau\sqrt{1 - \frac{2\sigma\left(\tau\tau_1\omega^2+1\right)}{\omega^2 c\rho r\tau}}}
\end{cases}
\tag{3.26}
$$

Note that the velocity in (3.24), (3.25) and (3.26) is higher than the Maxwell-Cattaneo velocity. In (3.24) it tends to U_0 when ω tends to infinity; instead (3.25) and (3.26) cannot be extrapolated to ω infinite, because of the condition that $\tau\tau_1\omega^2 \ll 1$ in (3.24) and $\tau_1^2 w^2 \ll 1$ in (3.26). It is also interesting to note that if τ_1 is zero, the speed in (3.26) has the same form as in (3.24), corresponding to $\tau = \tau_1$. On the other side, note that the characteristic attenuation lengths (recriprocal of k_s) are shorter than in the Maxwell-Cattaneo case.

4 Nonlinear heat waves

In this section we consider the propagation of the nonlinear waves, i.e. temperature waves of amplitude sufficiently large that nonlinear terms in $f(T)$ given by (2.3) cannot be neglected (we consider that longitudinal heat transfer is linear, namely that ρ, c and τ do not depend on T). By differentiating the first equation of (2.1) with respect to the time and using the second and the third equations, the following equation for the temperature is found

$$\rho c\left(\frac{\partial^2 T}{\partial t^2}+\tau_1\frac{\partial^3 T}{\partial t^3}\right)=\frac{\lambda}{\tau}\left(\frac{\partial^2}{\partial z^2}T+\tau_1\frac{\partial^2}{\partial z^2}\frac{\partial T}{\partial t}\right)-\frac{\rho c}{\tau}\left(\frac{\partial T}{\partial t}+\tau_1\frac{\partial^2 T}{\partial t^2}\right)-\frac{2}{\tau r}f(T)-\frac{2}{r}\frac{\partial f(T)}{\partial t}. \tag{4.1}$$

For $\tau_1 = \tau$ we find the same mathematical model proposed in Ref. [13] and [14], when $f(T)$ is given by (2.2) or (2.3), respectively.

4.1 Nonlinear waves for the Stefan-Boltzmann heat exchange

When $f(T)$ is given by (2.2), the nonlinear equation for the temperature (4.1) takes the form

$$\rho c\left(\frac{\partial^2 T}{\partial t^2}+\tau_1\frac{\partial^3 T}{\partial t^3}\right)=\frac{\lambda}{\tau}\left(\frac{\partial^2}{\partial z^2}T+\tau_1\frac{\partial^3 T}{\partial z^2\partial t}\right)-\frac{\rho c}{\tau}\left(\frac{\partial T}{\partial t}+\tau_1\frac{\partial^2 T}{\partial t^2}\right)-\frac{2}{r\tau}\sigma_{SB}(T^4-T_0^4)-\frac{2}{r}\sigma_{SB}4T^3\frac{\partial T}{\partial t} \tag{4.2}$$

Following the usual procedure for investigating on the propagation of nonlinear waves [13], we use the similarity variable

$$\xi = kz - \omega t \tag{4.3}$$

in order to transform our equation from a partial differential equation to an ordinary differential equation. Indeed, the transformation (4.3) leads to the rules: $\frac{\partial}{\partial t} = -\omega\frac{\partial}{\partial \xi}$ and $\frac{\partial}{\partial z} = k\frac{\partial}{\partial \xi}$. Thus, equation (4.2) becomes

$$\omega\tau_1\left(k^2\frac{\lambda}{\tau}-\rho c\omega^2\right)\frac{\partial^3 T}{\partial \xi^3}+\left(\rho c\omega^2-\frac{\lambda k^2}{\tau}+\frac{\rho c\tau_1\omega^2}{\tau}\right)\frac{\partial^2 T}{\partial \xi^2}-\frac{\rho c\omega}{\tau}\frac{\partial T}{\partial \xi}+\frac{2}{r\tau}\sigma_{SB}(T^4-T_0^4)-\frac{8}{r}\omega\sigma_{SB}T^3\frac{\partial T}{\partial \xi}=0 \tag{4.4}$$

If we replace $T = (T - T_0) + T_0$ in equation (4.4) and write $u = \dfrac{\Delta T}{T_0}$, we find

$$\begin{aligned}&\omega\tau_1\left(k^2\frac{\lambda}{\tau}-\rho c\omega^2\right)\frac{\partial^3 u}{\partial \xi^3}+\left(\rho c\omega^2-\frac{\lambda k^2}{\tau}+\frac{\rho c\tau_1\omega^2}{\tau}\right)\frac{\partial^2 u}{\partial \xi^2}-\frac{\rho c\omega}{\tau}\frac{\partial u}{\partial \xi}+\\&+\frac{2}{r\tau}\sigma_{SB}T_0^3(u^4+4u^3+6u^2+4u)-\frac{8}{r}\omega\sigma_{SB}T_0^3(u^3+3u^2+3u+1)\frac{\partial u}{\partial \xi}=0\end{aligned} \tag{4.5}$$

The general solution of (4.5) is a hard task to find, for this reason we will give some particular solutions.

4.1.1 Exact solutions from reduced first-order equation (4.5)

In this subsection we consider some nonlinear heat waves arising from equation (4.5). First, we set zero the first term of (4.5), namely the coefficient of the third-order term, and we find the following dispersion relation between k and ω:

$$\omega\tau_1\left(k^2 - \frac{c\rho\tau\omega^2}{\lambda}\right) = 0 \tag{4.6}$$

which leads to the solutions $\omega = 0$, $\omega = \pm k\sqrt{\frac{\lambda}{c\rho\tau}}$. The former refers to the stationary wave whereas the latter leads to speeds of the nonlinear waves:

$$v = \pm U_0 = \pm\sqrt{\frac{\lambda}{c\rho\tau}} \tag{4.7}$$

which is the Maxwell-Cattaneo velocity, already found for the propagation of linear waves.

Case $\omega = 0$

Let's consider first the case $\omega = 0$, which simplifies further the equation to

$$2\sigma_{SB}T_0^3\left(u^4 + 4u^3 + 6u^2 + 4u\right) - k^2\lambda r u'' = 0 \tag{4.8}$$

which we multiply by u' and integrate

$$u'^2 = 4\frac{\sigma_{SB}T_0^3}{k^2\lambda r}\left(\frac{1}{5}u^5 + u^4 + 2u^3 + 2u^2 + C_1\right) \tag{4.9}$$

with C_1 an integration constant. Note that (4.9) is exactly equation (A.51) exception made for the function u which is $u = T/T_0$ instead of $u = (T - T_0)/T_0$ and the independent variable z instead of $\xi = kz$, because $\omega = 0$. From the above equation (4.9) we find

$$\int \frac{1}{\sqrt{\frac{1}{5}u^5 + u^4 + 2u^3 + 2u^2 + C_1}}du = 2\sqrt{\frac{\sigma_{SB}T_0^3}{\lambda r}}z + C_2 \tag{4.10}$$

C_2 being a further integration constant.

