Solutions via double wave ansatz to the 1-D non-homogenous gas-dynamics equations

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Abstract

In this paper classes of double wave solutions of the 1D Euler system describing a ideal fluid in the non-homogeneous case have been determined. In order that the analytical procedure under interest to hold, suitable model laws for the source term involved in the governing model were characterized. Finally such a class of exact double wave solutions has been used for solving some problems of interest in nonlinear wave propagation.

Keywords: Hyperbolic systems. Double wave solutions. Riemann Problem. Nonlinear wave interactions.
MSC2010: 35L50, 35L60, 35N10

1 Introduction

Along the years many mathematical methods have been proposed in order to determine exact solutions of PDEs. Some of them make use of group properties of differential equations as, for instance, Lie-group analysis, non-classical methods, partially invariant solutions, weak symmetries, etc. (see [1, 2]). Others belong to the class of the so-called solutions with degenerate hodograph as multiple waves [3] (in particular simple waves and double waves) and generalized hodograph method [4, 5, 6]. A third group is based on the requirement that the governing equations satisfy some suitable additional conditions as, for instance, in the differential constraint method. Such an approach was first proposed and applied to gas-dynamics in [7, 8]. The main idea is to add to the governing system under interest some further differential equations which play the role of constraints because they select the class of special exact solutions admitted by the overderdetermined set of equations consisting of the original equations along with the additional differential constraints. The method is developed on two steps: first the compatibility of such an overdetermined system must be studied, next exact solutions of the full set of equations can be determined. On this subject many contributions have been given [9]-[14] as well as different problems of interest in the applicatons have been solved [15]-[21].

More recently an approach based on the combined use of the double wave theory as well as of the method of differential constraints has been developed in order to find special exact double wave solutions of quasilinear first order hyperbolic systems [22]. The main idea of the method is to reduce the problem of integrating the governing equations to that of solving a suitable 2×2 auxiliary system by requiring that the remaining equations specialize to conservation laws along with a differential constraint of the reduced 2×2 system under consideration. Therefore, according to the theory of differential constraints, such an approach leads to solve a 2×2 "ODE" system along with a differential constraint which selects the class of initial value problems compatible with the procedure at hand (see section 2.1 for more details). The resulting solutions are given in terms of one arbitrary function and such degree of freedom permitted to solve problems of interest in nonlinear wave interactions (see for instance [23]).

Within such a theoretical framework, in this paper we consider the non-homogeneous 1D Euler equations describing an ideal fluid which in Eulerian coordinates assume the form

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0, \tag{1}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + \frac{\partial p}{\partial x} = \rho f(\rho, u), \tag{2}$$

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = 0, \tag{3}$$

where ρ, u, s denote, respectively, the mass density, the velocity and the entropy density, whereas $p(\rho, s)$ is the pressure. Moreover the production term $f(\rho, u)$ represents the specific body force. The system (1) is strictly hyperbolic [24, 25] and its characteristic speeds are

$$\lambda_1 = u - c, \qquad \lambda_2 = u, \qquad \lambda_3 = u + c \tag{4}$$

where, as usual, $c = \sqrt{\frac{\partial p}{\partial \rho}}$ denotes the sound speed.

As far as we know, exact solutions of the Euler system (1) have been obtained in the homogeneous case and only for very few special cases (see [25]-[28]). Moreover some further results concerning the non-homogeneous 3-D case have been obtained in [29] . Therefore, the main aim of the present paper is to develop, along the lines of the analysis carried on in [22], a reduction procedure for determing classes of exact double wave solutions to (1)-(3) and, consequently, some nonlinear wave problems of relevant interest such as Riemann problems ([30]-[32]) and nonlinear wave interactions ([33]-[35]) are analysed. In particular, after reducing the full set of governing equations to a suitable 2×2 hyperbolic auxiliary system, following the idea developed in [17, 18], a Riemann problem will be solved as well as an exact analitical description of nonlinear waves interaction admitted by the governing system under interest will be given. Furthermore, possible functional forms of the pressure $p(\rho, s)$ as well as of the production term $f(\rho, u)$ allowing the reduction procedure under interest to hold are characterized.

The plan of the paper is the following. In Section 2 we outline the main steps of the method of differential constraints as well as of the reduction process developed in [22] in order to find particular double wave solutions to quasilinear hyperbolic non-homogeneous system of first order PDEs in one space dimension. In Section 3 we obtain classes of double waves admitted by the non-homogeneous model (1)-(3). In Section 4 we make use of the solutions obtained via double wave ansatz to solve a Riemann problem and to give an exact description of nonlinear simple wave interactions. Conclusions and final remarks are given in the last section while in the appendix we sketch a more general analysis concerning the differential compatibility between the differential constraints under interest and the corresponding 2×2 reduced systems.

2 On a class of double waves to quasilinear hyperbolic systems

In this section, for further convenience, we outline the main steps of methods of differential constraints as well as of the reduction approach proposed in [22] in order to find particular double wave solutions to a first order hyperbolic system of PDEs.

2.1 Differential Constraints

Let us consider the set of N equations

$$\mathbf{U}_t + A\left(\mathbf{U}\right)\mathbf{U}_x = \mathbf{B}\left(\mathbf{U}\right) \tag{5}$$

where $\mathbf{U} \in R^{\mathrm{N}}$ denotes the vector of the field variables, $\mathbf{B}\left(\mathbf{U}\right)$ a vector source and $A\left(\mathbf{U}\right)$ the $N \times N$ matrix coefficients. Furthermore we assume the system (5) is hyperbolic in the t direction, namely the N–th order matrix A is required to admit N real eigenvalues λ_i to which there correspond N right eigenvectors $\mathbf{d}^{(\lambda_i)}$ and N left eigenvectors $\mathbf{l}^{(\lambda_i)}$ spanning the Euclidean space E^{N}.

For strictly hyperbolic systems (i. e. $\lambda_i \neq \lambda_j, \forall i, j = 1, .., N$) it has been proved (see for instance [28] and references there quoted) that the more general first order differential constraint admitted by (5) must adopt the form

$$\mathbf{l}^{(\lambda_i)} \cdot \mathbf{U}_x = q_i\left(x, t, \mathbf{U}\right) \tag{6}$$

where q_i is a function which must be determined during the compatibility process. Let $M < N$ be the number of the differential constraints like (6) appended to (5). Once the compatibility of the resulting overdetermined set of equations is satisfied, then exact solutions of the governing system (5) can be obtained in terms of $N - M$ arbitrary functions. The case of relevant interest is when $M = N - 1$. In fact if we append to the governing system (5) the $N - 1$ differential constraints (6) with $i = 1, .., N - 1$, then the equations (5) can be rewritten under the form

$$\mathbf{U}_t + \lambda_{\mathrm{N}}\mathbf{U}_x = \mathbf{B} + \sum_{i=1}^{N-1} q_i\left(\lambda_{\mathrm{N}} - \lambda_i\right)\mathbf{d}^{(\lambda_i)} \tag{7}$$

so that the searched solutions can be obtained through integration along the characteristic curves associated to the eigenvalue λ_{N}. Therefore exact solutions of (7) are determined by solving a set of "ODEs". Moreover the initial data $\mathbf{U}(x, 0)$ associated to (7) must satisfy the $N-1$ constraint equations (6) (see [28]) which also select the class of initial value problems compatible with the reduction approach under interest. The corresponding class of exact solutions are called generalized simple waves because when in (5) and (6) the source $\mathbf{B}=0$ and $q_i=0$ they specialize to the classical simple wave solutions of a homogeneus hyperbolic model.

2.2 Double wave solutions

Here we consider the quasilinear hyperbolic system (5). Owing to the assumed hyperbolicity with respect to t, the equation

$$\det(\mathrm{A}-\lambda\,\mathrm{I})=0 \tag{8}$$

admits N real roots λ_i to which there corresponds a complete set of left and right eigenvectors $\mathbf{l}^{(\lambda_i)}$ and $\mathbf{d}^{(\lambda_i)}$ respectively

$$\mathbf{l}^{(\lambda_i)}\,(\mathrm{A}-\lambda_i\,\mathrm{I})=\mathbf{0},\quad (\mathrm{A}-\lambda_i\,\mathrm{I})\,\mathbf{d}^{(\lambda_i)}=\mathbf{0},\quad (i=1,..,N). \tag{9}$$

We set

$$\mathbf{U}=\begin{bmatrix}\mathbf{V}\\ \mathbf{W}\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}\mathbf{B}_1\\ \mathbf{B}_2\end{bmatrix},\quad \mathrm{A}=\begin{bmatrix}\mathrm{P} & \mathrm{Q}\\ \mathrm{R} & \mathrm{S}\end{bmatrix},\quad \mathbf{l}^{(\lambda)}=\begin{bmatrix}\widehat{\mathbf{l}}^{(\lambda)}, & \bar{\mathbf{l}}^{(\lambda)}\end{bmatrix} \tag{10}$$

being $\mathbf{V},\mathbf{B}_1,\widehat{\mathbf{l}}^{(\lambda)}\in R^2$, $\mathbf{W},\mathbf{B}_2,\bar{\mathbf{l}}^{(\lambda)}\in R^{N-2}$ and $\mathrm{P},\mathrm{Q},\mathrm{R},\mathrm{S}$ suitable matrix coefficients

$$\begin{aligned}&\mathrm{P}=\|P_{hk}\|,\quad \mathrm{Q}=\|Q_{hs}\|,\quad \mathrm{R}=\|R_{rk}\|,\quad \mathrm{S}=\|S_{rs}\|,\\ &h,k=1,2;\quad r,s=3,..,N.\end{aligned} \tag{11}$$

Looking for solutions of (5) under the form

$$\mathbf{U}=\mathbf{U}(\mathbf{V})=\begin{bmatrix}\mathbf{V}\\ \mathbf{W}(\mathbf{V})\end{bmatrix} \tag{12}$$

with $\mathbf{W}(\mathbf{V})$ sufficiently smooth functions of $\mathbf{V}$, we get the overdetermined system

$$\frac{\partial\mathbf{V}}{\partial t}+(\mathrm{P}+\mathrm{Q}\,\nabla\mathbf{W})\,\frac{\partial\mathbf{V}}{\partial x}=\mathbf{B}_1 \tag{13}$$

$$(\nabla\mathbf{W})\frac{\partial\mathbf{V}}{\partial t}+(\mathrm{R}+\mathrm{S}\,\nabla\mathbf{W})\,\frac{\partial\mathbf{V}}{\partial x}=\mathbf{B}_2 \tag{14}$$

where

$$\nabla\mathbf{W}=\left\|\frac{\partial W_r}{\partial V_k}\right\|,\quad r=3,...,N;\quad k=1,2.$$

The main goal of the reduction approach proposed in [22] was the characterization of exact solutions consistent with special evolution processes ruled by an auxiliary 2×2 hyperbolic system. Thus, for what concerns the hyperbolicity of the $N(N-1)/2$ reduced systems (13), obtained for each fixed choice of the field $\mathbf{V}$, it was proved the following

Proposition 1 *Let $\mathbf{U} = \mathbf{U}(\mathbf{V})$ a class of solutions to (5), then the hyperbolicity of (5) induces the hyperbolicity of at least one of the 2×2 reduced system in the new field variable $\mathbf{V}$.*

