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U. O. DOTTORATI

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IL DOTTORE
Danny Troia

IL COORDINATORE
Maria Carmela Lombardo

IL TUTOR
Marco D'Anna

EVENTUALE CO TUTOR

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The nearly Gorenstein property for numerical duplications and semitrivial extensions

Author:
DANNY TROIA

Supervisor:
Prof. Super MARCO D'ANNA

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Abstract

Department of Mathematics and Computer Sciences

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by DANNY TROIA

In this thesis we present and study the ideal duplication, a new construction within the class of the relative ideals of a numerical semigroup S, that, under specific assumptions, produces a relative ideal of the numerical duplication $S \bowtie^b E$, for some ideal E of S. We prove that every relative ideal of the numerical duplication can be uniquely written as the ideal duplication of two relative ideals of S; this allows us to better understand how the basic operations of the class of the relative ideals of $S \bowtie^b E$ work. In particular, we characterize the ideals E such that $S \bowtie^b E$ is nearly Gorenstein. With the aim to generalize this construction to commutative rings with unity, we introduce the semitrivial ideal extension, a construction that, starting with an ideal of a commutative ring R with unity and a submodule of a module M over R, under specific assumptions, produces an ideal of the semitrivial extension $R \ltimes_\phi M$. Using this tool we characterize a certain family of prime ideals of the semitrivial extension and we completely describe the family of the maximal ideals. Similarly as it was done for the numerical duplication, using the semitrivial ideal extension, we characterize the modules M such that $R \ltimes_\phi M$ is nearly Gorenstein.

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Introduction

Since the second half of the last century Gorenstein rings (and their generalizations) have been one of the most interesting and studied class of rings both in commutative algebra and algebraic geometry. One important aspect of the rings in this class is that every finitely generated module over a zero-dimensional Gorenstein ring R has the property that $\mathrm{Hom}_R(\mathrm{Hom}_R(M,R),R) \cong M$. Note that this property is true in the category of finitely generated vector spaces, but not for the category of finitely generated modules over a local Cohen-Macaulay ring.

A useful way to introduce Gorenstein rings is through the canonical module. Let R be a local Cohen-Macaulay ring; a canonical module is an R-module C such that $\mathrm{Hom}_R(\cdot, C)$ is a dualizing functor of the category of the finitely generated modules over R. For zero-dimensional rings such module exists and it is isomorphic to the injective envelope of the residue class field of R (see Proposition 1.16). For local Cohen-Macaulay rings of positive Krull dimension, the canonical module may not exist; however, if it does exists, then it is unique up to isomorphism (see Corollary 1.31) and the ring is said to be Gorenstein if it is a canonical module over itself.

In recent years, many authors introduced classes of Cohen-Macaulay rings which are close to be Gorenstein under some respects. In [5], the authors introduced the notion of almost Gorenstein rings for analytically unramified rings, subsequently in [13], the authors generalize this notion for any ring. Nearly Gorenstein rings were originally introduced in [14], and the definition is given through the trace of the canonical module. Any Gorenstein ring is both almost Gorenstein and nearly Gorenstein, moreover any one-dimensional almost Gorenstein ring is nearly Gorenstein.

Let R be a commutative ring with unity and M an R-module; the idealitazion, also called trivial extension, is a classical construction introduced by Nagata (see [**p. 2**, 18], [**Chapter VI**, **Section 25**, 15] and [10]) that produces a new ring containing an ideal isomorphic to M. Another more recent construction is the amalgamated duplication (see [7]) that, starting with a ring R and an ideal I, produces a new ring that, if $M = I$, has many properties coinciding with the idealization (e.g., they have the same Krull dimension and if I is a canonical ideal of a local Cohen–Macaulay ring, both of them give a Gorenstein ring). In [3], the authors unify these two constructions, by presenting them as particular quadratic quotients of the Rees algebra associated to I. More precisely, given a monic polynomial $t^2 + at + b \in R[t]$, and denoting $\mathcal{R}_+$ the Rees algebra associated to the ideal I, that is $\mathcal{R}_+ = \bigoplus_{n \geq 0} I^n t^n$, in [3], the authors study the ring $R(I)_{a,b} = \mathcal{R}_+ / I^2(t^2 + at + b)$, where $I^2(t^2 + at + b)$ is the contraction to $\mathcal{R}_+$ of the ideal generated by $t^2 + at + b$ in $R[t]$. Moreover, if R is local, some relevant properties, such as Gorensteinness, almost Gorensteinness and Cohen-Macaulay type, coincide for any member of the family cited above, depending only on R and I (see [4]).

If R is an algebroid branch with valuation semigroup $v(R) = S$, and $b \in R$ is an element of odd valuation, then $R(I)_{0,-b}$ is an algebroid branch whose valuation semigroup is a numerical semigroup strongly linked to S, called the numerical duplication (see [**Theorem 3.6**, 3]).

The numerical duplication is a construction introduced in [8] that, starting with a numerical semigroup S and a semigroup ideal $E \subset S$, produces a new numerical semigroup, denoted by $S \bowtie^b E$ (where b is any odd integer belonging to S). In [8] the authors give a characterization of the ideals producing a symmetric or an almost symmetric semigroup respectively; this characterization is essentially a specialization of the condition found for the family of the quotients of the Rees algebra and, as expected, does not depend on the integer b used to define the numerical duplication. A natural question that arises is if it is possible to find analogue results for nearly Gorenstein rings. More precisely, is it true that the nearly Gorenstein property for $R(I)_{a,b}$ is independent of a and b? Is it possible to characterize the ideals I such that $R(I)_{a,b}$ is nearly Gorenstein? Our approach to these questions started from numerical duplications, in order to get hints for the ring case. However, as we will see, our approach for numerical duplications proved to be efficient to answer the question also in a different class of rings, the so-called semitrivial extensions. This class contains the class of quadratic quotients of the Rees algebra of the form $R(I)_{0,b}$.

More precisely, in [20] we provided a systematical study of the relative ideals of a numerical duplication; this was done through a similar construction for relative ideals, that is called ideal duplication: given a numerical semigroup S, an odd number $b \in S$ and an ideal E, the ideal duplication starting from two relative ideals E_1, E_2 of S, under specific assumptions, produces a relative ideal $E_1 \bowtie^b E_2$ of the numerical duplication $S \bowtie^b E$. We proved that every relative ideal of the numerical duplication can be written, in a unique way, as the ideal duplication of two relative ideals of the semigroup (see Theorem 3.9).

The knowledge of the relative ideals of a numerical duplication allows to better understand its properties; in particular, it is possible to describe the trace of the numerical duplication (see Theorem 3.17) and, by this result, we characterize those ideals E, such that $S \bowtie^b E$ is nearly Gorenstein (see Corollary 3.18).

With the aim to generalize the ideal duplication and its applications to the ring case, the most natural way to do so is within a particular ring construction: the semitrivial extension (see [19] or [22]). The semitrivial extensions are a generalization of Nagata's idealization: let R be a commutative ring with unity (note that we do not need the commutative property to give the definition), M a module over R and $\phi \in \mathrm{Hom}_R(M \otimes_R M, R)$ such that ϕ is symmetric and associative, that is,

$$\phi(m \otimes m)' = \phi(m' \otimes m)$$

and

$$m\phi(m' \otimes m'') = \phi(m \otimes m')m''$$

for every $m, m', m'' \in M$. Then, the semitrivial extension of R by M and ϕ, denoted by $R \ltimes_\phi M$, is defined as the ring obtained from the abelian additive group $R \oplus M$ with product:

$$(r, m)(r', m') = (rr' + \phi(m \otimes m'), rm' + r'm).$$

Note that, if $\phi = 0$, then $R \ltimes_\phi M$ coincides with the trivial extension of R by M. It is not surprising that the assumption $\phi \neq 0$ gives more restrictions on the module M; hence the ring structure of $R \ltimes_\phi M$ is bound to be more subtle and complicated than the case of the Nagata's idealization (e.g. $0 \oplus M$ is not an ideal of $R \ltimes_\phi M$). However, more interesting examples are obtained through this construction. In particular, if I is an ideal of R and ϕ is the multiplication by some element $b \in R$, then $R \ltimes_\phi I = R(I)_{0,-b}$ (see Example 4.18); also, under specific assumptions, $R \ltimes_\phi M$ can be a domain (see Theorem 4.25) and it can even be a field (see Theorem 4.26).

Let I be an ideal of R and N a submodule of M. If $I \oplus N$ is an ideal of $R \ltimes_\phi M$, then we will write $I \oplus N = I \ltimes_\phi N$ and we will call it the semitrivial ideal extension of I by N and ϕ. The semitrivial ideal extension is quite a powerful tool, in fact it allows to characterize and describe the prime ideals which are the semitrivial extension of some prime ideal of R (see Proposition 4.36 and Theorem 4.43). Moreover, we give a precise description of all the maximal ideals of $R \ltimes_\phi M$ (see Theorem 4.44).

Finally, we characterize the modules M such that $R \ltimes_\phi M$ is generically Gorenstein and, since the trace of a $\mathbb{Z}_2$-graded module over $R \ltimes_\phi M$ is a semitrivial ideal extension (see Proposition **??**), we characterize the module M such that $R \ltimes_\phi M$ is nearly Gorenstein.

The structure of the thesis is the following: in Chapter 1, we present some basic results about injective modules; for example, we see how every module can be injected to an injective module (see Corollary 1.7) and we define the injective envelope (see Definition 1.10) which is an important tool that will be used to define the canonical module for zero-dimensional rings. Subsequently, after giving the definition of maximal Cohen-Macaulay modules, we define the canonical module for a local Cohen-Macaulay ring of any dimension (see Definition 1.23). In the last section, we define the trace of a module and we use it to define the class of nearly Gorenstein rings.

In Chapter 2, we give a brief overview about numerical semigroups and their properties. We specialize the notions of canonical module and trace for the numerical semigroup case, consequently we give the definition of nearly Gorenstein semigroups (see Definition 2.11).

In Chapter 3, we define the ideal duplication and we study its properties; afterwards we prove that every ideal of the numerical duplication can be written as the ideal duplication of two ideals of the starting numerical semigroup (see Theorem 3.9). In the last section, we give the conditions such that the numerical duplication is nearly Gorenstein (see Corollary 3.18).

In Chapter 4, since semitrivial extensions are essentially $\mathbb{Z}_2$-graded rings, we give some basic informations about this ring structure and we fix the notation that will be useful for the rest of the chapter. Therefore, after studying some general facts about semitrivial extensions (see e.g. Theorem 4.25), we define the semitrivial ideal extension and prove some basic results (see e.g. Proposition 4.29). Afterwards, we use the semitrivial ideal extension to describe the prime ideals which are a semitrivial ideal extension of a prime ideal of R and also we give a complete description of the maximal ideals. In the last section, we compute the trace of the canonical module and we characterize the semitrivial extensions that are generically Gorenstein and nearly Gorenstein.

Chapter 1

The canonical module

In the following, the rings considered will be always commutative and with unity.

In this chapter we present and study the canonical module. As it has been done in [9], we use a homological approach, therefore in the first section we give some basic results about the theory of injective module and develop some useful homological tool. In the second section, firstly, we define the canonical module for zerodimensional rings, then we give the definition for rings of any dimension. In the last section we define the trace of a module, we give some basic properties and, using the trace of the canonical module, we define nearly Gorenstein rings.

1.1 Injective modules

In this section, we present some basic results of the theory of injective modules. In the first part, we prove that every module can be embedded in an injective module. In the second part, we give the definition of essential extension of modules. Finally, we present the notion of injective hull and minimal resolution for a module. Most of the proofs of this section can be found in [9, Appendix 3.4], otherwise we will give the specific reference.

Let R be a ring.

Definition 1.1. *A R-module Q is said to be injective if for any monomorpshim of R-modules $\alpha : N \to M$ and for every homomorphism of R-modules $\beta : N \to Q$, there exists a homomorphism of R-modules $\gamma : M \to Q$ such that $\beta = \gamma\alpha$.*

$$\begin{array}{ccc} N & \overset{\alpha}{\hookrightarrow} & M \\ \downarrow{\scriptstyle \beta} & \swarrow{\scriptstyle \gamma} & \\ Q & & \end{array}$$

In order to determine that a module is injective, it is enough to check the case where α is the inclusion of an ideal of R.

Lemma 1.2. *Let Q be an R-module. If for every ideal $I \subset R$, every homomorphism $\beta : I \to Q$ extends to R, then Q is injective.*

$$\begin{array}{ccc} I & \subseteq & R \\ \downarrow{\scriptstyle \beta} & \swarrow{\scriptstyle \gamma} & \\ Q & & \end{array}$$

Injective $\mathbb{Z}$-modules (that is, injective abelian groups) are easy to describe:

Proposition 1.3. *An abelian group Q is injective if and only if it is divisible in the sense that for every $q \in Q$ and every $0 \neq n \in \mathbb{Z}$ there exists $q' \in Q$ such that $nq' = q$.*

Corollary 1.4. *If Q is an injective abelian group, and K is any subgroup, then Q/K is an injective abelian group.*

Corollary 1.5. *Every abelian group can be embedded in an injective abelian group.*

Lemma 1.6. *If R is an S-algebra, and Q' is an injective S-module, then $Q := \operatorname{Hom}_S(R, Q')$ is an injective R-module (the R-module structure comes via the action of R on the first factor of $\operatorname{Hom}_S(R, Q')$).*

Corollary 1.7. *Every R-module can be embedded to an injective R-module.*

Let M be an R-module; by Corollary 1.7, it follows that there exists an injective module Q_0 such that M can be embedded in Q_0. We may then embed the cokernel, Q_0/M, in an injective module Q_1. Continuing in this way, we get an injective resolution

$$0 \longrightarrow M \hookrightarrow Q_0 \longrightarrow Q_1 \longrightarrow Q_2 \longrightarrow \dots$$

of M; that is, an exact sequence of the given form in which all the Q_i are injective. In the final part of this section, we will show that every module has a minimal injective resolution; to this end, we need a preliminary definition.

Definition 1.8. *Let E be a R-module. We say that any submodule $M \subseteq E$ is essential, or that $M \subseteq E$ is an essential extension of M, if every nonzero submodule of E intersects M nontrivially.*

Proposition 1.9. *Let $M \subseteq F$ an extension of modules.*

1. *There exists a maximal submodule E of F containing M such that $M \subseteq F$ is essential.*

2. *If F is injective, then so is E.*

3. *There is, up to isomorphism, a unique essential extension E of M that is an injective R-module.*

Thanks to previous proposition, we can give the following definition.

Definition 1.10. *Let M be a R-module. We define the injective envelope of M the injective module $\mathcal{I}_R(M)$ such that the extension $M \subseteq \mathcal{I}_R(M)$ is essential.*

Definition 1.11. *We say that the injective resolution*

$$0 \longrightarrow M \longrightarrow Q_0 \xrightarrow{f_0} Q_1 \xrightarrow{f_1} Q_2 \xrightarrow{f_2} \dots$$

is a minimal injective resolution of M if:

- $Q_{n+1} = E(\operatorname{coker} f_{n-1})$
- $f_n = \pi_n i_n$, *where* $\pi_n : Q_n \to \operatorname{coker} f_{n-1}$ *is the canonical surjection and* $i_n : \operatorname{coker} f_{n-1} \to E(\operatorname{coker} f_{n-1})$ *is the canonical immersion.*

As an immediate consequence of Proposition 1.9, we have:

Corollary 1.12. *Any R-module M has a unique minimal injective resolution.*

Definition 1.13. *We define the injective dimension of M, denoted with $\operatorname{id}_R(M)$, the lenght of an injective minimal resolution.*

1.2 Definition of canonical module

There are many ways to introduce the canonical module; in this section we follow closely the method used in [9], which uses a homological approach. Most of the proofs of this section can be found in [9, Chapter 21], otherwise we will give the specific reference.

Definition 1.14. *Let R be a ring. On the category of the R-modules, a functor D is said to be dualizing if the following hold:*

1. *D is controvariant;*
2. *D is R-linear;*
3. *D is exact;*
4. *$D^2(M) \cong M$ for every R-module M.*

In the category of the finitely generated vector spaces over a field k, it is well known that $D := \mathrm{Hom}_k(\cdot, k)$ is dualizing. However, that is not true in general; e.g. for a finitely generated module M over a zero-dimensional local ring A, the equality $\mathrm{Hom}_A(\mathrm{Hom}_A(M, A), A) = M$ may not hold.

Proposition 1.15. *Let R be a ring and let D be a dualizing functor on the category of R-modules. Then:*

1. *If M is a simple module, then $D(M)$ is simple.*
2. *For any module M, M and $D(M)$ have the same lenght.*
3. *For any module M, $(0 : M) = (0 : D(M))$.*
4. *For any modules M and N, $\mathrm{Hom}_R(M, N) \cong \mathrm{Hom}_R(D(N), D(M))$. In particular, $\mathrm{End}(D(R)) \cong R$.*

Proposition 1.16. *Let $(A, \mathfrak{m})$ be a local zero-dimensional ring. If D is any dualizing functor from the category of finitely generated R-module to itself, then there is an isomorphism of functors $D(\cdot) \cong \mathrm{Hom}_A(\cdot, D(A))$. Further, $D(A)$ is isomorphic to the injective hull of $A/\mathfrak{m}$. Thus there is up to isomorphism at most one dualizing functor.*

Motivated by previous proposition, we give the following:

Definition 1.17. *Let A be a zero-dimensional local ring. We define the canonical module ω_A of A to be the injective hull of the residue class field of A.*

Proposition 1.18. *If A is a zero-dimensional local ring, then the functor*

$$D(M) := \mathrm{Hom}_A(M, \omega_A)$$

is dualizing on the category of finitely generated A-modules.

Corollary 1.19. *If A is local Artinian ring, then the annihilator of ω_A is 0; the lenght of ω_A is the same as the lenght of A; and the endomorphism ring of ω_A is A.*

Let B be a ring, if A is a B-algebra, then $\mathrm{Hom}_B(A, \cdot)$ can be naturally regarded as an A-module just by taking the action of the first component.

Proposition 1.20. *Let A be a zero-dimensional local ring. Suppose that for some local ring B, A is a B-algebra that is finitely generated as B-module and the maximal ideal of B maps into that of A. If E is the injective hull of the residue class field of B, then*

$$\omega_A = \mathrm{Hom}_B(A, E).$$

In particular, if B is also zero-dimensional, then $\omega_A = \mathrm{Hom}_B(A, \omega_B)$.

From the point of view of the duality, the simplest case is the following:

Definition 1.21. *A zero-dimensional local ring A is Gorenstein if $A \cong \omega_A$.*

Well known examples of Gorenstein rings are fields and complete intersections.

We recall that the socle of a finitely generated module R-module M, denoted by $\mathrm{Soc}(M)$, is defined as the sum of all the simple submodules of M. If R is a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue class field k, then $\mathrm{Soc}(M) = (0 :_M \mathfrak{m}) = \mathrm{Hom}_R(k, M)$. Let $\mathbf{x} = (x_1, \ldots, x_m)$ be a maximal M-sequence, we define the type of M to be the integer

$$\dim_k(\mathrm{Soc}(M/\mathbf{x}M)) = \dim_k(\mathrm{Hom}_R(k, M/\mathbf{x}M)) = \dim_k(\mathrm{Ext}^m_R(k, R))$$

In particular, the type of an artinian ring is equal to the dimension as k-space of $\mathrm{Soc}(R)$.

Property 5 and 6 of the following proposition are not in [9, Proposition 21.5], however the implications $3 \Rightarrow 5$, $5 \Rightarrow 6$ and $6 \Rightarrow 3$ are easy to prove.

Proposition 1.22. *Let A be a zero-dimensional local ring. The followings are equivalent:*

1. *A is Gorenstein.*
2. *A is injective as an A-module.*
3. *The socle of A is simple.*
4. *ω_A can be generated by one element.*
5. *The zero ideal is irriducible.*
6. *$t(A) = 1$.*

We now extend the definition of canonical module for Cohen-Macaulay rings not necessarily zero-dimensional. Following closely [9], we give a definition in terms of reduction to quotients.

