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PROPER K-BALL-CONTRACTIVE MAPPINGS IN $C_b^m[0,\infty)$

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Abstract. In this paper we deal with the Banach space $C_b^m[0,\infty)$ of all m-times continuously derivable, bounded with all derivatives up to the order m, real functions defined on $[0,+\infty)$. We prove, for any $\varepsilon > 0$, the existence of a new proper k-ball-contractive retraction with $k < 1+\varepsilon$ of the closed unit ball of the space onto its boundary, so that the Wośko constant $W_\gamma(C_b^m[0,\infty))$ is equal to 1.

1. Introduction

Given a Banach space X, we denote by $B(X) = \{x \in X : \|x\| \leq 1\}$ the closed unit ball and by $S(X) = \{x \in X : \|x\| = 1\}$ the unit sphere in X. It is well known that in any infinite-dimensional Banach space X there is a retraction from $B(X)$ onto $S(X)$, that is, a continuous mapping $R : B(X) \to S(X)$ such that $Rx = x$ for $x \in S(X)$. Moreover such a retraction can be chosen to be Lipschitzian [5] with $\|Rx - Ry\| \leq k_0\|x - y\|$, for some universal constant k_0. The optimal retraction problem, considered for the first time in [20], consists in the evaluation, in a given Banach space X, of the constant $k_0(X)$ which is the infimum of all k for which there exists a retraction of $B(X)$ onto $S(X)$ being Lipschitz with constant k. The problem has found a large interest in the literature. It is known $k_0(X) \geq 3$ for every space X. For evaluation of the constant in some specific Banach spaces we refer, among others, to results in [2, 6, 15, 23, 25, 26] and to the surveys on the subject [14, 21, 22].

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In this paper we are interested in the analogous problem which arises when we consider another metric property, namely measure of noncompactness, of the above retractions. Throughout we will consider γ to be the *Hausdorff measure of noncompactness*, i.e. for $A \subseteq X$ bounded, $\gamma(A)$ is the infimum of all $\varepsilon > 0$ such that A has a finite ε-net in X. We recall that the set function γ satisfies the following properties, for $A, B \subseteq X$ bounded, $K \subseteq X$ precompact and $\lambda \in \mathbb{R}$:

(i) $\gamma(A) = 0$ if and only if A is precompact;
(ii) $\gamma(\overline{co} A) = \gamma(A)$ (convex closure invariance);
(iii) $\gamma(A \cup B) = \max\{\gamma(A), \gamma(B)\}$ (maximum property);
(iv) $\gamma(A + K) = \gamma(A)$ (compact perturbations);
(v) $\gamma(\lambda A) = |\lambda|\gamma(A)$ (homogeneity);
(vi) $\gamma([0,1] \cdot A) = \gamma(A)$ (absorption invariance).

A continuous mapping $T : M \subset X \to X$ is said to be *k-ball-contractive* if $\gamma(TA) \leq k\gamma(A)$ for bounded $A \subseteq M$, and the *γ-norm*, $\gamma(T)$, of T is defined by

$$\gamma(T) = \inf\{k \geq 0 : \ \gamma(TA) \leq k\gamma(A) \text{ for bounded } A \subseteq M\}.$$

We will also consider $\omega(T) = \sup\{k \geq 0 : \ \gamma(TA) \geq k\gamma(A) \text{ for bounded } A \subseteq M\}$, which is called the *lower γ-norm* of T, the main reason is that $\omega(T) > 0$ implies T to be a *proper* mapping. Now the optimal retraction problem for k-ball-contractive mappings concerns the evaluation (see [4]) of the Wośko constant

$$W_\gamma(X) = \inf\{k \geq 1 : \exists \text{ a retraction } R : B(X) \to S(X) \text{ with } \gamma(R) \leq k\}.$$

The constant $W_\gamma(X)$ has been estimated in many Banach spaces X [3, 7, 10, 12, 13, 16, 27, 28]. In some spaces it has been proved $W_\gamma(X) = 1$, and in some cases the value 1 has been achieved [10, 12] with the construction of a 1-ball-contractive retraction. Actually it is an open problem whether or not $W_\gamma(X) = 1$ in any Banach space. The estimate of $W_\gamma(X)$, by means of retractions, eventually proper, leads to useful results for applications as, for instance, applications to theorems of Birkhoff-Kellogg type (see [3, 8, 9, 11, 17, 24]).

In [13] we have proved $W_\gamma(C^m[0,1]) = 1$, being $C^m[0,1]$ the Banach space of all m-times continuously derivable real functions defined on $[0,1]$, by constructing for any $\varepsilon > 0$ a proper k-ball-contractive retraction, with $k < 1 + \varepsilon$. Here we succeed to prove the same result in the Banach space $C_b^m[0,+\infty)$. In [13] we have followed a general scheme to construct a 1-ball-contractive mapping from the closed unit ball of $C^m[0,1]$ into itself and obtaining a retraction as the normalization of a compact perturbation of such a mapping. In the present framework we need a new original approach, which besides requires quite more technical proofs for intermediate results. We construct for any $p \in \mathbb{N}$ a mapping Q_p defined on $B(C_b^m[0,+\infty))$ taking values in $C_b^m[0,+\infty)$ which is $(1+\varepsilon_p)$-ball-contractive for some $\varepsilon_p > 0$ with $\lim_{p\to\infty} \varepsilon_p = 0$. After this we are in a position, for each $p \in \mathbb{N}$, to construct retractions, which will depend on some

$u > 0$, normalizing compact perturbations of Q_p. In such a way for any $\varepsilon > 0$ we can find a proper k-ball-contractive retraction, corresponding to some $p \in \mathbb{N}$ and $u > 0$, with $k < 1 + \varepsilon$, which, in turn, gives $W_\gamma(C_b^m[0,\infty)) = 1$. The paper is meant as a continuation of the research presented in [13].

2. The auxiliary function $\tilde{f}_{p,a}$ and the mapping Q_p

We denote by $C_b^m := C_b^m[0,+\infty)$ $(m \geq 1)$ the Banach space of all m-times continuously derivable, bounded with all derivatives up to the order m, functions $f : [0,+\infty) \to \mathbb{R}$, with the norm

$$\|f\|_m = \max\{\|f^{(s)}\|_\infty : \; s = 0, 1, \cdots, m\},$$

where, as usual, $f^{(0)} = f$ and $\|\cdot\|_\infty$ denotes the supremum norm. For a given compact interval $J \subset \mathbb{R}$ we denote by $C^m(J)$ the Banach space of m-times continuously derivable real functions defined on J, always endowed with the $\|\cdot\|_m$-norm. Let $p \in \mathbb{N}$ be given. For $f \in C_b^m$ and $a \in [1,2]$, we introduce the function $f_{p,a} \in C^m\left([1 - 1/\sqrt[p]{a}, 1]\right)$ defined in the following way

$$f_{p,a}(t) = \frac{1}{\sqrt[p]{a^m}} f\left(1 + \sqrt[p]{a}(t-1)\right) + \sum_{j=0}^{m-1} \frac{f^{(j)}(1)}{j!}\left(1 - \frac{1}{\sqrt[p]{a^{m-j}}}\right)(t-1)^j,$$

whose derivatives are

$$f_{p,a}^{(s)}(t) = \frac{1}{\sqrt[p]{a^{m-s}}} f^{(s)}(1 + \sqrt[p]{a}(t-1)) + \sum_{j=s}^{m-1} \frac{f^{(j)}(1)}{(j-s)!}\left(1 - \frac{1}{\sqrt[p]{a^{m-j}}}\right)(t-1)^{j-s},$$

for $s = 0, 1, \cdots, m-1$, and

$$f_{p,a}^{(m)}(t) = f^{(m)}(1 + \sqrt[p]{a}(t-1)).$$

Next we define the auxiliary function $\tilde{f}_{p,a} \in C_b^m$, an extension of $f_{p,a}$, by setting

$$\tilde{f}_{p,a}(t) = \begin{cases} \frac{1}{m!} f^{(m)}(0)\left(t - 1 + \frac{1}{\sqrt[p]{a}}\right)^m \\ \quad + \sum_{j=1}^{m} \frac{1}{(m-j)!} f_{p,a}^{(m-j)}\left(1 - \frac{1}{\sqrt[p]{a}}\right)\left(t - 1 + \frac{1}{\sqrt[p]{a}}\right)^{m-j} & \text{if } t \in \left[0, 1 - \frac{1}{\sqrt[p]{a}}\right) \\ f_{p,a}(t) & \text{if } t \in \left[1 - \frac{1}{\sqrt[p]{a}}, 1\right] \\ f(t) & \text{if } t \in (1, +\infty). \end{cases}$$

Notice that $\tilde{f}_{p,1} = f$. Moreover,

$$\tilde{f}_{p,a}^{(s)}(t) = \begin{cases} \frac{1}{(m-s)!} f^{(m)}(0)\left(t - 1 + \frac{1}{\sqrt[p]{a}}\right)^{m-s} \\ \quad + \sum_{j=1}^{m-s} \frac{1}{(m-s-j)!} f_{p,a}^{(m-j)}\left(1 - \frac{1}{\sqrt[p]{a}}\right)\left(t - 1 + \frac{1}{\sqrt[p]{a}}\right)^{m-s-j} & \text{if } t \in \left[0, 1 - \frac{1}{\sqrt[p]{a}}\right) \\ f_{p,a}^{(s)}(t) & \text{if } t \in \left[1 - \frac{1}{\sqrt[p]{a}}, 1\right] \\ f^{(s)}(t) & \text{if } t \in (1, +\infty), \end{cases}$$

for $s = 0, 1, \cdots, m-1$, and

$$\tilde{f}^{(m)}_{p,a}(t) = \begin{cases} f^{(m)}(0) & \text{if } t \in \left[0, 1 - \frac{1}{\sqrt[p]{a}}\right) \\ f^{(m)}_{p,a}(t) & \text{if } t \in \left[1 - \frac{1}{\sqrt[p]{a}}, 1\right] \\ f^{(m)}(t) & \text{if } t \in (1, +\infty). \end{cases}$$

Let us observe that, for $f \in C^m_b$, the norms $\left\|\tilde{f}^{(m)}_{p,a}\right\|_\infty$ and $\|f^{(m)}\|_\infty$ coincide.