Therefore, without loss of generality, we can assume as reduced 2×2 governing system the pair of equations (13) which results to be strictly hyperbolic so that the equation

$$\det\left(\mathrm{P} + \mathrm{Q}\,\nabla\mathbf{W} - \widetilde{\lambda}\,\mathrm{I}\right) = 0 \tag{15}$$

admits two real eigenvalues $\widetilde{\lambda}_1 \neq \widetilde{\lambda}_2$ with left and right eigenvectors $\widetilde{\mathbf{l}}^{(\widetilde{\lambda}_k)}$ and $\widetilde{\mathbf{d}}^{(\widetilde{\lambda}_k)}$ $(k = 1, 2)$ such that

$$\begin{aligned}
&\widetilde{\mathbf{l}}^{(\widetilde{\lambda}_k)}\left(\mathrm{P} + \mathrm{Q}\,\nabla\mathbf{W} - \widetilde{\lambda}_k\,\mathrm{I}\right) = \mathbf{0}, \quad \left(\mathrm{P} + \mathrm{Q}\,\nabla\mathbf{W} - \widetilde{\lambda}_k\,\mathrm{I}\right)\widetilde{\mathbf{d}}^{(\widetilde{\lambda}_k)} = \mathbf{0},\\
&\widetilde{\mathbf{l}}^{(\widetilde{\lambda}_h)} \cdot \widetilde{\mathbf{d}}^{(\widetilde{\lambda}_k)} = \delta_{hk}, \quad (h, k = 1, 2)
\end{aligned}$$

where δ_{hk} is the Kronecher tensor. The remaining $N-2$ equations (14) give further restrictions on the class of solutions (12) and, owing to the hyperbolicity of the auxiliary reduced system (13), can be recast in the following form

$$\omega_{r1}\left(\widetilde{\mathbf{l}}^{(\widetilde{\lambda}_1)} \cdot \frac{\partial\mathbf{V}}{\partial x}\right) + \omega_{r2}\left(\widetilde{\mathbf{l}}^{(\widetilde{\lambda}_2)} \cdot \frac{\partial\mathbf{V}}{\partial x}\right) = B_r - \sum_{h=1}^{2}\frac{\partial W_r}{\partial V_h}B_h \quad (r = 3, \ldots, N) \tag{16}$$

with $\omega_{rh} = \omega_{rh}(\mathbf{V}, \mathbf{W}(\mathbf{V}))$ suitable functions of $\mathbf{V}$. It follows that, for a fixed $\overline{r}$, the corresponding relation (16) is a supplementary equation to be satisfied by any solution of the 2×2 auxiliary reduced system iff $\omega_{\overline{r}1} = \omega_{\overline{r}2} = B_{\overline{r}} - \displaystyle\sum_{h=1}^{2}\frac{\partial W_{\overline{r}}}{\partial V_h}B_h = 0$ whereas, according to [28], it is a involutive differential constraint associated to the hyperbolic system (13) iff $\omega_{\overline{r}1} = 0$ or alternatively $\omega_{\overline{r}2} = 0$. In such a case relation (16) reduces to

$$\widetilde{\mathbf{l}}^{(\widetilde{\lambda}_2)} \cdot \frac{\partial\mathbf{V}}{\partial x} = \frac{1}{\omega_{\overline{r}\,2}}\left(B_{\overline{r}} - \sum_{h=1}^{2}\frac{\partial W_{\overline{r}}}{\partial V_h}B_h\right) = q_2\left(\mathbf{V}\right) \tag{17}$$

or

$$\widetilde{\mathbf{l}}^{(\widetilde{\lambda}_1)} \cdot \frac{\partial\mathbf{V}}{\partial x} = \frac{1}{\omega_{\overline{r}\,1}}\left(B_{\overline{r}} - \sum_{h=1}^{2}\frac{\partial W_{\overline{r}}}{\partial V_h}B_h\right) = q_1\left(\mathbf{V}\right). \tag{18}$$

Of course, according to the differential constraint method [7, 8], the relation (17) or (18) selects the class of initial value problem $\mathbf{V}(x, 0) = \mathbf{V}_0(x)$ compatible with the procedure under interest. In fact, it results

$$\widetilde{\mathbf{l}}^{(\widetilde{\lambda})}(\mathbf{V}_0(x)) \cdot \frac{\partial\mathbf{V}_0}{\partial x} = q(\mathbf{V}_0(x)). \tag{19}$$

In the following we will consider the case where one of the equations (16) is a differential constraint of (13) while the remaining $N-3$ relations are identically satisfied for all solutions of (13).

As far as the characteristic speeds $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ is concerned, it can be proved the following

Proposition 2 *Let $\widetilde{\lambda}(\mathbf{V})$ be a characteristic velocity associated to (13), and*

$$H_j(\widetilde{\lambda}) \neq 0 \quad \forall \widetilde{\lambda} \quad (3 \leq j \leq N) \tag{20}$$

where $H_j(\widetilde{\lambda})$ is the determinant of the matrix of order $N-2$ obtained from $\left(\mathrm{S} - \nabla \mathbf{W}\, \mathrm{Q} - \widetilde{\lambda}\, \mathrm{I}\right)$ when the $j-$th row is replaced by $\widetilde{\mathrm{l}}^{(\widetilde{\lambda})}\, \mathrm{Q}$. Then, under assumption

$$\omega_{r1} = \omega_{r2} = B_r - \sum_{h=1}^{2} \frac{\partial W_r}{\partial V_h} B_h = 0 \quad (r = 3, \ldots, N, \quad r \neq j) \tag{21}$$

the $j-th$ condition (16) reduces to a first order differential constraint associated to $\widetilde{\lambda}_k$ iff $\widetilde{\lambda}_k$ is not a characteristic velocity of the hyperbolic system (5) whereas the remaing eigenvalue belongs to the spectrum of λ's.

In particular, the following relations hold [22]

$$\det\left(\mathrm{A} - \widetilde{\lambda}_1\, \mathrm{I}\right) = \left(\widetilde{\lambda}_1 - \widetilde{\lambda}_2\right) \omega_{j1} H_j(\widetilde{\lambda}_1) \tag{22}$$

$$\det\left(\mathrm{A} - \widetilde{\lambda}_2\, \mathrm{I}\right) = \left(\widetilde{\lambda}_2 - \widetilde{\lambda}_1\right) \omega_{j2} H_j(\widetilde{\lambda}_2). \tag{23}$$

It turns out that in the case

$$H_j(\widetilde{\lambda}_1) = H_j(\widetilde{\lambda}_2) = 0 \tag{24}$$

both characteristic velocities of the hyperbolic reduced system (13) belong to the spectrum of $\lambda's$ and the $j-th$ condition (16) may reduce to a first order differential constraint associated both to $\widetilde{\lambda}_1$ or $\widetilde{\lambda}_2$.

Remark 1. The approach here sketched leads to determine exact solutions of (5) under the form (12) which belongs to the class of partially invariant (double wave) solutions. Of course in the case of a system involving two independent variables all the solutions are double waves [1]. Therefore, the reduction procedure here considered permits to characterize particular double wave solutions of the governing hyperbolic systems (5).

Remark 2. The key idea for determining particular double wave solutions of (5) by using the procedure developed in [22] is to require that one of the relations (16) specializes to a differential constraint of the reduced 2×2 system (13) while the remaining $N-3$ equations result to be identically satisfied for all solutions of (13) (i. e. they become conservation laws for the reduced 2×2 model). Hence, according to the general theory illustrated in section 2.1, classes of exact double wave solutions are determined by solving a 2×2 "ODE" system. Furthermore such solutions are characterized in terms of one arbitrary function so that classes on initial value problems can be solved. Therefore, the approach under interest here is not based on the study of the general compatibility between relations (16) with the reduced model (13) but on the requirement that equations (16) specialize to conservation laws and one differential constraint.

Remark 3. The method of differential constraints requires that the compatibility conditions between (5) and (6) are satisfied for all solutions of the resulting overdetermined system (5), (6) (i. e. $\forall\, \mathbf{U}_x$). If we do not invoke such a requirement (i. e. we look for particular solutions of (5) and (6) some of the field derivatives can be calculated from the consistency conditions so that new differential constraints must be appended to (5) and new compatibility conditions must be required. In the case here considered of a 2×2 system with one differential constraint, if we do not require that the compatibility conditions are satisfied $\forall\ \mathbf{U}_x$, we obtain a 2×2 model with two differential constraints. Such a possibility leads to determine exact solution which are Lie group invariants [21] and they are obtained in terms of arbitrary constants which usually cannot be useful for solving problems of interest in nonlinear wave propagation like Riemann problems or nonlinear wave interactions. Therefore in the analysis developed in section 3 we do not consider such a possibility.

3 Double waves to gas dynamics equations

Here, owing to the results sketched in the previous section, we look for double wave reduction of the model (1)-(3) in two different cases. Owing to the invertibility assumption, the wave parameters can be interchanged so that, first we'll investigate the possible reduction to a homogeneous 2×2 hyperbolic model along with the choice (ρ, s), next a reduction to a non-homogeneous system with (ρ, u) as dependent variables is considered.

I) Firstly, we consider double waves of the form

$$u = U(\rho, s) \tag{25}$$

where the function $U(\rho, s)$ will be determined in order that the reduction process be consistent. Substituting the ansatz (25) in the equations (1) and (3) we obtain

$$\frac{\partial \rho}{\partial t} + \left(U + \rho \frac{\partial U}{\partial \rho} \right) \frac{\partial \rho}{\partial x} + \rho \frac{\partial U}{\partial s} \frac{\partial s}{\partial x} = 0 \tag{26}$$

$$\frac{\partial s}{\partial t} + U \frac{\partial s}{\partial x} = 0 \tag{27}$$

whereas, taking (26) into account, equation (2) yields

$$\left(\frac{\partial p}{\partial \rho} - \rho^2 \left(\frac{\partial U}{\partial \rho} \right)^2 \right) \frac{\partial \rho}{\partial x} + \left(\frac{\partial p}{\partial s} - \rho^2 \frac{\partial U}{\partial \rho} \frac{\partial U}{\partial s} \right) \frac{\partial s}{\partial x} = \rho f(\rho, U) \tag{28}$$

According to Proposition 1, the hyperbolicity of the reduced auxiliary 2×2 system (26) is ensured by the hyperbolic character of the original model (1). In particular, the associated characteristic speeds along with the corresponding left and right eigenvectors of the matrix coefficients are

$$\begin{aligned} &\widetilde{\lambda}_1 = U, && \widetilde{\mathbf{l}}^{(\widetilde{\lambda}_1)} = \left[\begin{array}{cc} 0, & 1 \end{array} \right], && \widetilde{\mathbf{d}}^{(\widetilde{\lambda}_1)} = \left[\begin{array}{cc} \frac{\partial U}{\partial s}, & -\frac{\partial U}{\partial \rho} \end{array} \right]^T \\ &\widetilde{\lambda}_2 = U + \rho \frac{\partial U}{\partial \rho}, && \widetilde{\mathbf{l}}^{(\widetilde{\lambda}_2)} = \left[\begin{array}{cc} \frac{\partial U}{\partial \rho}, & \frac{\partial U}{\partial s} \end{array} \right], && \widetilde{\mathbf{d}}^{(\widetilde{\lambda}_2)} = \left[\begin{array}{cc} 1, & 0 \end{array} \right]^T, \end{aligned} \tag{29}$$

with $\frac{\partial U}{\partial \rho} \neq 0$. In the present case we have

$$\widetilde{\lambda}_1 = \lambda_2 \tag{30}$$

and, as far as the quantity $H_3(\widetilde{\lambda})$ involved in (20) is concerned

$$H_3(\widetilde{\lambda}_1) = 0, \qquad H_3(\widetilde{\lambda}_2) = \rho \tfrac{\partial U}{\partial \rho} \neq 0. \tag{31}$$

Owing to (31), the hypotesis (20) of the proposition 2 is not fulfilled. Therefore, in the present case, we have to require by direct inspection that relation (28) specializes to a differential constraint of (26), (27) associated to $\widetilde{\lambda}_1$ or $\widetilde{\lambda}_2$ which adopt, respectively, the form

$$\frac{\partial s}{\partial x} = q_1(\rho, s) \tag{i}$$

or

$$\frac{\partial U}{\partial \rho}\frac{\partial \rho}{\partial x} + \frac{\partial U}{\partial s}\frac{\partial s}{\partial x} = q_2(\rho, s) \tag{ii}$$

with $q_1(\rho, s)$ and $q_2(\rho, s)$ unknown functions to be determined. Hereafter, the two possibilities (i) or (ii) will be considered.

Case (i)

Firstly we consider double wave solutions satysfing the differential constraint (i), namely we require the condition (28) reduces to (i) so that, under assumption $\frac{\partial p}{\partial s} - \rho^2 \frac{\partial U}{\partial \rho}\frac{\partial U}{\partial s} \neq 0$, we get

$$\rho^2 \left(\frac{\partial U}{\partial \rho}\right)^2 - c^2 = 0 \tag{32}$$

$$f(\rho, U(\rho, s)) = \left(\frac{1}{\rho}\frac{\partial p}{\partial s} - \rho \frac{\partial U}{\partial \rho}\frac{\partial U}{\partial s}\right) q_1(\rho, s). \tag{33}$$

The compatibility between the equations (26), (27) and the constraint (i) gives rise to

$$\frac{\partial U}{\partial \rho}\left(\rho \frac{\partial q_1}{\partial \rho} - q_1\right)\left(\frac{\partial U}{\partial \rho}\frac{\partial \rho}{\partial x} + q_1 \frac{\partial U}{\partial s}\right) = 0 \tag{34}$$

which, bearing in mind that $\frac{\partial U}{\partial \rho} \neq 0$, in line with differential constraint theory, implies

$$\rho \frac{\partial q_1}{\partial \rho} - q_1 = 0 \tag{35}$$

whose general solution is

$$q_1(\rho, s) = \rho \Omega(s) \tag{36}$$

and $\Omega(s)$ arbitrary function of its argument. Therefore, from (32), (33) and (36), we have

$$U(\rho, s) = \pm \int \frac{c}{\rho} \, \mathrm{d}\, \rho + h(s) \tag{37}$$

$$f(\rho, U) = \left(\frac{\partial p}{\partial s} - \rho^2 \frac{\partial U}{\partial \rho}\frac{\partial U}{\partial s}\right) \Omega(s) \tag{38}$$

where h denotes an arbitrary function and, according to the choice of the sign in $(37)_1$, we have $\widetilde{\lambda}_2 = \lambda_1$ or $\widetilde{\lambda}_2 = \lambda_3$ as already observed. Once the pressure $p(\rho, s)$ is given, the relations (37) determine $u = U(\rho, s)$ and it also induces restrictions on the specific body force $f(\rho, u)$.

