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ABSTRACT

We obtain some exact solutions in the context of solitons, for heat conduction with inertia along a cylinder whose heat exchange with the environment is a non-linear function of the difference of temperatures of the cylinder and the environment, due to a flux-limiter behavior of the exchange. We study the consequences of heat transfer and information transfer along the wire, and we compare the situation with analogous solitons found in nonlinear lateral radiative exchange studied in some previous papers. We also find further exact solutions in terms of Weierstrass elliptic functions for the sake of completeness.

I. INTRODUCTION

The mathematical analysis of heat transport was, since the times of the original proposal by Fourier of his famous transport equation, a stimulus for mathematical methods. Since then, Fourier's equation with many different boundary conditions and source terms has been the subject of a huge analytical work. Since the 1970s, the interest in generalized forms of the Fourier equation, the Maxwell–Cattaneo–Vernotte equation, or the Guyer–Krumhansl equation has been outstanding in the analysis of heat transport. The physical relevance of these generalized heat transport equations comes from the experimental possibility of measuring very fast phenomena, requiring to take into account the relaxation time of the heat flux, or of manufacturing and controlling very small systems (nanosystems) whose size is comparable to the mean-free path of the heat carriers. The mentioned equations also require a mathematical attention because of their higher mathematical complexity, their higher number of boundary conditions and initial conditions, and the higher number of material parameters, potentially dependent on the temperature. In particular, the analysis of small amplitude temperature waves has received much attention, but in contrast, the attention toward the propagation of higher amplitude temperature waves, requiring non-linear methods, has been much less. This paper provides a contribution in this line, but considering nonlinear boundary conditions describing the heat exchange between the system and the environment.

Relaxational heat transfer allowing for inertial effects[1–6] is described by hyperbolic evolution equations, leading to finite speed for thermal pulses in bulk systems. In finite systems, the lateral heat exchange may modify some of the features of heat transport. In particular, nonlinear effects in the lateral heat exchange along thin wires may lead to soliton propagation of thermal signals, as it has been shown in the case of non-linear radiative exchange.[7–9] Here, we consider a different source of nonlinearity, namely, flux-limited heat exchange,[10–13] i.e., with nonlinear modification of Fourier's law such that the value of the heat flux for high temperature gradients does not indefinitely increase but has a maximum saturation value. This situation is expected to be found, for instance, when the wire is surrounded by a dense gas. Note that flux limiters are of interest not only in the particular case of heat transfer but also in high-resolution schemes of numerical modelization of other kinds of fluxes[14,15] to avoid spurious oscillations, which may appear in regions with high gradients.

In this paper, we follow the procedure used in Ref. 9, namely, we assume that heat transport along the wire is described by the Maxwell–Cattaneo equation, instead of the Fourier law, and we apply the auxiliary equation method[16–18] to find exact solutions of the investigated nonlinear equations in order to express them in terms of elliptic functions or some elementary functions. Further solutions can be found by means of the method proposed by Conte and Musette[19,20] or in terms of hyperelliptic functions.[21]

Soliton propagation is a topic of interest in many fields[22–29] because solitons can propagate without dispersion nor losses, a topic which has recently found interest in the emerging field of phononics,[30–32] where one aims at the possibility of transmitting and processing information by means of heat signals in contrast to electronics or photonics, where it is transmitted and processed by means of electric signals or optical signals. The aim of heat transmission and processing of information is nowadays possible because of recent developments in thermal diodes, thermal transistors, thermal logical-gates, and so on. Thus, transmission of heat signals is of interest not only from the perspective of energy transmission but also of information transmission. In this context, heat solitons have an additional interest since they could transmit a bit of information. The topic had been analyzed some years ago by Özişik *et al.*[7,8] in the context of the Maxwell–Cattaneo equation with lateral nonlinear radiative transfer. In Ref. 9, we used a different source of nonlinearity and studied the transmission velocity, the maximum rate of bit transmission, and the energy per bit, thus relating the energy point of view with the information point of view. In this paper, the physical origin and the mathematical form of the nonlinearity are different from those in radiative lateral heat transfer. Furthermore, solitons arising from nonlinearity in lateral heat transfer have a different origin from those arising from nonlinearities in the equation for bulk systems as, for instance, in heat transfer in the presence of chemical reactions, combustion, or phase transitions[33–36] or those arising from the consideration of microscopic non-linear effects in the lattice dynamics of the crystals along which heat is transported.[37,38]

This paper is organized as follows. In Sec. II, we present the mathematical model. In Sec. III, we explain the mathematical method used for searching solitons of our model. In Sec. IV, we present some nonlinear wave solutions traveling along a cylinder with a flux-limited heat exchange. In Sec. V, we find all the mathematical solutions in terms of traveling waves for high thermal conductivity and high relaxation time. Section VI is devoted to heat solitons of the sine-Gordon form, arising from an alternative version of the flux limiter being used. In Sec. VII, we present the conclusions of this paper.

II. THE MATHEMATICAL MODEL

In this section, we briefly consider the mathematical model given in Ref. 9, namely, heat propagation along a heat-conducting wire of radius r, composed of a material of mass density ρ and specific heat per unit mass c. The evolution equation for the temperature field T is obtained from the energy balance equation

$$\rho c \frac{\partial T}{\partial t} = -\frac{\partial q}{\partial z} - \frac{2}{r} q_t, \tag{2.1}$$

where $\mathbf{q} = q(z)\ \hat{\mathbf{z}}$ is the longitudinal heat transfer along the cylinder and q_t is the transverse heat exchange per unit area from the cylinder and the environment. We assume that $\mathbf{q}$ is described by the so-called Maxwell–Cattaneo equation[1–6]

$$\tau \frac{\partial q}{\partial t} = -q - \lambda \frac{\partial T}{\partial z}, \tag{2.2}$$

with τ being the relaxation time of the heat flux and λ being the thermal conductivity of the material.