Now, given $p \in \mathbb{N}$, making use of the auxiliary mapping $\tilde{f}_{p,a}$, we define the mapping $Q_p : B(C^m_b) \to C^m_b$ setting for $f \in B(C^m_b)$

$$Q_p f(t) = \tilde{f}_{p,a}(t) \quad \text{for } a = \frac{2}{1 + \|f\|_m} \quad \text{and} \quad t \in [0, +\infty). \tag{2.1}$$

3. Technical results on auxiliary functions of the type $\tilde{f}_{p,a}$

We begin this section with the following lemma which gives estimates, given $f \in C^m_b$, of the norm $\left\|\tilde{f}_{p,a}\right\|_m$. To this end, for $p \in \mathbb{N}$ we put

$$C_p = 1 - \frac{1}{\sqrt[p]{2}} + \left(1 + (m-1)\left(1 - \frac{1}{\sqrt[p]{2^m}}\right)\right)\left(1 + (m-1)\left(1 - \frac{1}{\sqrt[p]{2}}\right)\right) \tag{3.1}$$

and

$$D_p = 1 - m\left(1 - \frac{1}{\sqrt[p]{2^m}}\right), \tag{3.2}$$

we observe that D_p is positive for large enough p, and moreover

$$\lim_{p \to +\infty} C_p = \lim_{p \to +\infty} D_p = 1.$$

Lemma 3.1. *Let $p \in \mathbb{N}$, then*

$$D_p \|f\|_m \le \left\|\tilde{f}_{p,a}\right\|_m \le C_p \|f\|_m, \tag{3.3}$$

for all $f \in C^m_b$ and for all $a \in [1, 2]$.

Proof. Let $p \in \mathbb{N}$ and $f \in C^m_b$. Being the result obvious when $a = 1$, we assume all along this proof $a \in (1, 2]$ to be arbitrarily fixed. At first, we prove the right inequality of (3.3). To this end we will show

$$\left\|\tilde{f}^{(s)}_{p,a}\right\|_\infty \le C_p \|f\|_m, \quad \text{for } s = 0, 1, \ldots, m. \tag{3.4}$$

Since $\left\|\tilde{f}^{(m)}_{p,a}\right\|_\infty = \|f^{(m)}\|_\infty$, we immediately have

$$\left\|\tilde{f}^{(m)}_{p,a}\right\|_\infty \le \|f\|_m. \tag{3.5}$$

Assume now $s \in \{0, 1, \cdots, m-1\}$, then

$$\left\|\tilde{f}^{(s)}_{p,a}\right\|_\infty = \max\left\{ \max_{t \in \left[0, 1 - \frac{1}{\sqrt[p]{a}}\right]} \left|\tilde{f}^{(s)}_{p,a}(t)\right|, \left\|f^{(s)}_{p,a}\right\|_\infty, \sup_{t \in (1, +\infty)} |f^{(s)}(t)| \right\}. \tag{3.6}$$

Let us consider first $\left\|f_{p,a}^{(s)}\right\|_\infty$, we have

$$\begin{aligned}\left\|f_{p,a}^{(s)}\right\|_\infty &= \max_{t\in\left[1-\frac{1}{\sqrt[p]{a}},1\right]}\left|f_{p,a}^{(s)}(t)\right| \\ &\leq \max_{t\in\left[1-\frac{1}{\sqrt[p]{a}},1\right]}\left[\frac{1}{\sqrt[p]{a^{m-s}}}\left|f^{(s)}(1+\sqrt[p]{a}(t-1))\right| + \sum_{j=s}^{m-1}|f^{(j)}(1)|\ \left(1-\frac{1}{\sqrt[p]{a^{m-j}}}\right)\right] \\ &\leq \frac{1}{\sqrt[p]{a^{m-s}}}\max_{t\in[0,1]}|f^{(s)}(t)| + \|f\|_m\sum_{j=s}^{m-1}\left(1-\frac{1}{\sqrt[p]{a^{m-j}}}\right) \\ &\leq \left[\frac{1}{\sqrt[p]{a^{m-s}}} + \sum_{j=s}^{m-1}\left(1-\frac{1}{\sqrt[p]{a^{m-j}}}\right)\right]\|f\|_m.\end{aligned}$$

From the latter inequality we obtain at once

$$\left\|f_{p,a}^{(m-1)}\right\|_\infty \leq \|f\|_m, \tag{3.7}$$

while in the case $s = 0, 1, m-2$ we can write

$$\left\|f_{p,a}^{(s)}\right\|_\infty \leq \left[1 + \sum_{j=s+1}^{m-1}\left(1-\frac{1}{\sqrt[p]{a^{m-j}}}\right)\right]\|f\|_m. \tag{3.8}$$

Thus we have found, for $s = 0, 1, m-1$,

$$\left\|f_{p,a}^{(s)}\right\|_\infty \leq \left[1 + (m-1)\left(1-\frac{1}{\sqrt[p]{2^m}}\right)\right]\|f\|_m. \tag{3.9}$$

Next let $t \in \left[0, 1-\frac{1}{\sqrt[p]{a}}\right]$, then we have

$$\begin{aligned}\left|\tilde{f}_{p,a}^{(s)}(t)\right| &= \Bigg|\frac{1}{(m-s)!}f^{(m)}(0)\left(t-1+\frac{1}{\sqrt[p]{a}}\right)^{m-s} \\ &\quad + \sum_{j=1}^{m-s}\frac{1}{(m-s-j)!}f_{p,a}^{(m-j)}\left(1-\frac{1}{\sqrt[p]{a}}\right)\left(t-1+\frac{1}{\sqrt[p]{a}}\right)^{m-s-j}\Bigg| \\ &\leq \left(1-\frac{1}{\sqrt[p]{a}}\right)^{m-s}\|f\|_m + \sum_{j=1}^{m-s}\left|f_{p,a}^{(m-j)}\left(1-\frac{1}{\sqrt[p]{a}}\right)\right|\left(1-\frac{1}{\sqrt[p]{a}}\right)^{m-s-j}.\end{aligned}$$

Thus using (3.9) we obtain

$$\begin{aligned}\left|\tilde{f}_{p,a}^{(s)}(t)\right| &\leq \left[\left(1-\frac{1}{\sqrt[p]{a}}\right)^{m-s} + \left(1+(m-1)\left(1-\frac{1}{\sqrt[p]{2^m}}\right)\right)\sum_{j=1}^{m-s}\left(1-\frac{1}{\sqrt[p]{a}}\right)^{m-s-j}\right]\|f\|_m \\ &\leq \left[1-\frac{1}{\sqrt[p]{2}} + \left(1+(m-1)\left(1-\frac{1}{\sqrt[p]{2^m}}\right)\right)\left(1+(m-1)\left(1-\frac{1}{\sqrt[p]{2}}\right)\right)\right]\|f\|_m,\end{aligned}$$

that is,

$$\max_{t\in\left[0,1-\frac{1}{\sqrt[p]{a}}\right]} \left|\tilde{f}^{(s)}_{p,a}(t)\right| \le C_p\|f\|_m. \tag{3.10}$$

Looking at (3.9), we observe that $1+(m-1)\left(1-\frac{1}{\sqrt[p]{2^m}}\right) < C_p$. Therefore (3.9) and (3.10), taking into account that $\sup_{t\in(1,+\infty)}|f^{(s)}(t)| \le \|f\|_m$, imply

$$\left\|\tilde{f}^{(s)}_{p,a}\right\|_\infty \le C_p\|f\|_m,$$

for all $s = 0, 1, \cdots, m-1$. The latter, together with (3.5), gives us the right inequality of (3.3), with $C_p \ge 1$ and $\lim_{p\to+\infty} C_p = 1$.