Definition 1.23. *Let A be a local Cohen-Macaulay ring. A finitely generated A-module ω_A is a canonical module for A if there is a nonzerodivisor $x \in A$, that is also a nonzerodivisor on ω_A, such that $\omega_A / x\omega_A$ is a canonical module for $A/(x)$. The ring A is Gorenstein if A is itself a canonical module; that is, A is Gorenstein if there is a nonzerodivisor $x \in A$ such that $A/(x)$ is Gorenstein.*

The induction implicit in this definition terminates because $\dim A/(x) = \dim A - 1$. We may easily give an equivalent definition without the induction: ω_A is a canonical module for A if there is a maximal regular sequence $x_1, \ldots, x_d$ on A that is also an ω_A-sequence, and $\omega_A/(x_1, \ldots, x_d)\omega_A$ is the injective hull of the residue class field of $A/(x_1, \ldots, x_d)$. Similarly, A is Gorenstein if and only if $A/(x_1, \ldots, x_d)$ is a zero-dimensional Gorenstein ring.

Remark 1.24. There are three issues with these notions. First, it may seem that the definition of canonical module depends on the nonzerodivisor x. Second, the uniqueness of ω_A is not clear. Finally, by the definition, it is not obvious that a canonical module should exist. This last issue does not have positive answer in the general case. However, there are results to this matter which show that virtually every ring of interest in algebraic geometry and number theory have a canonical module.

For a simple example, it is easy to prove that any regular local ring is Gorenstein.

Proposition 1.25. *Any regular local ring A has canonical module, which is the ring itself. In particular, every regular local ring is Gorenstein.*

Proposition 1.26. *Let $(A, \mathfrak{m})$ be a local ring of dimension d, and let M be a finitely generated A-module. The following conditions are equivalent:*

1. *Every system of parameters in A is an M-sequence.*
2. *Some system of parameters in A is an M-sequence.*
3. $\mathrm{depth} M = d$.

If these conditions are satisfied, we say that M is a maximal Cohen-Macaulay module over A. Every element outside the minimal primes of A is a nonzerodivisor on M.

All finitely generated modules over a local zero-dimensional ring are maximal Cohen-macaulay modules. Moreover, if A is a local regular ring, by the Auslander-Buchsbaum formula, the maximal Cohen-Macaulay A-modules are exactly the free A-modules.

Proposition 1.27. *Let A be a local Cohen-Macaulay ring. If M is a maximal Cohen-Macaulay module of finite injective dimension, then $\mathrm{id}_A M = \dim A$. In addition, if $\dim A = 0$, then M is a direct sum of copies of ω_A, and $M \cong \omega_A$ if and only if $\mathrm{End}(M) = A$.*

Proposition 1.28. *Let A be a local Cohen-Macaulay ring of dimension d, and let M be a maximal Cohen-Macaulay module of finite injective dimension.*

1. *If N is a finitely generated module of depth e, then $\mathrm{Ext}^j_A(N, M) = 0$ for $j > d - e$.*
2. *If x is a nonzerodivisor on M, then x is a nonzerodivisor on $\mathrm{Hom}_A(N, M)$. If N is also a maximal Cohen-Macaulay module, then*

   $$\mathrm{Hom}_A(N, M)/x\mathrm{Hom}_A(N, M) \cong \mathrm{Hom}_{A/x}(N/xN, M/xM).$$

   by the homomorphism defined sending the class of a map $\varphi : N \to M$ to the map $N/xN \to M/xM$ induced by φ.

Proposition 1.29. *Let A be a local ring, and let M and N be finitely generated modules. Suppose that x is a nonzerodivisor on M and that x is in the maximal ideal of A. If $\varphi : N \to M$ is a map and $\psi : N/xN \to M/xM$ is the map induced by φ, then:*

1. *If ψ is surjective, then φ is surjective.*
2. *If ψ is injective, then φ is injective.*

Furthermore, if M and N are maximal Cohen-Macaulay modules, M has finite injective dimension, and $\psi : N/xN \to M/xM$ is any map, then there is a map inducing ψ.

The following theorem resolves the first two issues of Remark 1.24.

Theorem 1.30. *Let A be a local Cohen-Macaulay ring of dimension d, and let W be finitely generated A-module. W is a canonical module for A if and only if*

1. $\operatorname{depth} W = d$
2. *W is a module of finite injective dimension (necessarily equal to d).*
3. $\operatorname{End}(W) = A$

Corollary 1.31. *Let A be a local Cohen-Macaulay ring with a canonical module W. If M is any finitely generated maximal Cohen-Macaulay A-module of finite injective dimension, then M is a direct sum of copies of W. In particular, any two canonical modules of A are isomorphic.*

Henceforth we shall write ω_A for a canonical module of A (if one exists).

A proof for the following proposition, can be found in [6, Proposition 3.3.11].

Proposition 1.32. *Let A be a local and Cohen-Macaulay ring, and let ω_A be a canonical module. Denoted $\nu(\omega_A)$ the cardinality of a minimal set of generators of ω_A, we have:*

$$\nu(\omega_A) = t(A).$$

An immediate consequence of the previous proposition is the following:

Proposition 1.33. *Let A be a local and Cohen-Macaulay ring. The following facts are equivalent:*

1. *A is Gorenstein*
2. $t(A) = 1$

We now come to the question of existence.

Let R be an A-algebra, M a R-module and N an A-module. We endow $\operatorname{Hom}_R(M,N)$ with an A-module structure through the following product:

$$\cdot\ : R \times \operatorname{Hom}_R(M,N) \to \operatorname{Hom}_R(M,N)$$

$$r \cdot f(x) \mapsto f(rx),$$

for every $x \in M$.

Theorem 1.34. *Let R be a local Cohen-Macaulay ring with canonical module $\omega_{\mathcal{R}}$. If A is a local R-algebra that is finitely generated as R-module, and A is Cohen-Macaulay, then A has a canonical module. In fact, if $c = \dim R - \dim A$, then*

$$\omega_A \cong \operatorname{Ext}^c_R(A, \omega_R)$$

is a canonical module for A.

Proposition 1.35. *Let A be a local Cohen-Macaulay ring with canonical module ω_A. If $\mathfrak{p}$ is a prime ideal of A, then $(\omega_A)_\mathfrak{p}$ is a canonical module of $A_\mathfrak{p}$; in particular, if A is Gorenstein, then $A_\mathfrak{p}$ is Gorenstein.*

A proof of the following theorem can be found in [6, Theorem 3.3.6].

Theorem 1.36. *A local and Cohen-Macaulay ring has a canonical module if and only if is the homomorphic image of a Gorenstein ring.*

One could ask when a canonical module is isomorphic to an ideal of the ring, in order to answer this question we give the following

Definition 1.37. *A local and Cohen-Macaulay ring is said to be generically Gorenstein if the localization at any minimal prime ideal is Gorenstein.*

A proof of the following proposition can be found in [6, Proposition 3.3.18].

Proposition 1.38. *The canonical module of a local and Cohen-Macaulay ring A is isomorphic to an ideal if and only if A is generically Gorenstein.*

As an immediate consequence of the previous proposition, we get that any local and Cohen-Macaulay domain with canonical module has a canonical ideal.

1.3 Trace of the canonical module and nearly Gorenstein rings

In this section, we present the definitions of trace of a module and of nearly Gorenstein rings. Originally, the latter definition was introduced in [14], and all of the proofs of this section can be found there.

Let R be a ring.

Definition 1.39. *Let M be a R-module. We define the trace of M, denoted $\mathrm{Tr}_R(M)$, as the sum of the ideals $\varphi(M)$ with $\varphi \in \mathrm{Hom}_R(M,R)$. Thus,*

$$\mathrm{Tr}_R(M) = \sum_{\varphi \in \mathrm{Hom}_R(M,R)} \varphi(M).$$

When there is no risk of confusion about the ring we simply write $\mathrm{Tr}(M)$.

Proposition 1.40. *The trace satisfies the following properties:*

1. *If M_1 and M_2 are isomorphic R-modules, then $\mathrm{Tr}(M_1) \cong \mathrm{Tr}(M_2)$.*
2. *If M is an R-module of finite presentation, then $\mathrm{Tr}(M)R_P = \mathrm{Tr}(M_P)$, for every prime ideal of R.*

Let $I \subseteq R$ be an ideal of positive depth; we set

$$I^{-1} = \{x \in Q(R) \mid xI \subseteq R\},$$

where $Q(R)$ is the total fraction ring.

Lemma 1.41. *Let $I \subseteq R$ be an ideal of positive depth; then $\mathrm{Tr}(I) = I \cdot I^{-1}$. Moreover, if R is noetherian and $\mathrm{depth}(I) \geq 2$, then $\mathrm{Tr}(I) = I$.*

Proposition 1.42. *Let M and N be two R-modules. Then*

$$\mathrm{Tr}(M)\mathrm{Tr}(N) \subseteq \mathrm{Tr}(M \otimes_R N) \subseteq \mathrm{Tr}(M) \cap \mathrm{Tr}(N).$$

Proposition 1.43. *Let I and J be ideals of R of positive depth. Then*

$$\mathrm{Tr}(I)\mathrm{Tr}(J) \subseteq \mathrm{Tr}(IJ) \subseteq \mathrm{Tr}(I) \cap \mathrm{Tr}(J).$$

Proposition 1.44. *Let $\varphi : R' \to R$ a ring homomorphism, M an R-module and N an R'-module. Then:*

1. *If φ is surjective, then $(\mathrm{Tr}_{R'}(M))R \subseteq \mathrm{Tr}_R(M)$;*
2. *$(\mathrm{Tr}_{R'}(N))R \subseteq \mathrm{Tr}_R(N \otimes_{R'} R)$;*

Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring with canonical module ω_R. The trace of the canonical module describes the non-Gorenstein locus of R, i.e. those prime ideals $\mathfrak{p}$ such that the localizations of R at $\mathfrak{p}$ are not Gorenstein

Lemma 1.45. *Let $\mathfrak{p}$ be a prime ideal of R. Then $R_{\mathfrak{p}}$ is not Gorenstein if and only if $\mathrm{Tr}(\omega_R) \subseteq \mathfrak{p}$.*

Definition 1.46. *A local and Cohen-Macaulay ring R with canonical module ω_R is said to be nearly Gorenstein if $\mathrm{Tr}(\omega_R)$ contains the maximal ideal of R.*

Clearly, Gorenstein rings are nearly Gorenstein.

Proposition 1.47. *Let R be a nearly Gorenstein ring. Then:*

a. *$R_{\mathfrak{p}}$ is Gorenstein for every prime ideal $\mathfrak{p}$ not maximal.*

b. *Let $I = (x_1, \ldots, x_h)$ be a regular sequence of R; then R/I is nearly Gorenstein.*

Chapter 2

Numerical semigroups

In this chapter, we give some basic results about numerical semigroups. In particular, in the second section, we specialize the notion of canonical module and trace, given in Chapter 1, for numerical semigroups.

2.1 Definition of numerical semigroups and basic properties

In this section, we present some basic notions and results about numerical semigroups. Most of the proofs in this section can be found in [1], otherwise we will give specific reference.

Definition 2.1. *An additive submonoid $S \subseteq \mathbb{N}$ is said to be a numerical semigroup if $|\mathbb{N} \setminus S| < \infty$.*

We denote $G(S) = \mathbb{N} \setminus S$ and we call the elements of $G(S)$ the gaps of the numerical semigroup. The maximum of $G(S)$ is called Frobenius number and it is denoted $f(S)$. Clearly, for any $z \in \mathbb{Z}$, if $z \geq f(S) + 1$, then $z \in S$; the number $f(S) + 1$ is called conductor number and the set $C(S) = \{z \in \mathbb{Z} \mid z \geq f + 1\}$ is called the conductor ideal. It is easy to prove that, if $s \in S$, then $f(S) - s \notin S$.

Definition 2.2. *A numerical semigroup S is said to be symmetric if, for every $s \in \mathbb{Z}$, we have*

$$s \in S \Leftrightarrow f(S) - s \notin S.$$

A gap g is said to be a gap of the first type if $f(S) - g \in S \setminus \{0\}$, otherwise it is called of the second type. Among the gaps of the second type, we have:

Definition 2.3. *A gap g is said to be pseudo-frobenius if, for every $s \in S \setminus \{0\}$, $g + s \in S$.*

We denote $\mathrm{PF}(S)$ the set of the pseudo-frobenius numbers of S, and we define the type of S, denoted $t(S)$, to be the cardinality of $\mathrm{PF}(S)$.

Let $s_1, \ldots, s_d \in \mathbb{N}$, we denote $(s_1, \ldots, s_d) = \{\sum n_i s_i \mid n_i \in \mathbb{N}\}$, the monoid generated by the elements $s_1 \ldots, s_d$. Any submonoid of $(\mathbb{N}, +)$ is finitely generated. For a numerical semigroup we have:

Proposition 2.4. *Every numerical semigroup is finitely generated and has an unique set of minimal generators. Moreover, if $S = (s_1, \ldots, s_d)$, then*

$$S \text{ is a numerical semigroup } \Leftrightarrow \gcd(s_1, \ldots, s_d) = 1.$$

In the following, when we write $S = (s_1, \ldots, s_d)$, we will always assume that $(s_1 \ldots, s_d)$ is the minimal set of generators of S, and $s_1 < \ldots < s_d$. The number s_1 is the multiplicity of S, while d is the embedding-dimension. Clearly, $s_1 \leq d$ and, if the equality holds, S is said to be of minimal multiplicity (or maximal embedding-dimension).

Definition 2.5. *A subset I of $\mathbb{Z}$ is said to be a relative ideal of S if, for every $s \in S$, $s + I \subseteq I$, and there exists $z \in S$ such that $z + I \subseteq S$. If $I \subseteq S$, then I is said to be an ideal of S.*

The maximal ideal $M(S) = S \setminus \{0\}$ and $C(S)$ are easy examples of ideals of S. We can operate with relative ideals obtaining again relative ideals: if I and J are relative ideals of S, then $I + J = \{i + j \mid i \in I, j \in J\}$ and $I - J = \{z \in \mathbb{Z} \mid z + j \in I, \forall j \in J\}$ are relative ideals. Moreover, the union and intersection of relative ideals still give relative ideals. If I is a relative ideal of S, we define $m(I)$=min I, $f(I) = \max(\mathbb{Z} \setminus I)$ (it is well defined since $m(I) + C(S) \subseteq I$), $C(I) = \{f(I) + 1, \rightarrow\}$ the conductor ideal of I and $g(I) = |(\mathbb{Z} \setminus I) \cap \{m(I), m(I) + 1, \ldots, f(I)\}|$ (note that, if $I \subseteq \mathbb{N}$, then $|\mathbb{N} \setminus I| = g(I) + m(I)$). One can always shift a relative ideal I by adding to it an integer z: $z + I = \{z + i \mid i \in I\}$. It is obvious that the relation $I \sim I' \Leftrightarrow I' = z + I$ for some $z \in \mathbb{Z}$, is an equivalence relation. In every equivalence class there is exactly one representative $\tilde{I}$ such that $f(\tilde{I}) = f(S)$. This representative $\tilde{I}$ is obtained by adding $f(S) - f(I)$ to I. For all ideals I, we have that $C(S) \subseteq \tilde{I} \subseteq \mathbb{N}$. We now list some useful properties of the operations between relative ideals, whose proof is not difficult to check.

Lemma 2.6. *Let I, J, and H be relative ideals of S and $z \in \mathbb{Z}$. The following statements are true:*

1. *If $I \subseteq J$, then $I - H \subseteq J - H$ and $H - J \subseteq H - I$;*
2. $(I - J) + (J - H) \subseteq I - H$;
3. $(I - J) + H \subseteq (I + H) - J$;
4. $I - (J + H) = (I - J) - H$;
5. $I - (J \cup H) = (I - J) \cap (I - H)$;
6. $I + (J \cup H) = (I + J) \cup (I + H)$;
7. $(I \setminus J) + z = (I + z) \setminus (J + z)$;
8. $(I \cap J) + z = (I + z) \cap (J + z)$.

2.2 Definition of canonical ideal, trace of a numerical semigroup and ring semigroups

The set $\{z \in \mathbb{Z} \mid f(S) - z \notin S\}$, denoted by $K(S)$, is a relative ideal of S. The following well-known result is usually referred as Jäger Lemma, see [16, Hilfssatz 5].

Lemma 2.7. *For every relative ideal I of S,*

$$K(S) - I = \{x \in \mathbb{Z} \mid f(S) - x \notin I\}.$$

Some consequences of Jäger Lemma are the following:

Corollary 2.8. *The following properties hold for every numerical semigroup:*

1. *For every relative ideal I of S, $K(S) - (K(S) - I) = I$;*
2. $K(S) - K(S) = S$;
3. $K(S) - S = K(S)$;
4. *If I and J are relative ideals such that $I \subseteq J$, then $K(S) - J \subseteq K(S) - I$;*
5. *If I and J are relative ideals such that $I \subseteq J$, then $|J \setminus I| = |(K(S) - I) \setminus (K(S) - J)|$.*

Definition 2.9. *A relative ideal I of S is said to be canonical if there exists $x \in \mathbb{Z}$ such that $I = x + K(S)$.*

The relative ideal $K(S)$ is called the standard canonical ideal of S, and it is in fact the representative ideal of the canonical class $[K(S)]$ such that $f(K(S)) = f(S)$. It is straightforward to prove that $S \subseteq K(S) \subseteq \mathbb{N}$ and that S is symmetric if and only if $K(S) = S$. It can be proven (see [1, Theorem 3]) that a relative ideal I is canonical if and only if $I - (I - J) = J$, for every relative ideal J of S.

Let $S = (s_1, \dots, s_d)$ be a numerical semigroup and let k be an arbitrary field. Denoted by $k[[x]]$ the ring of formal power series, we define the semigroup ring associated to S as the ring $k[[S]] = k[[\{x^s \mid s \in S]] = k[[x^{s_1}, \dots, x^{s_d}]] \subseteq k[[x]]$. Let v be the discrete valuation of $k[[x]]$, then $v(k[[S]] \setminus \{0\}) = S$ and the valuations of fractional ideals of $K[[S]]$ are relative ideals of S.

It is not hard to prove that the extension $k[[S]] \subseteq k[[x]]$ is integral; by this fact, it follows that $k[[S]]$ is one-dimensional, local and Cohen-Macaulay with maximal $\mathfrak{m} = (x^{s_1}, \dots, x^{s_d})k[[S]]$, where $v(\mathfrak{m}) = M(S)$; moreover, since S is a numerical semigroup and hence it has a conductor, the field of fractions of $k[[S]]$ is $k((x))$. However, despite being Cohen-Macaulay, $k[[S]]$ is not regular (in fact it may not be even Gorenstein) unless $S = \mathbb{N}$. In [12], it is proven that the multiplicity and the embedding-dimension of the numerical semigroup coincide with the ones of the semigroup ring.

Since $k[[S]]$ is a homomorphic image of $k[[x_1, \dots, x_d]]$ which is a regular ring, by Theorem 1.34, $k[[S]]$ has a canonical module ω and, by Proposition 1.38, it is an ideal. By [9, Exercise 21.11], we have that ω can be generated by the elements x^{-g}, where $g \in \mathrm{PF}(S)$ and $v(\omega) = K(S) - f(S)$. Hence, $\mathrm{type}(R) = \mathrm{type}(S)$ and we get

Theorem 2.10. *S is symmetric if and only if $k[[S]]$ is Gorenstein.*

The original proof of the previous theorem can be found in [17].

By Lemma 1.41, $\mathrm{Tr}(\omega) = \omega\omega^{-1}$ and, since the trace is independent of shifts, we get that $v(\mathrm{Tr}(\omega)) = K(S) + (S - K(S))$. Hence, we define the trace of a numerical semigroup as $\mathrm{Tr}(S) = K(S) + K(S)^{-1}$, where $K(S)^{-1} = S - K(S)$ and it is usually called the anticanonical ideal.

Definition 2.11. *A numerical semigroup S is said to be nearly Gorenstein if its associated semigroup ring $k[[S]]$ is nearly Gorenstein. Hence, S is nearly Gorenstein if and only if $M(S) \subseteq K(S) + K(S)^{-1}$.*

Since Gorenstein rings are nearly Gorenstein, we have that symmetric numerical semigroups are nearly Gorenstein.

Chapter 3

The Ideal duplication

Let S be a numerical semigroup, $E \subset S$ an ideal of S and $b \in S$ an odd element of S. In [8, p. 153], the authors define the numerical duplication, $S \bowtie^b E$, of S with respect to E and b as the following subset of $\mathbb{N}$:

$$S \bowtie^b E = 2 \cdot S \cup (2 \cdot E + b).$$

It is straightforward to prove that $S \bowtie^b E$ is a numerical semigroup and $f(S \bowtie^b E) = 2f(E) + b$. It is also true that $S \bowtie^b E$ is symmetric if and only if E is a canonical ideal of S (see [8, Proposition 3.1.3]).