By differentiating Eq. (2.1) with respect to the time and using Eq. (2.2), the following hyperbolic differential equation for the temperature is obtained:

$$\tau \rho c \frac{\partial^2 T}{\partial t^2} = \lambda \frac{\partial^2 T}{\partial z^2} - \rho c \frac{\partial T}{\partial t} - \frac{2}{r}\left(q_t + \tau \frac{\partial q_t}{\partial t}\right). \tag{2.3}$$

If we assume that the last term in Eq. (2.3) is a function of the temperature, namely, $\frac{2}{r}\left(q_t + \tau \frac{\partial q_t}{\partial t}\right) \equiv g(T)$, then (2.3) becomes

$$\tau \rho c \frac{\partial^2 T}{\partial t^2} = \lambda \frac{\partial^2 T}{\partial z^2} - \rho c \frac{\partial T}{\partial t} - g(T). \tag{2.4}$$

The case $g(T) = 0$ in Eq. (2.4) corresponds to the telegraphist equation, which, for high frequencies and short wavelengths, leads to temperature waves propagating with speed $(\lambda/\rho c \tau)^{1/2}$.[1–6]

Equation (2.4) is invariant under the translation $T \to T - T_0 = \Delta T$ in such a way that Eq. (2.4) becomes

$$\tau \rho c \frac{\partial^2 \Delta T}{\partial t^2} = \lambda \frac{\partial^2 \Delta T}{\partial z^2} - \rho c \frac{\partial \Delta T}{\partial t} - g(\Delta T). \tag{2.5}$$

Now, we consider that the cylinder along which thermal signals are being transported is immersed in a dense gas and there is a flux-limited heat exchange, which satisfies a saturation behavior. Indeed, one of the remarkable nonlinear effects in heat transfer for high temperature gradients is the so-called heat flux saturation.[10,11] The physical idea is that given a particular density and temperature of a material, the heat flux cannot be indefinitely high but will be limited by a maximum heat flux value q_{max} of the order of the internal energy density per unit volume times the maximum speed of propagation of thermal signals, i.e., the propagation speed of thermal pulses, which is $v_{max} = (\lambda/\rho c\tau)^{1/2}$, with τ being the relaxation time of the heat flux and λ being the thermal conductivity. A particular proposal to describe this phenomenon is, for instance, given as

$$q_t + \tau \frac{\partial q_t}{\partial t} = \frac{\sigma \Delta T}{\sqrt{1 + \left(\frac{\sigma' \Delta T}{T}\right)^2}}, \tag{2.6}$$

with σ being the heat transfer coefficient through the wall separating the system and the enviroment and σ' being a coefficient related to the saturation. When ΔT, the temperature difference between the cylinder and the enviroment, and the relaxation time τ are small, (2.6) reduces to Newton's law for heat transfer, $q = \sigma \Delta T$. For ΔT relatively small, in such a way that the denominator is ~1, this equation is a relaxational generalization of Newton's law, analogous to the relaxational generalization of Fourier's law provided by the Maxwell–Cattaneo equation (2.2). For $\Delta T \to \infty$, one finds $\left(q_t + \tau \frac{\partial q_t}{\partial t}\right)_{sat} = \sigma T/\sigma'$ for the saturation value of q, which does not depend on ΔT. The relaxation time τ in Eq. (2.6) may be, in general, different from that in the Maxwell–Cattaneo equation (2.2) because the former is related to the collision of the phonons with the lateral surface and the latter to the collisions with the other phonons in the bulk. In this paper, we have followed the same approach used in Ref. 9 for a direct comparison between the two kinds of nonlinearities. The general case, with two different relaxation times, has been considered in Ref. 39.

Furthermore, note that (2.6) is only a particular modelization of the flux limiter. In the literature on flux limiters, there are a wide variety of forms, such as fractions of polynomials, for instance, $\frac{a(\nabla T)^2 + \nabla T}{(\nabla T)^2 + 1}$ or $\frac{a(\nabla T)^2 + \nabla T}{(\nabla T)^2 + b(\nabla T) + 1}$; these expressions are valid in one-dimensional problems, or in three-dimensional cases, it must be assumed that the terms in $(\nabla T)^2$ denote the modulus of a vector directed in the same direction as that of ∇T. In these two cases, the value for the fraction for small ∇T is ∇T, and the value at high ∇T is given by the coefficient a. In Sec. VI, we will still briefly consider another modelization of the flux limiter, having a sinusoidal form for the interpolation between 0 (small value) and 1 (saturation value) for the heat flux.

Now, we can consider the Taylor series of (2.6) around $\Delta T = 0$ truncated at the second degree, namely,

$$q_t + \tau \frac{\partial q_t}{\partial t} = \sigma \Delta T \left[1 - \frac{\sigma'^2}{2}\left(\frac{\Delta T}{T}\right)^2\right], \tag{2.7}$$

which shows a reduction in q with respect to Newton's law.

The nonlinear term $g(\Delta T)$, defined below (2.2), becomes

$$g(\Delta T) = \frac{2}{r}\left[\sigma \Delta T - \frac{\sigma \sigma'^2}{2} \Delta T \left(\frac{\Delta T}{T}\right)^2\right]. \tag{2.8}$$

Equation (2.5), together with (2.8), in the dimensionless form is given as follows:

$$\frac{\partial^2 u}{\partial t_1^2} = \frac{\partial^2 u}{\partial z_1^2} - \frac{\partial u}{\partial t_1} - \tilde{g}(u), \tag{2.9}$$

where we have used

$$t_1 = t/\tau, \quad z_1 = z\sqrt{\frac{\rho c}{\lambda \tau}}, \quad u = \frac{(T - T_0)}{T_0}, \tag{2.10}$$

and

$$\tilde{g}(u) = \frac{\tau}{\rho c T_0} g(T - T_0) = \frac{2\tau}{r\rho c}\left(\sigma u - \frac{\sigma \sigma'^2}{2} u^3\right). \tag{2.11}$$

If $\lambda \to \infty$ and $\tau \to \infty$ but λ/τ being finite, (2.4) leads to an equation, which in the dimensionless form is given as follows:

$$\frac{\partial^2 u}{\partial t_1^2} = \frac{\partial^2 u}{\partial z_1^2} - \tilde{g}(u). \tag{2.12}$$

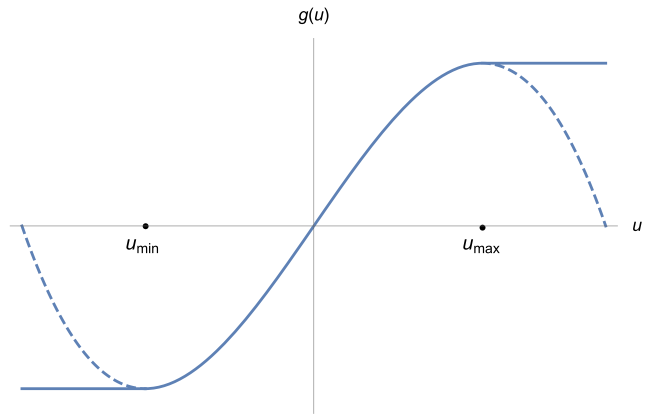

FIG. 1. Plot of $\tilde{g}(u) = \alpha u - \beta u^3$ given in (2.14). The horizontal lines indicate the saturation value of the heat flux; thus, our analysis is valid for perturbations less than the saturation value, where the approximation (2.7) is valid.