Now, we prove the left inequality of (3.3). By the definition of $\|\cdot\|_m$ there exists $s \in \{0, 1, \cdots, m\}$ such that

$$\|f\|_m = \|f^{(s)}\|_\infty. \tag{3.11}$$

If (3.11) holds with $s = m$, then we have

$$\|f\|_m = \|f^{(m)}\|_\infty = \left\|\tilde{f}^{(m)}_{p,a}\right\|_\infty \le \left\|\tilde{f}_{p,a}\right\|_m, \tag{3.12}$$

and we are done. When s, satisfying (3.11), is in $\{0, 1, \cdots, m-1\}$ we have two cases, either $\|f^{(s)}\|_\infty$ coincides with $\sup_{t\in(1,+\infty)}|f^{(s)}(t)|$ or with $\max_{t\in[0,1]}|f^{(s)}(t)|$. If $\|f^{(s)}\|_\infty = \sup_{t\in(1,+\infty)}|f^{(s)}(t)|$, then we immediately obtain

$$\|f\|_m = \|f^{(s)}\|_\infty = \sup_{t\in(1,+\infty)}|f^{(s)}(t)| \le \left\|\tilde{f}^{(s)}_{p,a}\right\|_\infty. \tag{3.13}$$

In the second case, when $\|f^{(s)}\|_\infty = \max_{t\in[0,1]}|f^{(s)}(t)|$, let us observe first that

$$\left|\sum_{j=s}^{m-1}\frac{f^{(j)}(1)}{(j-s)!}\left(1-\frac{1}{\sqrt[p]{a^{m-j}}}\right)(t-1)^{j-s}\right| \le (m-1)\left(1-\frac{1}{\sqrt[p]{2^m}}\right)\|f\|_m. \tag{3.14}$$

Indeed,

$$\begin{aligned}
&\left|\sum_{j=s}^{m-1}\frac{f^{(j)}(1)}{(j-s)!}\left(1-\frac{1}{\sqrt[p]{a^{m-j}}}\right)(t-1)^{j-s}\right| \\
&\qquad \le \|f\|_m\sum_{j=s}^{m-1}\left(1-\frac{1}{\sqrt[p]{a^{m-j}}}\right) \le (m-1)\left(1-\frac{1}{\sqrt[p]{a^m}}\right)\|f\|_m \\
&\qquad \le (m-1)\left(1-\frac{1}{\sqrt[p]{2^m}}\right)\|f\|_m.
\end{aligned}$$

Then, using (3.14), we have

$$\left\|\tilde{f}^{(s)}_{p,a}\right\|_\infty \geq \max_{t\in\left[1-\frac{1}{\sqrt[p]{a}},1\right]} \left|f^{(s)}_{p,a}(t)\right| \geq \max_{t\in\left[1-\frac{1}{\sqrt[p]{a}},1\right]} \Bigg[\left|\frac{1}{\sqrt[p]{a^{m-s}}} f^{(s)}(1+\sqrt[p]{a}(t-1))\right|$$

$$- \left|\sum_{j=s}^{m-1} \frac{f^{(j)}(1)}{(j-s)!}\left(1-\frac{1}{\sqrt[p]{a^{m-j}}}\right)(t-1)^{j-s}\right|\Bigg]$$

$$\geq \frac{1}{\sqrt[p]{2^m}}\|f^{(s)}\|_\infty - (m-1)\left(1-\frac{1}{\sqrt[p]{2^m}}\right)\|f\|_m$$

$$\geq \left[\frac{1}{\sqrt[p]{2^m}} - (m-1)\left(1-\frac{1}{\sqrt[p]{2^m}}\right)\right]\|f\|_m = D_p\|f\|_m.$$

The latter, together with (3.12) and (3.13), gives the left inequality of (3.4), and this completes the proof. □

The following corollary is, actually, a reformulation of Lemma 3.1.

COROLLARY 3.2. *Let $p \in \mathbb{N}$, then there exists $\varepsilon_p > 0$ such that*

$$(1-\varepsilon_p)\|f\|_m \leq \left\|\tilde{f}_{p,a}\right\|_m \leq (1+\varepsilon_p)\|f\|_m, \tag{3.15}$$

for all $f \in C^m_b$ and for all $a \in [1,2]$, with $\lim_{p\to\infty} \varepsilon_p = 0$.

REMARK 3.3. The case of the Banach space $C_b[0,+\infty)$ of all continuous and bounded functions $f : [0,+\infty) \to \mathbb{R}$ endowed with the supremum norm is studied in [7], where it is proved the existence, for every $\varepsilon > 0$, of a $(1+\varepsilon)$-ball-contractive retraction, so that $W_\gamma(C_b[0,+\infty)) = 1$. Here we reduce to the space $C_b[0,+\infty)$ by allowing $m = 0$. Then given $p \in \mathbb{N}$, $a \in [1,2]$ and $f \in C_b[0,+\infty)$ we can write

$$\tilde{f}_{p,a}(t) = \begin{cases} f(0) & \text{if } t \in \left[0, 1-\frac{1}{\sqrt[p]{a}}\right) \\ f(1+\sqrt[p]{a}(t-1)) & \text{if } t \in \left[1-\frac{1}{\sqrt[p]{a}}, 1\right] \\ f(t) & \text{if } t \in (1,+\infty), \end{cases}$$

hence $\|\tilde{f}_{p,a}\|_\infty = \|f\|_\infty$, for any $p \in \mathbb{N}$. This implies that we can consider $\tilde{f}_{1,a}$ as auxiliary function, then if we follow the main steps of the present paper, we will obtain again $W_\gamma(C_b[0,+\infty)) = 1$, but by means of proper retractions different from those constructed in [7].

REMARK 3.4. Let us notice that the case of the Banach space $C^m[0,1]$ can be deduced, from the present setting, restricting every mapping to the interval $[0,1]$. Then again we would obtain $W_\gamma(C^m[0,1]) = 1$, but by means of proper retractions different from those constructed in [13].

The following example shows that the inequality $\|\tilde{f}_{p,a}\|_m \leq \|f\|_m$, differently from [7] and [13], cannot be obtained in Lemma 3.3.

Example 3.5. Let $f \in C_b^1$ be defined as follows

$$f(t) = \begin{cases} -t^3 + 2t^2 - t + 1 & \text{if} \quad t \in [0,1] \\ \\ 1 & \text{if} \quad t \in (1, +\infty) \,. \end{cases}$$

Then $f(0) = f(1) = 1$, $f'(0) = -1$ and $\|f\|_\infty = \|f'\|_\infty = 1$. Consequently

$$\begin{aligned} \left|\tilde{f}_{k,a}(0)\right| &= \left|f'(0)\left(-1 + \tfrac{1}{\sqrt[p]{a}}\right) + f_{p,a}\left(1 - \tfrac{1}{\sqrt[p]{a}}\right)\right| \\ &= \left|f'(0)\left(-1 + \tfrac{1}{\sqrt[p]{a}}\right) + \tfrac{1}{\sqrt[p]{a}} f(0) + f(1)\left(1 - \tfrac{1}{\sqrt[p]{a}}\right)\right| \\ &= \left|\left(1 - \tfrac{1}{\sqrt[p]{a}}\right)\|f\|_\infty + \tfrac{1}{\sqrt[p]{a}}\|f\|_\infty + \left(1 - \tfrac{1}{\sqrt[p]{a}}\right)\|f\|_\infty\right| \\ &= \left(2 - \tfrac{1}{\sqrt[p]{a}}\right)\|f\|_1. \end{aligned}$$

Therefore we obtain

$$\|\tilde{f}_{p,a}\|_1 \geq \|\tilde{f}_{k,a}\|_\infty \geq \sup_{t \in \left[0, 1 - \frac{1}{\sqrt[p]{a}}\right]} \left|\tilde{f}_{k,a}(t)\right| \geq \left(2 - \frac{1}{\sqrt[p]{a}}\right)\|f\|_1,$$

that is, $\|\tilde{f}_{p,a}\|_1 > \|f\|_1$, which is our assert. More in general, it suffices $f \in C_b^1$ satisfies: $f(0) = f(1) = \|f\|_\infty$, $f'(0) = -\|f\|_\infty$ and $\|f'\|_\infty \leq \|f\|_\infty$ (which implies $\|f\|_1 = \|f\|_\infty$) to infer $\|\tilde{f}_{p,a}\|_1 > \|f\|_1$, as well.
The example can be suitably modified to carry out the case $m > 1$.

The following result shows that indeed the mapping Q_p, for p large, maps the unit ball into itselt.

Proposition 3.6. *The mapping Q_p, for sufficiently large $p \in \mathbb{N}$, maps $B(C_b^m)$ into itself.*

Proof. Let $f \in B(C_b^m)$. Put $\|f\|_m = w$, so $w \in [0,1]$ and $Q_p f(t) = \tilde{f}_{p,\frac{2}{1+w}}(t)$, for $t \in [0, +\infty)$. We have to show $\|Q_p f||_m \leq 1$, for sufficiently large $p \in \mathbb{N}$, which in view of (3.5) means to prove

$$\left\|\tilde{f}^{(s)}_{p,\frac{2}{1+w}}\right\|_\infty \leq 1 \quad \text{for } s = 0, 1, \ldots, m-1. \tag{3.16}$$

Having in mind (3.6), at first we consider $\left\|f^{(s)}_{p,\frac{2}{1+w}}\right\|_\infty$ and rewrite (3.7) and (3.8) for $a = \frac{2}{1+w}$. So on the one hand

$$\left\|f^{(m-1)}_{p,\frac{2}{1+w}}\right\|_\infty \leq w. \tag{3.17}$$

On the other hand, for $s = 0, 1, m-2$, we have

$$\left\|f^{(s)}_{p,\frac{2}{1+w}}\right\|_\infty \leq w\left[1 + \sum_{j=s+1}^{m-1}\left(1 - \sqrt[p]{\left(\frac{1+w}{2}\right)^{m-j}}\right)\right] \tag{3.18}$$

and, in such a case, we set

$$\varphi_{p,s}(w) = w\left[1 + \sum_{j=s+1}^{m-1}\left(1 - \sqrt[p]{\left(\frac{1+w}{2}\right)^{m-j}}\right)\right]. \tag{3.19}$$

Then $\varphi_{p,s}(0) = 0,\ \varphi_{p,s}(1) = 1$ and

$$\varphi'_{p,s}(w) = 1 + \sum_{j=s+1}^{m-1}\left(1 - \sqrt[p]{\left(\frac{1+w}{2}\right)^{m-j}}\right) - \frac{w}{1+w}\sum_{j=s+1}^{m-1}\frac{m-j}{p}\sqrt[p]{\left(\frac{1+w}{2}\right)^{m-j}}.$$