In this chapter, we define the ideal duplication, an operation between two relative ideals of S which gives, under specific assumptions, a relative ideal of the numerical duplication $S \bowtie^b E$. We will prove that every relative ideal of $S \bowtie^b E$ can be written, in a unique way, as the ideal duplication of two ideals of S. Thanks to this representation we will be able to better understand how the basic operations among relative ideals of $S \bowtie^b E$ work. Futhermore, we will characterize the ideals such that the duplication $S \bowtie^b E$ is nearly Gorenstein.

The results of this chapter are contained in the paper [20].

3.1 Definition of ideal duplication and basic results

In this section, we give the notations that we will use for the rest of the chapter and we prove some preliminaries results. After that, we give the definition of ideal duplication and we prove that every relative ideal of the duplication can be written as the ideal duplication of two ideals of S.

Notations

We fix the notation that we will use in this chapter: S is a numerical semigroup, f its Frobenius number, s_1 its multiplicity, $b \in S$ is an odd number, E is a proper ideal of S (i.e. an ideal such that $E \neq S$ or equivalently $0 \notin E$), $e = f(E) - f$, $\tilde{E} = E - e$, K is the standard canonical ideal of S, while M and C are respectively the maximal ideal and the conductor ideal of S. For any subset $A \subseteq \mathbb{Z}$, we also define $2 \cdot A = \{2a \mid a \in A\}$ (note that $2 \cdot S \neq 2S = S + S$ and $2 \cdot E \neq 2E = E + E$).

The following proposition will be useful later; for its proof we refer to [1, p. 34].

Proposition 3.1. *The following equality holds:*

$$K - M(S) = K \cup \{f(S)\}.$$

Some later proofs will make use of the following facts that we did not find in the literature:

Lemma 3.2. *Let I and J be two relative ideals of S. Then:*

1. $m(I+J) = m(I) + m(J)$;
2. $f(I-J) = f(I) - m(J)$.

Moreover, we have:

$$m(K-I) = f(S) - f(I)$$

Proof.
Property 1 is trivial. As for Property 2, notice that $f(I) - m(J) \notin I - J$, but at the same time if $x > 0$, then $f(I) - m(J) + x \in I - J$. For the last part, using Lemma 2.7, we have $f(S) - f(I) \in K - I$; also, if $z \in K - I$, then $f(S) - z \leq f(I)$, so $f(S) - f(I) \leq z$. □

Proposition 3.3. *Let I and J be two relative ideals of S. The following equality holds:*

$$I - J = (K - J) - (K - I).$$

Proof.
By Property 1 of Corollary 2.8 and by Property 4 of Lemma 2.6 we have:

$$I - J = (K - (K - I)) - J = K - ((K - I) + J) = (K - J) - (K - I).$$

□

Corollary 3.4. *Let I be a relative ideal of S. The following equality holds:*

$$S - (K - I) = I - K.$$

Proof.
Using Proposition 3.3 and both Property 1 and 3 of Corollary 2.8, it follows that

$$S - (K - I) = ((K - (K - I)) - (K - S)) = I - K.$$

□

Corollary 3.5. *Let I be a relative ideal of S. Then, $K - \tilde{I}$ is a numerical semigroup if and only if $K - \tilde{I} = I - I$.*

Proof.
If $K - \tilde{I}$ is a numerical semigroup, using Proposition 3.3 we get:

$$K - \tilde{I} = (K - \tilde{I}) - (K - \tilde{I}) = \tilde{I} - \tilde{I} = I - I.$$

The converse statement is trivial. □

Definition 3.6. *Let E_1 and E_2 be two relative ideals of S satisfying the following:*

1. $E_1 + E \subseteq E_2$;
2. $E_2 + b + E \subseteq E_1$.

We define the ideal duplication of E_1 and E_2 with respect to b the following set:

$$E_1 \bowtie^b E_2 := 2 \cdot E_1 \cup (2 \cdot E_2 + b).$$

In the following, when we write $E_1 \bowtie^b E_2$, we will always assume that E_1 and E_2 (the order is important) satisfy the conditions in Definition 3.6.

Proposition 3.7. *The ideal duplication $E_1 \bowtie^b E_2$ is a relative ideal of $S \bowtie^b E$. Moreover, if E_1 and E_2 are proper ideals of S with $E_2 \subseteq E$, then $E_1 \bowtie^b E_2$ is a proper ideal of $S \bowtie^b E$.*

Proof.
Let $x \in E_1 \bowtie^b E_2$ and $y \in S \bowtie^b E$. There are four possibilities:

i) Let $x = 2e_1$, with $e_1 \in E_1$ and $y = 2s$, with $s \in S$. We have:

$$2e_1 + 2s = 2(e_1 + s) \Rightarrow 2e_1 + 2s \in 2 \cdot E_1 \subseteq E_1 \bowtie^b E_2.$$

ii) Let $x = 2e_2 + b$, with $e_2 \in E_2$ and $y = 2s$, with $s \in S$. We have:

$$2e_2 + b + 2s = 2(e_2 + s) + b \Rightarrow 2e_2 + b + 2s \in 2 \cdot E_2 + b \subseteq E_1 \bowtie^b E_2.$$

iii) Let $x = 2e_1$, with $e_1 \in E_1$, and $y = 2e + b$, with $e \in E$.

$$2e_1 + 2e + b = 2(e_1 + e) + b \Rightarrow 2e_1 + 2e + b \in 2 \cdot E_2 + b \subseteq E_1 \bowtie^b E_2;$$

the fact that $e_1 + e \in E_2$ follows from Property 1 of Definition 3.6.

iv) Let $x = 2e_2 + b$, with $e_1 \in E_1$, and $y = 2e + b$, with $e \in E$. We have:

$$2e_2 + b + 2e + b = 2(e_2 + e + b) \Rightarrow 2e_2 + b + 2e + b \in 2 \cdot E_1 \subseteq E_1 \bowtie^b E_2;$$

the fact that $e_2 + e + b \in E_1$ follows from Property 2 of Definition 3.6.

Since E_1 and E_2 are relative ideals, they both have a minimum, hence $2 \cdot E_1 \cup (2 \cdot E_2 + b)$ has a minimum too. Since $S \bowtie^b E$ has a conductor, it follows trivially that $(E_1 \bowtie^b E_2) + s \subseteq S \bowtie^b E$, for some $s \in S \bowtie^b E$. □

Example 3.8. Let $S = \{0, 3, 6, \rightarrow\}$, $E = \{7, 8, 10, \rightarrow\}$, $b = 7$. We have

$$S \bowtie^7 E = \{0, 6, 12, 14, 16, 18, 20, 21, 22, 23, 24, 26, \rightarrow\}.$$

The sets $E_1 = \{3, 6, 9, \rightarrow\}$ and $E_2 = \{-2, 1, 4, \rightarrow\}$ are relative ideals of S.

- $E_1 \bowtie^7 E_2 = \{3, 6, 9, 12, 15, 17, \rightarrow\}$,
- $E_2 \bowtie^7 E_1 = \{-4, 2, 8, 10, 12, 13, 14, 16, 18, 19, 20, 22, 24, \rightarrow\}$.

$E_1 \bowtie^7 E_2$ is a relative ideal of $S \bowtie^7 E$, but $E_2 \bowtie^7 E_1$ is not (e.g. $-4 + 21 = 17 \notin E_2 \bowtie^7 E_1$). We deduce that, in general, in the ideal duplication the order is important.

If $P \subset \mathbb{Z}$ is a subset of even numbers, we call $\frac{P}{2}$ the set of the halves of the numbers in P. More precisely

$$\frac{P}{2} = \{z \in \mathbb{Z} \mid 2z \in P\}.$$

Theorem 3.9. *Let H be a relative ideal of $S \bowtie^b E$, then there exist and are unique E_1 and E_2 relative ideals of S such that:*

$$H = E_1 \bowtie^b E_2.$$

Moreover, if H is a proper ideal of $S \bowtie^b E$, then E_1 and E_2 are proper ideals of S with $E_2 \subseteq E$.

Proof.
Let P be the set of even numbers of H and let D be the set of odd numbers of H. It is trivial to prove:

$$H = 2 \cdot \frac{P}{2} \cup (2 \cdot \frac{D-b}{2} + b).$$

We want to prove the thesis for $E_1 = \frac{P}{2}$ and $E_2 = \frac{D-b}{2}$.

We need to show that $\frac{P}{2}$ is a relative ideal of S. Let $e_1 \in \frac{P}{2}$ and $s \in S$; we have:

$$e_1 + s \in \frac{P}{2} \Leftrightarrow 2(e_1 + s) \in P \Leftrightarrow 2e_1 + 2s \in P,$$

and the last relation is true since $P \subset H$, H is an ideal of $S \bowtie^b E$ and $2e_1 + 2s$ is even. Since E_1 is a relative ideal of S, it has a minimum, and therefore $\frac{P}{2}$ has a minimum too. Since $S \bowtie^b E$ has a conductor, it follows that $\frac{P}{2} + s \subseteq S \bowtie^b E$ for some $s \in S \bowtie^b E$.

We show now that $\frac{D-b}{2}$ is a relative ideal of S. Let $e_2 \in \frac{D-b}{2}$ and $s \in S$, we have:

$$e_2 + s \in \frac{D-b}{2} \Leftrightarrow 2e_2 + 2s + b \in D,$$

and the last relation is true since $2e_2 + b \in D$. Arguing similarly as we did for $\frac{P}{2}$, we get that $\frac{D-b}{2} + s \subseteq S \bowtie^b E$, for some $s \in S \bowtie^b E$.

We prove now that $\frac{P}{2}$ and $\frac{D-b}{2}$ satisfy Properties 1 and 2 of Definition 3.6. If $e_1 \in \frac{P}{2}$ and $e \in E$, then $2e_1 + (2e + b)$ is an odd element of H, hence $2e_1 + 2e + b \in D$ and it follows that $e_1 + e \in \frac{D-b}{2}$, so Property 1 is proven. If $e_2 \in E_2$, then $2e_2 + b + 2e + b$ is an even element of H. Hence $2e_2 + b + 2e + b \in P$ and it follows that $e_2 + b + e \in \frac{P}{2}$, so Property 2 is proven.

Finally we prove the unicity of E_1 and E_2. Let E_1' and E_2' be relative ideals of S such that $H = E_1' \bowtie^b E_2'$. If $e_1' \in E_1'$, then $2e_1'$ is an even element of H, so $e_1' \in \frac{P}{2}$. If instead $e_1 \in \frac{P}{2}$, then $2e_1$ is an even element of H, so $e_1 \in E_1'$. Similarly we prove that $E_2' = \frac{D-b}{2}$. □

In the following, if H is a relative ideal of $S \bowtie^b E$, then we will denote by $E_1(H)$ and $E_2(H)$, the unique relative ideals of S such that $H = E_1(H) \bowtie^b E_2(H)$, and we will call them the even and the odd component respectively. If there is not risk of misunderstanding, we will not write H within the brackets.

If we want to compute the even component of a relative ideal of $S \bowtie^b E$, we have to take the even numbers of the ideal and divide them by 2. Similarly, to compute the odd component, we have to take the odd numbers, subtract b and then divide them by 2; by Theorem 3.9 they are indeed relative ideals of S. With this mindset, it is trivial to compute $M(S \bowtie^b E) = M \bowtie^b E$ and $C(S \bowtie^b E) = (C(E) + \frac{b-1}{2}) \bowtie^b C(E)$. It is fairly more interesting though to compute the components of the standard canonical ideal of $S \bowtie^b E$. The following result was originally proven in [11, Lemma 2.2] without using the ideal duplication.

Proposition 3.10. *The following equality holds:*

$$K(S \bowtie^b E) = (K - \tilde{E}) \bowtie^b (K + e).$$

Proof.
Let x be even. We have:

$$x \in K(S \bowtie^b E) \Leftrightarrow 2f(E) + b - x \notin 2 \cdot E + b \Leftrightarrow 2f - x \notin 2 \cdot \tilde{E} \Leftrightarrow f - \frac{x}{2} \notin \tilde{E}.$$

Since $K - \tilde{E} = \{x \in \mathbb{Z} \mid f - x \notin \tilde{E}\}$, we get $\frac{x}{2} \in K - \tilde{E}$.

Let x be odd. We have:

$$x \in K(S \bowtie^b E) \Leftrightarrow 2f(E) + b - x \notin 2 \cdot S \Leftrightarrow 2f - (x - b - 2e) \notin 2 \cdot S \Leftrightarrow \frac{x - b}{2} \in K + e,$$

that is the thesis. □

We see now how the components behave under the basic operations between ideals.

Proposition 3.11. *Let I and J be two relative ideals of $S \bowtie^b E$. The following facts are true:*

a. *If $I \subseteq J$, then $E_i(I) \subseteq E_i(J)$, $i = 1, 2$;*

b. *$E_i(I \cup J) = E_i(I) \cup E_i(J)$, $i = 1, 2$;*

c. *$E_i(I \cap J) = E_i(I) \cap E_i(J)$, $i = 1, 2$.*

Proof.
a is trivial.

The proofs of b and c are similar, so we prove b as an example:

$$E_1(I \cup J) = \frac{P(I) \cup P(J)}{2} = \frac{P(I)}{2} \cup \frac{P(J)}{2} = E_1(I) \cup E_1(J),$$

where $P(I)$ and $P(J)$ are the even elements of I and J respectively. The proof for the odd component is similar. □

Proposition 3.12. *Let $x \in \mathbb{Z}$ be and let H be a relative ideal of $S \bowtie^b E$. If x is even, then*

$$E_1(H + x) = E_1(H) + \frac{x}{2} \text{ and } E_2(H + x) = E_2(H) + \frac{x}{2}.$$

If x is odd, then

$$E_1(H + x) = E_2(H) + \frac{x + b}{2} \text{ and } E_2(H + x) = E_1(H) + \frac{x - b}{2}.$$

Proof.
Let x be an even number and let $y \in E_1(H + x)$; this means that there exists $h \in H$ even such that $2y = h + x$, therefore $y = \frac{h}{2} + \frac{x}{2}$. Let $y \in E_2(H + x)$; this means that there exists $h \in H$ odd such that $2y + b = h + x$, therefore $y = \frac{h-b}{2} + \frac{x}{2}$.

Let x be an odd number and let $y \in E_1(H + x)$; this means that there exists $h \in H$ odd such that $2y = h + x$, hence $y = \frac{h-b}{2} + \frac{x+b}{2}$. Let $y \in E_2(H + x)$; this means that there exists $h \in H$ even such that $2y + b = h + x$, therefore $y = \frac{h}{2} + \frac{x-b}{2}$. □

Proposition 3.13. *Let I and J be two ideals of $S \ltimes^b E$, then the following equalities hold:*

$$E_1(I-J) = \big((E_1(I)-E_1(J)\big) \cap \big(E_2(I)-E_2(J)\big)$$

and

$$E_2(I-J) = \big(E_2(I)-E_1(J)\big) \cap \big(E_1(I)-(E_2(J)+b)\big).$$

Proof.
Let $x \in E_1(I-J)$; we first show that $x \in E_1(I)-E_1(J)$. For every $e_1 \in E_1(J)$, we have:

$$x+e_1 \in E_1(I) \Leftrightarrow 2x+2e_1 \in I.$$

The last relation is true since $2x \in I-J$ and $2e_1 \in J$. Similarly we prove that $x \in E_2(I)-E_2(J)$.

Conversely let $x \in \big(E_1(I)-E_1(J)\big) \cap \big(E_2(I)-E_2(J)\big)$ and $j \in J$.

i) If $j = 2e_1$, with $e_1 \in E_1(J)$, we have:

$$2x+j = 2x+2e_1 = 2(x+e_1) \in I \Leftrightarrow 2x \in I-J \Leftrightarrow x \in E_1(I-J);$$

ii) If $j = 2e_2+b$, with $e_2 \in E_2(J)$, we have:

$$2x+j = 2x+2e_2+b = 2(x+e_2)+b \in I \Leftrightarrow 2x \in I-J \Leftrightarrow x \in E_1(I-J).$$

The proof for the odd component is similar. □

Proposition 3.14. *Let I and J be two relative ideals of $S \ltimes^b E$, the following equalities hold:*

$$E_1(I+J) = \big(E_1(I)+E_1(J)\big) \cup \big(E_2(I)+E_2(J)+b\big)$$

and

$$E_2(I+J) = \big(E_1(I)+E_2(J)\big) \cup \big(E_2(I)+E_1(J)\big).$$

Proof.
Using Property 6 of Lemma 2.6, we get:

$$\begin{aligned} I+J &= \big[2\cdot E_1(I) \cup (2\cdot E_2(I)+b)\big] + \big[2\cdot E_1(J) \cup (2\cdot E_2(J)+b)\big] = \\ &= \big[2\cdot(E_1(I)+E_1(J))\big] \cup \big[2\cdot(E_2(I)+E_2(J)+b)\big] \cup \\ &\qquad \cup \big[2\cdot(E_1(I)+E_2(J))+b\big] \cup \big[2\cdot(E_2(I)+E_1(J))+b\big]. \end{aligned}$$

Since all the even numbers of the sum are within the first and the second set of the union and the odd numbers are within the last two sets of the union, we obtain the thesis. □

Proposition 3.15. *Let I and J be two relative ideals of $S \ltimes^b E$; then the following equality holds:*

$$I \setminus J = 2\cdot\big(E_1(I)\setminus E_1(J)\big) \cup \big(2\cdot(E_2(I)\setminus E_2(J))+b\big).$$

In particular, the cardinality of $I \setminus J$ is:

$$|E_1(I)\setminus E_1(J)| + |E_2(I)\setminus E_2(J)|.$$

Proof.
Let $x \in I \setminus J$. If x is even, then $\frac{x}{2} \in E_1(I)$, and at the same time, since $x \notin J$, it follows that $\frac{x}{2} \notin E_1(J)$. If x is odd the proof is similar.

For the last part of the thesis it suffices to notice that the functions $z \mapsto 2z$ and $z \mapsto 2z + b$, defined in $\mathbb{Z}$, are injective.

□

The following result was originally proven in [8, Proposition 3.1.2] without using the ideal duplication.

Corollary 3.16. *The following equality holds:*

$$g(S \bowtie^b E) = g(S) + g(E) + m(E) + \frac{b-1}{2}$$

Proof.
Since $\mathbb{N} = C(S \bowtie^b E) - (2f(E) + b + 1)$, by Proposition 3.12 we get $\mathbb{N} = \mathbb{N} \bowtie^b (\mathbb{N} - \frac{b-1}{2})$. We have that $\mathbb{N} - \frac{b-1}{2} = B \cup \mathbb{N}$, where $B = \{-\frac{b-1}{2}, \dots, -1\}$; moreover, $2 \cdot B + b$ is equal to the set of odd numbers between 1 and b. By Proposition 3.15 we have:

$$\mathbb{N} \setminus (S \bowtie^b E) = 2 \cdot (\mathbb{N} \setminus S) \cup \{1, 3, \dots, 2h+1, \dots, b, 2g_1 + b, \dots, 2g_n + b\},$$

where $\mathbb{N} \setminus E = \{g_1, \dots, g_n\}$ and $n = g(E) + m(E)$.

□

3.2 Nearly Gorenstein duplication and application of the ideal duplication

In the first part of this section, we present a characterization for $S \bowtie^b E$ to be nearly Gorenstein; furthermore we apply this result to study the nearly Gorensteinness for numerical duplications obtained by some particular classes of ideals.

In the last part, after introducing the definition of almost symmetric semigroup, we describe the pseudo-Frobenius numbers of the numerical duplication. This allows to produce a new characterization for $S \bowtie^b E$ to be almost symmetric. For the rest of this section, we exclude the trivial case $S = \mathbb{N}$.

Let I be a relative ideal of S. We define the trace of I the following ideal:

$$\mathrm{Tr}_S(I) = I + (S - I).$$

This definition is a specialization, for numerical semigroups, of the definition of trace given in Definition 1.39. The trace of S is defined as the trace of its standard canonical ideal, $\mathrm{Tr}(S) = \mathrm{Tr}_S(K) = K + (S - K)$.