Thus, Eqs. (2.9) and (2.12) become

$$\frac{\partial^2 u}{\partial t_1^2} - \frac{\partial^2 u}{\partial z_1^2} + a\frac{\partial u}{\partial t_1} = -\alpha u + \beta u^3, \tag{2.13}$$

where $\alpha = \frac{2\tau\sigma}{r\rho c}$, $\beta = \frac{\tau\sigma\sigma'^2}{r\rho c}$, and $a = 1$ stands for Eq. (2.9) and $a = 0$ stands for Eq. (2.12).

Our expression for $\tilde{g}(u)$ is

$$\tilde{g}(u) = \alpha u - \beta u^3. \tag{2.14}$$

In what follows, the above equations will be restricted to the range of values $u \in [u_{min}, u_{max}]$, with $u_{min} = -\frac{1}{\sqrt{3}}\frac{\sqrt{\alpha}}{\sqrt{\beta}}$ and $u_{max} = \frac{1}{\sqrt{3}}\frac{\sqrt{\alpha}}{\sqrt{\beta}}$ being the values of u corresponding to the first minimum and the first maximum of $\tilde{g}(u)$ [a plot of $\tilde{g}(u)$ is shown in Fig. 1], respectively, with the mentioned choice for $\tilde{g}(u)$. The kind of nonlinearity in (2.14) (in u^3) is different from that found in nonlinear radiative transfer (in u^2).[7,9] Thus, from a mathematical point of view, it is worth of analysis.

Now, we use the similarity variable $\xi = kz - \omega t$ in order to transform our equation from a partial differential equation to an ordinary differential equation and look for some traveling wave solutions by means of the auxiliary equation method.[16–18] With the choice (2.14) for $\tilde{g}(u)$, Eq. (2.13) becomes

$$(\omega^2 - k^2)\frac{\partial^2 u}{\partial \xi^2} - a\omega\frac{\partial u}{\partial \xi} + \alpha u - \beta u^3 = 0, \tag{2.15}$$

where we have used the parameter a in order to include both Eqs. (2.9) and (2.12).

III. AUXILIARY METHOD FOR TRAVELING WAVES

In this section, we recall the main steps of the auxiliary method,[16–18] which allows us to find some exact traveling wave solutions of the following 1 + 1 nonlinear equation:

$$E(z, t, u, u_z, u_t, \ldots) = 0. \tag{3.1}$$

This method allows us to find and/or to check the existence of a particular kind of exact solution of Eq. (3.1), which is the aim of the current paper.

The first step is to transform Eq. (3.1) in an ordinary nonlinear equation, $E(\xi, u, u_\xi, u_{\xi\xi}, \ldots) = 0$, by means of the transformation $\xi = kz - \omega t$, which is typical for searching for traveling wave solutions.

The second step is to choose for $u(\xi)$ a polynomial form

$$u(\xi) = \sum_{i=0}^{n} u_i\, y(\xi)^i, \tag{3.2}$$

where u_i are the constants to be determined and the functions $y(\xi)$ are solutions of the auxiliary equation. The first choice of the auxiliary equation is the Riccati equation[16]

$$y(\xi)' = 1 - y(\xi)^2, \tag{3.3}$$

which is solved by the function $y(\xi) = \tanh(\xi)$, having the form of a propagating front. Another interesting example is

$$y(\xi)'^2 = y(\xi)^2(1 - y(\xi)^2), \tag{3.4}$$

which has the solution $y(\xi) = \operatorname{sech}(\xi)$, having the form of a propagating pulse.

The third step of the method is to determine the coefficients u_i in expression (3.2). This is achieved after the introduction of (3.2) into (3.1) taking into account (3.3) or (3.4). The value of n [the maximum value of the exponents of $y(\xi)$ in (3.2)] is determined by balancing the higher-order linear term with the higher nonlinear term of Eq. (3.1).

It is interesting for the purpose of this paper to define the concept of solitons.

Definition: Soliton is a nonlinear traveling wave, $U(\xi = kx - \omega t)$, solution of a non-linear evolution equation (partial differential equation), which at each time is localized in a bounded domain (compact set) of $\mathbb{R}$ such that the size of the domain remains bounded in time.

IV. SOME SOLUTIONS FOR FLUX-LIMITED HEAT EXCHANGE

In this section, we find the exact traveling wave solutions of model (2.15), which includes the cases for $a = 0$, Eq. (2.12), and the case for $a = 1$, Eq. (2.9), both using the Riccati equation (3.3) in Sec. IV A or Eq. (3.4) in Sec. IV B.

A. Traveling fronts associated with the Riccati equation

In this section, we look for solutions of Eq. (2.15) of the form (3.2) with $y(\xi)$ solution of the Riccati equation (3.3). Thus, balancing the linear term $u_{\xi\xi} \sim y^{n+2}$ with the nonlinear term $u^3 \sim y^{3n}$ in (2.15), namely, $n + 2 = 3n$, we find that $n = 1$. Had we chosen $\tilde{g}(u) = u^\alpha$, we would have found $n = 2/(\alpha - 1)$: in particular, for $\alpha = 2$, the exponent $n = 2$, for $\alpha = 3$, then $n = 1$, and for $\alpha = 3/2$, then $n = 4$.

1. Case $a = 1$

The case $a = 1$ corresponds to the Maxwell–Cattaneo equation (2.2) for the longitudinal heat flux, leading to (2.3) after the heat exchange has been introduced.

In this case, we find the following solutions with conditions $9\alpha - 2 > 0$ and $\beta > 0$:

$$\text{Solution I}: \qquad u(z_1, t_1) = \frac{\sqrt{\alpha}}{2\sqrt{\beta}}(\tanh(kz_1 - \omega t_1) - 1), \tag{4.1}$$

where $k = \mp\frac{1}{4}\sqrt{\alpha}\sqrt{9\alpha - 2}$ and $\omega = \mp\frac{3\alpha}{4}$ and the velocity $v = \frac{\omega}{k} = \frac{3\sqrt{\alpha}}{\sqrt{9\alpha-2}}$,

$$\text{Solution II}: \qquad u(z_1, t_1) = \frac{\sqrt{\alpha}}{2\sqrt{\beta}}(\tanh(kz_1 - \omega t_1) + 1), \tag{4.2}$$

where $k = \mp\frac{1}{4}\sqrt{\alpha}\sqrt{9\alpha - 2}$ and $\omega = \pm\frac{3\alpha}{4}$ and the velocity $v = \frac{\omega}{k} = -\frac{3\sqrt{\alpha}}{\sqrt{9\alpha-2}}$,

$$\text{Solution III}: \qquad u(z_1, t_1) = -\frac{\sqrt{\alpha}}{2\sqrt{\beta}}(\tanh(kz_1 - \omega t_1) - 1), \tag{4.3}$$

where $k = \mp\frac{1}{4}\sqrt{\alpha}\sqrt{9\alpha - 2}$ and $\omega = -\frac{3\alpha}{4}$ and the velocity $v = \pm\frac{3\sqrt{\alpha}}{\sqrt{9\alpha-2}}$,

$$\text{Solution IV}: \qquad u(z_1, t_1) = -\frac{\sqrt{\alpha}}{2\sqrt{\beta}}(\tanh(kz_1 - \omega t_1) + 1), \tag{4.4}$$

where $k = \mp\frac{1}{4}\sqrt{\alpha}\sqrt{9\alpha - 2}$ and $\omega = \frac{3\alpha}{4}$ and the velocity $v = \mp\frac{3\sqrt{\alpha}}{\sqrt{9\alpha-2}}$.