As an example, hereafter we assume the following pressure law

$$p(\rho, s) = \Pi(s)\rho^{\gamma} \tag{39}$$

with $\gamma \neq 0, 1$ a real number and $\Pi(s)$ a function of the entropy s such that $\gamma\Pi(s) > 0 \quad \forall s$. For a polytropic gas $\gamma = \frac{C_p}{C_v}$ and $\Pi(s) = e^{\frac{s-s_0}{C_v}}$ being C_p, C_v the dimensionless specific heat capacities at constant pressure and volume respectively and s_0 a constant. Owing to (39), from relation (37) we obtain

$$u = U(\rho, s) = \pm \frac{2\sqrt{\gamma\Pi(s)}}{\gamma - 1}\rho^{\frac{\gamma-1}{2}} + h(s) \tag{40}$$

whereas, from (38), the production term f, evaluated at the class of solutions (40), becomes

$$f(\rho, U(\rho, s)) = \Omega(s)\left(\frac{\Pi'(s)\,\rho^{\gamma}}{\gamma - 1} \mp h'(s)\sqrt{\gamma\Pi(s)}\,\rho^{\frac{\gamma+1}{2}}\right) \tag{41}$$

and the prime denotes the derivative with respect to the indicated argument. As far as the functional form of the body force $f(\rho, u)$ is concerned, it will be obtained from (41) after inserting $s = s(\rho, u)$ determined through (40). In particular, if h_0 and $\Omega_0 < 0$ denote two real constants, a friction-like body force is obtained with the choice

$$\begin{cases} h(s) = h_0 \\ \Omega(s) = \Omega_0 \left(\frac{2\sqrt{\gamma\Pi(s)}}{\gamma-1}\right)^{\frac{2\gamma}{\gamma-1}} \frac{\gamma-1}{\Pi'(s)} \end{cases} \quad \Rightarrow \quad f(u) = \Omega_0 (u - h_0)^{\frac{2\gamma}{\gamma-1}}. \tag{42}$$

Finally, exact solutions of the original set of equations (1)-(3) endowed with (41), are obtained through (40) by solving the reduced 2×2 homogeneous hyperbolic model (26), (27). To this end we introduce the Riemann invariants [26, 27]

$$r^1 = \widetilde{\lambda}_1 = U(\rho, s), \qquad r^2 = s \tag{43}$$

and, by using the variable transformations (43), we recast the system (26), (27) under the form

$$\frac{\partial r^1}{\partial t} + \left(\frac{\gamma + 1}{2} r^1 + \frac{1 - \gamma}{2} h(r^2)\right)\frac{\partial r^1}{\partial x} = 0 \tag{44}$$

$$\frac{\partial r^2}{\partial t} + r^1 \frac{\partial r^2}{\partial x} = 0. \tag{45}$$

Thus, via the well known hodograph transformation

$$x = x(r^1, r^2), \qquad t = t(r^1, r^2), \qquad \left|\frac{\partial(x, t)}{\partial(r^1, r^2)}\right| \neq 0 \tag{46}$$

the system (44), (45) and in turn (26), (27), can be reduced to a pair of linear equations whose integration yields

$$x(r^1, r^2) = r^1 \frac{\partial \Lambda}{\partial r^1} - \Lambda + r^1 M(r^1) - m(r^1) \tag{47}$$

$$t(r^1, r^2) = \frac{\partial \Lambda}{\partial r^1} + M(r^1), \tag{48}$$

with

$$m(r^1) = \int M(r^1)\, \mathrm{d}\, r^1 \tag{49}$$

$$\Lambda(r^1, r^2) = \frac{(1-\gamma)}{2} \int \left(r^1 - h(r^2)\right)^{\frac{2}{1-\gamma}} L(r^2)\, \mathrm{d}\, r^2. \tag{50}$$

and $M(r^1)$, $L(r^2)$ arbitrary functions of their arguments which, according to (19), will be determined once initial or boundary data, selected by the constraint equation (i) endowed with (36), are given. We'll give further details in Section 4.2 where initial value problems will be solved in order to investigate nonlinear simple wave interactions.

Case (ii)

Next, in order that condition (28) reduces to (ii) we have to require

$$\frac{\partial p}{\partial \rho}\frac{\partial U}{\partial s} - \frac{\partial p}{\partial s}\frac{\partial U}{\partial \rho} = 0 \quad \Rightarrow \quad U = U(p). \tag{51}$$

Therefore, the source term f must adopt the form

$$f(\rho, U(p)) = \frac{q_2(\rho, s)}{\rho U'(p)} \left(1 - \rho^2 \frac{\partial p}{\partial \rho} (U'(p))^2\right) \tag{52}$$

so that the constraint (ii) specializes to

$$U'(p) \frac{\partial p}{\partial x} = q_2(\rho, s). \tag{53}$$

By requiring the compatibility conditions between the reduced 2×2 system (26), (27) and the constraint (53) we obtain

$$\begin{aligned} &\left(\rho U'(p) \left(\frac{\partial p}{\partial \rho}\frac{\partial q_2}{\partial s} - \frac{\partial p}{\partial s}\frac{\partial q_2}{\partial \rho}\right) + q_2 \left(\frac{\partial^2(\rho U)}{\partial s \partial \rho} - \frac{\frac{\partial p}{\partial s}}{\frac{\partial p}{\partial \rho}} \frac{\partial^2(\rho U)}{\partial \rho^2}\right)\right) \frac{\partial s}{\partial x} + \\ &+ \frac{q_2^2}{U'(p)\frac{\partial p}{\partial \rho}} \frac{\partial^2(\rho U)}{\partial \rho^2} = 0 \end{aligned} \tag{54}$$

and in turn

$$\frac{\partial^2(\rho U)}{\partial \rho^2} = 0 \tag{55}$$

$$\rho U'(p) \left(\frac{\partial p}{\partial \rho}\frac{\partial q_2}{\partial s} - \frac{\partial p}{\partial s}\frac{\partial q_2}{\partial \rho}\right) + q_2 \frac{\partial^2(\rho U)}{\partial \rho \partial s} = 0. \tag{56}$$

After some algebra, from (55) and (56) it follows

$$u = U(p) = -\frac{b(s)}{\rho} + k(s) \tag{57}$$

$$q_2(\rho, s) = \frac{\rho}{b(s)}\Omega(p) \tag{58}$$

with $k(s)$, $b(s)$ and $\Omega(p)$ arbitrary functions and, in turn, from (52) we get

$$f(\rho, U) = \frac{1 - b(s)U'(p)}{b(s)U'(p)}\Omega(p). \tag{59}$$

We notice that from (57) we deduce $s = s(\rho, u)$ and, once the arbitrary function $U(p)$ is given, $p = U^{-1}(u)$ so that the functional form of the body force $f(\rho, u)$ is obtained. In the particular case $U(p) = p$ and $b(s) = b_0 \neq 1$ we have

$$f(u) = \frac{1 - b_0}{b_0}\Omega(u). \tag{60}$$

Finally, as it is straightforward to ascertain, the reduced system (26), (27) is *completely exceptional* (CEX) [24] being both characteristic speeds $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ linearly degenerate [31]

$$\widetilde{\lambda}_1 = k(s) - \frac{1}{\rho}b(s), \qquad \widetilde{\lambda}_2 = k(s) \tag{61}$$

$$\nabla\widetilde{\lambda}_1 \cdot \widetilde{\mathbf{d}}^{(\widetilde{\lambda}_1)} = \nabla\widetilde{\lambda}_2 \cdot \widetilde{\mathbf{d}}^{(\widetilde{\lambda}_2)} = 0. \tag{62}$$

For the class of 2×2 homogeneous CEX system a wide literature is avaible and several problems concerning wave interaction phenomena have been solved, so that we do not go further with investigation of this case (see [20], [34]).

II) Here we look for double wave solution of system (1)-(3) in the form

$$s = F(\rho, u). \tag{63}$$

Substituting the ansatz (63) in (1) and (2), after few calculations, we have

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0 \tag{64}$$

$$\rho\frac{\partial u}{\partial t} + \left(\frac{\partial p}{\partial \rho} + \frac{\partial p}{\partial s}\frac{\partial F}{\partial \rho}\right)\frac{\partial \rho}{\partial x} + \left(\frac{\partial p}{\partial s}\frac{\partial F}{\partial u} + \rho u\right)\frac{\partial u}{\partial x} = \rho f(\rho, u) \tag{65}$$

while from (3) we get

$$\left(\frac{\partial p}{\partial \rho} + \frac{\partial p}{\partial s}\frac{\partial F}{\partial \rho}\right)\frac{\partial F}{\partial u}\frac{\partial \rho}{\partial x} + \left(\frac{\partial p}{\partial s}\left(\frac{\partial F}{\partial u}\right)^2 + \rho^2\frac{\partial F}{\partial \rho}\right)\frac{\partial u}{\partial x} = \rho f(\rho, u)\frac{\partial F}{\partial u}. \tag{66}$$

The characteristic speeds $\widetilde{\lambda}$ of the reduced system (64), (65) satisfy the following equation

$$\rho(u - \widetilde{\lambda})^2 + \frac{\partial p}{\partial s}\frac{\partial F}{\partial u}(u - \widetilde{\lambda}) - \rho\left(\frac{\partial p}{\partial \rho} + \frac{\partial p}{\partial s}\frac{\partial F}{\partial \rho}\right) = 0 \tag{67}$$

while $H_3(\widetilde{\lambda})$ assumes the form

$$H_3(\widetilde{\lambda}) = \frac{1}{\rho}\frac{\partial p}{\partial s}\, l_2^{(\widetilde{\lambda})}. \tag{68}$$

Then equation (66), under assuption $H_3(\widetilde{\lambda}) \neq 0$ reduces to a first order differential constraint associated to (64), (65) iff only one of the two characteristic velocity $\widetilde{\lambda}_1$ or $\widetilde{\lambda}_2$ belongs to the spectrum of $\lambda' s$. Therefore we next consider the following two cases

$$\widetilde{\lambda}_1 = u \pm c \tag{a}$$

or

$$\widetilde{\lambda}_1 = u. \tag{b}$$

Case (a)

Here we assume

$$c\frac{\partial F}{\partial u} \pm \rho\frac{\partial F}{\partial \rho} = 0, \tag{69}$$

so that equation (67) admits the following solutions

$$\widetilde{\lambda}_1 = u \pm c, \qquad \widetilde{\lambda}_2 = u \mp c + \frac{1}{\rho}\frac{\partial p}{\partial s}\frac{\partial F}{\partial u} \tag{70}$$

with associated left and right eingevectors

$$\begin{aligned} \widetilde{\mathbf{l}}^{(\widetilde{\lambda}_1)} &= \left[\begin{array}{cc} \pm c - \frac{1}{\rho}\frac{\partial p}{\partial s}\frac{\partial F}{\partial u}, & \rho \end{array}\right], & \widetilde{\mathbf{d}}^{(\widetilde{\lambda}_1)} &= \left[\begin{array}{cc} \rho, & \pm c \end{array}\right]^T \\ \widetilde{\mathbf{l}}^{(\widetilde{\lambda}_2)} &= \left[\begin{array}{cc} \mp c, & \rho \end{array}\right], & \widetilde{\mathbf{d}}^{(\widetilde{\lambda}_2)} &= \left[\begin{array}{cc} \rho, & \frac{1}{\rho}\frac{\partial p}{\partial s}\frac{\partial F}{\partial u} \mp c \end{array}\right]^T, \end{aligned} \tag{71}$$

such that

$$H_3(\widetilde{\lambda}_1) = H_3(\widetilde{\lambda}_2) = \frac{\partial p}{\partial s} \neq 0 \tag{72}$$

and, in turn,

$$F(\rho, u) = F(\eta), \qquad \eta = u \mp \int \frac{c}{\rho}\,\mathrm{d}\,\rho. \tag{73}$$

Therefore relation (66) can be a differential constraint of (64), (65) and it assumes the form

$$c\frac{\partial \rho}{\partial x} \mp \rho\frac{\partial u}{\partial x} = q(\rho, u) \tag{74}$$

where, by assuming $\rho\, c \mp \dfrac{\partial p}{\partial s}\dfrac{\partial F}{\partial u} \neq 0$, the function q is related to the source term $f(\rho, u)$ through

$$f(\rho, u) = \left(\frac{c}{\rho} \mp \frac{1}{\rho^2}\frac{\partial p}{\partial s}\frac{\partial F}{\partial u}\right) q(\rho, u). \tag{75}$$