The left-hand side in (4.10) is again an inversion Jacobi problem, which defines the hyper-elliptic functions of genus 2 when the 5 zeros of the polynomial in the root square are distinct. There are only two values of C_1 in which the polynomial $\frac{1}{5}u^5 + u^4 + 2u^3 + 2u^2 + C_1$ has one double zero: the case $C_1 = 0$ (which leads to the double zero $u = 0$) and the case $C_1 = -8/5$ (which leads to the double zero $u = -2$). For these two cases, the polynomial in the left-hand side in (4.10) has 4 different zeros, and the inversion Jacobi problem leads to elliptic solutions.

More precisely, for $C_1 = 0$ the exact solution (4.10) becomes

$$-\frac{\left[2u(u_3 - u_2)\sqrt{-\frac{(u-u_2)(u-u_3)}{(u_2-u_3)^2}}\sqrt{\frac{u}{u_3}}\Pi\left(1 - \frac{u_2}{u_3};\sin^{-1}\left(\sqrt{\frac{u_3-u}{u_3-u_2}}\right)\mid -\frac{u_2-u_3}{u_3}\right)\right]}{\left[\sqrt{\frac{u^5}{5} + u^4 + 2u^3 + 2u^2}u_3\right]} = 2\sqrt{\frac{\sigma_{SB}T_0^3}{\lambda r}}z + C_2 \tag{4.11}$$

where u_j is the $j-th$ zero of the polynomial $P[u]=u^3+5u^2+10u+10$, which are $u_1\simeq-2.65$, $u_2\simeq-1.17-i1.55$ and $u_3\simeq-1.17+i1.55$. The function Π is the elliptic integral of the third kind. The solution (4.11) can be written by inserting the zeros of the polynomial $P[u]$:

$$-\frac{(0.82\;-0.62i)u\sqrt{u(u+2.35)+3.77}\sqrt{(0.32\;-0.34i)u+(0.856\;-0.9i)}}{\sqrt{u^2\left(u\left(0.2u^2+u+2\right)+2\right)}}$$

$$\Pi\left(1.279\;-0.96i;\sin^{-1}\left(\sqrt{(0.\;+0.32i)u+(0.5\;+0.38i)}\right)\Big|\,1.05\;+i\right)=2\sqrt{\frac{\sigma_{SB}T_0^3}{\lambda r}}z+C_2 \tag{4.12}$$

For $C_1=-8/5$ the exact solution (4.10) becomes the elliptic function $u(z)$

$$\frac{2\sqrt{5}(u+2)(u_2-u_3)\sqrt{\frac{u-u_1+2}{u_3-u_1}}\sqrt{-\frac{(u-u_2+2)(u-u_3+2)}{(u_2-u_3)^2}}\Pi\left(1-\frac{u_2}{u_3};\sin^{-1}\left(\sqrt{\frac{u-u_3+2}{u_2-u_3}}\right)|\frac{u_2-u_3}{u_1-u_3}\right)}{\sqrt{(u+2)^2\left(u^3+u^2+2u-2\right)}u_3}=2\sqrt{\frac{\sigma_{SB}T_0^3}{\lambda r}}z+C_2 \tag{4.13}$$

where Π is the elliptic integral of the third kind and u_j is the $j-th$ zero of the polynomial $P[u]=u^3-5u^2+10u-10$, which are $u_1\simeq2.65$, $u_2\simeq1.17-i1.55$ and $u_3\simeq1.17+i1.55$. Note that the elliptic solution written for the case $\omega=0$ can be read as solutions of the equation (A.51) by changing the variables: $u=T/T_0$ instead of $u=(T-T_0)/T_0$ and z instead of $\xi=kz$.

Case $\omega=\pm k\sqrt{\dfrac{\lambda}{c\rho\tau}}$

Let's consider now the cases for $\omega\neq0$. We also assume that $\tau_1/\tau\ll1$ in such a way that the coefficient in $u_{\xi\xi}$ is zero too. Thus, equation (4.5) becomes

$$2\sigma_{SB}T_0^3u\left(u^3+4u^2+6u+4\right)-\frac{k\sqrt{\lambda}u'\left(c\rho r+8\sigma_{SB}\tau T_0^3+8\sigma_{SB}\tau T_0^3u^3+24\sigma_{SB}\tau T_0^3u^2+24\sigma_{SB}\tau T_0^3u\right)}{\sqrt{c}\sqrt{\rho}\tau^{1/2}}=0 \tag{4.14}$$

which yields the following exact solution implicitly defined

$$\frac{8\sqrt{c}\sqrt{\rho}\sigma_{SB}\sqrt{\tau}T_0^3}{k\sqrt{\lambda}}\xi+4c_1=c\rho r\left(\log\left(\frac{u}{u+2}\right)-2\tan^{-1}(u+1)\right)+8\sigma_{SB}\tau T_0^3\log\left((u^2+2u+2)u(u+2)\right) \tag{4.15}$$

with velocity given by (4.7), namely the Maxwell-Cattaneo velocity.

4.2 Nonlinear waves from the flux-limited heat exchange

Let's consider now (4.1) with the function $f(T)$ given by the flux limiter proposal (2.3). For the sake of simplicity, we use the same approximation for $f(T)$ used in (3.15), namely $f(T)\simeq\sigma(T-T_0)-\frac{\sigma\sigma'^2}{2T_0^2}(T-T_0)^3$, and the equation (4.1) becomes

$$\rho c\left(\frac{\partial^2T}{\partial t^2}+\tau_1\frac{\partial^3T}{\partial t^3}\right)=\frac{\lambda}{\tau}\left(\frac{\partial^2}{\partial z^2}T+\tau_1\frac{\partial^2}{\partial z^2}\frac{\partial T}{\partial t}\right)-\frac{\rho c}{\tau}\left(\frac{\partial T}{\partial t}+\tau_1\frac{\partial^2T}{\partial t^2}\right)$$

$$-\frac{2}{\tau r}\left(\sigma(T-T_0)-\frac{\sigma\sigma'^2}{2T_0^2}(T-T_0)^3\right)-\frac{2}{r}\left(\sigma-\frac{3\sigma\sigma'^2}{2T_0^2}(T-T_0)^2\right)\frac{\partial T}{\partial t}. \tag{4.16}$$