In these solutions, there are heat solitons going to the right ($v > 0$) and to the left ($v < 0$) with specific values of k, ω, and v and for amplitude $\frac{\sqrt{\alpha}}{2\sqrt{\beta}}$. On physical terms, $\tilde{g}(u)$ should be positive for $u > 0$ (i.e., heat goes from the cylinder to the enviroment for $T > T_0$ or vice versa) so that $\beta < \alpha$. If $\beta \ll \alpha$, the soliton will have a large amplitude.

2. Case $a = 0$

The case $a = 0$ corresponds to τ and λ in (2.4), very high but with a finite value for their ratio. This is the case of superfluid liquid helium, where second sound was observed for the first time. In this case, some of the solutions are the propagating fronts,

$$\text{Solution V}: \qquad u(z_1, t_1) = \frac{\sqrt{\alpha}}{\sqrt{\beta}} \tanh(kz_1 - \omega t_1), \tag{4.5}$$

where $\omega = \pm\frac{\sqrt{\alpha+2k^2}}{\sqrt{2}}$ and the velocity $v = \pm\frac{\sqrt{\alpha+2k^2}}{\sqrt{2}k}$,

$$\text{Solution VI}: \qquad u(z_1, t_1) = -\frac{\sqrt{\alpha}}{\sqrt{\beta}} \tanh(kz_1 - \omega t_1), \tag{4.6}$$

where $\omega = \pm\frac{\sqrt{\alpha+2k^2}}{\sqrt{2}}$ and the velocity $v = \pm\frac{\sqrt{\alpha+2k^2}}{\sqrt{2}k}$.

Although the profiles in (4.5) and (4.6) have the "tanh" form, such as those in (4.1)–(4.4), in (4.5) and (4.6), the soliton was limited to concrete values of the wavenumber k (related to the width of the soliton), whereas in (4.1)–(4.4), they exist for all k.

3. Heat flow

In (2.7), we have considered the expansion of the function (2.6) up to the third degree in series of $\Delta T/T$. The truncation to the third order is motivated by our purpose to investigate the influence of a weak nonlinearity (and the third degree is the first non-null term).

For the third-order truncation (2.13) of the series expansion for the function (2.6), we need to require that $\frac{\Delta T}{T} \ll 1$ or $u \ll 1$. Let us apply this requirement to solutions (4.1)–(4.6). Since all these solutions are simply a vertical translation of the function "tanh(...)" by unity, the following condition for solution (4.1) holds for solutions (4.2)–(4.6):

$$\left| \frac{\sqrt{\alpha}}{2\sqrt{\beta}} (\tanh(kz_1 - \omega t_1) - 1) \right| \leq \frac{\sqrt{\alpha}}{\sqrt{\beta}} < 1 \quad \Rightarrow \quad \frac{2}{\sigma'^2} < 1. \tag{4.7}$$

For the sake of illustration, let us consider the soliton solution (4.1) with the sign +. The function $\tilde{g}(u)$ has maximum and minimum at $u = u_{max} = \frac{1}{\sqrt{3}}\frac{\sqrt{\alpha}}{\sqrt{\beta}}$ and $u = u_{min} = -\frac{1}{\sqrt{3}}\frac{\sqrt{\alpha}}{\sqrt{\beta}}$, respectively. Thus, soliton (4.1) can be written as

$$u(z_1, t_1) = \frac{\sqrt{3}}{2} u_{max} (\tanh(kz_1 - \omega t_1) - 1). \tag{4.8}$$

If we choose $\beta = 4$ and $\alpha = 1$ in such a way that $u_{max} = \frac{1}{2\sqrt{3}}$, then we find from (4.1) the values $\omega = \mp\frac{3}{4}$ and $k = \mp\frac{\sqrt{7}}{4}$, which lead to the velocity $v = 3/\sqrt{7}$. Soliton (4.8) is plotted in Fig. 2 for $\omega = -\frac{3}{4}$ and $k = -\frac{\sqrt{7}}{4}$ and for $z_1 \in [-3, 10]$ and $t_1 \in [0, 10]$ and in Fig. 3 for $z_1 \in [-4, 11]$ and for three different values of the time, $t = 0$, $t = 3$, and $t = 6$.

The wave (4.8) can be written in the dimensional form as

$$\Delta T = \frac{\sqrt{3}}{2} \Delta T_{max} (\tanh(k_d z - \omega_d t) - 1), \tag{4.9}$$

with $\Delta T_{max} = \sqrt{\frac{2}{3}\frac{T_0}{\sigma'}}$ and the velocity

$$v = \frac{3\sqrt{2\tau\sigma}}{\sqrt{18\tau\sigma - 2r\rho c}}. \tag{4.10}$$

From a direct comparison among solutions (4.1)–(4.4), we note that they have the same shape given by "tanh(...)," eventually translated by a unit, exception made for the sign of the velocity of the waves, positive for (4.1), (4.3), and (4.4) and/or negative for (4.2)–(4.4).

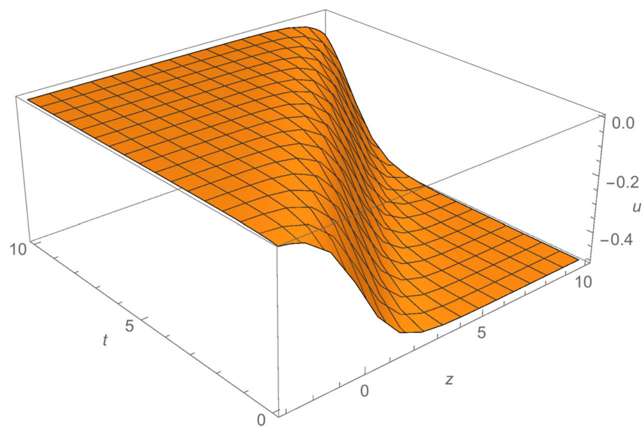

FIG. 2. Plot of (4.8) for $u_{max} = \frac{1}{2\sqrt{3}}$, $\beta = 1$, and $\alpha = 4$, and hence, $k = -\frac{\sqrt{7}}{4}$ and $\omega = -\frac{3}{4}$, which lead to $v = \frac{3}{\sqrt{7}}$.