The compatibility between (64), (65) and (74) gives

$$\left(\left(2\pm\frac{\partial c}{\partial s}\frac{\partial F}{\partial u}\right)q\mp c\frac{\partial q}{\partial u}-\rho\frac{\partial q}{\partial\rho}\right)\left(c\frac{\partial\rho}{\partial x}-q\right)=0 \tag{76}$$

which, in line with differential constraint theory, gives

$$\pm c\frac{\partial q}{\partial u}+\rho\frac{\partial q}{\partial\rho}=\left(2\pm\frac{\partial c}{\partial s}\frac{\partial F}{\partial u}\right)q, \tag{77}$$

so that we have

$$q(\rho,u)=\Omega(\eta)\exp\left(\int\frac{1}{\rho}\left(2\pm\frac{\partial c}{\partial s}\frac{\partial F}{\partial u}\right)\mathrm{d}\,\rho\right) \tag{78}$$

with $\Omega(\eta)$ an arbitrary function. Once the pressure law $p=p(\rho,s)$ is given, the body force f is obtained from (75) whereas the class of solutions $s=F(\rho,u)$ is fully determined by integrating the following system

$$\frac{\partial\rho}{\partial t}+\widetilde{\lambda}_1\frac{\partial\rho}{\partial x}=\pm q(\rho,u) \tag{79}$$

$$\frac{\partial u}{\partial t}+\widetilde{\lambda}_1\frac{\partial u}{\partial x}=0 \tag{80}$$

with initial data $\rho(x,0)=\rho_0(x)$ and $u(x,0)=u_0(x)$ satisfying the differential constraint (74) which specializes to

$$c(\rho_0(x),u_0(x))\frac{\mathrm{d}\,\rho_0(x)}{\mathrm{d}\,x}\mp\rho_0(x)\frac{\mathrm{d}\,u_0(x)}{\mathrm{d}\,x}=q(\rho_0(x),u_0(x)). \tag{81}$$

Case (b)

Let us assume

$$\frac{\partial p}{\partial\rho}+\frac{\partial p}{\partial s}\frac{\partial F}{\partial\rho}=0 \tag{82}$$

so that, from (67), we obtain

$$\widetilde{\lambda}_1=u,\qquad\widetilde{\lambda}_2=u+\frac{1}{\rho}\frac{\partial p}{\partial s}\frac{\partial F}{\partial u}, \tag{83}$$

with associated left and right eigenvectors

$$\begin{aligned}
\widetilde{\mathbf{l}}^{(\widetilde{\lambda}_1)}&=\left[\begin{array}{cc}-\frac{1}{\rho}\frac{\partial p}{\partial s}\frac{\partial F}{\partial u}, & \rho\end{array}\right], & \widetilde{\mathbf{d}}^{(\widetilde{\lambda}_1)}&=\left[\begin{array}{cc}0, & 1\end{array}\right]^T\\
\widetilde{\mathbf{l}}^{(\widetilde{\lambda}_2)}&=\left[\begin{array}{cc}1, & 0\end{array}\right], & \widetilde{\mathbf{d}}^{(\widetilde{\lambda}_2)}&=\left[\begin{array}{cc}\rho, & \frac{1}{\rho}\frac{\partial p}{\partial s}\frac{\partial F}{\partial u}\end{array}\right]^T
\end{aligned} \tag{84}$$

such that

$$H_3(\widetilde{\lambda}_1)=\frac{\partial p}{\partial s}\neq 0,\qquad H_3(\widetilde{\lambda}_2)=0. \tag{85}$$

After (85), condition (23) requires

$$\frac{1}{\rho}\frac{\partial p}{\partial s}\frac{\partial F}{\partial u} = \mp c, \tag{86}$$

so that (66) reduces to

$$\frac{\partial F}{\partial u} f(\rho, u) = 0. \tag{87}$$

The present case will not be further investigated because from (87) we get $f(\rho, u) = 0$ (homogeneous case) or, taking (86) into account, $c = 0$.

Remark 4. The approach proposed in [22] is based on the requirement that relation (28) and (66) are first order differential constraints of the corresponding 2×2 reduced systems. Therefore, taking into account the previous remarks, in section 3 we required that (28) and (66) assume the form (6). Of course, more generally, the compatibility of relations (28) or (66) with the 2×2 subsystems under interest can be studied . Such an approach is different from the one developed in [22] which is based on the method of differential constraints and, usually, it leads to results which are not useful for determining double wave solutions of interest in nonlinear wave propagations. In fact in some cases the analysis cannot be fully developed analytically, in others we are led to append to the corresponding 2×2 subsystem two first order differential constraints so that the resulting exact solutions are not determined in terms of arbitrary functions as in [22] but they are parameterized by arbitrary constants. Nevertheless a sketch of this different strategy is given in the appendix.

4 Nonlinear wave interactions

Here, by using the results obtained in the previous sections, our main aim is to study two problems of relevant interest within the framework of nonlinear wave propagation: Riemann problems and nonlinear simple wave interactions.

4.1 Riemann problem

In the case of system of conservation laws a Riemann Problem (RP), under the assumption of not large initial jumps, admits an unique solution in terms of constant states separated by rarefaction waves, shock waves and/or contact discontinuities [30]-[32]. However, a rarefaction wave is characterized by a simple wave solution which, in general, is not admitted by non homogeneous systems (balance laws). Furthermore, to study generalized Riemann problems (GRP) which are characterized by non constant discontinuous initial states, is a very hard task both in the homogenoeus as well as in the non homogeneous case. In fact for solving a GRP it is necessary to determine the exact solution of the governing equations for general initial data. Therefore only few cases of exact solution to RP and GRP for balance laws are known in the litterature. Within such a framework, a combined use of the approach developed first in [11] and later in [17, 18] with the reduction procedure developed in this paper will permit us to solve a RP as well as a GRP for the non homogeneous system (1)-(3).

To this end we will consider the results obtained in the case (a) of Section 3. We develop our analysis for $\tilde{\lambda}_1 = u + c$. Of course a similar procedure can be carried on in the remaing case $\tilde{\lambda}_1 = u - c$. Therefore from (73) and (78) we get

$$s = F(\eta), \qquad \eta = u - \int \frac{c\,(\rho, F(\eta))}{\rho} d\rho \tag{88}$$

$$q = \rho^2 \Omega(\eta) g(\rho, \eta), \qquad g = \exp\left(\int \left(\frac{1}{\rho} \frac{\partial c}{\partial s} \frac{\partial F}{\partial u} \right) d\rho \right) \tag{89}$$

In view of integrating equations (79), (80) and (81) it is more convenient to make use of the change o variable $u \leftrightarrow \eta$ so that the system (79), (80) assumes the form

$$\frac{\partial \rho}{\partial t} + (\eta + \sigma(\rho, \eta)) \frac{\partial \rho}{\partial x} = \rho^2 \Omega(\eta) g(\rho, \eta) \tag{90}$$

$$\frac{\partial \eta}{\partial t} + (\eta + \sigma(\rho, \eta)) \frac{\partial \eta}{\partial x} = -\frac{\rho c\,(\rho, \eta)\, \Omega(\eta) g(\rho, \eta)}{1 + F'(\eta) \int \left(\frac{1}{\rho} \frac{\partial c}{\partial s} \right) d\rho} \tag{91}$$

where

$$\sigma(\rho, \eta) = c\,(\rho, F(\eta)) + \int \frac{c}{\rho} d\rho \tag{92}$$

while equation (81) specializes to

$$\frac{\partial \eta}{\partial x} = -\frac{\rho \Omega(\eta) g(\rho, \eta)}{1 + F'(\eta) \int \left(\frac{1}{\rho} \frac{\partial c}{\partial s} \right) d\rho}. \tag{93}$$

First we consider the Riemann problem

$$\rho(x, 0) = \begin{cases} \rho_L & \text{for } x < 0 \\ \rho_R & \text{for } x > 0 \end{cases} ; \qquad \eta(x, 0) = \begin{cases} \eta_L & \text{for } x < 0 \\ \eta_R & \text{for } x > 0 \end{cases} \tag{94}$$

where ρ_L, ρ_R, η_L and η_R are constant equilibrium states of (90), (91) and (93) so that

$$\Omega(\eta_L) = \Omega(\eta_R) = 0. \tag{95}$$

By using the method of characteristics, owing to (95), integration of (90), (91) and (93) for $x < 0$ and $x > 0$ subjected to (94) leads, respectively, to

$$\begin{cases} \rho = \rho_L \\ \\ \eta = \eta_L \end{cases} \qquad \text{for } x < (\eta_L + \sigma_L(\rho_L, \eta_L))\, t \tag{96}$$

and

$$\begin{cases} \rho = \rho_R \\ \\ \eta = \eta_R \end{cases} \qquad \text{for } x > (\eta_R + \sigma_R(\rho_R, \eta_R))\, t \tag{97}$$

Since we are looking for a smooth solution connecting the left state (96) to the right state (97), the next step is to solve equations (90), (91) and (93) under the conditions

$$x(0) = 0, \quad \rho = \widehat{\rho}(a), \quad \eta = \widehat{\eta}(a) \tag{98}$$

where $a \in [0, 1]$ is a real parameter characterizing the characteristics of the fun starting from the origin in the (x, t) half- plane ($t \geq 0$). Moreover we require

$$\widehat{\rho}(0) = \rho_L; \quad \widehat{\eta}(0) = \eta_L; \quad \widehat{\rho}(1) = \rho_R; \quad \widehat{\eta}(1) = \eta_R. \tag{99}$$

In [17] the authors proved that the solution of an initial data like (98) reduces the differential constraint under interest in a homogeneous form. Therefore, in the present case from (93) we get

$$\frac{d\widehat{\eta}}{da} = 0 \tag{100}$$

so that, owing to conditions (99), we obtain

$$\widehat{\eta} = \eta_L = \eta_R \tag{101}$$

and in turn, taking (95) and (101) into account, integration of (90), (91) subject to (98) leads to

$$\begin{cases} \rho = \widehat{\rho}(a) \\ \eta = \widehat{\eta} \\ x = \left(\widehat{\eta} + \sigma\left(\widehat{\rho}(a), \widehat{\eta}\right)\right) t \end{cases} \tag{102}$$

Of course the non constant central state characterized by (102) is defined in the domain

$$\left(\eta_L + \sigma_L(\rho_L, \eta_L)\right) t \leq x \leq \left(\eta_R + \sigma_R(\rho_R, \eta_R)\right) t \tag{103}$$

and it connects smootly the left state (96) with the right state (97) if the following condition holds

$$\sigma_L(\rho_l, \eta_L) < \sigma_R(\rho_R, \eta_R). \tag{104}$$

Therefore relations (96), (97) and (102) along with the conditions (101) and (104) give the solution of the Riemann problem (94) for the system (90), (91) in terms of constant states separated by a generalized simple wave. Consequently, once the constitutive law $p(\rho, s)$ is given, then from (88), (96), (97) and (102) the solution of a Riemann problem for the full system (1)-(3) can be obtained. In such a case, taking (101) into account, the initial data must satisfy the condition (104) along with

$$u_L - \left(\int \frac{c}{\rho} d\rho\right)_L = u_R - \left(\int \frac{c}{\rho} d\rho\right)_R, \qquad s_L(F(\eta_L)) = s_R(F(\eta_R)). \tag{105}$$

We notice that, because of conditions (105), the solution of the RP under interest is characterized by an isoentropic flow with initial discontinuities for the mass density and velocity as it happens for the 2×2 isoentropic homogeneous fluid dynamics model. Therefore in order to study a more interesting case as well as to show the flexibility of the approach here developed, in the following we are going to solve the generalized Riemann problem

$$\rho(x, 0) = \begin{cases} \rho_l(x) & \text{for } x < 0 \\ \rho_r(x) & \text{for } x > 0 \end{cases}; \qquad u(x, 0) = \begin{cases} u_l(x) & \text{for } x < 0 \\ u_r(x) & \text{for } x > 0 \end{cases} \tag{106}$$

where ρ_l, ρ_r, u_l, and u_r are smooth functions. Moreover we set

$$\lim_{x\to 0^-}\left(\rho_l(x)\right)=\rho_L, \qquad \lim_{x\to 0^+}\left(\rho_r(x)\right)=\rho_R \tag{107}$$

$$\lim_{x\to 0^-}\left(u_l(x)\right)=u_L, \qquad \lim_{x\to 0^+}\left(u_r(x)\right)=u_R \tag{108}$$

with $\rho_L \neq \rho_R$ and $u_L \neq u_R$. As an example, we consider a polytropic real gas characterized by the pressure law (39) with $\Pi = e^{\frac{s-s_0}{C_v}}$. Furthermore in $(88)_1$ we choose

$$F(\eta)=C_v\ln\left(\frac{\eta^2}{\gamma r_0}\right)+s_0 \tag{109}$$

where r_0 is an arbitrary constant which, for further convenience, we assume to be positive. Therefore, from (88) and (89) we get

$$s=s_0+C_v\ln\left(\frac{\eta^2}{\gamma r_0}\right), \qquad \eta=\frac{r_0 u}{r_0+\frac{2}{\gamma-1}\rho^{\frac{\gamma-1}{2}}} \tag{110}$$

$$q=\rho^2\Omega(\eta)\left(1+\frac{2}{r_0\left(\gamma-1\right)}\rho^{\frac{\gamma-1}{2}}\right) \tag{111}$$