Theorem 3.17. *The following equalities hold:*

$$E_1\big(\mathrm{Tr}(S \bowtie^b E)\big) = (K - E) + (E - K)$$

and

$$E_2\big(\mathrm{Tr}(S \bowtie^b E)\big) = \big((K - E) + (E - (K - E))\big) \cup \big(K + (E - K)\big).$$

Proof.
First of all we prove that $E-(K-E)\subseteq S-(K+b)$. We have:

$$\big(E-(K-E)\big)+\big(K+b\big)\subseteq (E-K)+(K+b)\subseteq E+b\subseteq S.$$

Using this fact, Proposition 3.5 and Proposition 3.13 we have:

$$S\bowtie^b E-K(S\bowtie^b E)=(\tilde{E}-K)\bowtie^b(E-(K-\tilde{E})).$$

It is straightforward to prove that $(K+b)+(E-(K-E))\subseteq(K-E)+(E-K)$. Using this fact, Property 8 of Lemma 2.6, and Proposition 3.14, we get:

$$E_1\big(\mathrm{Tr}(S\bowtie^b E)\big)=(K-E)+(E-K).$$

With a similar argument, we can prove the equality for the odd component. □

As direct consequence of Theorem 3.17, we get the following characterization.

Corollary 3.18. *$S\bowtie^b E$ is nearly Gorenstein if and only if:*

$$M\subseteq(K-E)+(E-K)$$

and

$$E=\big((K-E)+(E-(K-E))\big)\cup\big(K+(E-K)\big).$$

In particular, the nearly Gorensteinness of $S\bowtie^b E$ does not depend on b or on shifts of E.

Example 3.19.

1. Let $S=\{0,3,5,6,8\rightarrow\}$, $E=\{10,13,\rightarrow\}$. We have:

   $$(K-E)+(E-K)=\{5,6,8,\rightarrow\}\subset M,$$

   but

   $$((K-E)+(E-(K-E)))\cup(K+(E-K))=\{10,13\rightarrow\}=E.$$

   Hence, the first condition of Corollary 3.18 is independent of the second one.

2. Let $S=\{0,4,7,8,10,\rightarrow\}$, $E=\{4,8,11,12,14,\rightarrow\}$. We have:

   $$(K-E)+(E-K)=\{4,7,8,10,\rightarrow\}=M;$$

   but

   $$((K-E)+(E-(K-E)))\cup(K+(E-K))=\{8,11,12,14,\rightarrow\}\subset E,$$

   Hence, the second condition of Corollary 3.18 is independent of the first one.

Proposition 3.20. *If $M\subseteq(K-E)+(E-K)$, then:*

1. *$M-M\supseteq(E-K)-(E-K)$;*

2. $K - M \supseteq E - (E - K)$.

In particular, $E - E \subseteq M - M$.

Proof.
Both statements 1 and 2 are consequence of Property 4 of Lemma 2.6, Property 1 of Corollary 2.8 and Proposition 3.3. The last part is true since:

$$E - E \subseteq E - (K + (E - K)) = (E - K) - (E - K) \subseteq M - M.$$

□

It is worth noticing that if $M = (K - E) + (E - K)$, then both 1 and 2 of the previous proposition become equalities.

We recall that s_1 denotes the multiplicity of S, i.e. the minimum of M.

Lemma 3.21. *If* $S \bowtie^b E$ *is nearly Gorenstein and not symmetric, then* $s_1 = m(\tilde{E} - K)$*, i.e.* $s_1 + K \subset \tilde{E} \subset K$.

Proof.
Clearly $s_1 = m(K - \tilde{E}) + m(\tilde{E} - K)$. Using Lemma 3.2, we get $m(K - \tilde{E}) = f(K) - f(\tilde{E}) = f - f = 0$. □

Note that, if $\tilde{E} \neq K$, then $s_1 \in \tilde{E} - K$ if and only if $s_1 = m(\tilde{E} - K)$.

We now apply the above results to some particular cases.

Corollary 3.22. *If S is symmetric, then:*

$$\mathrm{Tr}(S \bowtie^b E) = [(S - E) + E] \bowtie^b E.$$

In particular, if S is symmetric, then $S \bowtie^b E$ *is nearly Gorenstein if and only if* $\mathrm{Tr}_S(E) \supseteq M$.

Proof.
By Theorem 3.17, we see that $E - K \subseteq E_2(\mathrm{Tr}(S \bowtie^b E)) \subseteq E$. Since S is symmetric, then $E - K = E - S = E$, that is the thesis. □

Corollary 3.23. *If* $K - \tilde{E}$ *is a numerical semigroup, then:*

$$\mathrm{Tr}(S \bowtie^b E) = \mathrm{Tr}_S(E - E) \bowtie^b E.$$

Proof.
Using Theorem 3.17, by Corollary 3.4 and Corollary 3.5, if follows easily that

$$E_1(\mathrm{Tr}(S \bowtie^b E)) = (E - E) + (S - (E - E)).$$

Using Theorem 3.17 to compute the odd component of $\mathrm{Tr}(S \bowtie^b E)$, it suffices to show that Corollary 3.5 implies that:

$$(K - \tilde{E}) + (E - (K - \tilde{E})) = (E - E) + (E - (E - E)) = E.$$

□

Proposition 3.24. *The following equality holds:*

$$\mathrm{Tr}(S \bowtie^b M) = (\mathrm{Tr}(S) \cap M) \bowtie^b (\mathrm{Tr}(S) \cap M).$$

In particular, $S \bowtie^b M$ is nearly Gorenstein if and only if S is nearly Gorenstein.

Proof.
If S is symmetric, then, by Corollary 3.22, we get:

$$\mathrm{Tr}(S \bowtie^b M) = ((S - M) + M) \bowtie^b M = M \bowtie^b M = (\mathrm{Tr}(S) \cap M) \bowtie^b (\mathrm{Tr}(S) \cap M).$$

Assume S to be not symmetric, hence $S - K = M - K$. In fact, let $x \in S - K$ and $y \in K$. If $x + y = 0$, then $x < 0$ which is a contradiction because $S - K \subset S$. By Theorem 3.17, using Property 6 of Lemma 2.6 and Proposition 3.1, we get:

$$E_1(\mathrm{Tr}(S \bowtie^b M)) = (K - M) + (M - K) = (K \cup \{f\}) + (M - K) =$$

$$= (K + (M - K)) \cup ((M - K) + f).$$

Since $(M - K) + f \subseteq C \subseteq K + (M - K)$ and $M - K = S - K$, we get:

$$E_1(\mathrm{Tr}(S \bowtie^b M)) = K + (M - K) = K + (S - K) = \mathrm{Tr}(S) = \mathrm{Tr}(S) \cap M.$$

With a similar argument done at the beginning of the proof, we can prove that $M - (K - M) = S - (K - M)$ (this equality holds also in the case that S is symmetric). Using this fact, Corollary 3.4 and using Theorem 3.17 to compute the odd component of the trace of $S \bowtie^b M$, we get:

$$E_2(\mathrm{Tr}(S \bowtie^b M)) = ((K - M) + (M - K)) \cup (K + (M - K)) =$$

$$= (K - M) + (M - K) = E_1(\mathrm{Tr}(S \bowtie^b M)).$$

□

The following definition is a specialization of the notion of integrally closed ideal for numerical semigroup rings.

Definition 3.25. *An ideal I is said to be integrally closed if there exists $a \in S$ such that*

$$I = \{s \in M \mid s \geq a\}.$$

The maximal ideal M is clearly integrally closed.

Theorem 3.26. *If E is integrally closed and not maximal, then $S \bowtie^b E$ is nearly Gorenstein if and only if $s_1 = f + 1$.*

Proof.
Assume $s_1 \neq f + 1$. If $E = \{s \in M \mid s \geq a\}$, we have two possibilities:

1. If $a \leq f + 1$, then $\tilde{E} = E$. Since $s_1 \notin E$, by Lemma 3.21, $S \bowtie^b E$ is not nearly Gorenstein.

2. If $a > f+1$, then $\tilde{E} = E - e$. We show now that $s_1 + e \notin E$. If $a = f+1+x$, with $x > 0$, then $e = f + x - f = x$. If we suppose that $s_1 + x \in E$, then $s_1 + x \geq f + 1 + x$, and hence $s_1 \geq f+1$; it follows that $s_1 = f+1$, a contradiction. Since $s_1 + e \notin E$, we get $s_1 \notin E - e$, and once again, by Lemma 3.21, $S \bowtie^b E$ is not nearly Gorenstein.

Conversely if $s_1 = f+1$ we can write $S = \{0, s_1, \rightarrow\}$ and, since E is integrally closed, E is a shift of the maximal ideal. It is straightforward to prove that S is nearly Gorenstein, so by Proposition 3.24 and using the fact that the nearly Gorensteinness is independent of shifts of E, we have that $S \bowtie^b E$ is nearly Gorenstein. □

By the definition of pseudo-Frobenius numbers, it is obvious that

$$\mathrm{PF}(S) = (S - M) \setminus S.$$

If $S \neq \mathbb{N}$, then $S - M = M - M$, hence the pseudo-Frobenius numbers are exactly the numbers in $(M - M) \setminus S$.

Theorem 3.27. *The following equality holds:*

$$\mathrm{PF}(S \bowtie^b E) = 2 \cdot \left[((M-M) \cap (E-E)) \setminus S\right] \cup \left[2 \cdot ((E-M) \setminus E) + b\right].$$

In particular, $t(S \bowtie^b E) = |(M-M) \cap (E-E)| + |(E-M) \setminus E|$.

Proof.
Since $E - M \subseteq (S - E) - b$, by Proposition 3.13 it follows

$$(S \bowtie^b E) - (M \bowtie^b E) = \big((M-M) \cap (E-E)\big) \bowtie^b (E-M).$$

By Proposition 3.15 it follows

$$((S \bowtie^b E) - (M \bowtie^b E)) \setminus (S \bowtie^b E) = 2 \cdot \left[((M-M) \cap (E-E)) \setminus S\right] \cup \left[2 \cdot ((E-M) \setminus E) + b\right].$$

□

The formula for the type of $S \bowtie^b E$ was originally proven in [8, Proposition 3.5] without using the ideal duplication.

S is said to be almost symmetric if $\mathrm{PF}(S) = (K - M) \setminus S$ (i.e. $K - M$ is a numerical semigroup). This definition was originally introduced in [5] with the aim to generalize the notion of symmetric semigroups and consequently, with the introduction of almost Gorenstein rings, the notion of Gorenstein rings. In fact every symmetric semigroup is almost symmetric. Moreover, in [14, Proposition 6.1] the authors show that any one-dimensional almost Gorenstein ring is nearly Gorenstein, and therefore any almost symmetric semigroup is nearly Gorenstein.

Corollary 3.28. *$S \bowtie^b E$ is almost symmetric if and only if:*

$$K - \tilde{E} = (M - M) \cap (E - E)$$

and

$$K - M = \tilde{E} - M.$$

In particular, the almost symmetry of S does not depend on b or shifts of S.

Proof.
Since

$$(K(S \bowtie^b E) - M(S \bowtie^b E)) \setminus (S \bowtie^b E) = 2 \cdot \big((K - \tilde{E}) \setminus S\big) \cup \big(2 \cdot ((K+e) - M) + b\big),$$

by Theorem 3.27 and the definition of almost symmetric semigroup, the statement follows. □

The following result was originally proven in [8, Theorem 4.3]. We present an alternative proof using Corollary 3.28.

Theorem 3.29. *$S \bowtie^b E$ is almost symmetric if and only if*

$$K - (M - M) \subseteq \tilde{E} \subseteq K$$

and

$K - \tilde{E}$ is a numerical semigroup.

In particular, the almost symmetry of $S \bowtie^b E$ does not depend on b or shifts of E.

Proof.
If $S \bowtie^b E$ is almost symmetric, then, by the first equality of Corollary 3.28, $K - \tilde{E}$ is a numerical semigroup; furthermore, by Corollary 3.5, it is also true that $K - \tilde{E} = E - E$. Hence $K - \tilde{E} \subseteq M - M$, and therefore $K - (M - M) \subseteq \tilde{E} \subseteq K$.

Conversely, we want to prove that $S \bowtie^b E$ is almost symmetric using Corollary 3.28. Since $K - \tilde{E}$ is a numerical semigroup and $K - (M - M) \subseteq \tilde{E}$, by Corollary 3.5, it follows that $E - E \subseteq M - M$, therefore $K - \tilde{E} = (M - M) \cap (E - E)$. Since $\tilde{E} - M \subseteq K - M$ is easily proven, we only need to show the converse inclusion. By Property 5 of Lemma 2.6, we get:

$$K - M \subseteq K - ((M - M) + M) \subseteq (K - (M - M)) - M \subseteq \tilde{E} - M.$$

□

Using Theorem 3.29, it is trivial to prove the following:

Corollary 3.30. *$S \bowtie^b M$ is almost symmetric if and only if S is almost symmetric.*

In the case of almost symmetric numerical duplication, we are able to describe the pseudo-Frobenius numbers of $S \bowtie^b E$ with a greater degree of precision.

Theorem 3.31. *Suppose $S \bowtie^b E$ to be almost symmetric. Then, the following is true:*

$$\mathrm{PF}(S \bowtie^b E) = 2 \cdot \big[(E - E) \setminus S\big] \cup \big[2 \cdot ((K + e) \setminus E) + b\big] \cup \{2f(E) + b\}.$$

Moreover, if $\tilde{E} \neq K$, then $S \bowtie^b E$ always has at least one even pseudo-Frobenius number, and the even pseudo-Frobenius numbers do not depend on b or shifts of E.

Proof.
Using Property 7 of Lemma 2.6, by Corollary 3.28 and Proposition 3.1 we get:

$$(E-M)\setminus E = ((K-M)\setminus \tilde{E})+e = ((K+e)\cup\{f(E)\})\setminus E = ((K+e)\setminus E)\cup\{f(E)\}.$$

Using Theorem 3.27 and Proposition 3.20, we get the thesis.

For the last part, the only non-trivial thing to prove is that, if $\tilde{E}\neq K$, then $E-E\neq S$. In fact, if $E-E=S$, by Theorem 3.29 and Corollary 3.5, we would get that $K=K-(E-E)=K-(K-\tilde{E})=\tilde{E}$. □

The following result was originally proven in [8, Proposition 4.8]; we present an alternative proof using Theorem 3.31.

Corollary 3.32. *Suppose $S\bowtie^b E$ to be almost symmetric. Then we have:*

$$t(S\bowtie^b E) = 2|(E-E)\setminus S|+1 = 2|K\setminus\tilde{E}|+1.$$

In particular, the type of $S\bowtie^b E$ is always odd and $1\leq t(S)\leq 2t(S)+1$.

Proof.
By Property 7 of Corollary 2.6, we get that $|2\cdot((K+e)\setminus E)+b| = |K\setminus\tilde{E}|$. Moreover, by Property 5 of Corollary 2.8 and Corollary 3.5, it follows that $|(E-E)\setminus S| = |K\setminus\tilde{E}|$. Using Theorem 3.31, we get

$$t(S\bowtie^b E) = |(E-E)\setminus S|+|K\setminus\tilde{E}|+1 = 2|K\setminus\tilde{E}|+1.$$

The last part follows from the fact that, by Proposition 3.20, if $S\bowtie^b E$ is almost symmetric (or also if it is nearly Gorenstein), then $E-E\subseteq M-M$. □

In [8, p.159], the authors prove that it is possible to obtain any odd integer $x=2m+1$ in the range described in the previous corollary.

Example 3.33. The previous corollary states that, in almost symmetric numerical duplications, the type is always odd; hence, it is natural to ask if this fact is true in the nearly Gorenstein case. In general this is not the case, in fact take for example $S=\{0,4,7,8,10,\rightarrow\}$ and $E=\{3,6,7,10,11,13,\rightarrow\}$. Then $S\bowtie^b E$ is nearly Gorenstein, but $t(S\bowtie^b E)=2$. Furthermore, since $E-E=S$, the pseudo-Frobenius numbers are all odd.

Proposition 3.34. *Let E be a principal ideal (i.e. $E=s+S$, with $s\in S$). The following facts are equivalent:*

1. *S is symmetric;*
2. *$S\bowtie^b E$ is symmetric;*
3. *$S\bowtie^b E$ is almost symmetric;*
4. *$S\bowtie^b E$ is nearly Gorenstein.*

Proof.
The relations 1⇒ 2, 2 ⇒ 3 and 3 ⇒ 4 are trivial; we only need to prove that 4 ⇒ 1.

If $S \bowtie^b E$ is nearly Gorenstein, using Theorem 3.17 to compute the trace, we get:

$$\mathrm{Tr}(S \bowtie^b E) = \mathrm{Tr}(S) \bowtie^b (s + \mathrm{Tr}(S)) \supseteq M \bowtie^b (s + S).$$

Therefore $\mathrm{Tr}(S) = S$, and hence S is symmetric.

□

Example 3.35. We proved that, if E is principal, then $S \bowtie^b E$ is almost symmetric if and only if it is nearly Gorenstein. However, the claim is false for any 2-generated ideal.

Let $S = \{0, 6, 7, 9, 12, 13, 14, 15, 16, 18, \rightarrow\}$ and $E = \{6, 12, 13, 15, 18, \rightarrow\} = (6, 23) + S$. Since S is symmetric, using Corollary 3.22 to compute the trace, we get that $S \bowtie^b E$ is nearly Gorenstein. On the other hand, since $\tilde{E} = E \subset M = K - (M - M)$, by Theorem 3.29, S is not almost symmetric.

Example 3.36. Let $S = \{0, 4, 7, 8, 10, \rightarrow\}$. We have $K = \{0, 3, 4, 6, 7, 8, 10, \rightarrow\}$ and $K - (M - M) = M$. All the ideals between M and K are listed below:

- $\tilde{E}_0 = M$,
- $\tilde{E}_1 = \{0, 4, 7, 8, 10, \rightarrow\} = S$,
- $\tilde{E}_2 = \{3, 4, 7, 8, 10, \rightarrow\}$,
- $\tilde{E}_3 = \{4, 6, 7, 8, 10, \rightarrow\}$,
- $\tilde{E}_4 = \{0, 3, 4, 7, 8, 10, \rightarrow\}$,
- $\tilde{E}_5 = \{0, 4, 6, 7, 8, 10, \rightarrow\}$,
- $\tilde{E}_6 = \{3, 4, 6, 7, 8, 10, \rightarrow\}$,
- $\tilde{E}_7 = K$.

Among these ideals, only $\tilde{E}_1$ and $\tilde{E}_2$ do not give rise to nearly Gorenstein numerical duplications. While $\tilde{E}_4$ is the only one that gives rise to nearly Gorenstein numerical duplication but not almost symmetric (because $K - \tilde{E}_4$ is not a numerical semigroup).

Unfortunately, it is much harder to determinate the family of class of shifts for the nearly Gorensteinness of the numerical duplication. Thanks to Lemma 3.21, we can say that $\tilde{E}$ must be between $s_1 + K \subset \tilde{E} \subseteq K$ (do note that $s_1 + K \subset K - (M - M)$). In the example, since $s_1 + K = M \setminus \{13\}$, we can say that we already found all the possible classes of shifts for which $S \bowtie^b E$ is nearly Gorenstein.

Chapter 4

Semitrivial ideal extension

In this chapter we want to generalize the idea behind the ideal duplication, to rings. To this aim, we will need to work within the class of $\mathbb{Z}_2$-graded ring (and more in general $\mathbb{Z}_2$-graded modules).

The results of this chapter are contained in [21].

4.1 $\mathbb{Z}_2$-graded modules

In this section we introduce some basic facts about $\mathbb{Z}_2$-graded modules and we fix the notation, which will be useful for the whole chapter. Although most of the results are a specialization of the general graded case, we didn't find much about this subject in the literature.

Some useful notions before starting. Let I be a monoid, we recall that a ring A is said to be I-graded if the underlying additive group is a direct sum of abelian groups A_i such that $A_iA_j \subseteq A_{i+j}$ for every $i, j \in I$. Similarly, a module M over an I-graded ring A is said to be I-graded if M is the direct sum of abelian groups M_i such that $A_iM_j = M_{i+j}$ for every $i, j \in I$. The direct sum decomposition is usually referred to as grading.