With regard to, instead, solutions (4.5) and (4.6), they have the same shape as the ones of the case $a = 1$. Their velocity is

$$v = \pm\sqrt{\frac{\sigma + r\lambda k_d^2}{r\lambda}}. \tag{4.11}$$

In order to sustain the steady propagation of the solitonic fronts of the form (4.1)–(4.6), a heat flow must be continuously supplied to one end of the wire, given by

$$\dot{Q} = \rho c A v (T_{max} - T_0), \tag{4.12}$$

with $A = \pi r^2$ being the transversal area of the wire and v being the propagation speed given by (4.10) or (4.11). This heat depends on $(T_{max} - T_0)$, which corresponds in this case to the temperatures of the hot wall and the cold wall. Thus, this regime corresponds to ballistic propagation of heat, in contrast to Fourier's diffusive propagation.

B. Traveling waves associated with (3.4)

Let us consider again u of the form (3.2) with $y(\xi)$ satisfying Eq. (3.4), namely, $y'^2 = y^2(1 - y^2)$, having the solution $y(\xi) = \operatorname{sech}(\xi)$. Balancing the linear term $u_{\xi\xi} \sim y^{n+2}$ with the nonlinear term $u^3 \sim y^{3n}$ in (2.15), then we find again $n = 1$ for the maximum exponent of expansion (3.2).

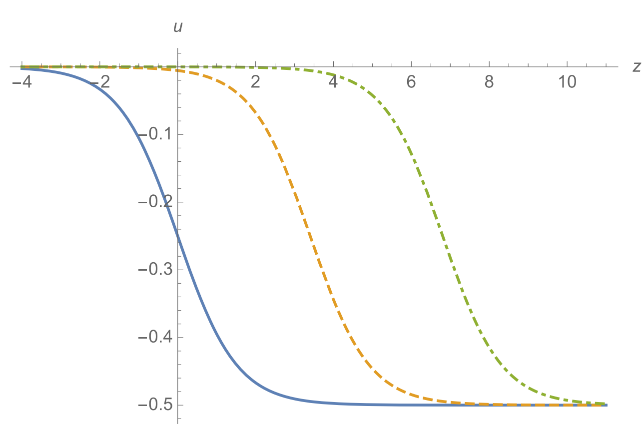

FIG. 3. Plot of (4.8) for $u_{max} = \frac{1}{2\sqrt{3}}$, $\beta = 1$, and $\alpha = 4$, and hence, $k = -\frac{\sqrt{7}}{4}$ and $\omega = -\frac{3}{4}$, which lead to $v = \frac{3}{\sqrt{7}}$, for three different values of the dimensionless time $t = 0$, $t = 3$, and $t = 6$.

1. Case a = 1

We find the following two stationary solitons:

$$\text{Solution VII}: \qquad u(z_1,t_1) = \pm\frac{\sqrt{2\alpha}}{\sqrt{\beta}}\operatorname{sech}(\sqrt{b}z_1), \tag{4.13}$$

where $\omega = 0$ and $\alpha = k^2$.

This means that if the initial imposed profile is (4.13), heat will not disperse along the cylinder but will stay confined around its initial position. Of course, this would not be possible with pure diffusion.

2. Case a = 0

$$\text{Solution VIII}: \qquad u(z_1,t_1) = \frac{\sqrt{2\alpha}}{\sqrt{\beta}}\operatorname{sech}(kz_1 - \omega t_1), \tag{4.14}$$

where $\omega = \pm\sqrt{k^2-\alpha}$ and the velocity $v = \pm\frac{\sqrt{k^2-\alpha}}{k}$. Thus, choosing k, the values for ω and v are determined (for given α).

3. Heat flow

Again, the condition for solutions (4.13) and (4.14), or (4.16), is that $u \ll 1$, namely,

$$\frac{\sqrt{2\alpha}}{\sqrt{\beta}} < 1 \quad \Rightarrow \quad \frac{4}{\sigma'^2} < 1, \tag{4.15}$$

which is a more stringent condition with respect to the "tanh"-type solution.

Again, we can write soliton (4.14) in terms of $u_{\max}$,

$$u(z_1,t_1) = \sqrt{6}u_{max}\operatorname{sech}(kz_1 - \omega t_1). \tag{4.16}$$

Choosing $\alpha = 1$, $\beta = 4$, and $k = 2$, then we find that $u_{max} = \frac{1}{2\sqrt{6}}$, $\omega = \sqrt{3}$, and $v = \frac{\sqrt{3}}{2}$. The plot of soliton (4.16) is shown in Figs. 4 and 5.

The soliton solution (4.14) in the dimensional form becomes

$$\Delta T = \sqrt{6}\Delta T_{max}\operatorname{sech}(k_d z - \omega_d t), \tag{4.17}$$

with $k_d = \sqrt{\frac{\rho c}{\lambda \tau}}k$, $\omega_d = \frac{\omega}{\tau}k$, and the velocity

$$v = \frac{k_d^2\lambda r - 2r}{r\lambda k_d^2}. \tag{4.18}$$

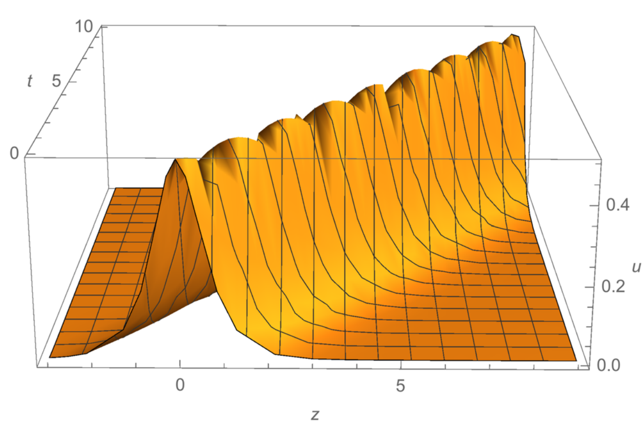

FIG. 4. Plot of (4.16) for $\alpha = 1$, $\beta = 4$, $k = 2$, $u_{max} = \frac{1}{2\sqrt{6}}$, $\omega = \sqrt{3}$, and $v = \frac{\sqrt{3}}{2}$.