Finally for simplicity we choose $\Omega(\eta)=k_0\eta$, where k_0 is an arbitrary constant. Of course different solutions of (106) can be obtained by choosing different forms of $\Omega(\eta)$. Therefore, in the present case, integration of (79), (80) and (81) gives

$$\rho=\frac{\rho_0(z)}{1-k_0 t u_0(z)\rho_0(z)}, \qquad u=u_0(z) \tag{112}$$

$$\frac{\rho_0'(z)}{r_0+\rho_0}-\frac{u_0'(z)}{u_0}=k_0\rho_0 \tag{113}$$

where $\rho(x,0)=\rho_0(x)$, $u(x,0)=u_0(x)$ and the variable z is defined by solving the equation

$$\frac{dx}{dt}=u\left(\frac{r_0+\frac{\gamma+1}{\gamma-1}\rho^{\frac{\gamma-1}{2}}}{r_0+\frac{2}{\gamma-1}\rho^{\frac{\gamma-1}{2}}}\right) \tag{114}$$

Once the gas index γ is assigned, by integrating (114) the characteristic curves associated to the solution (112) will be obtained. In the case of $\gamma=3$, from (114) we have

$$x=u_0(z)t-\frac{1}{r_0k_0}\ln\left(\frac{r_0+\rho_0(z)-r_0k_0tu_0(z)\rho_0(z)}{r_0+\rho_0(z)}\right)+z \tag{115}$$

In passing we notice that (114) can be solved also in the case of monoatomic gas $\left(\gamma=\frac{5}{3}\right)$ and for a diatomic gas $\left(\gamma=\frac{7}{5}\right)$. In such a cases the corresponding solution is cumbersome and therefore hereafter we point out our attention to the case $\gamma=3$.

The initial data (106) must satisfy the differential constraint (113), so that we get

$$u_r=\frac{u_R\left(r_0+\rho_r(x)\right)}{r_0+\rho_R}e^{-k_0\int_0^x\rho_r(x)dx}, \quad u_l=\frac{u_L\left(r_0+\rho_l(x)\right)}{r_0+\rho_L}e^{-k_0\int_0^x\rho_l(x)dx} \tag{116}$$

Therefore, taking (112), (113) and (115) into account, by developing the same procedure considered in this section for the RP (94), the following solution of the GRP (106) is obtained:
Left state

$$\rho = \frac{\rho_l(z)}{1 - k_0 t u_l(z)\rho_l(z)}, \qquad u = u_l(z) \tag{117}$$

$$x = u_l(z)t - \frac{1}{r_0 k_0}\ln\left(\frac{r_0 + \rho_l(z) - r_0 k_0 t u_l(z)\rho_l(z)}{r_0 + \rho_l(z)}\right) + z, \quad z < 0 \tag{118}$$

Central state

$$\rho = \frac{\hat{\rho}(a)}{1 - k_0 t \hat{u}(a)\hat{\rho}(a)}, \qquad u = \hat{u}(a) \tag{119}$$

$$x = \hat{u}(a)t - \frac{1}{r_0 k_0}\ln\left(\frac{r_0 + \hat{\rho}(a) - r_0 k_0 t \hat{u}(a)\hat{\rho}(a)}{r_0 + \hat{\rho}(a)}\right), \quad 0 \leq a \leq 1 \tag{120}$$

where

$$\frac{\hat{u}(a)}{r_0 + \hat{\rho}(a)} = \frac{u_L}{r_0 + \rho_L} = \frac{u_R}{r_0 + \rho_R} \tag{121}$$

Right state

$$\rho = \frac{\rho_r(z)}{1 - k_0 t u_r(z)\rho_r(z)}, \qquad u = u_r(z) \tag{122}$$

$$x = u_r(z)t - \frac{1}{r_0 k_0}\ln\left(\frac{r_0 + \rho_r(z) - r_0 k_0 t u_r(z)\rho_r(z)}{r_0 + \rho_r(z)}\right) + z, \ z > 0 \tag{123}$$

From (118) and (123), by setting, respectively,

$$\lim_{z \to 0^-} x = \beta_L(t) = u_L t - \frac{1}{r_0 k_0}\ln\left(\frac{r_0 + \rho_L - r_0 k_0 t u_L \rho_L}{r_0 + \rho_L}\right) \tag{124}$$

$$\lim_{z \to 0^+} x = \beta_R(t) = u_R t - \frac{1}{r_0 k_0}\ln\left(\frac{r_0 + \rho_R - r_0 k_0 t u_R \rho_R}{r_0 + \rho_R}\right) \tag{125}$$

the left state is defined for $x < \beta_L(t)$, the central state for $\beta_L(t) \leq x \leq \beta_R(t)$ and the right state for $x > \beta_R(t)$. Next, in order that the central state connects smoothly the left state with the right one, we require

$$\frac{d\lambda}{da} > 0 \tag{126}$$

where we denoted by $\lambda(a)$ the characteristc speed $\tilde{\lambda}_1 = u + c$ calculated in the central state, so that from (126) the following condition is obtained

$$c_0 \frac{d\hat{\rho}}{da} > 0, \qquad \text{where} \qquad c_0 = \frac{u_L}{r_0 + \rho_L} = \frac{u_R}{r_0 + \rho_R} \tag{127}$$

Moreover it is not difficult to ascertain that the characteristic of the central state given in (120) are well defined if

$$k_0 c_0 < 0 \tag{128}$$

Therefore, the solution of the GRP (106) is given by (117)-(123) if the conditions (127) and (128) hold. In particular from (127) and (128) two possible cases are obtained

$$c > 0, \quad \rho_L < \rho_R, \quad k_0 < 0 \quad \Rightarrow \quad 0 < u_L < u_R \tag{129}$$

$$c < 0, \quad \rho_L > \rho_R, \quad k_0 > 0 \quad \Rightarrow \quad u_L < u_R < 0 \tag{130}$$

As far as the entropy is concerned, from (110) we obtain

$$\eta = \frac{(r_0 - r_0 k_0 t u_l(z)\rho_l(z))\, u_l(z)}{r_0 - r_0 k_0 t u_l(z)\rho_l(z) + \rho_l(z)} \quad \text{for} \quad x < \beta_L(t) \tag{131}$$

$$\eta = \frac{(r_0 - r_0 k_0 t \hat{u}(a)\hat{\rho}(a))\, \hat{u}(a)}{r_0 - r_0 k_0 t \hat{u}(a)\hat{\rho}(a) + \hat{\rho}(a)} \quad \text{for} \quad \beta_L(t) \leq x \leq \beta_R(t) \tag{132}$$

$$\eta = \frac{(r_0 - r_0 k_0 t u_r(z)\rho_r(z))\, u_r(z)}{r_0 - r_0 k_0 t u_r(z)\rho_r(z) + \rho_r(z)} \quad \text{for} \quad x > \beta_R(t) \tag{133}$$

From (131) and (133) we get

$$\eta(x,0) = \begin{cases} \eta_l(x) = \frac{r_0 u_l(x)}{r_0 + \rho_l(x)} & \text{for} \quad x < 0 \\ \eta_r(x) = \frac{r_0 u_r(x)}{r_0 + \rho_r(x)} & \text{for} \quad x > 0 \end{cases} \tag{134}$$

so that

$$\lim_{x \to 0^-} \eta_l(x) = \lim_{x \to 0^+} \eta_r(x) = r_0 c_0 \tag{135}$$

Therefore, from (131)-(133) we get a non constant entropy state which is continuous $\forall t \geq 0$.

Finally, as far as the source term $f(\rho, u)$ is concerned, in the case concerning a polytropic gas, from (75) we get

$$f = \Omega(\eta) \left(\rho c(\rho, \eta) + \frac{F'(\eta)}{C_v(\gamma - 1)} \Pi(\eta) \rho^\gamma \right). \tag{136}$$

If we make the further assumptions $\Omega = k_0$, $\gamma = 3$ along with (109) as for the GRP here considered, we get

$$f(\rho, u) = \frac{k_0\,(3r_0 + \rho)\,\rho^2}{3r_0\,(r_0 + \rho)^2} u^2. \tag{137}$$

4.2 Simple wave interactions

In this section, following the analytical approach outlined in [20] for classes of 2×2 strictly hyperbolic and homogeneous systems, we make use of the exact solution (40),

(47) and (48) to describe nonlinear wave interaction processes ruled by (1)-(3) endowed with (41).

To this aim we consider the characteristic curves $C^{(\widetilde{\lambda}_1)}$ and $C^{(\widetilde{\lambda}_2)}$ defined, respectively, by

$$C^{(\widetilde{\lambda}_1)} : \frac{\mathrm{d}\,x}{\mathrm{d}\,t} = \widetilde{\lambda}_1 = r^1, \qquad C^{(\widetilde{\lambda}_2)} : \frac{\mathrm{d}\,x}{\mathrm{d}\,t} = \widetilde{\lambda}_2 = \frac{\gamma+1}{2} r^1 + h(r^2) \tag{138}$$

and we denote with $\alpha(x,t)$ and $\beta(x,t)$ the characteristic parameters satisfying

$$\frac{\partial \beta}{\partial t} + \widetilde{\lambda}_1 \frac{\partial \beta}{\partial x} = 0\,, \quad \frac{\partial \alpha}{\partial t} + \widetilde{\lambda}_2 \frac{\partial \alpha}{\partial x} = 0\,, \qquad \alpha(x,0) = \beta(x,0) = x \tag{139}$$

Then, owing to (44), (45) which express the invariance of the Riemann variables along the associated characteristic curves, we have $r^1 = r^1(\alpha)$ and $r^2 = r^2(\beta)$ and, in turn, from (47) and (48) the following representation of the solution in terms of the characteristic parameters is obtained

$$\begin{aligned} & x - \widetilde{\lambda}_1(\alpha)\, t = -\Lambda(\alpha,\beta) - m(\alpha) \\ & x - \widetilde{\lambda}_2(\alpha,\beta)\, t = \frac{1-\gamma}{2} \left(r^1(\alpha) - h(\beta) \right) \left(\frac{\partial \Lambda}{\partial r^1}(\alpha,\beta) + M(\alpha) \right) - \Lambda(\alpha,\beta) - m(\alpha) \end{aligned} \tag{140}$$

where, as usual, for a generic function $G(r^1, r^2)$ we denote $G(\alpha,\beta) = G(r^1(\alpha), r^2(\beta))$. Next we consider initial data for r^1 and r^2

$$r^1\,(x,0) = \mathcal{R}^1\,(x)\,, \qquad r^2\,(x,0) = \mathcal{R}^2\,(x)\,, \qquad -\infty < x < +\infty$$

obeying the constraint equation

$$\frac{\mathrm{d}\,\mathcal{R}^2(x)}{\mathrm{d}\,x} = \Omega(\mathcal{R}^2(x)) \left(\pm \frac{\gamma-1}{2} \, \frac{\mathcal{R}^1(x) - h(\mathcal{R}^2(x))}{\sqrt{\gamma \Pi(\mathcal{R}^2(x))}} \right)^{\frac{2}{\gamma-1}} \tag{141}$$

and, taking (140) into account, we determine the functions $M(\alpha)$, $m(\alpha)$, $\Lambda(\alpha,\beta)$, $\frac{\partial \Lambda}{\partial r^1}(\alpha,\beta)$ as follows

$$\Lambda(\alpha,\beta) = -\int_{x_0}^{\beta} \left(\frac{\mathcal{R}^1(\alpha) - h(x)}{\mathcal{R}^1(x) - h(x)} \right)^{\frac{2}{1-\gamma}} \,\mathrm{d}\,x \tag{142}$$

$$\frac{\partial \Lambda}{\partial r^1}(\alpha,\beta) = \frac{2}{\gamma-1} \int_{x_0}^{\beta} \left(\mathcal{R}^1(\alpha) - h(x) \right)^{\frac{1+\gamma}{1-\gamma}} \, \left(\mathcal{R}^1(x) - h(x) \right)^{\frac{2}{\gamma-1}} \mathrm{d}\,x \tag{143}$$

$$m(\alpha) = -\alpha - \Lambda(\alpha,\alpha), \qquad M(\alpha) = -\frac{\partial \Lambda}{\partial r^1}(\alpha,\alpha). \tag{144}$$