As the last term goes to zero for $p \to \infty$, uniformly with respect to w, we have that for p sufficiently large $\varphi'_{p,s}(w) > 0$ for all $w \in [0,1]$. Therefore, for such $p's$, $0 \le \varphi_{p,s}(w) \le 1$ for all $w \in [0,1]$, which together with (3.17) gives $\left\|f^{(s)}_{p,\frac{2}{1+w}}\right\|_\infty \le 1$, for all $s = 0,1,\dots,m-1$.
To prove (3.16), going back to (3.6), now we consider $\max_{t\in\left[0,1-\frac{1}{\sqrt[p]{a}}\right]}\left|\tilde{f}^{(s)}_{p,a}(t)\right|$.
Let $t \in \left[0, 1 - \sqrt[p]{\frac{1+w}{2}}\right]$, then

$$\begin{aligned}\left|\tilde{f}^{(s)}_{p,\frac{2}{1+w}}(t)\right| &\le w\left(1 - \sqrt[p]{\frac{1+w}{2}}\right)^{m-s}\\ &+ \sum_{j=1}^{m-s}\left|f^{(m-j)}_{p,\frac{2}{1+w}}\left(1 - \sqrt[p]{\frac{1+w}{2}}\right)\right|\left(1 - \sqrt[p]{\frac{1+w}{2}}\right)^{m-s-j}.\end{aligned}$$

From the latter, we deduce $\left|\tilde{f}^{(m-1)}_{p,\frac{2}{1+w}}(t)\right| \le w\left[2 - \sqrt[p]{\frac{1+w}{2}}\right]$, therefore, for p sufficiently large, $\left|\tilde{f}^{(m-1)}_{p,\frac{2}{1+w}}(t)\right| \le 1$. On the other hand, for $s = 0,1,\dots,m-2$,

$$\begin{aligned}\left|\tilde{f}^{(s)}_{p,\frac{2}{1+w}}(t)\right| &\le w\left(1 - \sqrt[p]{\frac{1+w}{2}}\right)^{m-s}\\ &+ \left|f^{(m-1)}_{p,\frac{2}{1+w}}\left(1 - \sqrt[p]{\frac{1+w}{2}}\right)\right|\left(1 - \sqrt[p]{\frac{1+w}{2}}\right)^{m-s-1}\\ &+ \sum_{j=2}^{m-s}\left|f^{(m-j)}_{p,\frac{2}{1+w}}\left(1 - \sqrt[p]{\frac{1+w}{2}}\right)\right|\left(1 - \sqrt[p]{\frac{1+w}{2}}\right)^{m-s-j},\end{aligned}$$

hence, using (3.17) and (3.18), the last together with (3.19), we can write

$$\begin{aligned}\left|\tilde f^{(s)}_{p,\frac{2}{1+w}}(t)\right| &\le w\left(1-\sqrt[p]{\frac{1+w}{2}}\right)^{m-s} + w\left(1-\sqrt[p]{\frac{1+w}{2}}\right)^{m-s-1}\\ &\quad+\sum_{j=2}^{m-s}\varphi_{p,m-j}(w)\left(1-\sqrt[p]{\frac{1+w}{2}}\right)^{m-s-j}\\ &=\varphi_{p,s}(w)+w\left(1-\sqrt[p]{\frac{1+w}{2}}\right)^{m-s} + w\left(1-\sqrt[p]{\frac{1+w}{2}}\right)^{m-s-1}\\ &\quad+\sum_{j=2}^{m-s-1}\varphi_{p,m-j}(w)\left(1-\sqrt[p]{\frac{1+w}{2}}\right)^{m-s-j}.\end{aligned}$$

Set

$$\begin{aligned}\psi_{p,s}(w)&=\varphi_{p,s}(w)+w\left(1-\sqrt[p]{\frac{1+w}{2}}\right)^{m-s} + w\left(1-\sqrt[p]{\frac{1+w}{2}}\right)^{m-s-1}\\ &\quad+\sum_{j=2}^{m-s-1}\varphi_{p,m-j}(w)\left(1-\sqrt[p]{\frac{1+w}{2}}\right)^{m-s-j}.\end{aligned}$$

Then $\psi_{p,s}(0)=0$ and $\psi_{p,s}(1)=1$. Computing the derivative

$$\begin{aligned}&\psi'_{p,s}(w)=\varphi'_{p,s}(w)+\left(1-\sqrt[p]{\frac{1+w}{2}}\right)^{m-s} + \left(1-\sqrt[p]{\frac{1+w}{2}}\right)^{m-s-1}\\ &-w\left(\frac{1+w}{2}\right)^{\frac{1}{k}-1}\frac{1}{2p}\Bigg[(m-s)\left(1-\sqrt[p]{\frac{1+w}{2}}\right)^{m-s-1}\\ &+(m-s-1)\left(1-\sqrt[p]{\frac{1+w}{2}}\right)^{m-s-2}\Bigg]+\sum_{j=2}^{m-s}\varphi'_{p,m-j}(w)\left(1-\sqrt[p]{\frac{1+w}{2}}\right)^{m-s-j}\\ &-\left(\frac{1+w}{2}\right)^{\frac{1}{k}-1}\frac{1}{2p}\sum_{j=2}^{m-s-1}\varphi_{p,m-j}(w)(m-s-j)\left(1-\sqrt[p]{\frac{1+w}{2}}\right)^{m-s-j-1}.\end{aligned}$$

As in the previous case, it can be seen that for p sufficiently large $\psi'_{p,s}(w)>0$ for all $w\in[0,1]$, which implies $\left|\tilde f^{(s)}_{p,\frac{2}{1+w}}(t)\right|\le 1$. Since $Q_pf(t)=f(t)$ for $t\in[1,+\infty)$, we infer $Q_pf\in B(C^m_b)$ for any p sufficiently large, as claimed. □

Before Lemma 3.8, which will allow us to deduce the continuity of the function Q_p, we need the following lemma.

Lemma 3.7. *Let $p\in\mathbb{N}$ be given. Let $f\in C^m_b$ and assume $\{a_n\}$ to be a sequence in $[1,2]$ such that $a_n\to a$, as $n\to+\infty$. Then, for any $s\in\{0,1,\cdots,m\}$*

we have

$$\left|\tilde{f}^{(s)}_{p,a_n}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)-\tilde{f}^{(s)}_{p,a}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)\right|\to 0 \tag{3.20}$$

and

$$\left|\tilde{f}^{(s)}_{p,a_n}\left(1-\frac{1}{\sqrt[p]{a}}\right)-\tilde{f}^{(s)}_{p,a}\left(1-\frac{1}{\sqrt[p]{a}}\right)\right|\to 0, \tag{3.21}$$

as $n\to+\infty$.

Proof. We will prove (3.20). To calculate the term $\tilde{f}^{(s)}_{p,a}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)$ we will take into account that

$$1-\frac{1}{\sqrt[p]{a_n}}\in\left[1-\frac{1}{\sqrt[p]{a}},1\right]\ \text{if } a\le a_n,\quad\text{and}\quad 1-\frac{1}{\sqrt[p]{a_n}}\in\left[0,1-\frac{1}{\sqrt[p]{a}}\right]\ \text{if } a_n\le a.$$

Assume first $s=m$. Then $\tilde{f}^{(m)}_{p,a_n}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)=f^{(m)}(0)$, and

$$\begin{aligned}
&\tilde{f}^{(m)}_{p,a}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)=f^{(m)}_{p,a}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)=f^{(m)}\left(1-\sqrt[p]{\frac{a}{a_n}}\right), && \text{if } a\le a_n,\\
&\tilde{f}^{(m)}_{p,a}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)=f^{(m)}(0), && \text{if } a_n\le a,
\end{aligned}$$

thus (3.20), trivial in the case $a_n\le a$, follows by the continuity of $f^{(m)}$ in the case $a\le a_n$. We consider now the case $s\in\{0,1,\cdots,m-1\}$. On the one hand

(3.22)

$$\begin{aligned}
\tilde{f}^{(s)}_{p,a_n}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)&=f^{(s)}_{p,a_n}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)\\
&=\frac{1}{\sqrt[p]{a_n^{m-s}}}f^{(s)}(0)+\sum_{j=s}^{m-1}\frac{f^{(j)}(1)}{(j-s)!}\left(1-\frac{1}{\sqrt[p]{a_n^{m-j}}}\right)\left(-\frac{1}{\sqrt[p]{a_n}^{m-j}}\right)^{j-s}.
\end{aligned}$$

Assume first $a\le a_n$, we have

(3.23)

$$\begin{aligned}
\tilde{f}^{(s)}_{p,a}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)&=f^{(s)}_{p,a}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)\\
&=\frac{1}{\sqrt[p]{a^{m-s}}}f^{(s)}\left(1+\sqrt[p]{\frac{a}{a_n}}\right)+\sum_{j=s}^{m-1}\frac{f^{(j)}(1)}{(j-s)!}\left(1-\frac{1}{\sqrt[p]{a^{m-j}}}\right)\left(-\frac{1}{\sqrt[p]{a^{m-j}}}\right)^{j-s}.
\end{aligned}$$