By means of (4.3), equation (4.16) becomes

$$\omega\tau_1\left(k^2\frac{\lambda}{\tau}-\rho c\omega^2\right)\frac{\partial^3 T}{\partial\xi^3}+\left(\rho c\omega^2-\frac{\lambda k^2}{\tau}+\frac{\rho c\tau_1\omega^2}{\tau}\right)\frac{\partial^2 T}{\partial\xi^2}-\frac{\rho c\omega}{\tau}\frac{\partial T}{\partial\xi}$$
$$+\frac{2}{\tau r}\left(\sigma(T-T_0)-\frac{\sigma\sigma'^2}{2T_0^2}(T-T_0)^3\right)-\frac{2\omega}{r}\left(\sigma-\frac{3\sigma\sigma'^2}{2T_0^2}(T-T_0)^2\right)\frac{\partial T}{\partial\xi}=0. \tag{4.17}$$

In the above equation (4.17), we can set $u=\dfrac{T-T_0}{T_0}$ and we find

$$\omega\tau_1\left(k^2\frac{\lambda}{\tau}-\rho c\omega^2\right)\frac{\partial^3 u}{\partial\xi^3}+\left(\rho c\omega^2-\frac{\lambda k^2}{\tau}+\frac{\rho c\tau_1\omega^2}{\tau}\right)\frac{\partial^2 u}{\partial\xi^2}-\frac{\rho c\omega}{\tau}\frac{\partial u}{\partial\xi}$$
$$+\frac{2\sigma}{\tau r}\left(u-\frac{\sigma'^2}{2}u^3\right)-\frac{2\sigma\omega}{r}\left(1-\frac{3\sigma'^2}{2}u^2\right)\frac{\partial u}{\partial\xi}=0. \tag{4.18}$$

As we have done with (4.5), we consider the case in which the coefficient of the first term in (4.18) (corresponding to the third-order derivatives of u) vanishes. This leads to $\omega=0$ and $\omega=\pm k\sqrt{\dfrac{\lambda}{c\rho\tau}}$.

By means of the first condition $\omega=0$ we find the stationary solution given by

$$\lambda k^2 r\frac{\partial^2 u}{\partial\xi^2}=2\sigma\left(u-\frac{\sigma'^2}{2}u^3\right). \tag{4.19}$$

Equation (4.19) can be integrated after multipling by $\dfrac{\partial u}{\partial\xi}$

$$\left(\frac{\partial u}{\partial z}\right)^2=-\frac{\sigma\sigma'^2}{2\lambda r}u^4+2\frac{\sigma}{\lambda r}u^2+2\frac{C_1}{\lambda r}. \tag{4.20}$$

where we have used that $\partial/\partial\xi=(1/k)\partial/\partial z$ because of the condition $\omega=0$.

The integration of equation (4.20) can be handled by the elliptic functions [35]. The general solution of the equation (4.20) is

$$u(z)=\frac{\sqrt{\frac{2C_1}{\lambda r}}\,\wp'(z;g_2,g_3)}{2\left(\wp(z;g_2,g_3)-\frac{\sigma}{6\lambda r}\right)^2+\frac{C_1\sigma\sigma'^2}{2\lambda^2 r^2}} \tag{4.21}$$

where $\wp(z;g_2,g_3)$ is the Weierstrass elliptic function of variable z and invariants g_2 and g_3 given by [35]

$$g_2=\frac{\sigma\left(\sigma-3C_1\sigma'^2\right)}{3\lambda^2 r^2}$$

and

$$g_3=-\frac{\sigma^2\left(\sigma+9C_1\sigma'^2\right)}{27\lambda^3 r^3}$$

and the discriminant $\Delta=g_2^3-27g_3^2$ is

$$\Delta=-\frac{C_1\sigma^3\sigma'^2\left(\sigma+C_1\sigma'^2\right)^2}{\lambda^6 r^6}$$

The discriminant Δ and the invariants g_2 and g_3 are useful to classify the solutions. For $\Delta \neq 0$ the Weierstrass elliptic function is a double-periodic elliptic function, whereas for $\Delta = 0$ it reduces to elementary functions of genus zero, as we will see below. The condition $\Delta = 0$ is fulfilled when

$$C_1 = 0; \qquad C_1 = -\frac{\sigma}{\sigma'^2}$$

which simplify the solution (4.21) accordingly.

In particular for $C_1 = 0$, the solution (4.21) becomes:

$$u(z) = \pm \frac{2i}{\sigma'} \operatorname{csch}\left(\sqrt{\sigma}\left(\frac{\sqrt{2}z}{\sqrt{\lambda r}} - 2c_2\right)\right) \tag{4.22}$$

with c_2 constant.
Instead, for $C_1 = -\frac{\sigma}{\sigma'^2}$ it is

$$u(z) = \pm \frac{i\sqrt{2}}{\sigma'} \tan\left(\frac{\sqrt{\sigma}z}{\sqrt{\lambda r}} \pm i\sqrt{2}c_2\sigma'\right) \tag{4.23}$$

Let's consider now the case $\omega = \pm k\sqrt{\frac{\lambda}{c\rho\tau}}$ which leads to the velocity $v = \pm U_0$. If we also assume that $\frac{\tau_1 k^2 \lambda}{\tau^2} \ll 1$, namely the term $\frac{\partial^2 u}{\partial \xi^2}$ in (4.18) is negligible, then (4.18) becomes:

$$-\frac{\rho c \omega}{\tau}\frac{\partial u}{\partial \xi} + \frac{2\sigma}{\tau r}\left(u - \frac{\sigma'^2}{2}u^3\right) - \frac{2\sigma\omega}{r}\left(1 - \frac{3\sigma'^2}{2}u^2\right)\frac{\partial u}{\partial \xi} = 0 \tag{4.24}$$

The solutions of (4.24) are

$$\frac{1}{4}(c\rho r - 4\sigma\tau)\log\left(2 - u(\xi)^2\sigma'^2\right) + \frac{1}{2}(-c\rho r - 2\sigma\tau)\log(u(\xi)) = -\frac{\sqrt{c}\sqrt{\rho}\sigma\sqrt{\tau}}{k\sqrt{\lambda}}\xi + c_3 \tag{4.25}$$

for $\omega = k\sqrt{\frac{\lambda}{c\rho\tau}}$ and c_3 constant, and

$$\frac{1}{4}(c\rho r - 4\sigma\tau)\log\left(2 - u(\xi)^2\sigma'^2\right) + \frac{1}{2}(-c\rho r - 2\sigma\tau)\log(u(\xi)) = \frac{\sqrt{c}\sqrt{\rho}\sigma\sqrt{\tau}}{k\sqrt{\lambda}}\xi + c_3 \tag{4.26}$$

for $\omega = -k\sqrt{\frac{\lambda}{c\rho\tau}}$ and c_3 constant.