In the following, without loss of generality, from (40) we consider

$$u = U(\rho,s) = -\frac{2\sqrt{\gamma \Pi(s)}}{\gamma-1} \rho^{\frac{\gamma-1}{2}} + h(s). \tag{145}$$

Our aim, here, is to describe in the (x,t) plane the interaction of two simple waves travelling along characteristic curves belonging to different families. Therefore, owing to (145)

$$U = \widetilde{\lambda}_1 > \widetilde{\lambda}_2 = U + \rho \frac{\partial U}{\partial \rho}, \tag{146}$$

so that the pulse travelling along $C^{(\widetilde{\lambda}_1)}$ occupies the region $x_1 \leq x \leq x_2$ whereas the pulse travelling along $C^{(\widetilde{\lambda}_2)}$ the region $x_3 \leq x \leq x_4$ (see figure 1). Both waves propagate into a region of constant state where $r^1 = r_0^1$ and $r^2 = r_0^2$. We also require that $\mathcal{R}^1(x)$ and $\mathcal{R}^2(x)$ are continuous. Therefore at $t = 0$ we have

$$\begin{gathered}
\mathcal{R}^2(x) = \begin{cases} \zeta(x) & x_1 \leq x \leq x_2 \\ r_0^2 & \text{otherwise} \end{cases} \\
\\
\mathcal{R}^1(x) = \begin{cases} \omega(x) & x_3 \leq x \leq x_4 \\ r_0^1 & \text{otherwise} \end{cases} \\
\\
\zeta(x_1) = \zeta(x_2) = r_0^2, \quad \omega(x_3) = \omega(x_4) = r_0^1.
\end{gathered} \tag{147}$$

where, taking (141) into account, we require $\Omega\left(r_0^2\right) = 0$.

Remark 5. As far as the role of the restriction (141) is concerned we remark that, once the functional form of $\Omega(s)$ is given and in turn, by means of inversion $s = s(\rho, u)$, the force $f(\rho, u)$ is obtained, equation (141) defines the initial datum $\mathcal{R}^1(x)$ (or $\mathcal{R}^2(x)$) in terms of remaing one. In a different way, the constraint equation (141) may be used, once the initial data $\mathcal{R}^1(x)$ and $\mathcal{R}^2(x)$ are given, to define the function $\Omega(x)$ and in turn the force f.

In the (x,t) plane, explicit evaluation of the characteristic parameters $x - \widetilde{\lambda}_1 t$ and $x - \widetilde{\lambda}_2 t$ allows us to describe the behavior of the emerging simple waves. In particular the simple wave regions I_R, II_R and I_S, II_S are adjacent, respectively, to the constant state $r^2 = r_0^2$ and $r^1 = r_0^1$ [27]. Therefore we have

Simple wave $r^2 = r_0^2, \quad r^1 = \omega(\alpha)$

$$\begin{cases} \text{REGION } I_R \\ \\ x_3 \leq \alpha \leq x_4, \quad \beta \geq x_2 \\ \\ x - \widetilde{\lambda}_2\, t = \alpha \end{cases} \qquad \begin{cases} \text{REGION } II_R \\ \\ x_3 \leq \alpha \leq x_4, \quad \beta \leq x_1 \\ \\ x - \widetilde{\lambda}_2\, t = \alpha + J_s(\alpha) \end{cases} \tag{148}$$

Simple wave $r^1 = r_0^1, \qquad r^2 = \zeta(\beta)$

$$\left\{ \begin{array}{l} \text{REGION } I_S \\ \\ x_1 \leq \beta \leq x_2, \;\; \alpha \leq x_3 \\ \\ x - \widetilde{\lambda}_1\, t = \beta \end{array} \right. \qquad \left\{ \begin{array}{l} \text{REGION } II_S \\ \\ x_1 \leq \beta \leq x_2, \quad \alpha \geq x_4 \\ \\ x - \widetilde{\lambda}_1\, t = \beta + J_r \end{array} \right. \tag{149}$$

with

$$\begin{aligned} J_r &= \int_{x_3}^{x_4} \left(\left(\frac{r_0^1 - h(r_0^2)}{\omega(x) - h(r_0^2)} \right)^{\frac{2}{1-\gamma}} - 1 \right) \mathrm{d}\, x\,, \\ J_s\,(\alpha) &= \int_{x_1}^{x_2} \frac{h(r_0^2) - h(\zeta(x))}{\omega(\alpha) - h(\zeta(x))} \left(\frac{\omega(\alpha) - h(\zeta(x))}{r_0^1 - h(\zeta(x))} \right)^{\frac{2}{1-\gamma}} \mathrm{d}\, x. \end{aligned} \tag{150}$$

From(148)–(150) it follows that the pulse travelling along $C^{(\widetilde{\lambda}_1)}$ traverses region I_S, it interacts with the $C^{(\widetilde{\lambda}_2)}$ travelling pulse and emerges in the region II_S as a simple wave identical with that produced by the following initial conditions at $t = 0$

$$\begin{aligned} \mathcal{R}^1(x) &= \left\{ \begin{array}{cc} \omega(x) & x_3 \leq x \leq x_4 \\ r_0^1 & \text{otherwise} \end{array} \right. \\ \mathcal{R}^2(x) &= \left\{ \begin{array}{cc} \zeta(x + J_r) & x_1 - J_r \leq x \leq x_2 - J_r \\ S_0 & \text{otherwise.} \end{array} \right. \end{aligned} \tag{151}$$

Therefore the pulse travelling along $C^{(\widetilde{\lambda}_1)}$ evolves as an hyperbolic wave but in the interaction process exhibit a soliton-like behavior being the only effect of the interaction a change in the origin of the original pulse [33, 34]. On the contrary, the pulse travelling along $C^{(\widetilde{\lambda}_2)}$ emerges in the region II_R as a simple wave with altered profile. The interaction product $J_s(\alpha)$ represents a quantitative measure of the distortion, it depends on the initial data (147) and it vanishes when $h(s) = h_0 = constant$ as in (42). In this latter case, as it is straightforward to ascertain, the 2×2 system (44), (45) partially decouples.

In order to better illustrate the wave behavior described hitherto, hereafter we choose

$$\left\{ \begin{array}{l} U(\rho, s) = h(s) - \frac{2\sqrt{\gamma \Pi(s)}}{\gamma - 1} \rho^{\frac{\gamma - 1}{2}} \\ h(s) = h_1 \sqrt{\Pi(s)} \end{array} \right. \tag{152}$$

with h_1 constant.

Next we perfom a numerical investigation of the system (26), (27) with initial data for the density $\rho(x,t)$ and the entropy $s(x,t)$ obtained from

$$\begin{aligned} \rho(x, 0) &= \left(\frac{\gamma - 1}{2\sqrt{\gamma}} \left(h_1 - \frac{\mathcal{R}^1(x)}{\sqrt{\Pi(\mathcal{R}^2(x))}} \right) \right)^{\frac{2}{\gamma - 1}} \\ s(x, 0) &= \mathcal{R}^2(x). \end{aligned} \tag{153}$$

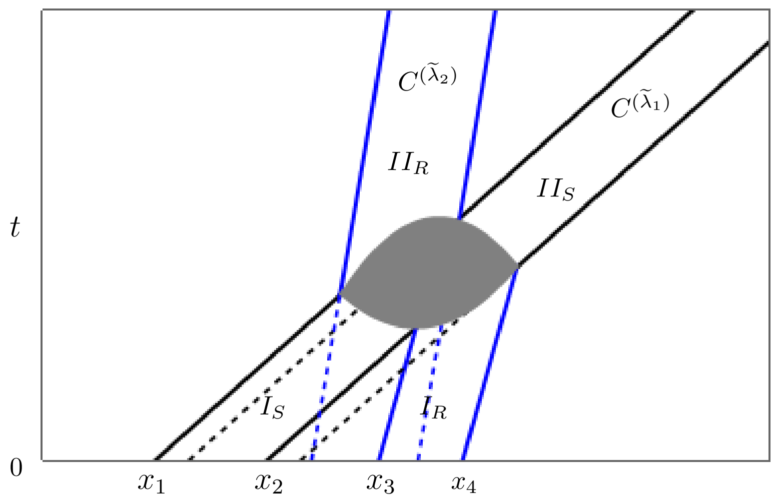

Figure 1: Qualitative behavior in the (x,t)-plane of the interaction between two simple waves travelling along different characteristic curves. The initial data for $\mathcal{R}^1(x)$ and $\mathcal{R}^2(x)$ are as in (147).

Moreover the initial datum for the velocity $u(x,t)$ is given by

$$u(x,0) = \mathcal{R}^1(x). \tag{154}$$

The numerical solution showed in figure 2 is obtained with the following choice

$$\begin{aligned} \mathcal{R}^1(x) &= 4 - 0.9\,\mathrm{sech}\,(0.02(x-600)) \\ \mathcal{R}^2(x) &= 2 + 0.6\,\mathrm{sech}\,(0.02\,(x+300))\,, \end{aligned} \tag{155}$$

and simulates two simple waves travelling along different families of characteristic curves. We notice that to the initial value problem (153) for the auxiliary 2×2 reduced model (26), (27) there corresponds through (154) an initial value problem for the full governing system (1)-(3). Such behaviours are in agreement with the theoretical results obtained in the present section. Finally in figure 3 we show the corresponding plot of the force f obtained from (41), (141) and (153)-(155).

5 Conclusion

In this paper we considered the Euler system describing the one dimensional flow of a ideal fluid with a source term. Following the procedure proposed in [22], a class of double wave solutions of the governing model (1)–(3) has been determined. The approach developed in this article permits to reduce the problem of integrating a hyperbolic full systems to that of solving a reduced 2×2 sub-system. Since we can always choose such a 2×2 reduced model so that it results to be hyperbolic, then the obtained solutions can be used for studying problems of interest in nonlinear wave propagation.

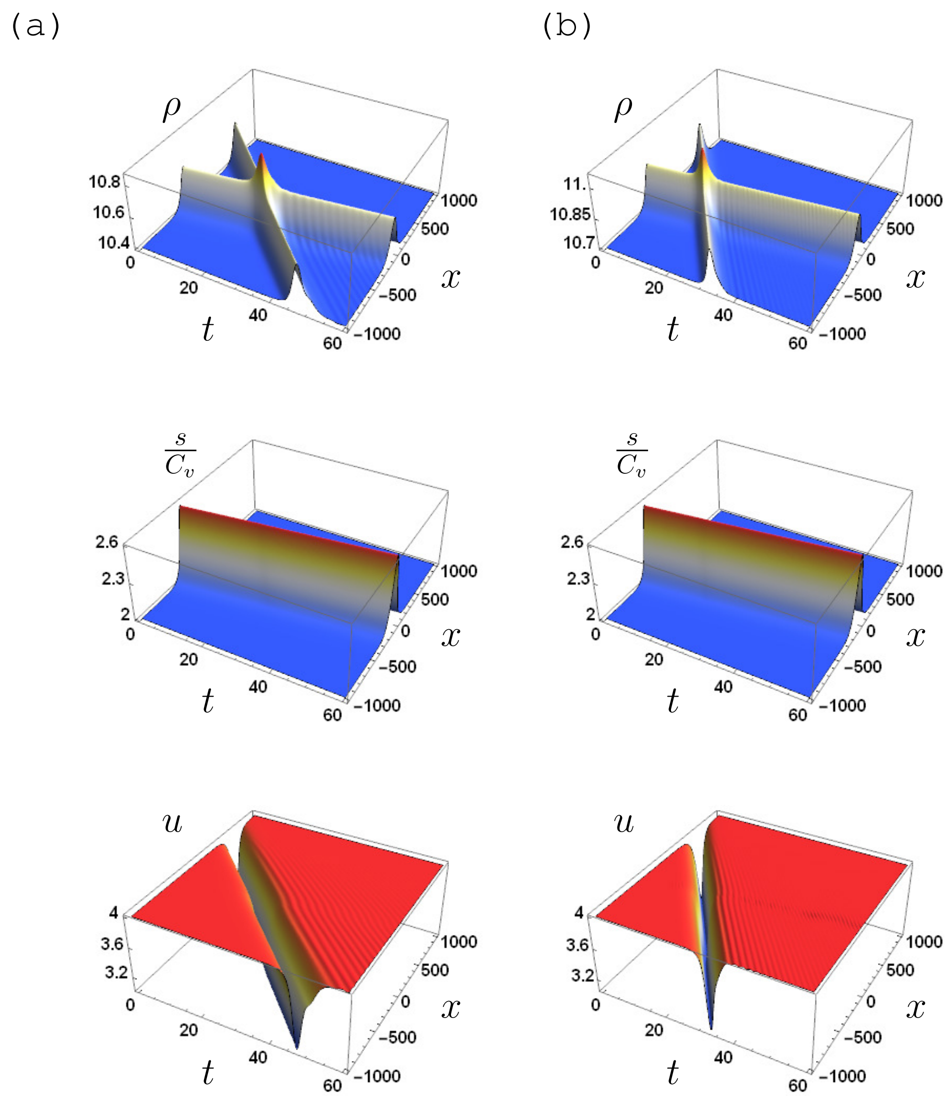

Figure 2: Simulation of two interacting simple waves. The numerical solution of equations (26) with (141) is obtained with initial data (155), $\gamma = 3$, $\;h_1 = 20$ and the choice: (a) $\Pi(s) = s^2$; (b) $\Pi(s) = e^{\frac{s}{C_v}}$.