Thus, using (3.22) and (3.23), we find

$$\begin{aligned}
&\left|\tilde{f}^{(s)}_{p,a_n}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)-\tilde{f}^{(s)}_{p,a}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)\right|\le\\
&\left|\frac{1}{\sqrt[p]{a_n^{m-s}}}f^{(s)}(0)-\frac{1}{\sqrt[p]{a^{m-s}}}f^{(s)}\left(1-\sqrt{\frac{a}{a_n}}\right)\right|+\|f\|_m\sum_{j=s}^{m-1}\left|-\frac{1}{\sqrt[p]{a_n^{m-j}}}+\frac{1}{\sqrt[p]{a^{m-j}}}\right|,
\end{aligned}$$

where the right-hand side of the latter inequality goes to zero, as $n \to +\infty$ due to the hypothesis that $a_n \to a$ and the continuity of $f^{(s)}$, so (3.20) follows in the case under consideration.
Now, for $a_n \leq a$ we have

(3.24)
$$\begin{aligned}\tilde{f}^{(s)}_{p,a}\left(1-\frac{1}{\sqrt[p]{a_n}}\right) &= \frac{1}{(m-s)!}f^{(m)}(0)\left(-\frac{1}{\sqrt[p]{a_n}}+\frac{1}{\sqrt[p]{a}}\right)^{m-s}\\ &\quad+\sum_{j=1}^{m-s}\frac{1}{(m-s-j)!}f^{(m-j)}_{p,a}\left(1-\frac{1}{\sqrt[p]{a}}\right)\left(-\frac{1}{\sqrt[p]{a_n}}+\frac{1}{\sqrt[p]{a}}\right)^{m-s-j}\\ &= \frac{1}{(m-s)!}f^{(m)}(0)\left(-\frac{1}{\sqrt[p]{a_n}}+\frac{1}{\sqrt[p]{a}}\right)^{m-s}\\ +\sum_{j=1}^{m-s-1}\frac{1}{(m-s-j)!}&f^{(m-j)}_{p,a}\left(1-\frac{1}{\sqrt[p]{a}}\right)\left(-\frac{1}{\sqrt[p]{a_n}}+\frac{1}{\sqrt[p]{a}}\right)^{m-s-j} - f^{(s)}_{p,a}\left(1-\frac{1}{\sqrt[p]{a}}\right),\end{aligned}$$

where

$$f^{(s)}_{p,a}\left(1-\frac{1}{\sqrt[p]{a}}\right) = \frac{1}{\sqrt[p]{a^{m-s}}}f^{(s)}(0)+\sum_{j=s}^{m-1}\frac{f^{(j)}(1)}{(j-s)!}\left(1-\frac{1}{\sqrt[p]{a^{m-j}}}\right)\left(-\frac{1}{\sqrt[p]{a}}\right)^{j-s}.$$

Using (3.22) and (3.24) we obtain

$$\begin{aligned}&\left|\tilde{f}^{(s)}_{p,a_n}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)-\tilde{f}^{(s)}_{p,a}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)\right|\\ &\leq |f^{(m)}(0)|\left|-\frac{1}{\sqrt[p]{a_n}}+\frac{1}{\sqrt[p]{a}}\right|^{m-s}+\sum_{j=1}^{m-s-1}\left|f^{(m-j)}_{p,a}\left(1-\frac{1}{\sqrt[p]{a}}\right)\right|\left|-\frac{1}{\sqrt[p]{a_n}}+\frac{1}{\sqrt[p]{a}}\right|^{m-s-j}\\ &\qquad+\left|\frac{1}{\sqrt[p]{a_n^{m-s}}}f^{(s)}(0)+\sum_{j=s}^{m-1}\frac{f^{(j)}(1)}{(j-s)!}\left(1-\frac{1}{\sqrt[p]{a_n^{m-j}}}\right)\left(-\frac{1}{\sqrt[p]{a_n}}\right)^{j-s}\right.\\ &\qquad\left.-\frac{1}{\sqrt[p]{a^{m-s}}}f^{(s)}(0)+\sum_{j=s}^{m-1}\frac{f^{(j)}(1)}{(j-s)!}\left(1-\frac{1}{\sqrt[p]{a^{m-j}}}\right)\left(-\frac{1}{\sqrt[p]{a}}\right)^{j-s}\right|\\ &\leq \|f\|_m\left|-\frac{1}{\sqrt[p]{a_n}}+\frac{1}{\sqrt[p]{a}}\right|^{m-s}+\sum_{j=1}^{m-s-1}\|f^{(m-j)}_{p,a}\|_\infty\left|-\frac{1}{\sqrt[p]{a_n}}+\frac{1}{\sqrt[p]{a}}\right|^{m-s-j}\\ &\qquad+\|f\|_m\left|\frac{1}{\sqrt[p]{a_n^{m-s}}}-\frac{1}{\sqrt[p]{a^{m-s}}}\right|+\|f\|_m\sum_{j=s}^{m-1}\left|-\frac{1}{\sqrt[p]{a_n^{m-j}}}+\frac{1}{\sqrt[p]{a^{m-j}}}\right|.\end{aligned}$$

Set $C'_p = 1 + (m-1)\left(1 - \frac{1}{\sqrt[p]{2^m}}\right)$, then in view of (3.9) we obtain

$$\begin{aligned}\left|\tilde{f}^{(s)}_{p,a_n}\left(1-\frac{1}{\sqrt[p]{a_n}}\right) - \tilde{f}^{(s)}_{p,a}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)\right| &\le \|f\|_m \left|-\frac{1}{\sqrt[p]{a_n}} + \frac{1}{\sqrt[p]{a}}\right|^{m-s} \\ &+ C'_p\|f\|_m \sum_{j=1}^{m-s-1}\left|-\frac{1}{\sqrt[p]{a_n}} + \frac{1}{\sqrt[p]{a}}\right|^{m-s-j} \\ &+ \|f\|_m \left|\frac{1}{\sqrt[p]{a_n^{m-s}}} - \frac{1}{\sqrt[p]{a^{m-s}}}\right| + \|f\|_m \sum_{j=s}^{m-1}\left|-\frac{1}{\sqrt[p]{a_n^{m-j}}} + \frac{1}{\sqrt[p]{a^{m-j}}}\right|\end{aligned}$$

and we get (3.20) since the right-hand side of the above inequality goes to zero, due to the fact that $a_n \to a$, as $n \to +\infty$. The proof of (3.21) is similar, so the proof is complete. $\square$

Lemma 3.8. *Let $p \in \mathbb{N}$ be given. Let $f \in C^m_b$, and $\{a_n\}$ a sequence in $[1,2]$ such that $a_n \to a$, as $n \to +\infty$. Then*

$$\lim_{n\to+\infty} \|\tilde{f}_{p,a_n} - \tilde{f}_{p,a}\|_m = 0.$$

Proof. Let $p \in \mathbb{N}$ be fixed. The assert for $f = 0$ is immediate, so we assume $f \in C^m_b$ and $f \neq 0$. We prove, for any $s \in \{0, 1, \cdots, m\}$,

$$\left\|\tilde{f}^{(s)}_{p,a_n} - \tilde{f}^{(s)}_{p,a}\right\|_\infty \to 0, \quad \text{as } n \to +\infty, \tag{3.25}$$

and this will give the thesis. Let $\varepsilon > 0$ be given. Preliminarily, since $f^{(s)}$ is uniformly continuous on $[0,1]$, we find $\delta > 0$ such that, for any $s \in \{0, 1, \cdots, m\}$,

$$|f^{(s)}(t_1) - f^{(s)}(t_2)| \le \varepsilon \tag{3.26}$$

for $t_1, t_2 \in [0,1]$ and $|t_2 - t_1| \le \delta$.
To prove (3.25), we will evaluate $\left|\tilde{f}^{(s)}_{p,a_n}(t) - \tilde{f}^{(s)}_{p,a}(t)\right|$ separately in each of the following cases:

(i) $t \in \left[0, 1 - 1/\sqrt[p]{a_n}\right] \cap [0, 1 - 1/\sqrt[p]{a}]$;
(ii) either $t \in \left[1 - 1/\sqrt[p]{a}, 1 - 1/\sqrt[p]{a_n}\right]$ if $a \le a_n$, or $t \in \left[1 - 1/\sqrt[p]{a_n}, 1 - 1/\sqrt[p]{a}\right]$ if $a_n \le a$;
(iii) $t \in \left[\max\left\{1 - 1/\sqrt[p]{a}, 1 - 1/\sqrt[p]{a_n}\right\},\ 1\right]$;
(iv) $t \in (1, +\infty)$.

For $s = m$, the evaluation of $\left|\tilde{f}^{(m)}_{p,a_n}(t) - \tilde{f}^{(m)}_{p,a}(t)\right|$ will be almost immediate:
(i) and (iv) For $t \in \left(\left[0, 1 - 1/\sqrt[p]{a_n}\right] \cap [0, 1 - 1/\sqrt[p]{a}]\right) \cup (1, +\infty)$ we have

$$\left|\tilde{f}^{(m)}_{p,a_n}(t) - \tilde{f}^{(m)}_{p,a}(t)\right| = 0. \tag{3.27}$$

(ii) Assume $a \le a_n$ and let $t \in \left[1 - 1/\sqrt[p]{a}, 1 - 1/\sqrt[p]{a_n}\right]$. Choose $\bar{n} \in \mathbb{N}$ such that $\left|1 - \sqrt[p]{a/a_n}\right| \le \delta$ for $n > \bar{n}$. Let $n > \bar{n}$, then we obtain

$$\left|\tilde{f}^{(m)}_{p,a_n}(t) - \tilde{f}^{(m)}_{p,a}(t)\right| = \left|f^{(m)}(0) - f^{(m)}(1 + \sqrt[p]{a}(t-1))\right| \le \varepsilon \tag{3.28}$$

as in view of (3.26), denoting $t_1 = 0$ and $t_2 = 1 + \sqrt[p]{a}(t-1)$, we have

$$|t_2 - t_1| = \left|1 + \sqrt[p]{a}(t-1)\right| \leq \left|1 - \sqrt[p]{\frac{a}{a_n}}\right| \leq \delta.$$

The case $a_n \leq a$ similar.