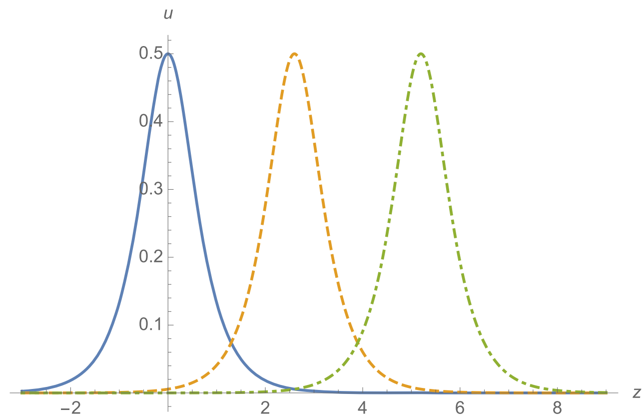

FIG. 5. Plot of (4.16) for $\alpha = 1$, $\beta = 4$, $k = 2$, $u_{max} = \frac{1}{2\sqrt{6}}$, $\omega = \sqrt{3}$, and $v = \frac{\sqrt{3}}{2}$ for three different values of the dimensionless time $t = 0$, $t = 3$, and $t = 6$.

In this case, the heat flow will be given by

$$\dot{Q} = \dot{n} \times \frac{Energy}{soliton}, \tag{4.19}$$

with $\dot{n}$ being the number of solitons produced per unit time.

V. EXACT SOLUTIONS OF (2.15) WITH $a = 0$ IN TERMS OF THE WEIERSTRASS ELLIPTIC FUNCTION

In this section, we consider other exact solutions of Eq. (2.15) with $a = 0$, namely, in the moving frame of reference, which means traveling waves in terms of Weierstrass elliptic funcions,[40] which may be of practical interest, for instance, to choose the best one for sending information with highest speed, with least energy, or so on.

Let us multiply both sides of Eq. (2.15) by $\frac{\partial u}{\partial \xi}$, and the integration of both sides leads to

$$\left(\frac{\partial u}{\partial \xi}\right)^2 = \frac{\beta}{2(\omega^2 - k^2)}\left[u^4 - \frac{2\alpha}{\beta}u^2 + \frac{2\gamma}{\beta}\right], \tag{5.1}$$

which, in general, has elliptic functions as solutions. The new constant γ arises from the integration.

In order to find all the solutions of (5.1), let us transform the above equation in the standard equation (5.4) whose solution is the Weierstrass function $\wp(\xi, g_2, g_3)$.

The first step is to reduce the degree of the polynomial on the right-hand side of Eq. (5.1). For this reason, we choose one zero of this polynomial, for instance, $u_1 = \sqrt{\frac{\alpha}{\beta} + \sqrt{\frac{\alpha^2}{\beta^2} - \frac{2\gamma}{\beta}}}$, and we consider the transformation $w = \frac{1}{u - u_1}$, which leads zero to infinite. After substituting this new variable w into Eq. (5.1), we find that

$$\left(\frac{\partial w}{\partial \xi}\right)^2 = a_3 w^3 + a_2 w^2 + a_1 w + a_0, \tag{5.2}$$

with $a_3 = 4Au_1B$, $a_2 = 2A(B + 2u_1^2)$, $a_1 = 4Au_1$, and $a_0 = A$, where $A = \frac{\beta}{2(\omega^2 - k^2)}$, u_1 is given above Eq. (5.2), and $B = \sqrt{\frac{\alpha^2}{\beta^2} - \frac{2\gamma}{\beta}}$.

By means of further transformation

$$w = \frac{4}{a_3}\wp(\xi) - \frac{a_2}{3a_3}. \tag{5.3}$$

Equation (5.2) becomes

$$\wp'^2(\xi) = 4\wp^3(\xi) - g_2\wp(\xi) - g_3, \tag{5.4}$$

where g_2 and g_3 are the well-known constants of the Weierstrass elliptic function $\wp(\xi; g_2, g_3)$,

$$g_2 = \frac{a_2^2 - 3a_1 a_3}{12} = \frac{\alpha^2 + 6\gamma\beta}{12(\omega^2 - k^2)^2} \tag{5.5}$$

and

$$g_3 = \frac{9a_1a_2a_3 - 2a_2^3 - 27a_0a_3^2}{432} = \frac{\alpha(\alpha^2 - 18\gamma\beta)}{216(\omega^2 - k^2)^3}. \tag{5.6}$$

The Weiestrass elliptic function is also characterized by the discriminant

$$\Delta = g_2^3 - 27g_3^2 = \frac{\beta\gamma(\alpha^2 - 2\gamma\beta)^2}{32(\omega^2 - k^2)^6}, \tag{5.7}$$

which allows us to distinguish the cases for $\Delta \neq 0$ and $\Delta = 0$.

A. Case $\Delta \neq 0$

In this case, the Weiestrass elliptic function $\wp(\xi, g_2, g_3)$ does not degenerate and the final solution of (5.1) is the elliptic function,

$$u = u_1 + \frac{6Au_1B}{6\wp(\xi) - A(B + 2u_1^2)}, \tag{5.8}$$

with the coefficients A, u_1, and B defined above.

B. Case $\Delta = 0$

In this case, we have the degenerate cases of the Weierstrass function, namely, the cases where it degenerates to the elementary functions, which happens when the discriminant Δ, defined in (5.7), is zero. Condition $\Delta = 0$ is satisfied for $\gamma = \frac{\alpha^2}{2\beta}$ or $\gamma = 0$.

The condition $\gamma = \frac{\alpha^2}{2\beta}$ leads to $B = 0$, and hence, u_1 becomes $u_1 = \sqrt{\frac{\alpha}{\beta}}$. The final solution from (5.8) is hence a constant,

$$u = u_1 = \sqrt{\frac{\alpha}{\beta}}. \tag{5.9}$$

Let us consider now the cases for $\gamma = 0$. The two constants g_2 and g_3 become

$$g_2 = \frac{\alpha^2}{12(\omega^2 - k^2)^2}, \qquad g_3 = \frac{\alpha^3}{216(\omega^2 - k^2)^3}, \tag{5.10}$$

where $g_2 > 0$ and g_3 may be positive or negative according to the sign of $\frac{\alpha}{(\omega^2 - k^2)}$. Thus, we define the parameter $c = \frac{\alpha}{12(\omega^2 - k^2)}$ in order to distinguish two subcases: the case $c > 0$ for $g_2 > 0$ and $g_3 > 0$ and the case $c < 0$ for $g_2 > 0$ and $g_3 < 0$.[41]

Case $c > 0$ ($g_2 > 0$ and $g_3 > 0$).