4.3 Existence of soliton waves

In the above subsections we have found some nonlinear heat waves for our model, stationary or propagating with the Maxwell-Cattaneo velocity. Here we analyze the existence of solitons, because of their peculiarity to travel without changing their shape. Indeed, soliton is a non-linear travelling wave, $U(\xi = kx - \omega t)$, solution of a non-linear evolution equation (partial differential equation), which at every moment of time is localized in a bounded domain of space, such that the size of the domain remains bounded in time.

In our previous papers [13, 14] we have found solitons, but assuming that $\tau_1 = \tau$ in such a way that equation (4.2) becomes:

$$\rho c \frac{\partial^2 T}{\partial t^2} + = \frac{\lambda}{\tau} \frac{\partial^2}{\partial z^2} T - \frac{\rho c}{\tau} \frac{\partial T}{\partial t} - \frac{2}{\tau r} f(T) \tag{4.27}$$

and, in the limit case that $\lambda \to \infty$, $\tau \to \infty$ but λ/τ finite, to the following equation

$$\rho c \frac{\partial^2 T}{\partial t^2} + = \frac{\lambda}{\tau} \frac{\partial^2}{\partial z^2} T - \frac{2}{\tau r} f(T) \tag{4.28}$$

Now, we apply the the Auxiliary equation method [36], [37], [38] to the nonlinear equation (4.5) for the case of Stefan-Boltzmann heat exchange and to the nonlinear equation (4.18) for the flux-limited heat exchange in order to search the "sech" and "tanh"-type soliton (the same method used in [13, 14]).

4.3.1 Auxiliary method for travelling waves

In this section we recall the main steps of the Auxiliary method [36], [37], [38], which allows to find some exact travelling wave solutions of the $1+1$ nonlinear equation:

$$E(z, t, u, u_z, u_t, ...) = 0. \tag{4.29}$$

The first step is to transform equation (4.29) in an ordinary nonlinear equation, $E(\xi, u, u_\xi, u_{\xi\xi}, ...) = 0$, by means of the transformation $\xi = kz - \omega t$, which is typical for searching for travelling wave solutions.

The second step is to choose for $u(\xi)$ a polynomial form

$$u(\xi) = \sum_{i=0}^{n} u_i y(\xi)^i, \tag{4.30}$$

where u_i are constants to be determined and the functions $y(\xi)$ are solutions of the auxiliary equation. The first choice of the auxiliary equation is the Riccati equation [36]:

$$y(\xi)' = 1 - y(\xi)^2, \tag{4.31}$$

which is solved by the function $y(\xi) = \tanh(\xi)$, having the form of a propagating front. Another interesting example is

$$y(\xi)'^2 = y(\xi)^2(1 - y(\xi)^2) \tag{4.32}$$

which has the solution $y(\xi) = \mathrm{sech}(\xi)$, having the form of a propagating pulse.

The third step of the method is to determine the coefficients u_i in the expression (4.30). This is achieved after the introduction of (4.30) into (4.29) taking into account of (4.31) or (4.32). The value of n (the maximum value of the exponents of $y(\xi)$ in (4.30)) is determined by balancing the higher-order linear term with the higher nonlinear term of the equation (4.29).

4.3.2 Soliton waves

First of all we have applied the Auxiliary equation method to the nonlinear equation (4.5), both with the auxiliary equation (4.31) and (4.32), as well as the case for $\lambda \to \infty$ and $\tau \to \infty$ but keeping λ/τ finite. The result was that they are not satisfied by the solution (4.30).

Let's follow the same procedure for the equation (4.18), namely for the flux-limited exchange case. By means of the auxiliary equation (4.32) we find the two stationary soliton solutions

$$u(z) = \pm \frac{2}{\sigma'} \operatorname{sech}(kz) \tag{4.33}$$

with the frequency $\omega = 0$ (a non-propagating solution) and $k^2 = \dfrac{2\sigma}{r\lambda}$.

We apply the method to the same equation but with the auxiliary equation (4.31) and we find

$$u(z,t) = \mp \frac{1}{\sqrt{2}\sigma'} \tanh(kz - \omega t) \pm \frac{1}{\sqrt{2}\sigma'} \tag{4.34}$$

with $\lambda = 0$, $\omega = -\dfrac{1}{6\tau_1}$ and $\rho = \dfrac{9\sigma\tau_1}{cr}$, and

$$u(z,t) = \pm \frac{1}{\sqrt{2}\sigma'} \tanh(kz - \omega t) \pm \frac{1}{\sqrt{2}\sigma'} \tag{4.35}$$

with $\lambda = 0$, $\omega = \dfrac{1}{6\tau_1}$ and $\rho = \dfrac{9\sigma\tau_1}{cr}$.

We have also found two stationary soliton solutions

$$u(z) = \pm \frac{\sqrt{2}}{\sigma'} \tanh(kz) \tag{4.36}$$

with the further condition that $k^2 = -\dfrac{\sigma}{\lambda r}$. The positive parameters σ (the heat exchange coefficient), λ (thermal conductivity) and r (the radius of the cylinder) lead to a imaginary wavenumber k. In this case, the function $\tanh(kz)$ becomes $\mathrm{i}\tan(\Im(k)z)$ ($\Im$ being the imaginary part of k).

We have also found these other solitons:

$$u(z) = \mp \frac{1}{\sqrt{2}\sigma'} \tanh(-\omega t) \pm \frac{1}{\sqrt{2}\sigma'} \tag{4.37}$$

with $\omega = -\dfrac{1}{6\tau_1}$ and $\rho = \dfrac{9\sigma\tau_1}{cr}$, and

$$u(z) = \pm \frac{1}{\sqrt{2}\sigma'} \tanh(-\omega t) \pm \frac{1}{\sqrt{2}\sigma'} \tag{4.38}$$

with $\omega = \dfrac{1}{6\tau_1}$ and $\rho = \dfrac{9\sigma\tau_1}{cr}$.