Notation

In a $\mathbb{Z}_2$-graded module M over a $\mathbb{Z}_2$-graded ring A, for any element $m \in M$, we denote by $m_0 \in M_0$ the homogeneous component of degree 0 and by $m_1 \in M_1$ the homogeneous component of degree 1. Let $G \subseteq M$; then we set $G_0 = \{g_0 \mid \exists g \in G : g = g_0 + g_1\}$ and $G_1 = \{g_1 \mid \exists g \in G : g = g_0 + g_1\}$; clearly, it is true that $G \cap M_0 \subseteq G_0$ and $G \cap M_1 \subseteq G_1$, while the converse may not be true. G is said to be homogeneous if, for every $g \in G$, both $g_0 \in G$ and $g_1 \in G$. Obviously, if G is an additive subgroup of M, then G is homogeneous if and only if $G \cap M_0 + G \cap M_1 = G = G_0 + G_1$, and, if G is also a submodule of M, then G is homogeneous if and only if it is a graded submodule of M.

Since A_0 is a subring of A, M is an A_0-module in the natural way, and $M \cong M_0 \oplus M_1$ as A_0-modules. Let $m \in M$; the functions $m \mapsto m_0$ and $m \mapsto m_1$ are the canonical projection of M (as an A_0-module) to M_0 and M_1 respectevely, moreover, for every $a \in A$, $(am)_0 = a_0m_0 + a_1m_1$ and $(am)_1 = a_0m_1 + a_1m_0$.

Example 4.1.

1. Any $\mathbb{N}$-graded ring A can be endowed with a natural $\mathbb{Z}_2$-grading; $A = A_{\bar{0}} + A_{\bar{1}}$, where $A_{\bar{0}} = \oplus_{h\in\mathbb{N}} B_{2h}$ and $A_{\bar{1}} = \oplus_{h\in\mathbb{N}} B_{2h+1}$.

2. Let M be an R-module. We recall that the Nagata's idealization of R by M, denoted by $R \ltimes M$, is defined as the ring obtained from the abelian additive group $R \oplus M$ with product $(r, m)(r', m') = (rr', rm' + r'm)$. It is easy to prove that $R \ltimes M$ is a $\mathbb{Z}_2$-graded ring with grading $R \oplus M$.

We recall that a module M over a ring A is said to be torsion-free if, for every nonzero divisor element r, $rm = 0$ implies $m = 0$ for every $m \in M$. Also, let a be an element of A, then a is said to be M-regular if $am = 0$ implies $m = 0$ for every $m \in M$. An A-regular element is called regular.

Proposition 4.2. *Let M be a $\mathbb{Z}_2$-graded module over a $\mathbb{Z}_2$-graded graded ring A. Assume that M_1 is a torsion-free module over A_0 (we treat M_1 as an A_0-module in the obvious way) and take a nonzero divisor element $a_0 \in A_0$. Then, a_0 is M_0-regular if and only if it is M-regular.*

Proof. Let $m = m_0 + m_1 \in M$; then $a_0 m = 0$ implies that $a_0 m_0 = a_0 m_1 = 0$ and hence, by hypothesis, $m_0 = m_1 = 0$.

The converse statement is trivial. □

Remark 4.3. Let M be a $\mathbb{Z}_2$-graded module over a $\mathbb{Z}_2$-graded graded ring A. Let $G \subseteq A$ be an additive subgroup of A and let $N, N' \subseteq M$ be additive subgroups of M. We denote by GN the additive subgroup of M generated by the elements of the form gn with $g \in G$ and $n \in N$. If G is an A_0-submodule of A and N is an A_0-submodule of M, then GN is an A_0-submodule of M. We set $(N :_A N') = \{a \in A \mid aN' \subseteq N\}$, which is an additive subgroup of A. If N and N' are A_0-submodules M, then $(N :_A N')$ is an A_0-submodule of M. Also, we denote $(N :_{A_0} N') = (N :_A N') \cap A_0$ and $(N :_{A_1} N') = (N :_A N') \cap A_1$. If N and N' are A_0-modules, then $(N :_{A_0} N')$ is an ideal of A_0 and $(N :_{A_1} N')$ is a submodule of A_1 over A_0.

Proposition 4.4. *Let $I \subseteq A$ be a homogeneous additive subgroup of A and let $N, N' \subseteq M$ be homogeneous additive subgroups of M. Then $N + N'$, $N \cap N'$ and IN are homogeneous additive subgroups of M; also $(N :_A N')$ is a homogeneous additive subgroup of A. Moreover:*

1. $(N + N')_0 = N_0 + N'_0$ *and* $(N + N')_1 = N_1 + N'_1$.
2. $(N \cap N')_0 = N_0 \cap N'_0$ *and* $(N \cap N')_1 = N_1 \cap N'_1$.
3. $(IN)_0 = I_0 N_0 + I_1 N_1$ *and* $(IN)_1 = I_0 N_1 + I_1 N_0$.
4. $(N :_A N')_0 = (N_0 :_{A_0} N'_0) \cap (N_1 :_{A_0} N'_1)$ *and* $(N :_A N')_1 = (N_0 :_{A_1} N'_1) \cap (N_1 :_{A_1} N'_0)$.

Proof. The only non-trivial thing to prove is that $(N :_A N')$ is homogeneous. Let $a \in (N :_A N')$ and $n' \in N'$; since both N and N' are homogeneous, we get:

$$a_0 n' = a_0 n'_0 + a_0 n'_1 = (an'_0)_0 + (an'_1)_1 \in N.$$

Since $(N :_A N')$ is an additive subgroup of A, $a_1 \in (N :_A N')$, hence $(N :_A N')$ is homogeneous. □

Proposition 4.5. *Let M be a $\mathbb{Z}_2$-graded module over a $\mathbb{Z}_2$-graded ring A and let $R \subseteq A$ be a homogeneous subring. An R-module $N \subseteq M$ is homogeneous if and only if it can be generated by a homogeneous set.*

Proof. Let G be a set of generators for N, then, since N is homogenoeus, $G_0 \cup G_1 \subseteq N$, and this is a homogeneous set of generators for N.

The converse is trivial. □

Proposition 4.6. *Let M and N be $\mathbb{Z}_2$-graded modules over a $\mathbb{Z}_2$-graded ring A. Denoted by $H = \mathrm{Hom}_{A_0}(M, N)$, $H_0 = \{f \in H \mid f(M_i) \subseteq N_i\}$ and $H_1 = \{f \in H \mid f(M_i) \subseteq N_{i+1}\}$, then H is a $\mathbb{Z}_2$-graded module over A with grading $H_0 \oplus H_1$. Moreover:*

1. $H_0 \cong \mathrm{Hom}_{A_0}(M_0, N_0) \oplus \mathrm{Hom}_{A_0}(M_1, N_1)$ *as* A_0*-modules.*

2. $H_1 \cong \mathrm{Hom}_{A_0}(M_0, N_1) \oplus \mathrm{Hom}_{A_0}(M_1, N_0)$ *as* A_0*-modules.*

Proof. Clearly, H_0 and H_1 are additive subgroups of H such that $H_0 \cap H_1 = 0$, also $A_i H_j \subseteq H_{i+j}$, with $i, j \in \mathbb{Z}_2$. The only non-trivial thing left to prove is that $H = H_0 + H_1$. Let $f \in H$; we define the maps:

$$f_0 : M \to N,$$

$$m \mapsto (f(m_0))_0 + (f(m_1))_1$$

and

$$f_1 : M \to N,$$

$$m \mapsto (f(m_0))_1 + (f(m_1))_0;$$

it is easy to check that $f_0 \in H_0$, $f_1 \in H_1$ and $f = f_0 + f_1$.

To prove Property 1, consider the isomorphism:

$$H_0 \to \mathrm{Hom}_{A_0}(M_0, N_0) \oplus \mathrm{Hom}_{A_0}(M_1, N_1),$$

$$f \mapsto (f|_{M_0}, f|_{M_1}).$$

A similar argument can be used to prove Property 2. □

The A_0-module $\mathrm{Hom}_{A_0}(M, N)$ can be $\mathbb{Z}_2$-graded even if N is not $\mathbb{Z}_2$-graded.

Proposition 4.7. *Let M be a $\mathbb{Z}_2$-graded module over a $\mathbb{Z}_2$-graded ring A and N an A_0-module. Denoted by $G = \mathrm{Hom}_{A_0}(M, N)$, $G_0 = \{f \in G \mid f(M_1) = 0\}$ and $G_1 = \{f \in H \mid f(M_0) = 0\}$, then G is a $\mathbb{Z}_2$-graded module over A with grading $G_0 \oplus G_1$. Moreover:*

1. $G_0 \cong \mathrm{Hom}_{A_0}(M_0, N)$ *as* A_0*-modules.*

2. $G_1 \cong \mathrm{Hom}_{A_0}(M_1, N)$ *as* A_0*-modules.*

Proof. Clearly, G_0 and G_1 are additive subgroups of G such that $G_0 \cap G_1 = 0$, also $A_i G_j \subseteq G_{i+j}$, with $i, j \in \mathbb{Z}_2$. The only non-trivial thing left to prove is that $G = G_0 + G_1$. Let $f \in G$; we define the maps:

$$f_0 : M \to N,$$

$$m \mapsto f|_{M_0}(m_0)$$

and

$$f_1 : M \to N,$$

$$m \mapsto f|_{M_1}(m_1);$$

it is easy to check that $f_0 \in G_0$, $f_1 \in G_1$ and $f = f_0 + f_1$.

To prove Property 1, consider the isomorphism:

$$G_0 \to \mathrm{Hom}_{A_0}(M_0, N),$$

$$f \mapsto f|_{M_0}.$$

A similar argument can be used to prove Property 2. □

Remark 4.8. Let M be a $\mathbb{Z}_2$-graded module over $\mathbb{Z}_2$-graded ring A. Hence $M_0 \oplus M_1$ is by definition a module over A_0 and A, the latter with the product:

$$a \cdot (m_0, m_1) = (a_0 m_0 + a_1 m_1, a_1 m_1 + a_1 m_0),$$

for every $a \in A$, $m_0 \in M_0$ and $m_1 \in M_1$. If we suppose that M_0 and M_1 are isomorphic to respectevily to N_0 and N_1 as A_0-module, then we can endow an A-module structure to $N_0 \oplus N_1$ through the following action:

$$a \bullet (n_0, n_1) := (f(a_0 m_0 + a_1 m_1), g(a_0 m_1 + a_1 m_0)),$$

where $f : M_0 \to N_0$ and $g : M_1 \to N_1$ are isomorphisms. It is not hard to prove that $(N_0 \oplus N_1, \bullet)$ is $\mathbb{Z}_2$-graded over A, and $M \cong N_0 \oplus N_1$ as graded A-modules.

Let $a = a_0 + a_1 \in A$ be an element of a $\mathbb{Z}_2$-graded ring A; then we denote by $\overline{a} = a_0 - a_1$ the conjugate of a. If G is a subset of A, then we denote by $\overline{G} = \{\overline{g} \mid g \in G\}$ the the conjugate of G, and G is said to be conjugable if $G = \overline{G}$. It is not hard to prove that if G is conjugable, then also $A \setminus G$ is conjugable. If G is a graded subgroup of A, then G is conjugable, while the converse may be false. The function $f : A \to A$ defined as $f(a) = a_0 - a_1$, is a graded isomorphism such that $f^2 = id_A$. Hence, the conjugate of a prime (maximal) ideal is still a prime (maximal) ideal; also, the sets of all zerodivisors and units are conjugable.

It is interesting to study the case of conjugable prime ideals which are not homogeneous.

Proposition 4.9. *Let $\mathfrak{p}$ be a prime ideal of A. Then $\mathfrak{p}$ is conjugable if and only one and only one of the following is true:*

1. *$\mathfrak{p}$ is homogeneous;*
2. *$\mathfrak{p}$ is not homogeneous and* $\mathrm{char}(A/\mathfrak{p}) = 2$.

Proof. Suppose that $\mathfrak{p}$ is not homogeneous. Take $x + y \in \mathfrak{p}$ with $x \in A_0$, $y \in A_1$ and assume that $x \notin \mathfrak{p}$. Since $\mathfrak{p}$ is conjugable, $x - y \in \mathfrak{p}$, hence $x + x = 2x \in \mathfrak{p}$, that is $2 \in \mathfrak{p}$. We proved that $\mathrm{char}(A/\mathfrak{p}) = 2$. If we suppose that $y \notin \mathfrak{p}$, with a similar argument, we arrive at the same conclusion.

Conversely, if $\mathfrak{p}$ is homogeneous, then $\mathfrak{p}$ is conjugable. Assume $\mathfrak{p}$ not homogeneous and such that $\mathrm{ch}(A/\mathfrak{p}) = 2$. Let $a \in \mathfrak{p}$, then $a + \overline{a} = a_0 + a_0 \in \mathfrak{p}$, hence $\overline{a} \in \mathfrak{p}$. □

We define the modulus of an element $a \in A$ as $|a| := a\overline{a} = a_0^2 - a_1^2 \in A_0$. Clearly, a is a unit of A if and only if $|a|$ is a unit of A_0. If $|a|$ is a zerodivisor of A_0, then a is a zerodivisor of A. If a is a regular element, then $|a|$ is a regular element of A_0. If A_1 is a torsion-free module over A_0, by Proposition 4.2, the converse statements hold:

Lemma 4.10. *Let A be a $\mathbb{Z}_2$-graded ring such that A_1 is a torsion-free module over A_0. Then, for every $a \in A$, a is a regular (zerodivisor) element of A if and only if $|a|$ is a regular (zerodivisor) element of A_0.*

We see now how the grading of a $\mathbb{Z}_2$-graded module behaves under the usual operations.

Proposition 4.11. *Let M be a $\mathbb{Z}_2$-graded module over a $\mathbb{Z}_2$-graded ring A and let $N \subseteq M$ be a homogeneous submodule of M. Then M/N is a $\mathbb{Z}_2$-graded module over A with grading:*

$$M_0/N_0 \oplus M_1/N_1.$$

Proof. Consider the subgroups $G_0 = \{g_0 + N \mid g_0 \in M_0\}$ and $G_1 = \{g_1 + N \mid g_1 \in M_1\}$. Clearly, $M/N = G_0 \oplus G_1$, moreover $G_0 \cong M_0/N_0$ and $G_1 \cong M_1/N_1$ as A_0-modules. The thesis is a consequence of Remark 4.8. □

To an extent, we can do the same thing for the localization: let T be a multiplicative closed set of A_0, then it is easy to prove that $T^{-1}M$ is a $\mathbb{Z}_2$-graded module over $T^{-1}A_0$ (here the ring is considered trivially graded) with grading:

$$T^{-1}M_0 \oplus T^{-1}M_1.$$

In particular, if $M = A$ and $A_1 \neq 0$, then $T^{-1}A$ is a $\mathbb{Z}_2$-graded ring with grading $T^{-1}A_0 \oplus T^{-1}A_1$. This form of localization, however, is not sharp enough for what we will need, thus some more assumptions are necessary.

Proposition 4.12. *Let A be a $\mathbb{Z}_2$-graded ring and $S \subseteq A$ a conjugable multiplicatively closed set of A. Then $S^{-1}A$ is a $\mathbb{Z}_2$-graded ring with grading:*

$$(S \cap A_0)^{-1}A_0 \oplus (S \cap A_0)^{-1}A_1.$$

Proof. We need to prove that S and $S \cap A_0$ have the same saturation in A. To this end, it suffices to show that for every $s \in S$, there exists $t \in A$ such that $st \in S \cap A_0$. Since S is conjugable, $\bar{s} \in S$, hence $s\bar{s} = |s| \in S \cap A_0$. □

Proposition 4.13. *Let M be a $\mathbb{Z}_2$-graded module over a $\mathbb{Z}_2$-graded ring A and let $S \subseteq A$ be a conjugable multiplicatively closed set of A. Then $S^{-1}M$ is a $\mathbb{Z}_2$-graded module over the $\mathbb{Z}_2$-graded ring $S^{-1}A$ (endowed with the grading of Proposition 4.12) with grading:*

$$(S \cap A_0)^{-1}M_0 \oplus (S \cap A_0)^{-1}M_1.$$

Proof. Using Proposition 4.12, we get:

$$S^{-1}M = S^{-1}A \otimes_A M = (S \cap A_0)^{-1}A \otimes_A M = (S \cap A_0)^{-1}M.$$

□

Remark 4.14. For any homogeneous ideal $\mathfrak{a}$ of a $\mathbb{Z}_2$-graded ring A, the canonical surjection $\pi : A \to A/\mathfrak{a}$, is a graded homomorphism where $A/\mathfrak{a}$ is endowed with the $\mathbb{Z}_2$-grading of Proposition 4.11.

For any conjugable multiplicatively closed set S of A, the homomorphism $f : A \to S^{-1}A$, defined as $f(a) = \frac{a}{1}$, is a graded homomorphism where $S^{-1}A$ is endowed with the $\mathbb{Z}_2$-grading of Proposition 4.12.

Corollary 4.15. *Let A be a ring and $Q(A)$ its total fraction ring. If A is a $\mathbb{Z}_2$-graded ring, $Q(A)$ is also a $\mathbb{Z}_2$-graded ring. Moreover, for any multiplicatively closed set S conjugable and composed of nonzero divisor elements, the ring $S^{-1}A$ is a homogeneous subring of $Q(A)$.*

Proof. Since the set of nonzero divisors of A is conjugable, by Proposition 4.12, follows that $Q(A)$ is $\mathbb{Z}_2$-graded. □

Let M be a $\mathbb{Z}_2$-graded module over a $\mathbb{Z}_2$-graded ring A. The following definition will be useful later.

Definition 4.16. *Let $N \subseteq M$ be a submodule, we define the homogeneous body of N, denoted with $\mathcal{B}(N)$, the largest homogeneous A-module cointained in N.*

The homogeneous body does exist because, by Proposition 4.5, the module generated by union of homogeneous sets is homogeneous. Also, it is easy to prove that $\mathcal{B}(N) = N \cap M_0 + N \cap M_1$.

4.2 General properties of the semitrivial extension

With the purpose of giving a natural generalization of the idealization of Nagata, it is introduced the class of semitrivial extensions (see [19] or [22]). In this section we give some generalities about this construction, that we will use in the rest of the chapter.

Definition 4.17. *Let R be a commutative ring with unity, M an R-module and let $\phi : M \otimes_R M \to R$ be an symmetric and associative R-module homomorphism, that is,*

$$\phi(m \otimes m)' = \phi(m' \otimes m)$$

and

$$m\phi(m' \otimes m'') = \phi(m \otimes m')m''$$

for every $m, m', m'' \in M$. Then, we define the semitrivial extension of R by M and ϕ, denoted by $R \ltimes_\phi M$, the ring obtained from the abelian additive group $R \oplus M$, endowed with the product:

$$(r, m)(r', m') = (rr' + \phi(m \otimes m), rm' + r'm).$$

It is clear that $R \ltimes_\phi M$ is a commutative ring with unity $(1, 0)$ and, since $R \subseteq R \ltimes_\phi M$ is integral, they have the same Krull dimension. Trivial extensions correspond to the special case $\phi = 0$. If $\phi \neq 0$, then $0 \oplus M \subseteq R \ltimes_\phi M$ is not an ideal of $R \ltimes_\phi M$; the ideal generated by $0 \oplus M$ is $\mathrm{Im}\phi \oplus M$.

Any semitrivial extension is a $\mathbb{Z}_2$-graded ring, and in fact the two concepts are essentially the same thing. This is true beacuse any $\mathbb{Z}_2$-graded ring is the semitrivial extension of the subring composed of the homogeneous elements of degree zero by the module composed of the homogeneous elements of degree 1 and the product of A. In symbols, any $\mathbb{Z}_2$-graded ring A can be written as $A = A_0 \ltimes_\phi A_1$, where $\phi \in \mathrm{Hom}_{A_0}(A_1 \otimes_{A_0} A_1, A_0)$ is the product in A.

Example 4.18.

1. The semitrivial extension of a ring R with itself is isomorphic to $R[t]/(t^2 + r)$, for some $r \in R$. To prove this, note that $\mathrm{Hom}_R(R \otimes_R R, R) \cong R$, thus ϕ can be indentified with a element of R, say r_ϕ, and $\phi(r_1 \otimes r_2) = r_1 r_2 r_\phi$ for every $r_1, r_2 \in R$. Consider the homomorphism:

$$f : R \ltimes_\phi R \to R[t]/(t^2 - r_\phi),$$

$$(r_1, r_2) \mapsto r_1 + r_2 t^*,$$

where $t^* = t + (t^2 - r_\phi)$. Since ${t^*}^2 = r_\phi$, it follows easily that f is an isomorphism.