(iii) Let $t \in \left[\max\left\{1 - 1/\sqrt[p]{a}, 1 - 1/\sqrt[p]{a_n}\right\}, \ 1\right]$, then $1 - t < 1$. Choose $\bar{n} \in \mathbb{N}$ such that $|\sqrt[p]{a} - \sqrt[p]{a_n}| \leq \delta$ for $n > \bar{n}$. Thus, for $n > \bar{n}$, we find

(3.29)
$$\left|\tilde{f}^{(m)}_{p,a_n}(t) - \tilde{f}^{(m)}_{p,a}(t)\right| = \left|f^{(m)}(1 + \sqrt[p]{a_n}(t-1)) - f^{(m)}(1 + \sqrt[p]{a}(t-1))\right| \leq \varepsilon,$$

since, as before by (3.26), denoting $t_1 = 1 + \sqrt[p]{a_n}(t-1)$ and $t_2 = 1 + \sqrt[p]{a}(t-1)$, we have

$$|t_2 - t_1| = \left|(\sqrt[p]{a} - \sqrt[p]{a_n})(t-1)\right| \leq \ |\sqrt[p]{a} - \sqrt[p]{a_n}| \leq \delta.$$

Then (3.27), (3.28), (3.29) and the arbitrariness of ε imply (3.25) when $s = m$.

Now we assume $s \in \{0, 1, \cdots, m-1\}$ and again we examine separately each of the cases $(i) - (iv)$:

(i) Let $t \in \left[0, 1 - 1/\sqrt[p]{a_n}\right] \cap [0, 1 - 1/\sqrt[p]{a}]$. Then we have

$$\begin{aligned}\tilde{f}^{(s)}_{p,a}(t) = {} & \frac{1}{(m-s)!} f^{(m)}(0)\left(t - 1 + \frac{1}{\sqrt[p]{a}}\right)^{m-s} \\ & + \sum_{j=1}^{m-s} \frac{1}{(m-s-j)!} f^{(m-j)}_{p,a}\left(1 - \frac{1}{\sqrt[p]{a}}\right)\left(t - 1 + \frac{1}{\sqrt[p]{a}}\right)^{m-s-j},\end{aligned}$$

and analogous formula gives $\tilde{f}^{(s)}_{p,a_n}(t)$, so that, adding and subtracting

$$f^{(m-j)}_{p,a_n}\left(1 - \frac{1}{\sqrt[p]{a_n}}\right)\left(t - 1 + \frac{1}{\sqrt[p]{a}}\right)^{m-s-j},$$

inside the summation sign, we obtain

$$\begin{aligned}\left|\tilde{f}^{(s)}_{p,a_n}(t) - \tilde{f}^{(s)}_{p,a}(t)\right| \leq {} & \frac{1}{(m-s)!}\left|f^{(m)}(0)\right|\left|\left(t - 1 + \frac{1}{\sqrt[p]{a_n}}\right)^{m-s} - \left(t - 1 + \frac{1}{\sqrt[p]{a}}\right)^{m-s}\right| \\ + \sum_{j=1}^{m-s} \frac{1}{(m-s-j)!} & \left|f^{(m-j)}_{p,a_n}\left(1 - \frac{1}{\sqrt[p]{a_n}}\right)\left(t - 1 + \frac{1}{\sqrt[p]{a_n}}\right)^{m-s-j}\right. \\ & \left. - f^{(m-j)}_{p,a_n}\left(1 - \frac{1}{\sqrt[p]{a_n}}\right)\left(t - 1 + \frac{1}{\sqrt[p]{a}}\right)^{m-s-j}\right| \\ + \sum_{j=1}^{m-s} \frac{1}{(m-s-j)!} & \left|f^{(m-j)}_{p,a_n}\left(1 - \frac{1}{\sqrt[p]{a_n}}\right)\left(t - 1 + \frac{1}{\sqrt[p]{a}}\right)^{m-s-j}\right. \\ & \left. - f^{(m-j)}_{p,a}\left(1 - \frac{1}{\sqrt[p]{a}}\right)\left(t - 1 + \frac{1}{\sqrt[p]{a}}\right)^{m-s-j}\right|\end{aligned}$$

$$\leq \frac{1}{(m-s)!}\left|f^{(m)}(0)\right|\left|\left(t-1+\frac{1}{\sqrt[p]{a_n}}\right)^{m-s}-\left(t-1+\frac{1}{\sqrt[p]{a}}\right)^{m-s}\right|$$
$$+\sum_{j=1}^{m-s}\frac{1}{(m-s-j)!}\left|f_{p,a_n}^{(m-j)}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)\right|\left|\left(t-1+\frac{1}{\sqrt[p]{a_n}}\right)^{m-s-j}-\left(t-1+\frac{1}{\sqrt[p]{a}}\right)^{m-s-j}\right|$$
$$+\sum_{j=1}^{m-s}\left|f_{p,a_n}^{(m-j)}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)-f_{p,a}^{(m-j)}\left(1-\frac{1}{\sqrt[p]{a}}\right)\right|\left|t-1+\frac{1}{\sqrt[p]{a}}\right|^{m-s-j}$$

Let us notice that, due to the fact that $\max\left\{t-1+1/\sqrt[p]{a},\ t-1+1/\sqrt[p]{a_n}\right\}\leq 1$, we have (we will apply it for $i=m-s$ and $i=m-s-j$)

$$\left|\left(t-1+\frac{1}{\sqrt[p]{a_n}}\right)^{i}-\left(t-1+\frac{1}{\sqrt[p]{a}}\right)^{i}\right|\leq\left|\frac{1}{\sqrt[p]{a_n}}-\frac{1}{\sqrt[p]{a}}\right|i.$$

Consequently we have

$$\left|\tilde{f}_{p,a_n}^{(s)}(t)-\tilde{f}_{p,a}^{(s)}(t)\right|$$
$$\leq\|f\|_m\left|\frac{1}{\sqrt[p]{a_n}}-\frac{1}{\sqrt[p]{a}}\right|+\sum_{j=1}^{m-s}\left|f_{p,a_n}^{(m-j)}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)\right|\left|\frac{1}{\sqrt[p]{a_n}}-\frac{1}{\sqrt[p]{a}}\right|$$
$$+\sum_{j=1}^{m-s}\left|f_{p,a_n}^{(m-j)}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)-f_{p,a}^{(m-j)}\left(1-\frac{1}{\sqrt[p]{a}}\right)\right|$$
$$\leq\|f\|_m\left|\frac{1}{\sqrt[p]{a_n}}-\frac{1}{\sqrt[p]{a}}\right|+\sum_{j=1}^{m-s}\left|f_{p,a_n}^{(m-j)}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)\right|\left|\frac{1}{\sqrt[p]{a_n}}-\frac{1}{\sqrt[p]{a}}\right|$$
$$\sum_{j=1}^{m-s}\left(\left|f_{p,a_n}^{(m-j)}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)-f_{p,a}^{(m-j)}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)\right|\right.$$
$$\left.+\left|f_{p,a}^{(m-j)}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)-f_{p,a}^{(m-j)}\left(1-\frac{1}{\sqrt[p]{a}}\right)\right|\right).$$

Looking at the last term of the above chain of inequalities we see that it does not depend on t and goes to zero, as $n\to+\infty$. Indeed, the first addend $\|f\|_m\left|\frac{1}{\sqrt[p]{a_n}}-\frac{1}{\sqrt[p]{a}}\right|\to 0$ since by hypothesis $a_n\to a$ as $n\to+\infty$. Using Lemma 3.1 we have

$$\sum_{j=1}^{m-s}\left|f_{p,a_n}^{(m-j)}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)\right|\left|\frac{1}{\sqrt[p]{a_n}}-\frac{1}{\sqrt[p]{a}}\right|\leq(m-s)C_p\|f\|_m\left|\frac{1}{\sqrt[p]{a_n}}-\frac{1}{\sqrt[p]{a}}\right|,$$

which again goes to zero as $n\to+\infty$. For the third addend we have, by Lemma 3.7,

$$\lim_{n\to+\infty}\sum_{j=1}^{m-s}\left|f_{p,a_n}^{(m-j)}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)-f_{p,a_n}^{(m-j)}\left(1-\frac{1}{\sqrt[p]{a}}\right)\right|=0,$$

and, by the continuity of $f_{p,a}^{(m-j)}$ at the point $\left(1-\frac{1}{\sqrt[p]{a}}\right)$,

$$\lim_{n\to+\infty}\sum_{j=1}^{m-s}\left|f_{p,a}^{(m-j)}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)-f_{p,a}^{(m-j)}\left(1-\frac{1}{\sqrt[p]{a}}\right)\right|=0.$$

Therefore we obtain, as desired,

$$\max_{[0,1-1/\sqrt[p]{a_n}]\cap[0,1-1/\sqrt[p]{a}]}\left|\tilde{f}_{p,a_n}^{(s)}(t)-\tilde{f}_{p,a}^{(s)}(t)\right|\to 0,\quad \text{as } n\to+\infty.$$