In the limit case $\lambda \to \infty$, $\tau \to \infty$ but λ/τ finite, equation (4.18) becomes

$$\omega\tau_1 \left(k^2 \frac{\lambda}{\tau} - \rho c \omega^2\right) \frac{\partial^3 u}{\partial \xi^3} + \left(\rho c \omega^2 - \frac{\lambda k^2}{\tau}\right) \frac{\partial^2 u}{\partial \xi^2} - \frac{2\sigma\omega}{r} \left(1 - \frac{3\sigma'^2}{2} u^2\right) \frac{\partial u}{\partial \xi} = 0. \tag{4.39}$$

When we apply the Auxiliary equation method to the (4.39), only the auxiliary equation (4.31) leads to solitons. More precisely, we find the following soliton solutions for (4.39):

$$u(z,t) = \pm \frac{2\sqrt{6}\tau_1\omega}{\sqrt{\sigma'^2(12\tau_1^2\omega^2+1)}}\tanh(kz-\omega t) \pm \frac{\sqrt{\frac{2}{3}}}{\sqrt{\sigma'^2\left(12\tau_1^2\omega^2+1\right)}} \tag{4.40}$$

with

$$k = \pm\sqrt{\frac{\tau\omega^2\left(12\tau_1\left(c\rho r\tau_1\omega^2-\sigma\right)+c\rho r\right)}{\lambda r\left(12\tau_1^2\omega^2+1\right)}} \tag{4.41}$$

which leads to the following velocity

$$v = \frac{\omega}{k} = \pm\sqrt{\frac{\lambda r\left(12\tau_1^2\omega^2+1\right)}{\tau\left(12\tau_1\left(c\rho r\tau_1\omega^2-\sigma\right)+c\rho r\right)}}$$

and the soliton

$$u(z,t) = \pm \frac{2\sqrt{6}\tau_1\omega}{\sqrt{\sigma'^2(12\tau_1^2\omega^2+1)}}\tanh(-\omega t) \pm \frac{\sqrt{\frac{2}{3}}}{\sqrt{\sigma'^2\left(12\tau_1^2\omega^2+1\right)}} \tag{4.42}$$

with $k=0$ and

$$\omega^2 = \frac{12\sigma\tau_1 - c\rho r}{12c\rho r\tau_1^2}$$

In this subsection we have found the soliton-like solutions of the nonlinear evolution equations. The solitons (4.33) and (4.36) are stationary solutions, independent of time, and for this reasons they are not propagating waves. The soliton solutions (4.37), (4.38) and (4.42) are instead independent of the variable z, but they depend on t, which mimic localized vibrations. The soliton (4.34) requires a null thermal conductivity, which may be a strong constraints for the existing materials, but it may be useful for future materials. For our aims, the most interesting solitons are given by (4.40), which have been found assuming high thermal conductivity λ and relaxation time τ but finite λ/τ.

The existence of the soliton (4.40) is related to the existence of the root square in (4.41), namely the requirements

$$\tau_1^2\omega^2 + \frac{1}{12} \geq \frac{\sigma\tau_1}{c\rho r},$$

which leads to two solitons: one propagating forward (+) and one propagating backwards (−).

5 Applications

In the previous sections we have found both some linear and nonlinear waves for the proposed model (2.1) with two relaxational times.

In this section we plot some of the nonlinear waves for suitable choice of the parameters.

5.1 The Stefan-Boltzmann law

Let's consider a concrete example to illustrate these results. Let's choose a Si thin wire with the same parameters used in Ref. [13]: diameter of the wire $d = 15000$ nm, temperature $T_0 = 300$ K, and $\sigma_{SB} = 5.67 \times 10^{-8}\mathrm{W}/(\mathrm{m}^2\mathrm{K}^4)$, $\rho = 2330\mathrm{kg}/\mathrm{m}^3$, $\lambda = 148\mathrm{W}/(\mathrm{m\ K})$, $c = 700\mathrm{J}/(\mathrm{kg\,K})$ and $\tau = 50\mathrm{ps} = 5 \cdot 10^{-11}\mathrm{s}$. Thus, the nonlinear thermal wave (4.15) becomes

$$\frac{9.09113 \times 10^{13}}{k}\xi = 12.23\left(\log\left(\frac{u}{u+2}\right) - 2\tan^{-1}(u+1)\right) + 6.1236\times 10^{6}\log\left((u^2+2u+2)u(u+2)\right) \tag{5.43}$$

where we have chosen $c_1 = 0$ and with the Maxwell-Cattaneo velocity U_0 given in (4.7)

$$v = \pm\sqrt{\frac{\lambda}{c\rho\tau}} = 13.47 \text{ m/s} \tag{5.44}$$

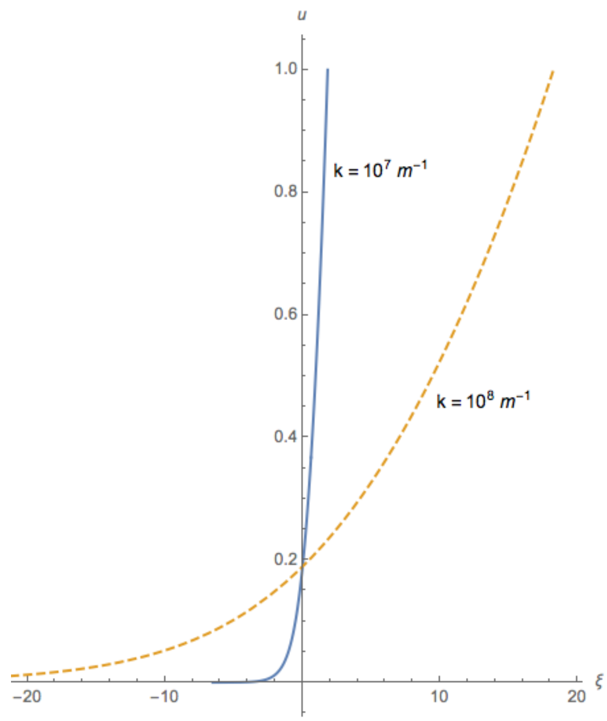

Figure 1: Plot of (5.43) for two values of the wavenumbers: $k = 10^7$ m^{-1} and $k = 10^8$ m^{-1}.

The plot of the above solution (5.43) is given in Figure 1 for $k = 10^7$ m^{-1} and $k = 10^8$ m^{-1}. It is interesting to note the steep increase of the plot around $\xi = 0$ (which is the center of the wave).

5.2 Flux-limited exchange

In this subsection we consider the same example used in the previous subsection, and hence the same parameters. We assume also that $\omega = \sqrt{3}$ as chosen in [14] and $\sigma' = 1$ for the sake of simplicity.