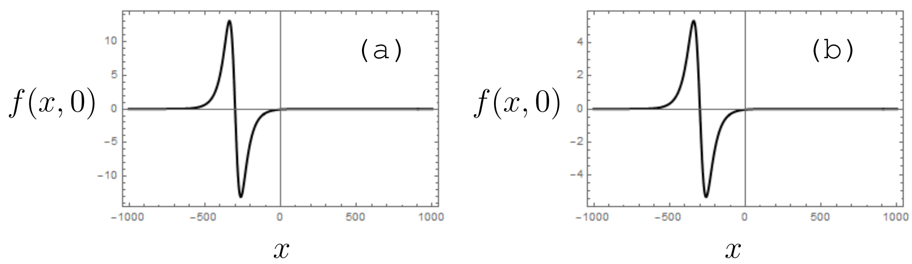

Figure 3: Plot of the body force f

In fact, first a Riemann problem for the non-homogeneous Euler system under consideration has been solved by means of a generalized rarefaction wave and an isentropic flow has been characterized. Next nonlinear wave interaction problems have been considered. In particular an exact analytical description of the interaction of two simple waves travelling along different families of characteristics was given and in one case a soliton-like behaviour was determined. Such behaviours are in a completely full agreement with the numeric results carried on in Section 4.2. Furthemore, the reduction procedure here adopted requires that the source term involved in the balance equation of the linear momentum assumes some special functional forms which in some cases are in agreement with the standard friction terms.

As far as the exact description of nonlinear wave interaction processes is concerned, we remark that it is fully developed for 2×2 strictly hyperbolic models but, unfortunately, such an analysis cannot be in general applied to quasilinear hyperbolic systems involving more dependent and/or independent variables although special wave interaction problems were solved [23, 35].

Finally, we point out that, although in the present study we confined ourselves to the 1D non-homogeneous Euler system, the extension to the 3D case is actually under investigation.

Acknowledgments

This work was supported by INdAM-GNFM and by Italian MIUR through project PRIN2017 "Multiscale phenomena in Continuum Mechanics: singular limits, off-equilibrium and transitions"(project number: 2017YBKNCE).

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Appendix

Our aim here is to sketch a general analysis of the compatibility conditions obtained in Section 3 which permits, in principle, a full classification of the exact solutions of the gas-dynamic equations obtained via the double wave ansatz here under interest.

Firstly, we consider the assumption $u = U(\rho, s)$ which leads to the equations (26), (27) and (28). Since $f \neq 0$, from (28) the following two cases arise

$$\begin{cases} \frac{\partial p}{\partial \rho} - \rho^2 \left(\frac{\partial U}{\partial \rho}\right)^2 = 0 \\ \\ \frac{\partial p}{\partial s} - \rho^2 \frac{\partial U}{\partial \rho}\frac{\partial U}{\partial s} \neq 0 \end{cases} \implies \begin{cases} \frac{\partial s}{\partial x} = q_1(\rho, s) \\ \\ f(\rho, U) = \left(\frac{1}{\rho}\frac{\partial p}{\partial s} - \rho \frac{\partial U}{\partial \rho}\frac{\partial U}{\partial s}\right) q_1(\rho, s) \end{cases} \tag{A1}$$

or

$$\frac{\partial p}{\partial \rho} - \rho^2 \left(\frac{\partial U}{\partial \rho}\right)^2 \neq 0 \quad \implies \quad \frac{\partial \rho}{\partial x} = \frac{\rho f(\rho, U) - \left(\frac{\partial p}{\partial s} - \rho^2 \frac{\partial U}{\partial \rho}\frac{\partial U}{\partial s}\right)\frac{\partial s}{\partial x}}{\frac{\partial p}{\partial \rho} - \rho^2 \left(\frac{\partial U}{\partial \rho}\right)^2}. \tag{A2}$$

The case (A1) was considered in Section 3 (see case (i)). In fact relation (28) reduces to the equation

$$\frac{\partial s}{\partial x} = q_1(\rho, s) \tag{i}$$

which is the first order differential constraint of (26), (27) associated to the eigenvalue $\widetilde{\lambda}_1 = U$. By requiring the compatibility between (26), (27) and (i) we get

$$\frac{\partial U}{\partial \rho}\left(\rho \frac{\partial q_1}{\partial \rho} - q_1\right)\left(\frac{\partial U}{\partial \rho}\frac{\partial \rho}{\partial x} + q_1 \frac{\partial U}{\partial s}\right) = 0 \tag{34}$$

In Section 3, according to the method of differential constraints, we required that relation (34) is satisfied for all solutions of (26), (27) (i.e. $\forall\, \frac{\partial \rho}{\partial x}$) so that condition (35) is obtained.

Since $\frac{\partial U}{\partial \rho} \neq 0$ because of the assumed hyperbolicity of the corresponding reduced 2×2 system, in order to complete our analysis, now we require

$$\frac{\partial U}{\partial \rho}\frac{\partial \rho}{\partial x} + q_1 \frac{\partial U}{\partial s} = 0 \tag{A3}$$

so that, taking (26), (27) into account, the following 2×2 system endowed with two differential constraint is obtained

$$\begin{aligned} &\tfrac{\partial \rho}{\partial t} + U(\rho, s)\tfrac{\partial \rho}{\partial x} = 0, && \tfrac{\partial s}{\partial t} + U(\rho, s)\tfrac{\partial s}{\partial x} = 0 \\ &\tfrac{\partial \rho}{\partial x} = -\frac{q_1(\rho,s)\,\frac{\partial U}{\partial s}}{\frac{\partial U}{\partial \rho}}, && \tfrac{\partial s}{\partial x} = q_1(\rho, s). \end{aligned} \tag{A4}$$

It is straigthforward to ascertain that equations (A4) are compatible (i. e. their differential consistencies are identically satisfied). Therefore from (A4) we have $\frac{\partial U}{\partial x} = \frac{\partial U}{\partial t} = 0$ and, in turn, the following travelling wave solution of the non-homogeneous Euler equations parameterized by arbitrary constants is obtained

$$u = U(\rho, s) = const, \quad \rho = \rho(s(z)), \quad \frac{\mathrm{d}\, s}{\mathrm{d}\, z} = q_1(\rho(s), s), \quad z = x - Ut \tag{A5}$$

where $U(\rho, s)$ and $q_1(\rho, s)$ can be determined from (A1) once the response functions $p(\rho, s)$ and $f(\rho, u)$ are assigned.

Next we consider the case (A2). The corresponding differential compatibility with the governing subsystem (26), (27) lead to

$$\left(\tfrac{\partial U}{\partial \rho}\right)^2 \left(\Upsilon \tfrac{\partial^2 s}{\partial x^2} + \left(\tfrac{\partial \Upsilon}{\partial s} + \Gamma \tfrac{\partial \Upsilon}{\partial \rho} - \Upsilon \tfrac{\partial \Gamma}{\partial \rho}\right)\left(\tfrac{\partial s}{\partial x}\right)^2\right) - q_2^2 \tfrac{\partial^2 (\rho U)}{\partial \rho^2} + \Theta \tfrac{\partial s}{\partial x} = 0 \tag{A6}$$

where $q_2(\rho, s)$ is related to the source term f through

$$f(\rho, U(\rho, s)) = \frac{q_2(\rho, s)}{\rho \frac{\partial U}{\partial \rho}} \left(\frac{\partial p}{\partial \rho} - \rho^2 \left(\frac{\partial U}{\partial \rho}\right)^2 \right)$$

while

$$\Upsilon(\rho, s) = \frac{\rho\left(\frac{\partial p}{\partial s}\frac{\partial U}{\partial \rho} - \frac{\partial U}{\partial s}\frac{\partial p}{\partial \rho}\right)}{\frac{\partial p}{\partial \rho} - \rho^2\left(\frac{\partial U}{\partial \rho}\right)^2}, \qquad \Gamma(\rho, s) = \frac{-\frac{\partial p}{\partial s} + \rho^2 \frac{\partial U}{\partial \rho}\frac{\partial U}{\partial s}}{\frac{\partial p}{\partial \rho} - \rho^2\left(\frac{\partial U}{\partial \rho}\right)^2},$$

$$\Theta(\rho, s) = \rho \tfrac{\partial U}{\partial \rho}\left(\tfrac{\partial U}{\partial s}\tfrac{\partial q_2}{\partial \rho} - \tfrac{\partial U}{\partial \rho}\tfrac{\partial q_2}{\partial s}\right) + q_2 \left(\tfrac{\partial (\Upsilon \frac{\partial U}{\partial \rho})}{\partial \rho} + \tfrac{\partial U}{\partial \rho}\left(\rho \tfrac{\partial U}{\partial \rho}\tfrac{\partial \Gamma}{\partial \rho} - 2\Gamma \tfrac{\partial U}{\partial \rho} - \tfrac{\partial U}{\partial s}\right)\right).$$

If

$$\Upsilon = 0 \quad \Longrightarrow \quad U = U(p)$$

we recover the case (ii) considered in Section 3 so that (A2) represents the first order differential constraint of (26), (27) associated to the eigenvalue $\widetilde{\lambda}_2 = U + \rho\frac{\partial U}{\partial \rho}$. In such a case, the differential compatibility between (A6) and (26), (27) has been already investigated (see (55), (56)) according to the differential constraint method (i. e. by requiring $\Theta(\rho,s)=\frac{\partial^2(\rho U)}{\partial\rho^2}=0$). Therefore here we require $\Theta(\rho,s)\neq 0$ which leads to

$$\frac{\partial s}{\partial x}=\frac{q_2^2}{\Theta(\rho,s)}\frac{\partial^2(\rho U)}{\partial\rho^2} \tag{A7}$$

so that the 2×2 model (26), (27) is endowed with the differential constraints (A2) and (A7). In such a case, the further compatibility condition $\frac{\partial^2 s}{\partial x\partial t}=\frac{\partial^2 s}{\partial t\partial x}$ gives

$$\frac{\partial p}{\partial\rho}\frac{\partial}{\partial s}\left(\frac{1}{q_2}\right)-\frac{\partial p}{\partial s}\frac{\partial}{\partial\rho}\left(\frac{1}{q_2}\right)+\frac{1}{q_2}\left(\frac{1}{\rho}\frac{\partial p}{\partial s}-\frac{\partial^2 p}{\partial s\partial\rho}+\frac{\frac{\partial p}{\partial s}}{\frac{\partial p}{\partial\rho}}\frac{\partial^2 p}{\partial\rho^2}\right)=\frac{\frac{\partial p}{\partial\rho}\,\frac{\partial^2(\rho U)}{\partial\rho^2}}{\rho^2\theta_0(s)\left(\frac{\partial U}{\partial\rho}\right)^2} \tag{A8}$$

with $\theta_0(s)\neq 0$ an arbitrary function. Once $p(\rho,s)$ and $U(p)$ are given, further integration of condition (A8), allows us to obtain $q_2(\rho,s)$ and, in turn, $f(\rho,U)$. Furthermore $\rho(x,t)$ and $s(x,t)$ are determined in terms of arbitrary constants as solutions of

$$\begin{aligned}
&\frac{\partial\rho}{\partial t}+U\frac{\partial\rho}{\partial x}=-\rho q_2, && \frac{\partial s}{\partial t}+U\frac{\partial s}{\partial x}=0\\
&\frac{\partial\rho}{\partial x}=\frac{q_2}{\frac{\partial U}{\partial\rho}}-\frac{q_2^2\frac{\partial p}{\partial s}}{\Theta(\rho,s)\frac{\partial p}{\partial\rho}}\frac{\partial^2(\rho U)}{\partial\rho^2}, && \frac{\partial s}{\partial x}=\frac{q_2^2}{\Theta(\rho,s)}\frac{\partial^2(\rho U)}{\partial\rho^2}.
\end{aligned}$$