2. Let R be a ring and I an ideal of R; if $a, b \in R$, denoting by $\mathcal{R}_+ = \bigoplus_{n \geq 0} I^n t^n$, we set

$$R(I)_{a,b} = \mathcal{R}_+ / I^2(t^2 + at + b),$$

where $I^2(t^2+at+b)$ is the contraction to $\mathcal{R}_+$ of the ideal generated by $t^2 + at + b$ in $R[t]$, the quadratic quotient of the Rees algebra of R associated to I. $R(I)_{0,b}$ is clearly the semitrivial extension of R by I and ϕ, where ϕ is defined as $\phi(i \otimes i') = -ii'b$.

Let R be a ring and M a module over R. We denote:

$$\Phi(M) = \{\phi \in \mathrm{Hom}_R(M \otimes M, R) \mid \phi \text{ is symmetric and associative } \}.$$

Obviously $\Phi(M)$ is a submodule of $\mathrm{Hom}_R(M \otimes_R M, R)$. It is hardly surprising that semitrivial extensions are bound to be more subtle and complicated than trivial extensions, especially concerning the nature of the homomorphism ϕ and the relations that it brings between M and R. Thus, assuming $\Phi(M) \neq 0$ will inevitably cause M to be less "general" with respect to the case of the Nagata's idealization. An important first example is the following lemma, which was originally proved in [22]:

Lemma 4.19. *If $\Phi(M) \neq 0$ and M is free, then $M \cong R$.*

Proof. Let $\{m_i \mid i \in I\}$ be a base of M. Consider $m_i \neq m_j$ and take the linear combination

$$\phi(m_i \otimes m_j)m_i - \phi(m_i \otimes m_i)m_j = \phi(m_i \otimes m_j)m_i - \phi(m_i \otimes m_j)m_i = 0.$$

This implies that $\phi(m_i \otimes m_j) = \phi(m_i \otimes m_i) = 0$, hence the thesis. □

An obvious consequence of the previous lemma is that if R is a field and $\Phi(M) \neq 0$, then $M \cong R$.

Lemma 4.20. *If the following conditions are satisfied:*

1. *M is torsion-free,*
2. *For some $\phi \in \Phi(M)$, $\mathrm{Im}(\phi)$ contains a nonzero divisor,*

then there exists $n \in \mathbb{N}$ such that M is isomorphic to a submodule of R^n. In particular, if the following condition holds:

2′. *There exists $x \otimes y \in M \otimes M$ such that $\phi(x \otimes y)$ is a nonzero divisor of R,*

then M is isomorphic to an ideal of R.

Proof. Let $a = x_1 \otimes y_1 + \ldots + x_n \otimes y_n \in M \otimes M$ such that $\phi(a)$ is a nonzero divisor of R. Consider the following homomorphism:

$$f_{(y_1,\ldots,y_n)} : M \to R^n,$$

$$m \mapsto (\phi(m \otimes y_1), \ldots, \phi(m \otimes y_n)).$$

Suppose that $f_{(y_1,\ldots,y_n)}(m) = 0$; then $\phi(m \otimes y_i) = 0$ and $0 = \phi(m \otimes y_i)x_i = \phi(x_i \otimes y_i)m$ for $i = 1, \ldots, n$. Hence, $\phi(a)m = (\sum_{i=1}^n \phi(x_i \otimes y_i))m = 0$, that is $m = 0$.

If 2′ is true, then $a = x \otimes y$, and we can chose the injective homomorphism

$$f_y : M \to R,$$

$$m \mapsto \phi(m \otimes y).$$

□

The previous Lemma allows us to say that the semitrivial extensions of R by a torsion-free module over a domain are essentially the semitrivial extensions of R by an ideal.

Remark 4.21. The following general fact is well-known: if I an J are ideals of a commutative ring A with unity and depth$I > 0$, then

$$\mathrm{Hom}_A(I, J) \cong (J :_{Q(A)} I),$$

where $(J :_{Q(A)} I) = \{x \in Q(A) \mid xI \subseteq J\}$ is a fractional ideal of A. The equality still holds even if I and J are both fractional ideals of A.

Assume that $M \cong I$ for some ideal of R and that I has positive depth. Then

$$\mathrm{Hom}_R(I \otimes_R I, R) = \mathrm{Hom}_R(I, \mathrm{Hom}_R(I, R)) = \mathrm{Hom}_R(I, I^{-1}) = (I^{-1} :_{Q(R)} I).$$

We conclude that ϕ can be indentified with the multiplication by an element $x \in (I^{-1} :_{Q(R)} I)$ and $\phi(i \otimes i') = ii'x$.

Proposition 4.22. *Assume that R is a noetherian domain. If $\phi \in \Phi(M) \setminus \{0\}$ and M is torsione-free with* depth$(M) \geq 2$*, then $M \cong I$ for some ideal I of R and $R \ltimes_\phi M \cong R(I)_{0,r}$ for some $r \in R$.*

Proof. By Lemma 4.20, we get that M is isomorphic to some ideal I of R. Since depth$(M) \geq 2$, by [**Exercise 1.2.24**, 6], $\mathrm{Hom}_R(I \otimes I, R) = \mathrm{Hom}_R(I, \mathrm{Hom}_R(I, R)) = \mathrm{Hom}_R(I, R) = R$. Hence, ϕ is the multiplication by some element r of R, proving the thesis. □

If we assume that ϕ is surjective, using the same homomorphism defined in the proof of Lemma 4.20, we get that M is a submodule of R^n, for some $n \in \mathbb{N}$. Moreover:

Lemma 4.23. *Assume that for some $\phi \in \Phi(M)$, $\mathrm{Im}\phi = R$. Then, if R is local, $M \cong R$.*

Proof. Let $a = x_1 \otimes y_1 + \ldots + x_n \otimes y_n \in M \otimes M$ be an element such that $\phi(a)$ is a unit of R. Since R is local, there must be a $x_i \otimes y_i$ such that $\phi(x_i \otimes y_i)$ is a unit. Let u be the inverse of $\phi(x_i \otimes y_i)$ and consider the injection f_{y_i} defined in the proof of Lemma 4.20. We claim that f_{y_i} is also surjective. Let $r \in R$, then $r = r\phi(x_i \otimes y_i)u = \phi(rux_i \otimes y_i) = f_{y_i}(rux_i)$. □

The following result is proved in [**Proposition 1**, 19]:

Proposition 4.24. *$R \ltimes_\phi M$ is noetherian (Artinian) if and only if R is noetherian (artinian) and M is finitely generated.*

Theorem 4.25. *Assume $\Phi(M) \neq 0$. Then $R \ltimes_\phi M$ is a domain if and only if:*

1. *R is a domain,*
2. *M is isomorphic to an ideal of R.*
3. *For every $x \in R \ltimes_\phi M$, $|x| = 0$ implies $x = 0$.*

Proof. Since $R \ltimes_\phi M$ is a domain, it follows easily that R is a domain. Let $r \in R \setminus \{0\}$ and $m \in M$, then $rm = 0$ implies that $(r, 0)(0, m) = 0$ and hence $m = 0$, proving that M is torsion-free. Using Proposition 4.20, we prove 2. Property 3 is a consequence of Lemma 4.10.

Conversely, let $(r,m) \in R \ltimes_\phi M$ be a zerodivisor. Since M is isomorphic to an ideal of R, it is torsion-free, consequently, by Lemma 4.10, $|(r,m)|$ is a zerodivisor of R. Since R is a domain, it follows that $|(r,m)| = 0$ and Property 3 implies that $(r,m) = 0$. □

The following Theorem was originally prove in [22] using a different proof.

Theorem 4.26. *Assume $\Phi(M) \neq 0$. Then $R \ltimes_\phi M$ is a field if and only if:*

1. *R is a field (hence $M \cong R$).*
2. *For every $r \in R$, $|(r,1)| \neq 0$.*

Proof. Since $R \ltimes_\phi M$ is a field, then R is a domain and, since $R \subseteq R \ltimes_\phi M$ is an integral extension, it follows that R is a field. Therefore, by Proposition 4.19, $M \cong R$. Property 3 of Proposition 4.25 implies Property 2.

Conversely, let $(r,r') \in R \ltimes_\phi R$ a non zero element. If $r' = 0$, then Property 1 implies that (r,r') is an unit. Assume $r' \neq 0$, and let $u \in R$ be its inverse. Then, using Property 2, we get:

$$|(r,r')(u,0)| = |(ru,1)| \neq 0.$$

Since R is a field, $|(ru,1)|$ is a unit of R and hence (r,r') is an unit of $R \ltimes_\phi R$. □

4.3 The semitrivial ideal extension

With the goal in mind to generalize the ideal duplication for rings, it seems that the most natural occurence of it is in the case of semitrival extensions. Thus, in this section, we define the semitrivial ideal extension, and using this tool we study the homogeneous ideals of $R \ltimes_\phi M$ and their operations.

Let $I \subseteq R$ be an ideal of R and let $N \subseteq M$ be a submodule of M. It is natural to ask on what conditions the additive subgroup $I \oplus N \subseteq R \oplus M$ is an ideal of $R \ltimes_\phi M$. We denote $N^{\perp,\phi} I = \{n \in N \mid \phi(n \otimes M) \in I\}$. If $N^{\perp,\phi} I = N$, then we say that N is ϕ-perpendicular to I. If $I = (0)$, then we shall write $N^{\perp,\phi}$ instead. If $M^{\perp,\phi} = 0$, we say that ϕ is non-degenerate. An easy check shows that:

Proposition 4.27. *Let $I \subseteq R$ be an ideal of R and let $N \subseteq M$ be a submodule of M. Then $I \oplus N$ is an ideal of $R \ltimes_\phi M$ if and only if the following properties are satisfied:*

1. *$IM \subseteq N$;*
2. *N is ϕ-pependicular to I.*

In the following, when I and N satisfy the properties of the previous proposition, we shall write $I \ltimes_\phi N$ instead of $I \oplus N$, and we will call it the semitrivial ideal extension of I by N and ϕ.

Let J be an ideal of $R \ltimes_\phi M$, then we say that J is a semitrivial ideal extension of some ideal I of R, if there exists a submodule of M, say N, such that $J = I \ltimes_\phi N$ the semitrivial ideal extension of I by N and ϕ.

Any homogeneous ideal H of $R \ltimes_\phi M$ can be written as $H = H \cap R \oplus H \cap M$; denote $I = H \cap R$ and $N = H \cap M$, we have that $H = I \ltimes_\phi N$ where $IM \subseteq N$ and N is ϕ-perpendicular to I.

Example 4.28. Let $R = k[[S]]$ where k is a field, the semigroup ring associated to S, M a monomial ideal of R and ϕ_m is the multiplication by the monomial x^m, where $m \in S$ is odd. By [**Theorem 3.4**, 3], $R \ltimes_{\phi_m} M = R(M)_{0,-x^m}$ is a numerical semigroup ring vith valuation $v(R) \bowtie^m v(M)$. Let I_1 and I_2 be ideals of R such that:

1. $I_1 M \subseteq I_2$,
2. I_2 is ϕ_m-perpendicular to I_1.

Denote $E_1 = v(I_1)$ and $E_2 = v(I_2)$, it is easy to prove that they satisfy the following:

1' $E_1 + v(M) \subseteq E_2$,

2' $E_2 + v(M) + m \subseteq E_1$.

Also we get $v(I_1 \ltimes_{\phi_m} I_2) = E_1 \bowtie^m E_2$, i.e. the ideal duplication of the valuation ideals.

Let N_1 and N_2 be submodules of M; we denote $N_1 \cdot_\phi N_2$ the ideal of R generated by the elements of the form $\phi(n_1 \otimes n_2)$ where $n_1 \in N_1$ and $n_2 \in N_2$. Let I be an ideal of R and N a R-submodule of M. We denote $(I :_\phi N) = \{m \in M \mid \phi(m \otimes N) \in I\}$. The following proposition is essentially a rewrite of Proposition 4.4 for the semitrivial extension case.

Proposition 4.29. *Let I_1, I_2 be ideals of R and N_1, N_2 submodule of M. Then:*

1. $I_1 \ltimes_\phi N_1 + I_2 \ltimes_\phi N_2 = (I_1 + I_2) \ltimes_\phi (N_1 + N_2)$.
2. $I_1 \ltimes_\phi N_1 \cap I_2 \ltimes_\phi N_2 = (I_1 \cap I_2) \ltimes_\phi (N_1 \cap N_2)$.
3. $(I_1 \ltimes_\phi N_1)(I_2 \ltimes_\phi N_2) = (I_1 I_2 + N_1 \cdot_\phi N_2) \ltimes_\phi (I_1 N_2 + I_2 N_1)$.
4. $(I_1 \ltimes_\phi N_1) : (I_2 \ltimes_\phi N_2) = \big((I_1 :_R I_2) \cap (N_1 :_R N_2)\big) \ltimes_\phi \big((I_1 :_\phi N_2) \cap (N_1 :_M I_2)\big)$.

Proof. Property 1 and 2 are trivial. To prove Property 3 note that $N_1 N_2$, as defined in Remark 4.3, is equal to $N_1 \cdot_\phi N_2$. To prove Property 4 note that $(I_1 :_M N_2)$, as defined in Remark 4.3, is equal to $(I_1 :_\phi N_2)$. □

Proposition 4.30. *Let $S \subseteq R \ltimes_\phi M$ a conjugable multiplicatively closed set of $R \ltimes_\phi M$, denote $T = S \cap R$, we have:*

$$S^{-1}(R \ltimes_\phi M) = T^{-1}R \ltimes_{T^{-1}\phi} T^{-1}M,$$

where $T^{-1}\phi : (T^{-1}R \otimes_R M) \otimes_R M \to T^{-1}R$ is the homomorphism defined as $T^{-1}\phi(\frac{m}{t} \otimes m') = \frac{\phi(m \otimes m')}{t}$, for every $m, m' \in M$ and $t \in T$. Moreover, $\mathrm{Im}(T^{-1}\phi) = T^{-1}(\mathrm{Im}\phi)$.

Proof. By Proposition 4.12, $S^{-1}(R \ltimes_\phi M) = T^{-1}(R \ltimes_\phi M)$ is a $\mathbb{Z}_2$-graded ring with grading $T^{-1}R \oplus T^{-1}M$. It is a standard exercise to prove that the following function

$$T^{-1}(R \ltimes_\phi M) \to T^{-1}R \ltimes_{T^{-1}\phi} T^{-1}M,$$

$$\frac{(r,m)}{(t,0)} \mapsto \left(\frac{r}{t}, \frac{m}{t}\right),$$

is an isomorphism. □

Corollary 4.31. *Assume M to be torsion-free, then*

$$Q(R \ltimes_\phi M) = Q(R) \ltimes_{Q(\phi)} (Q(R) \otimes_R M),$$

where $Q(\phi) : (Q(R) \otimes_R M) \otimes_R M \to Q(R)$ is the homomorphism defined in Proposition 4.30.

Proof. To prove the thesis is enough to show that, if S denotes the set of the nonzero divisors of $R \ltimes_\phi M$, then $T = S \cap R$ is the set of nonzero divisors of R. Since M is torsion-free, this is easily seen. □

Denote $Q(M) = Q(R) \otimes_R M$. The previous Corollary states that if M is torsion-free, then $Q(R \ltimes_\phi M) = Q(R) \ltimes_{Q(\phi)} Q(M)$. Let $I \subseteq Q(R)$ a R-submodule of $Q(R)$ and $N \subseteq Q(M)$ a R-submodule of $Q(M)$; we ask on what conditions $I \oplus N$ is a fractional ideal of $R \ltimes_\phi M$.

Proposition 4.32. *Assume that M is torsion-free. Let $I \subseteq Q(R)$ be an R-submodule of $Q(R)$ and let $N \subseteq Q(M)$ be an R-submodule of $Q(M)$. Then $I \oplus N$ is a fractional ideal of $R \ltimes_\phi M$ if and only if the following properties are satisfied:*

1. *$IM \subseteq N$,*
2. *For every $n \in N$, $Q(\phi)(n \otimes M) \in I$.*
3. *I is a fractional ideal of R,*
4. *There exists a nonzero divisor $r \in R$ such that $rN \subseteq M$.*

Proof. It is easy to check that $I \oplus N$ is a R-submodule of $Q(R) \ltimes_{Q(\phi)} Q(M)$ if and only if Properties 1 and 2 are satisfied. Suppose that $I \oplus N$ is a fractional ideal of $R \ltimes_\phi M$, then there exists a nonzero divisor $x \in R \ltimes_\phi M$ such that $x(I \oplus N) \subseteq R \ltimes_\phi M$. Consequently, it is true that $|x|(I \oplus N) = |x|I \oplus |x|N \subseteq R \ltimes_\phi M$. By Lemma 4.10, both Property 3 R and Property 4 follow.

Conversely, let $r_1, r_2 \in R$ be nonzero divisors such that $r_1 I \subseteq R$ and $r_2 N \subseteq M$. Then $r_1 r_2 \in R$ is a nonzero divisor such that $r_1 r_2 (I \oplus N) = r_2(r_1 I) \oplus r_1(r_2 N) \subseteq R \ltimes_\phi M$. □

In the following, when I and N satisfy the properties of the previous proposition, we shall write $I \ltimes_\phi N$ instead of $I \oplus N$. Note that this notation is not ambiguos because, if I is an ideal of R and N is a submodule of M, then I is an R-submodule of $Q(R)$ and N is R-submodule of M.

The following Proposition is a generalization of Proposition 4.29.

Proposition 4.33. *Assume that M is torsion-free. Let I_1, I_2 be fractional ideals of R and N_1, N_2 submodule of M such that there exist $x_1, x_2 \in R$ nonzero divisors element of R such that $x_1 N, x_2 N \subseteq M$. Then:*

1. *$I_1 \ltimes_\phi N_1 + I_2 \ltimes_\phi N_2 = (I_1 + I_2) \ltimes_\phi (N_1 + N_2)$.*
2. *$I_1 \ltimes_\phi N_1 \cap I_2 \ltimes_\phi N_2 = (I_1 \cap I_2) \ltimes_\phi (N_1 \cap N_2)$.*
3. *$(I_1 \ltimes_\phi N_1)(I_2 \ltimes_\phi N_2) = (I_1 I_2 + N_1 \cdot_{Q(\phi)} N_2) \ltimes_\phi (I_1 N_2 + I_2 N_1)$.*
4. *$(I_1 \ltimes_\phi N_1) :_{Q(R) \ltimes_{Q(\phi)} Q(M)} (I_2 \ltimes_\phi N_2) =$*

$$= ((I_1 :_{Q(R)} I_2) \cap (N_1 :_{Q(R)} N_2)) \ltimes_{Q(\phi)} ((I_1 :_{Q(\phi)} N_2) \cap (N_1 :_{Q(M)} I_2)).$$

Proof. Then proof is a rewrite of Proposition 4.4. Similarly as we did for the proof of Proposition 4.29, note that $N_1 N_2$ (as defined in Remark 4.3) is equal to $N_1 \cdot_{Q(\phi)} N_2$ and $(I_1 :_{Q(M)} N_2)$ (as defined in Remark 4.3) is equal to $(I_1 :_{Q(\phi)} N_2)$. □

The following proposition is essentially a rewrite of Proposition 4.11

Proposition 4.34. *Let $H = I \ltimes_\phi N$ be a semitrivial ideal extension of I, then*

$$\frac{R \ltimes_\phi M}{I \ltimes_\phi N} = R/I \ltimes_{\phi^H} M/N,$$

where $\phi^H : M/N \otimes_{R/I} M/N \to R/I$ is defined as $\phi^H((m+N) \otimes (m'+N)) = \phi(m \otimes m') + I$.

Proof. By Proposition 4.11, $(R \ltimes_\phi M)/(I \ltimes_\phi N)$ is a $\mathbb{Z}_2$-graded ring with grading $R/I \oplus M/N$. It is a standard exercise to prove that the following function

$$\frac{R \ltimes_\phi M}{I \ltimes_\phi N} \to R/I \ltimes_{\phi^H} M/N,$$

$$(r, m) + I \ltimes_\phi N \mapsto (r + I, m + N),$$

is an isomorphism. □

4.4 The primes of the semitrivial extension

In this section, we use the semitrivial ideal extension to describe the homogeneous primes of the semitrivial extension, and we find under which conditions a prime ideal of the semitrivial extension is homogeneous. Also, we give a complete description of the maximal ideals.