The nonlinear heat waves (4.25) becomes

$$-2.38 \times 10^{-6} k \left(-6.14 \log u(\xi) + 3.03 \log \left(2 - u(\xi)^2\right)\right) = \xi \tag{5.45}$$

where we have chosen $c_1 = 0$. The velocity of this wave follows from the dispersion relation $\omega = \pm k\sqrt{\dfrac{\lambda}{c\rho\tau}}$, which leads to the same velocity (4.7) found for the previos case and hence to the velocity (5.44), the Maxwell-Cattaneo velocity U_0, namely $v = 13.47$ m/s. The plot of the above solution (5.45) is given in Figure 2. Note that the solution looks like a dark soliton solution.

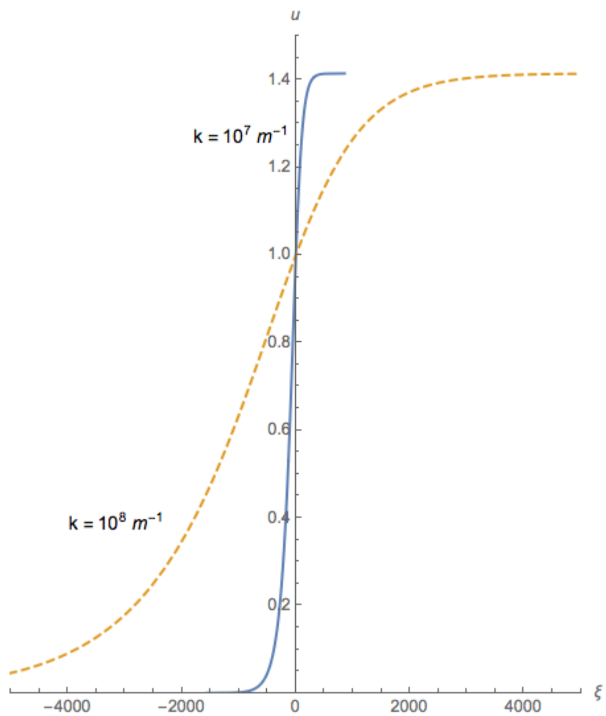

Figure 2: Plot of (5.45) for $\omega = \sqrt{3}$ s, and for two values of the wavenumbers: $k = 10^7$ m^{-1} and $k = 10^8$ m^{-1}.

Finally, we also consider the solitons (4.40) found in Section 4.3. For both solitons we have two speeds

$$v = \frac{\omega}{k} = \pm\sqrt{\frac{\lambda r \left(12\tau_1^2\omega^2 + 1\right)}{\tau\left(12\tau_1\left(c\rho r\tau_1\omega^2 - \sigma\right) + c\rho r\right)}}, \tag{5.46}$$

which allows the soliton to propagate forward and backwards. In the previous section we have found the condition $\tau_1^2\omega^2 + \frac{1}{12} \geq \frac{\sigma\tau_1}{c\rho r}$ for the existence of the speeds (5.46) and hence of the solitons.

For the sake of simplicity, we consider the soliton (4.40) with positive sign, which we may write

$$u(z,t) = \frac{\Delta T}{T_0} = \frac{1}{\sqrt{\sigma'^2(12\tau_1^2\omega^2+1)}}\left[2\sqrt{6}\tau_1\omega \tanh(kz - \omega t) + \sqrt{\frac{2}{3}}\right], \tag{5.47}$$

with k given by (4.41). The following arguments holds for the other solitons. Because of the two horizontal asymptotes of tanh, we assume that one of the asymptote is $u = 0$, namely $T = T_0$, with T_0 the equilibrium temperature. Thus, we find $\tau_1\omega = 1/6$. Let's assume instead that the second asymptote is $u = 0.1$, namely that $T = T_0 + \frac{1}{10}T_0$. This means that the parameter σ' is $\sigma' = 10\sqrt{2}$. The other parameters are those chosen in Ref. [13] for the Si thin wire: diameter of the wire $d = 15000$ nm, temperature $T_0 = 300$ K, $\sigma_{SB} = 5.67 \times 10^{-8}\mathrm{W}/(\mathrm{m}^2\mathrm{K}^4)$, $\rho = 2330\mathrm{kg}/\mathrm{m}^3$, $\lambda = 148\mathrm{W}/(\mathrm{m\ K})$ and $c = 700\mathrm{J}/(\mathrm{kg\,K})$. We instead choose for the two relaxation times $\tau = 5000\mathrm{ps} = 5 \cdot 10^{-9}\mathrm{s}$ and $\tau_1 = 500\mathrm{ps} = 5 \cdot 10^{-10}\mathrm{s}$ in such a way $\tau_1/\tau = 10^{-1}$, and the frequency $\omega = 1/(6\tau_1)$.

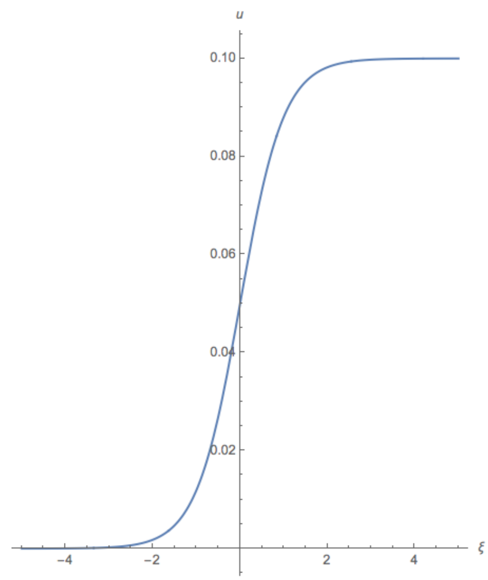

Figure 3: Plot of (5.48) with velocity $v = 151432$ m/s.

Inserting the parameters of our example, the expression of the soliton as function of ξ is

$$u(\xi) = \frac{1}{20}\left(1 + \tanh(\xi)\right) \tag{5.48}$$

with velocity $v = 151432$ m/s. The plot of the above dark soliton (5.48) is given in Figure 3. The same soliton, given in (5.47), is shown in Figure 4 in three dimensional space ztu (left) and in two dimensional space zu for three different times $t = 0$, $t = 0.01$ s and $t = 0.02$ s (right). Note that in the latter figure the soliton propagates to the right, and the temperature change from the $T_0 + \frac{1}{10}T_0$ to the equilibrium temperature T_0. The same conclusions can be achieved choosing dark soliton with negative sign or with negative speed (propagating to the left).