In this subcase, $g_2 = 12c^2 > 0$ and $g_3 = 8c^3 > 0$, and we find the solution from (5.8),

$$u = u_1 + \frac{6Au_1B}{6\wp(\xi) - A(B + 2u_1^2)} = \sqrt{\frac{2\alpha}{\beta}} + \frac{6\alpha\sqrt{\frac{2\alpha}{\beta}}}{-5\alpha + 12(\omega^2 - k^2)\left[-c + \frac{3c}{\sin(\sqrt{3c}\xi)}\right]}. \tag{5.11}$$

Case $c < 0$ ($g_2 > 0$ and $g_3 < 0$).

In this subcase, $g_2 = 12c^2 > 0$ and $g_3 = 8c^3 < 0$, and we find the solution from (5.8),

$$u = u_1 + \frac{6Au_1B}{6\wp(\xi) - A(B + 2u_1^2)} = \sqrt{\frac{2\alpha}{\beta}} + \frac{6\alpha\sqrt{\frac{2\alpha}{\beta}}}{-5\alpha + 12(\omega^2 - k^2)\left[-c - \frac{3c}{\sinh(\sqrt{3|c|}\xi)}\right]}. \tag{5.12}$$

VI. SINE-GORDON BEHAVIOR

In (2.14), we have assumed an interpolation of the form $\alpha u - \beta u^3$ between the usual Newton's law behavior, or, in fact, with its relaxational generalization as a Maxwell–Cattaneo equation (when one has only u), and the saturation value for the heat flux. It is conceptually interesting to consider other possible interpolations. One possibility of special mathematical interest is considering a sinusoidal function, between 0 and 1, with the linear approximation corresponding to Newton's law and for 1 corresponding to the saturation value of the heat flux. The ensuing equation has the form of the well-known sine-Gordon equation (which was proposed in 1862 by Edmond Bour in the study of surfaces of constant negative and later in crystal dislocations[42]). In this way, we may explore how a same physical problem—namely, lateral heat transport with heat flux saturation—may have different solutions when one uses different interpolation models. In fact, the main conclusion is that the nonlinearity of this physical situation allows for some solitonic transport in some circumstances, although the particular form of the solitons may depend on the mathematical interpolation.

In this section, if we choose $g_1(u)$ as (see Fig. 6)

$$g_1(u) = m^2 \sin(u) \tag{6.1}$$

in (2.12), we recover the sine-Gordon equation[42–44]

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial z^2} + m^2 \sin(u) = 0. \tag{6.2}$$

Up to the third order, $\sin(u) \sim u - \frac{1}{3}u^3$ so that up this approximation, $\tilde{g}(u)$ has the form taken in (2.14) with $\alpha = 1$ and $\beta = 1/3$, but here, we have additional terms.

The physical motivation for taking $g_1(u) = m^2 \sin(u)$ (with $u \in [-\pi/2, \pi/2]$) is to consider saturation effects in the lateral heat transfer by assuming that $g_1(u) = m^2$ for $u < -\pi/2$ and $u > \pi/2$. For small values of u and for $\tau \frac{\partial q_t}{\partial t} \ll q_t$, we get $g_1(u) \simeq q_t = m^2 u$. In explicit physical terms, this corresponds to assuming that the heat exchange rate per unit area through the lateral surface is proportional to the difference in temperature $T - T_0$, according to Newton's law for heat transfer, with m^2 being the heat transfer coefficient. Instead, replacing u by $\sin(u)$ (for $u \in [-\pi/2, \pi/2]$), we are describing in a special simplified way the saturation of the heat flux with m^2 being the saturation value of the heat flux.

The sine-Gordon equation (6.2) is widely known in mathematical physics, and it has solitonic solution of the form

$$u(z_1, t_1) = 4 \arctan(\exp(m\gamma(z_1 - vt_1))), \tag{6.3}$$

with $\gamma^2 = (1 - v^2)^{-1}$.[45–47] The solution with the positive (negative) sign of γ is, respectively, called a *kink* (or *anti-kink*). Thermal solitons described by sine-Gordon equations, such as (2.9) and (2.12), with $g(u) = m^2 \sin(u)$ have been considered in Ref. 48, in connection with equilibrium production of kink–antikink pairs in sine-Gordon systems subject to thermal fluctuations. The motivations and physical hypothesis of Ref. 48 are thus different from ours, but they contribute to the general physical interest of these mathematical solutions.

Indeed, the solitons considered here have a macroscopic origin based on the nonlinearity of lateral heat transfer. Instead, the thermal solitons mentioned in the bibliography usually refer to solitons in the bulk, and arising from microscopic equations, as, for instance, solitons in harmonic waves or in several kinds of nonlinear waves.

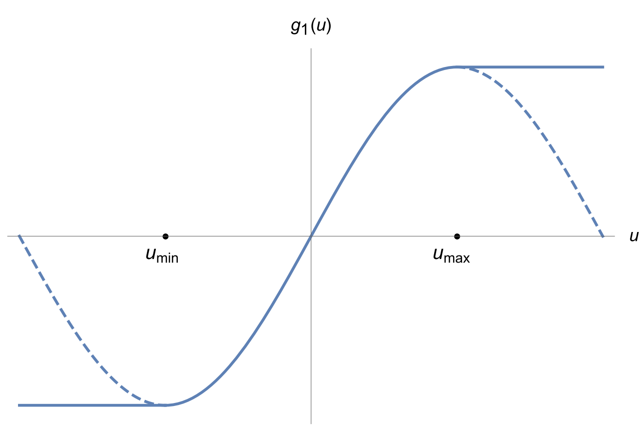

FIG. 6. Plot of $g_1(u) = m^2 \sin(u)$ given in (6.1). The horizontal lines indicate the saturation value of the heat flux.

VII. CONCLUSIONS

We have shown some special cases of heat solitons and fronts propagating along a heat-conducting cylinder when the lateral heat transfer between the cylinder and the environment is not given by linear Newton's law of heat transfer (namely, heat flow proportional to $T - T_0$) but by a nonlinear expression of the form (2.8), coming as a truncation of the flux-limiter expression of the form (2.7). Some of these solitonic solutions those having the form of localized travelling pulses as "sech" are especially interesting for transmitting heat energy without external loss nor internal dispersion because both effects (nonlinear effects and dispersion) compensate each other. This allows us to transport heat packets to long distance. These solutions could have interest in the so-called phononics for computations based on heat pulses rather than on electric pulse.[30,31] Heat rectifiers (diodes) and amplifiers (transistors), which are usually considered in steady state situations, should also be considered in the context of soliton heat signals to take advantage of these new possibilities in the context of soliton heat signals.