(ii) We assume $a\le a_n$ and $t\in\left[1-1/\sqrt[p]{a},1-1/\sqrt[p]{a_n}\right]$. We can write

(3.30)

$$\begin{aligned}\left|\tilde{f}_{p,a_n}^{(s)}(t)-\tilde{f}_{p,a}^{(s)}(t)\right| &\le \left|\tilde{f}_{p,a_n}^{(s)}(t)-\tilde{f}_{p,a_n}^{(s)}\left(1-\frac{1}{\sqrt[p]{a}}\right)\right|\\ &+\left|\tilde{f}_{p,a_n}^{(s)}\left(1-\frac{1}{\sqrt[p]{a}}\right)-\tilde{f}_{p,a}^{(s)}\left(1-\frac{1}{\sqrt[p]{a}}\right)\right|+\left|\tilde{f}_{p,a}^{(s)}\left(1-\frac{1}{\sqrt[p]{a}}\right)-\tilde{f}_{p,a}^{(s)}(t)\right|.\end{aligned}$$

We look, separately at each of the three terms of the right-hand side of (3.30). In view of Lemma 3.7, the second term goes to zero, i.e.

$$\left|\tilde{f}_{p,a_n}^{(s)}\left(1-\frac{1}{\sqrt[p]{a}}\right)-\tilde{f}_{p,a}^{(s)}\left(1-\frac{1}{\sqrt[p]{a}}\right)\right|\to 0,\ \text{as } n\to\infty.$$

Looking at the first term we have:

$$\begin{aligned}\tilde{f}_{p,a_n}^{(s)}(t) &= \frac{1}{(m-s)!}f^{(m)}(0)\left(t-1+\frac{1}{\sqrt[p]{a_n}}\right)^{m-s}\\ &+\sum_{j=1}^{m-s}\frac{1}{(m-s-j)!}f_{p,a_n}^{(m-j)}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)\left(t-1+\frac{1}{\sqrt[p]{a_n}}\right)^{m-s-j},\end{aligned}$$

$$\begin{aligned}\tilde{f}_{p,a_n}^{(s)}\left(1-\frac{1}{\sqrt[p]{a}}\right) &= \frac{1}{(m-s)!}f^{(m)}(0)\left(\frac{1}{\sqrt[p]{a_n}}-\frac{1}{\sqrt[p]{a}}\right)^{m-s}\\ &+\sum_{j=1}^{m-s}\frac{1}{(m-s-j)!}f_{p,a_n}^{(m-j)}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)\left(\frac{1}{\sqrt[p]{a_n}}-\frac{1}{\sqrt[p]{a}}\right)^{m-s-j}\end{aligned}$$

and

$$\left|t-1+\frac{1}{\sqrt[p]{a_n}}\right|\le\left|\frac{1}{\sqrt[p]{a_n}}-\frac{1}{\sqrt[p]{a}}\right|.$$

Therefore, also in view of Lemma 3.1, we have

$$\begin{aligned}\left|\tilde{f}_{p,a_n}^{(s)}(t)-\tilde{f}_{p,a_n}^{(s)}\left(1-\frac{1}{\sqrt[p]{a}}\right)\right| &\le 2|f^{(m)}(0)|\left|\frac{1}{\sqrt[p]{a_n}}-\frac{1}{\sqrt[p]{a}}\right|^{m-s}\\ &+2\sum_{j=1}^{m-s-1}\frac{1}{(m-s-j)!}\left|f_{p,a_n}^{(m-j)}\left(1-\frac{1}{\sqrt[p]{a_n}}\right)\right|\left|\frac{1}{\sqrt[p]{a}}-\frac{1}{\sqrt[p]{a_n}}\right|^{m-s-j}\\ &\le 2|f^{(m)}(0)|\left|\frac{1}{\sqrt[p]{a_n}}-\frac{1}{\sqrt[p]{a}}\right|^{m-s}+(m-s-1)C_p\|f\|_m\left|\frac{1}{\sqrt[p]{a_n}}-\frac{1}{\sqrt[p]{a}}\right|^{m-s-j}\end{aligned}$$

which shows that the first term of the right-hand side of (3.30) goes to 0, as $n \to \infty$, independently on t.
As for the third term, since

$$\tilde{f}_{p,a}^{(s)}\left(1-\frac{1}{\sqrt[p]{a}}\right) = f_{p,a}^{(s)}\left(1-\frac{1}{\sqrt[p]{a}}\right) = \frac{1}{\sqrt[p]{a^{m-s}}} f^{(s)}(0) + \sum_{j=s}^{m-1} \frac{f^{(j)}(1)}{(j-s)!}\left(1-\frac{1}{\sqrt[p]{a^{m-j}}}\right)\left(-\frac{1}{\sqrt[p]{a}}\right)^{j-s},$$

$$\tilde{f}_{p,a}^{(s)}(t) = f_{p,a}^{(s)}(t) = \frac{1}{\sqrt[p]{a^{m-s}}} f^{(s)}(1+\sqrt[p]{a}(t-1)) - \sum_{j=s}^{m-1} \frac{f^{(j)}(1)}{(j-s)!}\left(1-\frac{1}{\sqrt[p]{a^{m-j}}}\right)(t-1)^{j-s},$$

we have

$$\left|\tilde{f}_{p,a}^{(s)}\left(1-\frac{1}{\sqrt[p]{a}}\right) - \tilde{f}_{p,a}^{(s)}(t)\right| \le \left|f^{(s)}(0) - f^{(s)}(1+\sqrt[p]{a}(t-1))\right| + \sum_{j=s+1}^{m-1} \frac{|f^{(j)}(1)|}{(j-s)!}\left|1-\frac{1}{\sqrt[p]{a^{m-j}}}\right|\left|\left(-\frac{1}{\sqrt[p]{a}}\right)^{j-s} - (t-1)^{j-s}\right|.$$

Then, due to the fact that

$$\left|\left(-\frac{1}{\sqrt[p]{a}}\right)^{j-s} - (t-1)^{j-s}\right| \le \left|t-1+\frac{1}{\sqrt[p]{a}}\right|(j-s) \le \left|\frac{1}{\sqrt[p]{a}} - \frac{1}{\sqrt[p]{a_n}}\right|(j-s),$$

we infer

$$\left|\tilde{f}_{p,a}^{(s)}\left(1-\frac{1}{\sqrt[p]{a}}\right) - \tilde{f}_{p,a}^{(s)}(t)\right| \le \left|f^{(s)}(0) - f^{(s)}(1+\sqrt[p]{a}(t-1))\right| + \|f\|\left|\frac{1}{\sqrt[p]{a}} - \frac{1}{\sqrt[p]{a_n}}\right|(m-s-1).$$

Therefore, using the hypothesis $a_n \to a$ as $n \to \infty$ and the uniform continuity of $f^{(s)}$, as in (3.28), we obtain

$$\max_{\left[1-1/\sqrt[p]{a},1-1/\sqrt[p]{a_n}\right]}\left|\tilde{f}_{p,a_n}^{(s)}(t) - \tilde{f}_{p,a}^{(s)}(t)\right| \to 0 \quad \text{as } n \to +\infty.$$

The case $a_n \le a$ and $t \in \left[1-1/\sqrt[p]{a_n}, 1-1/\sqrt[p]{a}\right]$ can be carried out similarly.
(iii) Let $t \in \left[\max\left\{1-1/\sqrt[p]{a}, 1-1/\sqrt[p]{a_n}\right\},\ 1\right]$, $(\ 1-t<1)$ then

$$\tilde{f}_{p,a_n}^{(s)}(t) = f_{p,a_n}^{(s)}(t) = \frac{1}{\sqrt[p]{a_n^{m-s}}} f^{(s)}\left(1+\sqrt[p]{a_n}(t-1)\right) + \sum_{j=s}^{m-1} \frac{f^{(j)}(1)}{(j-s)!}\left(1-\frac{1}{\sqrt[p]{a_n^{m-j}}}\right)(t-1)^{j-s}$$

and analogous formula gives $\tilde{f}^{(s)}_{p,a}(t)$. So we have

$$\left|\tilde{f}^{(s)}_{p,a_n}(t)-\tilde{f}^{(s)}_{p,a}(t)\right|$$
$$\leq\left|\frac{1}{\sqrt[p]{a_n^{m-s}}}f^{(s)}\left(1+\sqrt[p]{a_n}(t-1)\right)-\frac{1}{\sqrt[p]{a^{m-s}}}f^{(s)}\left(1+\sqrt[p]{a}(t-1)\right)\right|$$
$$+\left|\sum_{j=s}^{m-1}\frac{f^{(j)}(1)}{(j-s)!}\left(1-\frac{1}{\sqrt[p]{a_n^{m-j}}}\right)(t-1)^{j-s}-\sum_{j=s}^{m-1}\frac{f^{(j)}(1)}{(j-s)!}\left(1-\frac{1}{\sqrt[p]{a^{m-j}}}\right)(t-1)^{j-s}\right|$$

and adding and subtracting $\frac{1}{\sqrt[p]{a^{m-s}}}f^{(s)}\left(1+\sqrt[p]{a_n}(t-1)\right)$ we get

$$\left|\tilde{f}^{(s)}_{p,a_n}(t)-\tilde{f}^{(s)}_{p,a}(t)\right|\leq\left|\frac{1}{\sqrt[p]{a_n^{m-s}}}f^{(s)}\left(1+\sqrt[p]{a_n}(t-1)\right)-\frac{1}{\sqrt[p]{a^{m-s}}}f^{(s)}\left(1+\sqrt[p]{a_n}(t-1)\right)\right|$$
$$+\left|\frac{1}{\sqrt[p]{a^{m-s}}}f^{(s)}\left(1+\sqrt[p]{a_n}(t-1)\right)-\frac{1}{\sqrt[p]{a^{m-s}}}f^{(s)}\left(1+\sqrt[p]{a}(t-1)\right)\right|$$
$$+\sum_{j=s}^{m-1}\frac{|f^{(j)}(1)|}{(j-s)!}|t-1|^{j}\left|\frac{1}{\sqrt[p]{a_n^{m-j}}}-\frac{1}{\sqrt[p]{a^{m-j}}}\right|$$
$$\leq\|f\|_m\left|\frac{1}{\sqrt[p]{a_n^{m-s}}}-\frac{1}{\sqrt[p]{a^{m-s}}}\right|+\left|f^{(s)}\left(1+\sqrt[p]{a_n}(t-1)\right)-f^{(s)}\left(1+\sqrt[p]{a}(t-1)\right)\right|$$
$$+\|f\|_m\sum_{j=s}^{m-1}\left|\frac{1}{\sqrt[p]{a_n^{m-j}}}-\frac{1}{\sqrt[p]{a^{m-j}}}\right|.$$