6 Conclusions

In this paper we have considered a mathematical model for studying linear and nonlinear waves along thin wires with lateral heat exchange with the enviroment assuming that both longitudinal heat flux and lateral heat flux satisfy relaxational constitutive equations, with two different relaxation times, τ and τ_1 respectively. From the mathematical point of view, this means that the two relaxation times are high enough that the first terms in the second and in the third equation of (2.1) are not negligible. From the physical point of view, this means that the collision time of phonons (given by τ when heat propagates along the wire) has

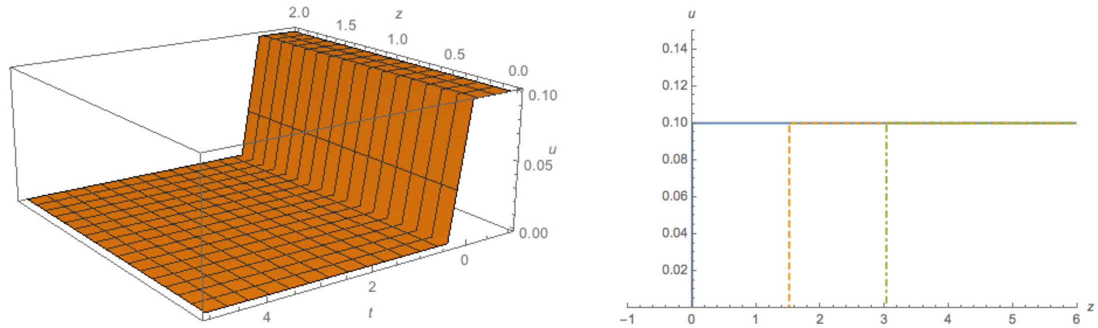

Figure 4: Plot of (5.47) in three dimensional space (left), and in the plane zu for three different times $t = 0$, $t = 0.01$ s and $t = 0.02$ s (right).

to be comparable to the time ω^{-1} (where ω is the frequency of the heat wave), and the time of emission of photons at the surface of the wire, or the time of collisions of particles in the environment against the wall of the wire (given by τ_1 when heat transport across the walls) has to be comparable to the time ω^{-1}, respectively. If these times are experimentally known, then the results obtained in this paper can be experimentally checked. In Section 5 we have used $\tau = 50\text{ps} = 5 \cdot 10^{-11}\text{s}$ for the sake of simplicity. This value is of the order of the collision time of phonons in Si at room temperature.

The model (2.1) generalizes those proposed in our previous papers [13, 14] where we considered the relaxation time τ equal for both longitudinal and lateral heat flux. Here, we have enlarged our analysis to the more realistic physical situation but more involved mathematical problem where τ_1 and τ may be different. We have also considered that the source term for the transversal heat exchange can be given by the Stefan-Boltzmann law or by a nonlinear flux-limited heat exchange.

The paper considers the propagation of linear waves in Section 3, where we have seen that in principle the perturbation of the inhomogeneous stationary solution would be possible but it is quite hard to handle. For this reasons we have considered the perturbation of the homogeneous stationary solution, which leads to the velocity and attenuation of the waves given in (3.9) for the Stefan-Boltzmann law and to (3.21) for nonlinear flux-limited heat exchange. There are two conditions for the existence of these perturbations, which are given in (3.10) and (3.22), respectively.

For both situations we can consider four main cases, namely: $\omega\tau \gg 1$ (high-frequency asymptotic limit); and three high-frequency (but not infinite) cases, corresponding to $\tau = \tau_1$; $\tau_1 \gg \tau$ and $\tau \gg \tau_1$, respectively given by expressions (3.12), (3.13) and (3.14) for the Stefan-Boltzmann law, and (3.24), (3.25) and (3.26) for the flux-limited lateral exchange. In both situations we note that the velocity of the linear waves is always higher than the Maxwell-Cattaneo velocity, exception made for the high frequency limit $\omega \gg 1$, where the velocity is the Maxwell-Cattaneo velocity.

In Section 4 we have looked for nonlinear waves. From the mathematical point of view, the partial differential equation (4.2), or the corresponding ordinary differential equation in moving reference frame, are very difficult to handle for finding the general solution. For this reasons, we have applied some mathematical tools for searching nonlinear exact solution and in particular soliton solutions. We have found some interesting solutions, some of them solitons. We have then applied in Section 5 some of them to a Si thin wire with diameter

$d = 15000$ nm, temperature $T_0 = 300$ K, and $\sigma_{SB} = 5.67 \times 10^{-8}\mathrm{W/(m^2K^4)}$, $\rho = 2330\mathrm{kg/m^3}$, $\lambda = 148\mathrm{W/(m\ K)}$, $c = 700\mathrm{J/(kg\,K)}$ and $\tau = 50\mathrm{ps} = 5 \cdot 10^{-11}\mathrm{s}$. For information transport to relatively long distances it is necessary that the corresponding solitons are stable; thus, in the future, the stability of the solitons found here could be considered, following analogous lines as those studying the stability of optical solitons in references [39, 40, 41].

It is interesting to note that we have found the nonlinear waves (4.15), (4.25) and (4.26) which propagate with the Maxwell-Cattaneo velocity U_0, which is the same velocity found for the high frequency linear perturbation of the homogeneous stationary solution.

By means of the auxiliary equation method we have also found soliton waves. Apart from the stationary solitons, in the limit case $\lambda \to \infty$, $\tau \to \infty$ but λ/τ finite, we have also found the soliton (4.40), plotted in Figure 3, which could be a candidate for information transmission, namely, to transmit a bit if information.

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A Appendix A

In this appendix we discuss the stationary solution of the system (3.1) with the two proposals for $f(T)$, given by (2.2) or by (2.3).

A.1 Stationary solution for the Stefan-Boltzmann law

We first consider the temperature (3.2), where we have used the Stefan-Boltzmann law. We start from

$$\frac{\partial^2}{\partial z^2}T - \frac{2}{\lambda r}\sigma_{SB}(T^4 - T_0^4) = 0 \tag{A.49}$$

Equation(A.49) has more complicate solutions apart from constant solution $T = T_0$.

Let's write (3.2) in terms of $u = \dfrac{T}{T_0}$, namely

$$\frac{\partial^2}{\partial z^2}u - \frac{2}{\lambda r}\sigma_{SB}T_0^3(u^4 - 1) = 0 \tag{A.50}$$

then if we multiply (A.50) by $\dfrac{\partial}{\partial z}u$ and integrate, we find

$$\left(\frac{\partial}{\partial z}u\right)^2 = \frac{4}{5}\frac{\sigma_{SB}}{\lambda r}T_0^3\left(u^5 - 5u + 5c_1\right) \tag{A.51}$$

Equation (A.51) regards the classical inversion Jacobi's problem, which in general defines the funtions $\arcsin(x)$, the elliptic functions such as the Weierstrass function [42, 35] and more in general the hyperelliptic functions [43, 44]. The righ-hand side of equation (A.51) is given by the polynomial $P[u] = \frac{4}{5}\frac{\sigma_{SB}}{\lambda r}T_0^3\left(u^5 - 5u + 5c_1\right)$ of 5 degree with 5 distinct zeros. This implies that (A.51) implicitly defines a hyperelliptic function of genus 2. There are two values for the constant c_1 for which two of the 5 zeros of $P[u]$ are equal. It happens for $5c_1 = 4$ and for $5c_1 = -4$. For these values of c_1 (A.51) degenerates from the hyperelliptic functions (genus 2) to the elliptic functions (genus 1), but never to the simplest functions of genus 0, for the elementary functions. For this reasons, we avoid to report here the exact elliptic solutions of equation (A.51). In the nonlinear section of this paper 5.1 we will find again equation (A.51), but in that situation we will write the exact solution in terms of the elliptic functions.