We have considered a Maxwell–Cattaneo law for the heat transport along the cylinder (i.e., Fourier's law plus a relaxation term accounting for some inertia of heat) and lateral heat transfer given by a flux-limiter law (i.e., with a saturation value for the heat flux for high temperature differences), thus leading to nonlinear terms in the lateral heat exchange. In Ref. 9, we have considered the radiative transfer, another source of nonlinear lateral heat exchange. In such a case, instead of having $\tilde{g}(u) = \alpha u - \beta u^3$ [see Eq. (2.11)], one has, up to the second order, $\tilde{g}(u) = b(u + 4u^2)$, with $b = \frac{2}{r}\frac{\tau\sigma_{SB}T_0^3}{\rho c}$ and σ_{SB} being Stefan–Boltzmann's constant. A useful comparison between the two models can be achieved considering, for instance, two exact solutions obtained in the two papers. In Ref. 9, we saw that a strategy for sending information by means of localized heat pulses (a pulse corresponding to state 1 of a bit, and the absence of pulse to state 0 of a bit, as it occurs in optical fiber) could be based on the soliton

$$\Delta T(z,t) = T_0 \mathrm{sech}^2(k_d z - \omega_d t) - \frac{2}{3}, \tag{7.1}$$

with $\omega_d^2 = \frac{2\sigma_{SB}T_0^3 + r\lambda k_d^2}{r\tau\rho c}$ and the velocity

$$v = \omega_d/k_d = \sqrt{\frac{\lambda}{\tau\rho c}}\sqrt{1 + \frac{2\sigma_{SB}T_0^3}{r\lambda k_d^2}}. \tag{7.2}$$

The bright soliton (7.1) can be compared to the soliton (4.14) obtained in this paper, namely,

$$\Delta T = \frac{2}{\sigma'} T_0 \mathrm{sech}(k_d z - \omega_d t), \tag{7.3}$$

with $\omega_d^2 = \frac{k_d^2\lambda r - 2\sigma}{r\rho c\tau}$ and the velocity

$$v^2 = \frac{k_d^2\lambda r - 2\sigma}{r\rho c\tau k_d^2}. \tag{7.4}$$

In Ref. 9, we have considered some typical parameters used in optical trasmission, namely, the velocity of the soliton, which in our case is given by (7.2) or (7.4); the temporal width of a soliton τ_A; the Full Width at Half Maximum (FWHM), as sketched in Fig. 1 of Ref. 9; the Rate of Information Transfer (RIT), namely, RIT $= \frac{1}{\tau_A}$; and the energy per bit (the energy carried by the soliton).

The values of these parameters for the soliton (7.1) are as follows:

$$\tau_A = 2\ln(1+\sqrt{2})\sqrt{\frac{r\tau\rho c}{2\sigma_{SB}T_0^3 + r\lambda k_d^2}}, \tag{7.5}$$

the rate of information transfer (i.e., bits/s),

$$\mathrm{RIT} = \frac{1}{2\ln(1+\sqrt{2})}\sqrt{\frac{2\sigma_{SB}T_0^3 + r\lambda k_d^2}{r\tau\rho c}}, \tag{7.6}$$

and the initial energy per bit carried by the soliton at $t = 0$,

$$\frac{Energy}{bit} = \frac{\rho c T_0 \pi r^2}{k_d}\left[\sqrt{2} - \frac{4}{3}\ln(1+\sqrt{2})\right] = 0.239\frac{\rho c T_0 \pi r^2}{k_d}. \tag{7.7}$$

Now, we can evaluate the same parameters for the soliton (7.3). For the width τ_A, we evaluate the width of the function $U(z,t) = \operatorname{sech}(k_d z - \omega_d t)$ at half of the maximum value $U(0,0) = 1$, which is at the distance $z = \frac{\omega_d}{k_d} t$ or at the time $t = \frac{k_d}{\omega_d} z$. Hence, τ_A is the double time obtained from the equation $\operatorname{sech}(\omega_d t) = \frac{1}{2}$, namely,

$$\tau_A = \frac{2}{\omega_d}\operatorname{sech}^{-1}\left(\frac{1}{2}\right) = \frac{2}{\omega_d}\ln(2+\sqrt{3}) = 2\ \ln(2+\sqrt{3})\sqrt{\frac{r\rho c\tau}{k_d^2\lambda r - 2\sigma}} \tag{7.8}$$

from which we find the RIT,

$$\mathrm{RIT} = \frac{1}{\tau_A} = \frac{1}{2\ln(2+\sqrt{3})}\sqrt{\frac{k_d^2\lambda r - 2\sigma}{r\rho c\tau}}. \tag{7.9}$$

The same arguments to find τ_A lead to spatial width of the soliton, namely, $\frac{2}{k_d}\ln(2+\sqrt{3})$, which we can use to evaluate the initial energy per bit carried by the soliton at $t = 0$,

$$\frac{Energy}{bit} = \int_{-\frac{1}{k_d}\ln(2+\sqrt{3})}^{\frac{1}{k_d}\ln(2+\sqrt{3})} \frac{2}{\sigma'} T_0 \operatorname{sech}(k_d z)\ \pi r^2\ dz = \frac{4}{3}\pi^2\frac{T_0 r^2}{k_d\sigma'}. \tag{7.10}$$

A remarkable qualitative difference between the "radiative" soliton and the "flux-limiter" soliton refers to the Rate of Information Transfer (RIT) for a given value of k_d: for the first soliton, RIT increases when r is reduced, whereas for the second soliton, RIT becomes an imaginary quantity (i.e., the capacity of sending solitons becomes truncated) for r too small, namely, for $r = 2\sigma/k_d^2\lambda$.

The choice $\tilde{g}(u) = \alpha u$, namely, linear Newton's law, does not lead to soliton solution but to dispersive and dissipative waves. Indeed, substituting $u = A\exp[\mathrm{i}(kx - \omega t)]$ into Eq. (2.3) with $\tilde{g}(u) = bu$, $\omega = \omega_r + \mathrm{i}\omega_i$, and k being real, the following dispersion relation is found:

$$\omega^2 - k^2 + \mathrm{i}\omega - \alpha = 0. \tag{7.11}$$

The solutions of the relation (7.11) are as follows: (1) $\omega_r = 0$ and $\omega_s = \frac{-1\pm\sqrt{1-4(k^2+\alpha)}}{2}$, which lead to a null group velocity, and (2) $\omega_s = -\frac{1}{2}$ and $\omega_r = \pm\sqrt{k^2 + \alpha - \frac{1}{2}}$ with the group velocity $v_g = d\omega_r/dk = \frac{k}{\omega_r} = \frac{1}{v_f}$, with v_f being the phase velocity. In both cases, we find dissipative waves.

In Sec. V, we consider (2.15) with $a = 0$ and we integrate it in order to also find all the exact solutions in terms of elliptic functions. The aim of this section is to find all the travelling nonlinear waves of the equation for future purposes.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

All authors contributed equally to this work.

DATA AVAILABILITY

The data that support the findings of this study are available within the article.

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