Let R be a commutative ring with unity, M an R-module and $\phi \in \Phi(M) \setminus \{0\}$. Let I be an ideal of R, then we denote with $\mathcal{E}_\phi(I) \subseteq R \ltimes_\phi M$ the family of all the semitrivial ideal extensions of I.

Proposition 4.35. *The family $\mathcal{E}_\phi(I)$ has a minimum and maximum with respect to the inclusion; they are:*

$$\min(\mathcal{E}_\phi(I)) = I \ltimes_\phi IM \text{ and } \max(\mathcal{E}_\phi(I)) = I \ltimes_\phi M^{\perp,\phi} I.$$

Proof. An easy computation shows that $I(R \ltimes_\phi M) = I \ltimes_\phi IM$, thus $I \ltimes_\phi IM$ is necessarily the minimum of $\mathcal{E}_\phi(I)$. Let $I \ltimes_\phi N \in \mathcal{E}_\phi(I)$, then $N = N^{\perp,\phi} I \subseteq M^{\perp,\phi} I$, proving the thesis. □

In the following, for the sake of simplicity, we shall denote $\mathcal{M}(I) = \max(\mathcal{E}_\phi(I))$.

Let H be an ideal of $R \ltimes_\phi M$. We recall that the homogeneous body of H, denoted by $\mathcal{B}(H)$, is the largest homogeneous ideal of $R \ltimes_\phi M$ contained in H and it is equal to $H \cap R \oplus H \cap M$.

Proposition 4.36. *Let $\mathfrak{p}$ be a prime ideal of R and $\mathfrak{q}$ a prime ideal of $R \ltimes_\phi M$. Then $\mathfrak{q} \cap R = \mathfrak{p}$ if and only if $\mathcal{B}(\mathfrak{q}) = \mathcal{M}(\mathfrak{p}) = \mathfrak{p} \ltimes_\phi M^{\perp,\phi}\mathfrak{p}$.*

Proof. Since $\mathcal{B}(\mathfrak{q}) = \mathfrak{q} \cap R \oplus \mathfrak{q} \cap M = \mathfrak{p} \oplus \mathfrak{q} \cap M$, then $\mathcal{B}(\mathfrak{q}) \in \mathcal{E}_\phi(\mathfrak{p})$, that is $\mathcal{B}(\mathfrak{q}) \subseteq \mathcal{M}(\mathfrak{p})$. Let $m \in M^{\perp,\phi}\mathfrak{p}$, we get:

$$(0, m)^2 = (\phi(m \otimes m), 0) \in \mathcal{M}(\mathfrak{p}) \cap R = \mathfrak{p} = \mathfrak{q} \cap R \subseteq \mathfrak{q} \Rightarrow (0, m) \in \mathfrak{q}.$$

The inclusion $\mathcal{M}(\mathfrak{p}) \subseteq \mathcal{B}(\mathfrak{q})$ follows immediately.

Conversely, since $\mathcal{B}(\mathfrak{q}) \cap R = \mathcal{M}(\mathfrak{p}) \cap R$, follows that $\mathfrak{q} \cap R = \mathcal{B}(\mathfrak{q}) \cap R = \mathcal{M}(\mathfrak{p}) \cap R = \mathfrak{p}$. □

Theorem 4.37. *Let $\mathfrak{p}$ be a prime ideal of R and let $\mathfrak{q}$ be a prime ideal $R \ltimes_\phi M$. If $\mathfrak{q}$ is a semitrivial ideal extension of $\mathfrak{p}$, then we have*

$$\mathfrak{q} = \mathfrak{p} \ltimes_\phi M^{\perp,\phi}\mathfrak{p}.$$

Proof. Since $\mathfrak{q}$ is a semitrivial ideal extension, $\mathfrak{q}$ is a homogeneous ideal and hence $\mathfrak{q} = \mathcal{B}(\mathfrak{q})$. □

Proposition 4.38. *Assume R to be local with maximal ideal $\mathfrak{m}$ and residue class field k. If* $\mathrm{Im}\phi \subseteq \mathfrak{m}$, *then $R \ltimes_\phi M$ is local with homogeneous maximal ideal $\mathfrak{m} \ltimes_\phi M$ and residue class field k.*

Proof. It easy to prove that the condition $\mathrm{Im}\phi \subseteq I$ is equivalent to $M^{\perp,\phi}I = M$, for every ideal I of R. By Proposition 4.34, $(R \ltimes_\phi M)/\mathcal{M}(\mathfrak{m}) = k$, hence $\mathcal{M}(\mathfrak{m})$ is maximal. Let $\mathfrak{n}$ be a maximal ideal of $R \ltimes_\phi M$; since R is local and $R \subseteq R \ltimes_\phi M$ is integral, $\mathfrak{n}$ lies over $\mathfrak{m}$, therefore, by Proposition 4.36, it follows that $\mathcal{M}(\mathfrak{m}) \subseteq \mathfrak{n}$, that is the thesis. □

Proposition 4.39. *Let $\mathfrak{p}$ be a prime ideal of R. Then there are at most two prime ideals (one the conjugate of the other) of $R \ltimes_\phi M$ whose contraction on R is $\mathfrak{p}$.*

Proof. First, we prove the thesis for the maximal ideal $\mathfrak{m}$ in the R local case. Since $R \subseteq R \ltimes_\phi M$ is integral, there exists a maximal ideal $\mathfrak{n}$ of $R \ltimes_\phi M$ such that $\mathfrak{n} \cap R = \mathfrak{m}$. Denote $S = R \ltimes_\phi M \setminus (\mathfrak{n} \cup \overline{\mathfrak{n}})$, by Proposition 4.30, we get:

$$S^{-1}(R \ltimes_\phi M) = R_\mathfrak{m} \ltimes_{\phi_\mathfrak{m}} M_\mathfrak{m} = R \ltimes_\phi M,$$

where the last equality is true because R is local. From this follows that S is contained in the set of units of $R \ltimes_\phi M$, hence $\mathfrak{n}$ and $\overline{\mathfrak{n}}$ are the only possible maximal ideals of $R \ltimes_\phi M$.

Let $\mathcal{F}$ be the family of all the prime ideals of $R \ltimes_\phi M$ lying over $\mathfrak{p}$. Denote $S = R \ltimes_\phi M \setminus \cup\mathcal{F}$, $\cup\mathcal{F}$ is conjugable because, if $\mathfrak{q}$ is a prime lying over $\mathfrak{p}$, then this is also true for $\overline{\mathfrak{q}}$. Denote $T = S \cap R = R \setminus \mathfrak{p}$, once again by Proposition 4.30, $S^{-1}(R \ltimes_\phi M) = R_\mathfrak{p} \ltimes_{\phi_\mathfrak{p}} M_\mathfrak{p}$. Since $R_\mathfrak{p}$ is local, by the first part of the proof, we get that $S^{-1}(R \ltimes_\phi M)$ is a ring with at most two maximal ideals, hence the cardinality of $\mathcal{F}$ is at most two. □

Remark 4.40. Let $\mathfrak{p}$ be a prime ideal of R. Let $\mathfrak{q}$ and $\overline{\mathfrak{q}}$ be the prime ideals of $R \ltimes_\phi M$ which lie over $\mathfrak{p}$, then we denote $S_\mathfrak{p} = R \ltimes_\phi M \setminus (\mathfrak{q} \cup \overline{\mathfrak{q}})$ and $\phi_\mathfrak{p} = S_\mathfrak{p}^{-1}\phi$ defined as in Proposition 4.30. The set $S_\mathfrak{p}$ is multiplicatively closed and conjugable; also

$$S_\mathfrak{p}^{-1}(R \ltimes_\phi M) = R_\mathfrak{p} \ltimes_{\phi_\mathfrak{p}} M_\mathfrak{p}$$

is a semilocal ring with at most two maximal ideals: $S_\mathfrak{p}^{-1}\mathfrak{q}$ and $S_\mathfrak{p}^{-1}\overline{\mathfrak{q}}$. Moreover, the following equalities hold:

1. $(S_\mathfrak{p}^{-1}(R \ltimes_\phi M))_{S_\mathfrak{p}^{-1}\mathfrak{q}} = (R \ltimes_\phi M)_\mathfrak{p}$,
2. $(S_\mathfrak{p}^{-1}(R \ltimes_\phi M))_{S_\mathfrak{p}^{-1}\overline{\mathfrak{q}}} = (R \ltimes_\phi M)_{\overline{\mathfrak{q}}}$.

If $\mathrm{Im}\phi \not\subseteq \mathfrak{p}$, then $\phi_{\mathfrak{p}}$ is surjective and, by Proposition 4.23, $S_{\mathfrak{p}}^{-1}(R \ltimes_{\phi} M) = R_{\mathfrak{p}} \ltimes_{\phi_{\mathfrak{p}}} R_{\mathfrak{p}}$. If instead $\mathrm{Im}\phi \subseteq \mathfrak{p}$, by Proposition 4.38, it follows that $S_{\mathfrak{p}}^{-1}(R \ltimes_{\phi} M)$ is local with homogeneous maximal ideal $\mathfrak{p}R_{\mathfrak{p}} \ltimes_{\phi_{\mathfrak{p}}} M_{\mathfrak{p}}$.

Clearly, a prime ideal of $R \ltimes_{\phi} M$ may not be homogeneous: e.g. $\mathbb{R}[t]/(t^2-2) = \mathbb{R} \ltimes_{\varphi_2} \mathbb{R}$ (where φ_2 is the product by 2) has two maximal ideals $(t-\sqrt{2})$ and $(t+\sqrt{2})$, which are not homogeneous.

Remark 4.41. Let $\mathfrak{p}$ a prime ideal of R, we denote $p) = R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}} = Q(R/\mathfrak{p})$. Assume $\mathrm{Im}\phi \subseteq \mathfrak{p}$, this is equivalent to say that $M^{\perp,\phi}\mathfrak{p} = M$. By Proposition 4.34, we get:

$$(R \ltimes_{\phi} M)/\mathcal{M}(\mathfrak{p}) = R/\mathfrak{p},$$

in particular $\mathcal{M}(\mathfrak{p})$ is prime and $Q((R \ltimes_{\phi} M)/\mathcal{M}(\mathfrak{p})) = k(\mathfrak{p})$.

If $\mathrm{Im}\phi \not\subseteq \mathfrak{p}$, then $M/M^{\perp,\phi}\mathfrak{p}$ is a torsion-free module over $R/\mathfrak{p}$. In fact, let $0 \neq r+\mathfrak{p} \in R/\mathfrak{p}$ and $m + M^{\perp,\phi}\mathfrak{p} \in M/M^{\perp,\phi}\mathfrak{p}$. If $(r+\mathfrak{p})(m+M^{\perp,\phi}\mathfrak{p}) = 0$, then $r\phi(m \otimes M) \in \mathfrak{p}$, hence $m \in M^{\perp,\phi}\mathfrak{p}$. By Proposition 4.20, $M/M^{\perp,\phi}\mathfrak{p}$ is isomorphic to an ideal of $R/\mathfrak{p}$. We denote $\phi^{\mathfrak{p}} = \phi^{\mathcal{M}(\mathfrak{p})}$ the homomorphism defined in Corollary 4.31 and $\varphi_{\mathfrak{p}} = Q(\phi^{\mathfrak{p}})$ the homomorphism defined in Proposition 4.11. By Proposition 4.11, we get:

$$Q((R \ltimes_{\phi} M)/\mathcal{M}(\mathfrak{p})) = k(\mathfrak{p}) \ltimes_{\varphi_{\mathfrak{p}}} k(\mathfrak{p})\,,$$

Note that $\varphi_{\mathfrak{p}} \in \mathrm{Hom}_{k(\mathfrak{p})}(k(\mathfrak{p}), k(\mathfrak{p})) = k(\mathfrak{p})$. In the following, $\varphi_{\mathfrak{p}}$ indicates, with slight abuse, the product by $\varphi_{\mathfrak{p}}$ and an element of $k(\mathfrak{p})$; thus $k(\mathfrak{p}) \ltimes_{\varphi_{\mathfrak{p}}} k(\mathfrak{p}) \cong k(\mathfrak{p})[t]/(t^2 - \varphi_{\mathfrak{p}})$.

The previous remark allows us to state the following proposition:

Proposition 4.42. *Let $\mathfrak{p}$ be a prime ideal of R, then we have:*

1. *If $\mathrm{Im}\phi \subseteq \mathfrak{p}$, then*
$$Q\left(\frac{R \ltimes_{\phi} M}{\mathcal{M}(\mathfrak{p})}\right) = k(\mathfrak{p}).$$

2. *If $\mathrm{Im}\phi \not\subseteq \mathfrak{p}$, then*
$$Q\left(\frac{R \ltimes_{\phi} M}{\mathcal{M}(\mathfrak{p})}\right) = k(\mathfrak{p}) \ltimes_{\varphi_{\mathfrak{p}}} (\mathfrak{p}) \cong k(\mathfrak{p})[t]/(t^2 - \varphi_{\mathfrak{p}}).$$

Theorem 4.43. *Let $\mathfrak{p}$ be a prime ideal of R and let $\mathfrak{q}$ be a prime ideal of $R \ltimes_{\phi} M$ which lies over $\mathfrak{p}$. Then $\mathfrak{q}$ is a semitrivial ideal extension of $\mathfrak{p}$ if and only if the equation $t^2 - \varphi_{\mathfrak{p}} = 0$ has no nonzero solutions in $k(\mathfrak{p})$.*

Proof. We recall that, by Proposition 4.36 and by the hypothesis, it follows that $\mathcal{M}(\mathfrak{p}) \subseteq \mathfrak{q}$. We have two possibilities:

1. If $\mathrm{Im}\phi \subseteq \mathfrak{p}$, by Remark 4.41, $\mathcal{M}(\mathfrak{p})$ is prime, then by incomparability, $\mathfrak{q} = \mathcal{M}(\mathfrak{p})$. Therefore, $\mathfrak{q}$ is a semitrivial ideal extension of $\mathfrak{p}$ and at the same time, by Remark 4.41, $\varphi_{\mathfrak{p}} = 0$.

2. If $\mathrm{Im}\phi \not\subseteq \mathfrak{p}$, then by Proposition 4.42, we get:
$$\mathfrak{q} = \mathcal{M}(\mathfrak{p}) \Leftrightarrow Q\left(\frac{R \ltimes_{\phi} M}{\mathcal{M}(\mathfrak{p})}\right) \cong k(\mathfrak{p})[t]/(t^2 - \varphi_{\mathfrak{p}}) \text{ is a field}\,.$$

□

We are able to give a complete description of the maximal ideals of $R \ltimes_\phi M$.

Theorem 4.44. *Let $\mathfrak{n}$ be a maximal ideal of $R \ltimes_\phi M$ and assume that it is not a semitrivial ideal extension of $m = n \cap R$. Then*

$$\mathfrak{n} = \mathcal{M}(\mathfrak{m}) + ((\alpha, m)),$$

where $\alpha + \mathcal{M}(\mathfrak{m}) \in k(\mathfrak{m})$ is a solution of $t^2 - \varphi_\mathfrak{m} = 0$ and $m \in M$ is such that $m \equiv 1 \pmod{M^{\perp,\phi}\mathfrak{m}}$.

Proof. Let $\mathfrak{n}$ be a maximal ideal of $R \ltimes_\phi M$ and assume that it is not a semitrivial ideal extension of $\mathfrak{n} \cap R = \mathfrak{m}$. By Remark 4.41, $(R \ltimes_\phi M)/\mathcal{M}(\mathfrak{m}) \cong k(\mathfrak{m})[t]/(t^2 - \varphi_\mathfrak{m})$. By Theorem 4.43, there exists $\alpha + \mathcal{M}(\mathfrak{m}) \in k(\mathfrak{m})$ such that $(\alpha + \mathcal{M}(\mathfrak{m}))^2 = \varphi_\mathfrak{m}$; futhermore $(\alpha - t)$ and and $(\alpha + t)$ are the maximal (not necessarily distinct) ideals of $k(\mathfrak{m})[t]/(t^2 - \varphi_\mathfrak{m})$. Let $m \in M$ such that $m \equiv 1 \pmod{M^{\perp,\phi}\mathfrak{m}}$, since the ideals $\mathcal{M}(\mathfrak{m}) + ((\alpha, m))$ and $\mathcal{M}(\mathfrak{m}) + ((\alpha, -m))$ project to the maximal ideals of $k(\mathfrak{m})[t]/(t^2 - \varphi_\mathfrak{m})$ respectively, they are exactly $\mathfrak{n}$ and $\bar{\mathfrak{n}}$. □

In the last part of this section, we want to give a precise description of the semitrivial ideal extensions of $R \ltimes_\phi R$, in the case that R is a PID (which is, up to isomorphism, the only semitrivial extension by a torsion-free module).

Let $\mathcal{U}$ be the set of units of R. In R we define the following relation:

$$r \sim r' \Leftrightarrow r' = ur \text{ for some } u \in \mathcal{U}.$$

It is easy to prove that $\sim$ is an equivalence relation and, for every $r \in R$, the equivalence class of r is $[r] = \{ur \mid u \in \mathcal{U}\}$; clearly $[0] = \{0\}$ and $[1] = \mathcal{U}$.

Let $r \in R \setminus \{0\}$; we define

$$\mathrm{Div}(r) = \big\{[x] \mid x \in R : x \text{ divides } r\big\}$$

and we denote by $d(r) = |\mathrm{Div}(r)|$. Note that $d(r) \geq 1$; also note that $d(r) = 1$ if and only if r is a unit and $d(r) = 2$ if and only if r is irreducible.

We recall that, if R is an UFD, then

$$(r_1 :_R r_2) = \left(\frac{r_1}{gcd(r_1, r_2)}\right)$$

for every $r_1, r_2 \in R \setminus \{0\}$.

Theorem 4.45. *Assume that R is a PID and that ϕ is the multiplication by some element $b \in R \setminus \{0\}$. Consider the semitrivial extension $R \ltimes_\phi R$ and take $I = (r) \neq (0)$ an ideal of R. Denoted by $g_r = gcd(r, b)$ and $a_r = \frac{r}{g_r}$; if $\mathrm{Div}(g_r) = \{[\lambda_1] = [g_r], [\lambda_2], \ldots, [\lambda_{n-1}], [\lambda_n] = [1]\}$, then*

$$\mathcal{E}_\phi(I) = \{I \ltimes_\phi I, I \ltimes_\phi (a_r\lambda_2), \ldots, I \ltimes_\phi (a_r\lambda_{n-1}), I \ltimes_\phi (a_r)\}.$$

In particular, $|\mathcal{E}_\phi(I)| = d(g_r)$. Moreover, let $\mathfrak{p} = (p)$ be a prime ideal of R, we have:

1. *If p divides b, then $\mathcal{E}_\phi(\mathfrak{p}) = \{\mathfrak{p} \ltimes_\phi \mathfrak{p}, \mathfrak{p} \ltimes_\phi R\}$, and $\mathfrak{p} \ltimes_\phi R$ is the only prime ideal of $R \ltimes_\phi R$ lying over $\mathfrak{p}$.*

2. *If p does not divide b, then $\mathcal{E}_\phi(\mathfrak{p}) = \{\mathfrak{p} \ltimes_\phi \mathfrak{p}\}$, and $\mathfrak{p} \ltimes_\phi \mathfrak{p}$ is the only prime ideal of $R \ltimes_\phi R$ lying over $\mathfrak{p}$ if and only if the equation $t^2 - (b + \mathfrak{p}) = 0$ has no nonzero solutions in $k(\mathfrak{p})$. Otherwise, if $\alpha + \mathfrak{p} \in k(\mathfrak{p})$ is such solution, then $\mathfrak{p} \ltimes_\phi \mathfrak{p} + ((\alpha, 1))$ and its conjugate are the only prime ideals lying over $\mathfrak{p}$.*

Proof. Let $J = (x)$ an ideal of R such that $I \ltimes_\phi J \in \mathcal{E}_\phi(I)$, by Proposition 4.27, it follows that $I \subseteq J$ and $J^{\perp,\phi} I = (I :_R b) \cap J = (r :_R b) \cap J = (a_r) \cap J = J$; hence x divides r and a_r divides x. We claim that this is equivalent to say that $x \sim a_r \lambda_i$, for some $i \in \{1, \dots, n\}$. In fact, since a_r divides x, it follows that $x = a_r \lambda$ for some $\lambda \in R$ and, since x divides r, it follows that $r = \mu x$ for some $\mu \in R$; hence $g_r = (r/x)\lambda = \mu\lambda$, that is $[\lambda] \in \mathrm{Div}(g_r)$. Conversely, if $x = u a_r \lambda_i$ with $u \in \mathcal{U}$ and $i \in \{1, \dots, n\}$, then a_r divides x and, since $r = u^{-1}(g_r/\lambda_i)x$, it follows that x divides r.