A.2 Stationary solution for the flux-limiter heat exchange

Now we consider the temperature (3.16), where we have used the flux-limiter heat exchange. We start from

$$\frac{\partial^2}{\partial z^2}T - \frac{2\sigma}{\lambda r}\left[(T - T_0) - \frac{\sigma'^2}{2T_0^2}(T - T_0)^3\right] = 0 \tag{A.52}$$

By setting $\frac{T - T_0}{T_0} = u$, the above equation becomes

$$\frac{\partial^2}{\partial z^2}u + \frac{2\sigma}{\lambda r}\left[-u + \frac{\sigma'^2}{2}u^3\right] = 0. \tag{A.53}$$

Multiplying the above equation (A.53) by $\partial u/\partial z$ and integrating both sides, we find

$$\left(\frac{\partial}{\partial z}u\right)^2 + \frac{2\sigma}{\lambda r}\left(-u^2 + \frac{\sigma'^2}{4}u^4\right) + 2c_1 = 0 \tag{A.54}$$

where c_1 appears from the integration.

The solution (A.54) is an elliptic function because the polynomial $P(u) = -\frac{2\sigma}{\lambda r}\left(-u^2 + \frac{\sigma'^2}{4}u^4\right) + 2c_1$ has 4 different zeros exception made for the degenerate cases, namely when the 4 zeros are not distinct, which happens for $c_1 = 0$ and $c_1 = \frac{\sigma}{\lambda r \sigma'^2}$. Following the same choice for the equation (A.51), we avoid to write the exact solutions of (A.54) in terms of the elliptic functions, apart from the degenerate cases to the elementary functions.

Case $c_1 = 0$

In this case we find for the steady-state profile of $u(z)$

$$u(z) = \pm\frac{2i}{\sigma'}\coth\left(\sqrt{\sigma}\left(\frac{\sqrt{2}z}{\sqrt{\lambda}\sqrt{r}} - 2c_2\right)\right)\sqrt{\operatorname{sech}^2\left(\sqrt{\sigma}\left(\frac{\sqrt{2}z}{\sqrt{\lambda}\sqrt{r}} - 2c_2\right)\right)} \tag{A.55}$$

which leads to the steady state temperature profile

$$T = \pm\frac{2T_0 i}{\sigma'}\coth\left(\sqrt{\sigma}\left(\frac{\sqrt{2}z}{\sqrt{\lambda}\sqrt{r}} - 2c_2\right)\right)\sqrt{\operatorname{sech}^2\left(\sqrt{\sigma}\left(\frac{\sqrt{2}z}{\sqrt{\lambda}\sqrt{r}} - 2c_2\right)\right)} + T_0 \tag{A.56}$$

$$q = -\lambda \frac{d}{dz} T = \mp \frac{\sqrt{2} T_0}{\sigma'} \frac{\sqrt{\lambda\sigma}}{\sqrt{r}} \operatorname{sech} \left(\frac{\sqrt{\sigma} z}{\sqrt{\lambda}\sqrt{r}} - \sqrt{2} c_2 \sigma' \right)^2 \tag{A.57}$$

$$q_t = f(T) = \pm\sigma \frac{2T_0 i}{\sigma'} \coth\left(\sqrt{\sigma}\left(\frac{\sqrt{2}z}{\sqrt{\lambda}\sqrt{r}} - 2c_2\right)\right) \sqrt{\operatorname{sech}^2\left(\sqrt{\sigma}\left(\frac{\sqrt{2}z}{\sqrt{\lambda}\sqrt{r}} - 2c_2\right)\right)}$$

$$\mp \frac{\sigma\sigma'^2}{2T_0^2} \left[\frac{2T_0 i}{\sigma'} \coth\left(\sqrt{\sigma}\left(\frac{\sqrt{2}z}{\sqrt{\lambda}\sqrt{r}} - 2c_2\right)\right) \sqrt{\operatorname{sech}^2\left(\sqrt{\sigma}\left(\frac{\sqrt{2}z}{\sqrt{\lambda}\sqrt{r}} - 2c_2\right)\right)} \right]^3 \tag{A.58}$$

Case $c_1 = \dfrac{\sigma}{\lambda r \sigma'^2}$

In this case we find

$$u(z) = \pm \frac{i\sqrt{2} \tan\left(\frac{\sqrt{\sigma} z}{\sqrt{\lambda}\sqrt{r}} \pm i\sqrt{2} c_2 \sigma'\right)}{\sigma'} \tag{A.59}$$

which leads to the steady state

$$T = \pm \frac{i\sqrt{2} T_0 \tan\left(\frac{\sqrt{\sigma} z}{\sqrt{\lambda}\sqrt{r}} \pm i\sqrt{2} c_2 \sigma'\right)}{\sigma'} + T_0 \tag{A.60}$$

$$q = -\lambda \frac{d}{dz} T = \mp \frac{i\sqrt{2}\sqrt{\lambda}\sqrt{\sigma} T_0 \sec^2\left(\frac{\sqrt{\sigma} z}{\sqrt{\lambda}\sqrt{r}} \pm i\sqrt{2} c_2 \sigma'\right)}{\sqrt{r}\sigma'} \tag{A.61}$$

$$q_t = f(T) = \pm\sigma \frac{i\sqrt{2} T_0 \tan\left(\frac{\sqrt{\sigma} z}{\sqrt{\lambda}\sqrt{r}} \pm i\sqrt{2} c_2 \sigma'\right)}{\sigma'} \mp \frac{\sigma\sigma'^2}{2T_0^2} \left(\frac{i\sqrt{2} T_0 \tan\left(\frac{\sqrt{\sigma} z}{\sqrt{\lambda}\sqrt{r}} \pm i\sqrt{2} c_2 \sigma'\right)}{\sigma'} \right)^3 \tag{A.62}$$

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