Finally we assume $\frac{\partial p}{\partial s}\frac{\partial U}{\partial\rho}-\frac{\partial U}{\partial s}\frac{\partial p}{\partial\rho}\neq 0$ (i.e. $\Upsilon\neq 0$) so that from (A6) we obtain $\frac{\partial^2 s}{\partial x^2}$ and, by requiring the equality of the third order mixed derivatives, the following compatibility condition is obtained

$$\Xi_1\left(\frac{\partial s}{\partial x}\right)^3+\Xi_2\left(\frac{\partial s}{\partial x}\right)^2+\Xi_3\frac{\partial s}{\partial x}+\Xi_4=0 \tag{A9}$$

where we set

$$\begin{aligned}
\Xi_1 =\;& \Upsilon\frac{\partial}{\partial\rho}\left(\frac{\partial\Gamma}{\partial\rho}-\frac{\Gamma}{\rho}-\frac{1}{\Upsilon}\frac{\partial\Upsilon}{\partial s}-\frac{\Gamma}{\Upsilon}\frac{\partial\Upsilon}{\partial\rho}\right),\\
\Xi_2 =\;& -\frac{\partial}{\partial\rho}\left(\frac{\Theta}{\left(\frac{\partial U}{\partial\rho}\right)^2}\right)+\frac{\Theta}{\left(\frac{\partial U}{\partial\rho}\right)^2}\left(\frac{1}{\Upsilon}\frac{\partial\Upsilon}{\partial\rho}+\frac{2}{\rho}\right)+\frac{\partial q_2}{\partial s}+\Gamma\frac{\partial q_2}{\partial\rho}\\
&-q_2\left(\rho\frac{\partial}{\partial\rho}\left(\frac{\partial\Gamma}{\partial\rho}-\frac{1}{\Upsilon}\frac{\partial\Upsilon}{\partial s}-\frac{\Gamma}{\Upsilon}\frac{\partial\Upsilon}{\partial\rho}\right)+\frac{1}{\frac{\partial U}{\partial\rho}}\frac{\partial}{\partial\rho}\left(\frac{\Upsilon}{\rho}\right)\right),\\
\Xi_3 =\;& q_2^2\left(\frac{\partial}{\partial\rho}\left(\frac{\frac{\partial^2(\rho U)}{\partial\rho^2}}{(\frac{\partial U}{\partial\rho})^2}\right)+\frac{\frac{\partial\Upsilon}{\partial\rho}\frac{\partial^2(\rho U)}{\partial\rho^2}}{\Upsilon(\frac{\partial U}{\partial\rho})^2}\right)+\rho^3 q_2\frac{\partial}{\partial\rho}\left(\frac{\Theta}{\rho^2\Upsilon(\frac{\partial U}{\partial\rho})^2}\right)\\
&+\frac{q_2}{(\frac{\partial U}{\partial\rho})^2}\left(3\frac{\partial U}{\partial\rho}+2\rho\frac{\partial^2 U}{\partial\rho^2}\right)\frac{\partial q_2}{\partial\rho},\\
\Xi_4 =\;& -\rho^3 q_2\frac{\partial}{\partial\rho}\left(\frac{q_2^2\frac{\partial^2(\rho U)}{\partial\rho^2}}{\rho^2\Upsilon(\frac{\partial U}{\partial\rho})^2}\right).
\end{aligned}$$

As far as relation (A9) is concerned, further analysis depends on Ξ_1,Ξ_2,Ξ_3 and Ξ_4. In the general case, when these coefficients do not simultaneously vanish, (A9) defines implicitly the derivative $\frac{\partial s}{\partial x}$ and two further compatibility conditions arising by differentiating (A9), respectively, with respect t or x will lead to fourth order polynomials in

$\frac{\partial s}{\partial x}$. Neverthless such an analysis goes beyond the aims of this paper so that we do not go further.

Now we consider the class of solutions $s = F(\rho, u)$ which leads to the equations (64), (65) and (66). Since $f\frac{\partial F}{\partial u} \neq 0$, without loss of generality we can assume $\frac{\partial p}{\partial \rho} + \frac{\partial p}{\partial s}\frac{\partial F}{\partial \rho} \neq 0$ (see case (b)), so that (66) specializes to

$$\frac{\partial \rho}{\partial x} = \frac{\rho f(\rho,u)\frac{\partial F}{\partial u} - \left(\frac{\partial p}{\partial s}\left(\frac{\partial F}{\partial u}\right)^2 + \rho^2\frac{\partial F}{\partial \rho}\right)\frac{\partial u}{\partial x}}{\frac{\partial F}{\partial u}\left(\frac{\partial p}{\partial \rho} + \frac{\partial p}{\partial s}\frac{\partial F}{\partial \rho}\right)}. \tag{A10}$$

The differential consistency between (A10) and (64), (65) leads to

$$\Phi\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial \Phi}{\partial u} - \Psi\frac{\partial \Phi}{\partial \rho} + \Phi\frac{\partial \Psi}{\partial \rho}\right)\left(\frac{\partial u}{\partial x}\right)^2 + \Delta\frac{\partial u}{\partial x} = 0 \tag{A11}$$

where

$$\Phi = \frac{\rho\left(\rho^2\left(\frac{\partial F}{\partial \rho}\right)^2 - \frac{\partial p}{\partial \rho}\left(\frac{\partial F}{\partial u}\right)^2\right)}{\left(\frac{\partial F}{\partial u}\right)^2\left(\frac{\partial p}{\partial \rho} + \frac{\partial p}{\partial s}\frac{\partial F}{\partial \rho}\right)}, \qquad \Psi = \frac{\frac{\partial p}{\partial s}\left(\frac{\partial F}{\partial u}\right)^2 + \rho^2\frac{\partial F}{\partial \rho}}{\frac{\partial F}{\partial u}\left(\frac{\partial p}{\partial \rho} + \frac{\partial p}{\partial s}\frac{\partial F}{\partial \rho}\right)},$$

$$\Delta = \frac{\rho}{c\frac{\partial F}{\partial u}}\left(\frac{\partial F}{\partial u}\frac{\partial q}{\partial \rho} - \frac{\partial F}{\partial \rho}\frac{\partial q}{\partial u}\right) + \frac{q}{c}\left(\Psi\frac{\partial}{\partial \rho}\left(\frac{\rho\frac{\partial F}{\partial \rho}}{\frac{\partial F}{\partial u}}\right) - 2 - \frac{\rho\frac{\partial^2 p}{\partial \rho^2}}{2\frac{\partial p}{\partial \rho}}\right).$$

with, as usual, $c = \sqrt{\frac{\partial p}{\partial \rho}}$ and $q(\rho, u)$ is related to the source term f through

$$f(\rho,u) = \frac{q}{\rho\, c}\left(\frac{\partial p}{\partial \rho} + \frac{\partial p}{\partial s}\frac{\partial F}{\partial \rho}\right).$$

If $\Phi = 0$ we recover relation (69) considered in Section 3

$$c\frac{\partial F}{\partial u} \pm \rho\frac{\partial F}{\partial \rho} = 0,$$

so that (A10) reduces to (74) and it is, in fact, the first order differential constraint of (64), (65) associated to the eigenvalue $\widetilde{\lambda}_2$. In such a case the resulting compatibility condition (A11) has been already investigated by requiring the vanishing of the coefficient of $\frac{\partial u}{\partial x}$ (case (a)).

Here we consider also the possibility $\frac{\partial u}{\partial x} = 0$ so that solutions of non-homogeneous gas-dynamic equations are obtained in terms of arbitrary constants by solving the corresponding 2×2 reduced system along with two differential

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} = 0, \qquad && \frac{\partial u}{\partial t} = 0 \\ &\frac{\partial \rho}{\partial x} = \frac{q(\rho,u)}{c(\rho,s)}, && \frac{\partial u}{\partial x} = 0 \end{aligned}$$

with $q(\rho, u)$ and $s = F(\rho, u)$ determined once $p(\rho, s)$ and $f(\rho, u)$ are given. The resulting exact solution is characterized by a travelling wave along with constant fluid velocity.

Next we assume $c\frac{\partial F}{\partial u} \pm \rho\frac{\partial F}{\partial \rho} \neq 0$ (i.e. $\Phi \neq 0$) so that from (A11) we obtain $\frac{\partial^2 u}{\partial x^2}$ and by requiring a further compatibility condition we obtain

$$\Sigma_1 \left(\frac{\partial u}{\partial x}\right)^2 + \Sigma_2 \frac{\partial u}{\partial x} + \Sigma_3 = 0 \tag{A12}$$

where

$$\begin{aligned}
\Sigma_1 = \; & \left(\tfrac{\partial \Phi}{\partial \rho} - \tfrac{1}{\Psi}\tfrac{\partial \Phi}{\partial u}\right)\left(2\tfrac{\Psi}{\Phi} - \tfrac{1}{\Phi}\tfrac{\partial \Phi}{\partial u} - \tfrac{\Phi+\rho}{\Phi}\left(\tfrac{\partial \Psi}{\partial \rho} - \tfrac{1}{\Psi}\tfrac{\partial \Psi}{\partial u}\right)\right) - \tfrac{\Phi+\rho}{\Psi}\tfrac{\partial}{\partial u}\left(\tfrac{\partial \Psi}{\partial \rho} - \tfrac{1}{\Psi}\tfrac{\partial \Psi}{\partial u}\right) \\
& + \left(\tfrac{\partial \Psi}{\partial \rho} - \tfrac{1}{\Psi}\tfrac{\partial \Psi}{\partial u}\right)\left(\tfrac{\partial \Phi}{\partial \rho} - \tfrac{1}{\Psi}\tfrac{\partial \Phi}{\partial u} - \tfrac{\Phi+\rho}{\Psi^2}\tfrac{\partial \Psi}{\partial u} + 1\right) + (\Phi+\rho)\tfrac{\partial}{\partial \rho}\left(\tfrac{\partial \Psi}{\partial \rho} - \tfrac{1}{\Psi}\tfrac{\partial \Psi}{\partial u}\right) \\
& + \tfrac{\partial}{\partial u}\left(\tfrac{\partial \Phi}{\partial \rho} - \tfrac{1}{\Phi}\tfrac{\partial \Psi}{\partial u}\right) - 2\tfrac{\partial \Psi}{\partial \rho} - \Phi\tfrac{\partial^2 \Psi}{\partial \rho^2}\,, \\
\Sigma_2 = \; & \tfrac{1}{c}\tfrac{\partial}{\partial \rho}\left(\tfrac{\Phi+\rho}{\Psi}\right)\left(\Psi\tfrac{\partial q}{\partial \rho} - \tfrac{\partial q}{\partial u}\right) + \tfrac{\Delta}{\Phi}\left(\tfrac{2}{\Psi}\tfrac{\partial \Phi}{\partial u} - \tfrac{\partial \Phi}{\partial \rho} + 2\tfrac{\Phi+\rho}{\Psi}\tfrac{\partial}{\partial u}\left(\tfrac{\partial \Psi}{\partial \rho} - \tfrac{1}{\Psi}\tfrac{\partial \Psi}{\partial u}\right) - 4\right) \\
& - \tfrac{\partial \Delta}{\partial \rho} + \tfrac{q}{c}\left(\Psi\tfrac{\partial^2}{\partial \rho^2}\left(\tfrac{\Phi+\rho}{\Psi}\right) - \tfrac{\partial^2}{\partial \rho \partial u}\left(\tfrac{\Phi+\rho}{\Psi}\right) - \tfrac{\Phi+\rho}{\Psi}\tfrac{\partial}{\partial \rho}\left(\tfrac{\Psi}{\Phi}\tfrac{\partial \Phi}{\partial \rho} - \tfrac{1}{\Phi}\tfrac{\partial \Phi}{\partial u} - \tfrac{\partial \Psi}{\partial \rho}\right)\right), \\
\Sigma_3 = \; & \tfrac{q}{c}\tfrac{\partial}{\partial \rho}\left(\tfrac{q}{c}\tfrac{\partial}{\partial \rho}\left(\tfrac{\Phi+\rho}{\Psi}\right) + \tfrac{\Delta}{\Phi}\left(u - \tfrac{\Phi+\rho}{\Psi}\right)\right)
\end{aligned}$$

Further analysis of condition (A12) depends on Σ_1, Σ_2 and Σ_3. Also in the present case when Σ_1, Σ_2 and Σ_3 do not simultaneously vanish, (A12) defines implicitly $\frac{\partial u}{\partial x}$ so that further compatibility condition leads to a third order polynomial in $\frac{\partial u}{\partial x}$.