We prove now the last part of the theorem:

1. If p divides b, then $g_p = p$ and $a_p = 1$; hence $\mathcal{E}_\phi(\mathfrak{p}) = \{\mathfrak{p} \ltimes_\phi \mathfrak{p}, \mathfrak{p} \ltimes_\phi R\}$. The thesis is a consequence of Theorem 4.43.

2. If p does not divide b, then $g_p = 1$ and $a_p = p$; hence $\mathcal{E}_\phi(\mathfrak{p}) = \{\mathfrak{p} \ltimes_\phi \mathfrak{p}\}$. The thesis is a consequence of Theorem 4.43 and Theorem 4.44.

□

4.5 The generically Gorenstein and nearly Gorenstein properties

Let $(R, \mathfrak{m}, k)$ be a noetherian local ring, M a finitely generated module over R and $\phi \in \Phi(M) \setminus \{0\}$ such that $\mathrm{Im}\phi \subseteq \mathfrak{m}$ (if $\mathrm{Im}\phi \nsubseteq \mathfrak{m}$, then $M \cong R$). By Proposition 4.24 and 4.38, $R \ltimes_\phi M$ is a noetherian local ring with maximal $\mathfrak{m} \ltimes_\phi M$. In this section, we want to find the conditions such that $R \ltimes_\phi M$ is generically Gorenstein and nearly Gorenstein. However, in this generality, the question is not well posed, since canonical modules are defined for Cohen-Macaulay rings, thus we first need to find the conditions such that $R \ltimes_\phi M$ is Cohen-Macaulay. A proof of the following proposition can be found in [**Corollary 5.3**, 22].

Proposition 4.46. *$R \ltimes_\phi M$ is Cohen-Macaulay if and only if R is Cohen-Macaulay and M is maximal Cohen-Macaulay.*

If R has canonical module ω_R and M is maximal Cohen-Macaulay, then, by Proposition 1.34, $R \ltimes_\phi M$ has a canonical module C which is isomorphic as $R \ltimes_\phi M$-modules to:

$$C \cong \mathrm{Hom}_R(R \ltimes_\phi M, \omega_R).$$

Therefore, since $R \ltimes_\phi M = R \oplus M$, by Proposition 4.7 and Remark 4.8, we get:

Proposition 4.47. *Assume that R has canonical module ω_R and M is maximal Cohen-Macaulay. The canonical module C of $R \ltimes_\phi M$ is a $\mathbb{Z}_2$-graded module with grading:*

$$\omega_R \oplus \mathrm{Hom}_R(M, \omega_R).$$

Let $(r', m') \in R \ltimes_\phi M$ and let $f = (f_0, f_1) \in \mathrm{Hom}_R(R \ltimes_\phi M, \omega_R)$, where $f_0 \in \mathrm{Hom}_R(R, \omega_R)$ and $f_1 \in \mathrm{Hom}_R(M, \omega_R)$. Note that, by Remark 4.8,

$$(r', m') \cdot f = (r', m') \cdot (f_0, f_1) = (r' \cdot f_0 + m' \cdot f_1, r' \cdot f_1 + m' \cdot f_0),$$

where $(m' \cdot f_1)(r) = f_1(rm')$ for every $r \in R$ and $(m' \cdot f_0)(m) = f_0(\phi(m \otimes m'))$ for every $m \in M$.

Proposition 4.48. *Let R be Cohen-Macaulay, assume that M is maximal Cohen-Macaulay and $\mathrm{Im}\phi$ can be generated by a regular sequence. If $R \ltimes_\phi M$ has a canonical module C, then $R/\mathrm{Im}\phi$ has a canonical module $\omega_{R/\mathrm{Im}\phi}$. In fact, if $c = \mathrm{depth}(\mathrm{Im}\phi)$, then*

$$\omega_{R/\mathrm{Im}\phi} \cong \mathrm{Ext}^c_{R \ltimes_\phi M}(R/\mathrm{Im}\phi, C).$$

Proof. Note that $R/\mathrm{Im}\phi$ can be endowed with a $R \ltimes_\phi M$-algebra structure through the following action:

$$(r_1, m) \bullet (r_2 + \mathrm{Im}\phi) := r_1 r_2 + \mathrm{Im}\phi,$$

for every $(r_1, m) \in R \ltimes_\phi M$ and $r_2 + \mathrm{Im}\phi \in R/\mathrm{Im}\phi$. Since $R/\mathrm{Im}\phi$ is a local Cohen-Macaulay R-algebra whose Krull dimension is equal to $\dim R - c$, by Theorem 1.34, it follows the thesis. □

A complete characterization for the artinian semitrivial extensions which are Gorenstein is given in [22, Theorem 4.3].

Theorem 4.49. *Assume that R is artinian. $R \ltimes_\phi M$ is Gorenstein if and only if either:*

1. *$M \cong \mathcal{I}_R(k)$ or*
2. *R is Gorenstein and ϕ is non-degenerate.*

Let $M^* = \mathrm{Hom}_R(M, R)$. We define the adjoint of ϕ, denoted ϕ^a, as following:

$$\phi^a : M \to M^*$$

$$m \mapsto \phi^a(m),$$

where $\phi^a(m)(m') = \phi(m \otimes m')$ for every $m' \in M$.

The proof of the following theorem can be found in [22, Corollary 5.11]:

Theorem 4.50. *$R \ltimes_\phi M$ is Gorenstein if and only if either:*

1. *$R \ltimes M$ is Gorenstein or*
2. *R is Gorenstein, M is maximal Cohen-Macaulay and ϕ^a is an isomorphism.*

We now characterize generically Gorenstein semitrivial extensions.

Theorem 4.51. *Assume that R is Cohen-Macaulay and M is maximal Cohen-Macaulay. $R \ltimes_\phi M$ is generically Gorenstein if and only if, for every $\mathfrak{p} \in \mathrm{Ass}_R(0)$, either one of the following is satisfied:*

1. *If $\mathfrak{p} \not\supseteq \mathrm{Im}\phi$, then $R_\mathfrak{p}$ is Gorenstein.*
2. *If $\mathfrak{p} \supseteq \mathrm{Im}\phi$, then $M_\mathfrak{p} = \mathcal{I}(k(\mathfrak{p}))$ or $R_\mathfrak{p}$ is Gorenstein and $\phi_\mathfrak{p}$ is non degenerate.*

Proof. Let $\mathfrak{q}$ a prime ideal of $R \ltimes_\phi M$ such that $\mathfrak{q} \cap R = \mathfrak{p}$, by incomparability, $\mathfrak{q}$ is also a minimal prime ideal of $R \ltimes_\phi M$. If $\mathfrak{p} \not\supseteq \mathrm{Im}\phi$, then $\mathrm{Im}\phi_\mathfrak{p} = \mathrm{Im}\phi R_\mathfrak{p} = R_\mathfrak{p}$ and, by Lemma 4.23, $M_\mathfrak{p} \cong R_\mathfrak{p}$; hence, $S_\mathfrak{p}^{-1}(R \ltimes_\phi M) = R_\mathfrak{p} \ltimes_{\phi_\mathfrak{p}} R_\mathfrak{p}$. By Remark 4.40, $(S_\mathfrak{p}^{-1}(R \ltimes_\phi M))_{S_\mathfrak{p}^{-1}\mathfrak{q}} = (R \ltimes_\phi M)_\mathfrak{q}$ and $(S_\mathfrak{p}^{-1}(R \ltimes_\phi M))_{S_\mathfrak{p}^{-1}\bar{\mathfrak{q}}} = (R \ltimes_\phi M)_{\bar{\mathfrak{q}}}$, therefore

$R_\mathfrak{p} \ltimes_{\phi_\mathfrak{p}} R_\mathfrak{p} = R_\mathfrak{p}[t]/(t^2 + r)$ is Gorenstein for some $r \in R$, that is, by [**Remark 4.5**, 22], $R_\mathfrak{p}$ is Gorenstein. If $\mathfrak{p} \supseteq \mathrm{Im}\phi$, then, by Remark 4.40, $S_\mathfrak{p}^{-1}(R \ltimes_\phi M) = R_\mathfrak{p} \ltimes_\phi M_\mathfrak{p}$ is a local artinian ring. Using Theorem 4.49, we get the thesis.

Conversely, let $\mathfrak{q}$ be a minimal prime ideal of $R \ltimes_\phi M$ and let $\mathfrak{p} = \mathfrak{q} \cap R$, by incomparability $\mathfrak{p}$ is a minimal prime of R. If $\mathfrak{p} \not\supseteq \mathrm{Im}\phi$, then $\mathrm{Im}\phi_\mathfrak{p} = \mathrm{Im}\phi R_\mathfrak{p} = R_\mathfrak{p}$ and, using Lemma 4.23, we get that $M_\mathfrak{p} \cong R_\mathfrak{p}$. Hence, $S_\mathfrak{p}^{-1}(R \ltimes_\phi M) = R_\mathfrak{p} \ltimes_{\phi_\mathfrak{p}} R_\mathfrak{p}$. By Remark 4.40, $(S_\mathfrak{p}^{-1}(R \ltimes_\phi M))_{S_\mathfrak{p}^{-1}\mathfrak{q}} = (R \ltimes_\phi M)_\mathfrak{q}$ and $(S_\mathfrak{p}^{-1}(R \ltimes_\phi M))_{S_\mathfrak{p}^{-1}\bar{\mathfrak{q}}} = (R \ltimes_\phi M)_{\bar{\mathfrak{q}}}$, therefore, since $R_\mathfrak{p}$ is Gorenstein, it follows that $R_\mathfrak{p} \ltimes_\phi R_\mathfrak{p} = R_\mathfrak{p}[t]/(t^2 + r)$ is Gorenstein for some $r \in R$, hence both $(R \ltimes_\phi M)_\mathfrak{q}$ and $(R \ltimes_\phi M)_{\bar{\mathfrak{q}}}$ are Gorenstein. If $\mathfrak{p} \supseteq \mathrm{Im}\phi$, then, by Remark 4.40, $S_\mathfrak{p}^{-1}(R \ltimes_\phi M) = R_\mathfrak{p} \ltimes_\phi M_\mathfrak{p}$ is a local artinian ring. Using Theorem 4.49, we get the thesis. □

Corollary 4.52. *In the same hypothesis of Theorem 4.51. If* $\mathrm{depth}(\mathrm{Im}\phi) > 0$, *then R is Generically Gorenstein if and only if $R \ltimes_\phi M$ is generically Gorenstein.*

Proof. If $\mathrm{depth}(\mathrm{Im}\phi) > 0$, then there are no associated primes of 0 that contain $\mathrm{Im}\phi$. □

Proposition 4.53. *Assume that R is Cohen-Macaulay of positive Krull dimension with canonical module ω_R and M is maximal Cohen-Macaulay. If $R \ltimes_\phi M$ is generically Gorenstein, then R is generically Gorenstein.*

Proof. Let C be the canonical module of $R \ltimes_\phi M$, by Proposition 1.34, $C \cong \omega_R \oplus \mathrm{Hom}_R(M, \omega_R)$. Since $R \ltimes_\phi M$ is generically Gorenstein, C must be isomorphic to an ideal of $R \ltimes_\phi M$. We recall that $\mathrm{Hom}_{R \ltimes_\phi M}(C, C') = R \ltimes_\phi M$ for any canonical module C' of $R \ltimes_\phi M$, therefore there must be a regular element $x \in R \ltimes_\phi M$ such that the multiplication by x is an injective homomorphism and xC is an ideal of $R \ltimes_\phi M$. From this fact follows that $|x|C$ is also a canonical ideal of $R \ltimes_\phi M$ and,

$$|x|C = |x|\omega_R \ltimes_\phi |x|\mathrm{Hom}_R(M, \omega_R).$$

By Lemma 4.10, the multiplication by $|x|$ is an injective R-linear and $R \ltimes_\phi M$-linear map, therefore we get that ω_R is isomorphic to an ideal of R. □

Proposition 4.54. *Assume that R is Cohen-Macaulay of positive Krull dimension with canonical ideal ω_R and assume that M is isomorphic to a maximal Cohen-Macaulay ideal of R, say I. Then $R \ltimes_\phi I$ is generically Gorenstein and the canonical module is isomorphic to the fractional ideal:*

$$C = \omega_R \ltimes_\phi (\omega_R :_{Q(R)} I).$$

Proof. Since I is maximal Cohen-Macaulay and R has positive Krull-dimension, I has positive depth. By Remark 4.21, it follows $\mathrm{Hom}_R(I, \omega_R) \cong (\omega_R :_{Q(R)} I)$. □

We now characterize nearly Gorenstein semitrivial extensions.

Theorem 4.55. *Assume that R is Cohen-Macaulay and M is maximal Cohen-Macaulay. $R \ltimes_\phi M$ is nearly Gorenstein if and only if for every prime ideal $\mathfrak{p}$ not maximal, one of the following conditions is satisifed:*

1. *If $\mathfrak{p} \not\supseteq \mathrm{Im}\phi$, then $R_\mathfrak{p}[t]/(t^2 - r_{\phi_\mathfrak{p}})$ is Gorenstein, where $r_{\phi_\mathfrak{p}}$ is the identification of $\phi_\mathfrak{p}$ as an element of $R_\mathfrak{p}$.*
2. *If $\mathfrak{p} \supseteq \mathrm{Im}\phi$, then either $R_\mathfrak{p} \ltimes M_\mathfrak{p}$ is Gorenstein or $R_\mathfrak{p}$ is Gorenstein and $\phi_\mathfrak{p}^a$ is an isomorphism.*

Proof. The thesis can be proven with a similar argument done for the proof of Theorem 4.51 and using Theorem 4.50. □

Let N be a $\mathbb{Z}_2$-graded module over $A = R \ltimes_\phi M$, in the same notations of Proposition 4.6, we have

$$\mathrm{Tr}_A(N) = \sum_{h \in \mathrm{Hom}_A(N,A)} h(N) = \sum_{h_0 \in H_0} h_0(N) \oplus \sum_{h_1 \in H_1} h_1(N) =$$

$$= \left(\sum_{h_0 \in H_0} h_0(N_0) \oplus \sum_{h_0 \in H_0} h_0(N_1) \right) \oplus \left(\sum_{h_1 \in H_1} h_1(N_0) \oplus \sum_{h_1 \in H_1} h_1(N_1) \right) =$$

$$= \left(\sum_{h_0 \in H_0} h_0(N_0) \oplus \sum_{h_1 \in H_1} h_1(N_1) \right) \ltimes_\phi \left(\sum_{h_1 \in H_1} h_1(N_0) \oplus \sum_{h_0 \in H_0} h_0(N_1) \right).$$

Hence, we deduce that the trace of a $\mathbb{Z}_2$-graded module is a semitrivial ideal extension by some ideal of R and this ideal will be denoted as

$$\mathfrak{t}_N = \sum_{h_0 \in H_0} h_0(N_0) \oplus \sum_{h_1 \in H_1} h_1(N_1);$$

we also denote $T_N = \sum_{h_1 \in H_1} h_1(N_0) \oplus \sum_{h_0 \in H_0} h_0(N_1)$, clearly $T_N^{\perp,\phi} \mathfrak{t}_N = T_N$. We proved the following proposition:

Proposition 4.56. *Let N be a $\mathbb{Z}_2$-graded module over a semitrivial extension $A = R \ltimes_\phi M$. We have*

$$\mathrm{Tr}_A(N) = \mathfrak{t}_N \ltimes_\phi T_N.$$

Using the previous proposition, we give another characteritazion of nearly Gorenstein semitrivial extensions.

Theorem 4.57. *Assume that R is Cohen-Macaulay and M is maximal Cohen-Macaulay. $R \ltimes_\phi M$ is nearly Gorenstein if and only if $\mathfrak{m} \subseteq \mathfrak{t}_\omega$ and $M = T_\omega$.*

Let I and J be two fractional ideals of R; for the sake of simplicity, in the sequel, we will write $(I : J)$ instead of $(I :_{Q(R)} J)$.

Proposition 4.58. *Let ω_R be a canonical ideal of R and let I, J_1, J_2 fractional ideals of R. Then the followings hold:*

1. $(J_1 : J_2) = (\omega_R : J_2) : (\omega_R : J_1)$
2. $R : (\omega_R : I) = (I : \omega_R)$.

Proof. Property 1 is a simple generalization of Proposition 3.3 that uses [**Exercise 1.12**, 2] for the case of fractional ideals. Using Property 1, similarly as we did for the proof of Proposition 3.4, we can prove Property 2. □

Theorem 4.59. *Assume that R is Cohen-Macaulay of positive Krull dimension with canonical module and M is isomorphic to a maximal Cohen-Macaulay ideal of R, say I. Suppose that ϕ is the multiplication by some element $b \in R$. Then $R \ltimes_\phi I = R(I)_{0,-b}$ is nearly Gorenstein if and only if the following conditions are true:*

1. *R is generically Gorenstein,*

2. *For any canonical ideal ω of R, we have:*

$$\mathfrak{m} \subseteq \omega(I : (\omega : I)) + b(\omega : I)(I : \omega)$$

and

$$I = \omega(I : \omega) + (\omega : I)(I : (\omega : I)).$$

Proof. Since $R \ltimes_\phi I$ is nearly Gorenstein an of positive Krull dimension, it follows that it is generically Gorenstein. By Proposition 4.53, R is generically Gorenstein. Let ω a canonical ideal of R, using Proposition 4.33 we now compute the trace of the canonical module like we did for the numerical semigroup case. Note that, by definition of ϕ, we have that for any fractional ideal J_1 and J_2 of R, $J_1 \cdot_\phi J_2 = J_1 J_2 b$ and $(J_1 :_\phi J_2) = (J_1 : bJ_2)$. Let $C = \omega \ltimes_\phi (\omega : I)$ a canonical fractional ideal of $R \ltimes_\phi I$, since the trace is invariant by shifts (note that any canonical ideal has depth 1), $\mathrm{Tr}_R(C) = CC^{-1}$. By Property 2 of Proposition 4.58, we have:

- $(I : (\omega : I)) \subseteq (R : (\omega : I)) = (I : \omega) \subseteq \omega^{-1}$;
- $(I : \omega) = (R : (\omega : I)) \subseteq (R : (b\omega : I))$.

Consequently, we get:

$$C^{-1} = \big(I : (\omega : I)\big) \ltimes_\phi (I : \omega).$$

Using the formula of the product of Proposition 4.33, we get:

$$CC^{-1} =$$

$$= \big[\omega(I : (\omega : I)) + b(\omega : I)(I : \omega)\big] \ltimes_\phi \big[\omega(I : \omega) + (\omega : I)(I : (\omega : I))\big].$$

Since $\mathfrak{m} \ltimes_\phi I$ is the maximal ideal of $R \ltimes_\phi I$, it follows the thesis. □

Remark 4.60. One could note that the nearly Gorensteiness of $R(I)_{0,-b}$ is dependent on $b \in R$, which is different from the numerical duplication case. It seems that, if we want that the nearly Gorensteiness does not depend on b, we need to add some more hypotheses.

Assume that R is a one-dimensional Cohen-Macaulay ring with canonical module isomorphic to a fractional ideal ω such that $R \subseteq \omega \subseteq \overline{R}$, where $\overline{R}$ is the integral closure of R and let I be a maximal Cohen-Macaulay ideal. In [4], the authors show that, if zR is the minimal reduction of $(\omega : I)$, then $R(I)_{0,-b}$ has canonical module isomorphic to $C = \frac{1}{z}(\omega : I) \ltimes_\phi \frac{1}{z}\omega$ and it is such that $R(I)_{0,-b} \subseteq C \subseteq \overline{R(I)_{0,-b}}$. With a proof which is virtually identical to the proof of 3.18, using Proposition 4.33, one can prove the following

Theorem 4.61. *In the same notations of Remark 4.60, $R(I)_{0,-b}$ is nearly Gorenstein if and only if, for any canonical ideal ω the following conditions are true:*

$$\mathfrak{m} \subseteq (\omega : I)(I : \omega)$$

and

$$I = \omega(I : \omega) + (\omega : I)(I : (\omega : I)).$$

In particular, the nearly Gorensteiness of $R(I)_{0,-b}$ does not depend on b.

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