Now, using the hypothesis $a_n\to a$ as $n\to\infty$ and the uniform continuity , as in (3.29) of $f^{(s)}$, we obtain

$$\max_{[\max\{1-1/\sqrt[p]{a},1-1/\sqrt[p]{a_n}\},\ 1]}\left|\tilde{f}^{(s)}_{p,a_n}(t)-\tilde{f}^{(s)}_{p,a}(t)\right|\to 0\quad\text{as } n\to+\infty.$$

(iv) If $t>1$ we have $\left|\tilde{f}^{(s)}_{p,a_n}(t)-\tilde{f}^{(s)}_{p,a}(t)\right|=0$.
The proof is complete. □

4. The mapping Q_p

In this section, first we prove that, for $p\in\mathbb{N}$, the mapping Q_p is C_p-ball-contractive, with C_p given in (3.1). In other words, in view of Corollary 3.2, there exists $\varepsilon_p>0$, with $\lim_{p\to\infty}\varepsilon_p=0$, such that Q_p is $(1+\varepsilon_p)$-ball-contractive. Then we prove that Q_p, at least for large p, has positive lower γ-norm.

Proposition 4.1. *For any $p\in\mathbb{N}$, the mapping Q_p is C_p-ball-contractive.*

Proof. Let $\{f_n\}$ be a sequence in $B(C_b^m)$ and f a function in $B(C_b^m)$ such that $\|f_n-f\|_m\to 0$, as $n\to+\infty$. Then, by definition of Q_p,

$$\|Q_pf_n-Q_pf\|_m=\|(\tilde{f}_n)_{p,a_n}-\tilde{f}_{p,a}\|_m,$$

for $a_n = \frac{2}{1+\|f_n\|_m}$ and $a = \frac{2}{1+\|f\|_m}$, so that $a_n \in [1,2]$ for each $n \in \mathbb{N}$, $a \in [1,2]$ and $a_n \to a$, as $n \to +\infty$. Since by the hypothesis and Lemma 3.8 we have

$$\begin{aligned}\|(\tilde{f}_n)_{p,a_n} - \tilde{f}_{p,a}\|_m &\le \|(\tilde{f}_n)_{p,a_n} - \tilde{f}_{p,a_n}\|_m + \|\tilde{f}_{p,a_n} - \tilde{f}_{p,a}\|_m \\ &= \|(\tilde{f}_n - \tilde{f})_{p,a_n}\|_m + \|\tilde{f}_{p,a_n} - \tilde{f}_{p,a}\|_m \\ &\le C_p\|f_n - f\|_m + \|\tilde{f}_{p,a_n} - \tilde{f}_{p,a}\|_m \to 0,\end{aligned}$$

we obtain that the mapping Q_p is continuous. To conclude we have to show that for $M \subseteq B(C_b^m)$

$$\gamma(Q_pM) \le C_p\gamma(M).$$

First we observe that for $\varphi \in C_b^m$ the subset $A_{p,\varphi} = \{\tilde{\varphi}_{p,a} : a \in [1,2]\}$ of C_b^m is compact. Indeed, if $\{\tilde{\varphi}_{p,a_n}\}$ is a sequence of elements in $A_{p,\varphi}$ and $\{a_{n_k}\}$ a subsequence of $\{a_n\}$ which is convergent, say to a, then by Lemma 3.8 we have $\|\tilde{\varphi}_{p,a_{n_k}} - \tilde{\varphi}_{p,a}\|_m \to 0$. Now let $\alpha > \gamma(M)$. Let $\{\varphi_1, \cdots, \varphi_l\}$ be an α-net for M in C_b^m. Then the set $A_k = \bigcup_{i=1}^{l} A_{p,\varphi_i}$ is a compact subset of C_b^m. Thus, given $\delta > 0$ we choose a δ-net $\{\psi_1, \cdots, \psi_p\}$ for A_k in C_b^m.

For $g \in Q_pM$ arbitrarily fixed, let $f \in M$ such that $Q_pf = g$. Then let $i \in \{1, \cdots, l\}$ be such that $\|f - \varphi_i\|_m \le \alpha$ and $j \in \{1, \cdots, p\}$ be such that

$$\|(\tilde{\varphi}_i)_{p,a} - \psi_j\|_m \le \delta, \qquad \text{for } a = \frac{2}{1+\|f\|_m}.$$

Then by Lemma 3.1 we obtain

$$\begin{aligned}\|g - \psi_j\|_m = \|Q_pf - \psi_j\|_m &= \|\tilde{f}_{p,a} - \psi_j\|_m \\ &\le \|\tilde{f}_{p,a} - (\tilde{\varphi}_i)_{p,a}\| + \|(\tilde{\varphi}_i)_{p,a} - \psi_j\|_m \\ &\le C_p\|f - \varphi_l\ \|_m + \delta \le C_p\alpha + \delta,\end{aligned}$$

that is, $\gamma(Q_pM) \le C_p\alpha + \delta$. The arbitrariness of δ gives the desired result $\gamma(Q_pM) \le C_p\gamma(M)$. □

Our next aim is to prove $\frac{D_p}{m+1}\gamma(M) \le \gamma(Q_pM)$, for $M \subseteq B(C_b^m)$. To this end, given $p \in \mathbb{N}$, $g \in C_b^m$ and $a \in [1,2]$ we introduce $g^{p,a} : [0,+\infty) \to \mathbb{R}$, in such a way to have $g^{p,a} \in C_b^m$, by setting

$$g^{p,a}(t) = \begin{cases} \sqrt[p]{a^m}\, g\left(1 + \frac{1}{\sqrt[p]{a}}(t-1)\right) + \sum_{j=0}^{m-1} \frac{g^{(j)}(1)}{j!}\left(1 - \sqrt[p]{a^{m-j}}\right)(t-1)^j & \text{if } t \in [0,1] \\ g(t) & \text{if } t \in (1,+\infty). \end{cases}$$

Computing the derivatives

$$\left(g^{p,a}\right)^{(s)}(t) = \begin{cases} \sqrt[p]{a^{m-s}}\, g^{(s)}\left(1 + \frac{1}{\sqrt[p]{a}}(t-1)\right) + \\ \qquad \sum_{j=s}^{m-1} \frac{g^{(j)}(1)}{(j-s)!}\left(1 - \sqrt[p]{a^{m-j}}\right)(t-1)^{j-s} & \text{if} \quad t \in [0,1] \\ g^{(s)}(t) & \text{if} \quad t \in (1,+\infty), \end{cases}$$

for $s \in \{0, 1, \cdots, m-1\}$ and

$$(g^{p,a})^{(m)}(t) = \begin{cases} g^{(m)}\left(1 + \frac{1}{\sqrt[p]{a}}(t-1)\right) & \text{if } t \in [0,1] \\ g^{(m)}(t) & \text{if } t \in (1, +\infty). \end{cases}$$

We need the following lemma.

Lemma 4.2. *Let $p \in \mathbb{N}$. Let $f \in B(C_b^m)$, $g \in C_b^m$ and $a \in [1,2]$. Then*

$$\left\| \tilde{f}_{p,a} - (\widetilde{g^{p,a}})_{p,a} \right\|_m \le (m+1) \left\| \tilde{f}_{p,a} - g \right\|_m .$$

Proof. Let $p \in \mathbb{N}$. Let $f \in B(C_b^m)$, $g \in C_b^m$ and $a \in [1,2]$. Let us write explicitly $(\widetilde{g^{p,a}})_{p,a}$, we have

$$(\widetilde{g^{p,a}})_{p,a}(t) = \begin{cases} \sum_{i=0}^{m} \frac{1}{i!}\, g^{(i)}\left(1 - \frac{1}{\sqrt[p]{a}}\right)\left(t - 1 + \frac{1}{\sqrt[p]{a}}\right)^i & \text{if } t \in \left[0, 1 - \frac{1}{\sqrt[p]{a}}\right] \\ g(t) & \text{if } t \in \left(1 - \frac{1}{\sqrt[p]{a}}, +\infty\right). \end{cases}$$

Moreover, for $s = 1, \cdots, m$, we have

(4.1)

$$(\widetilde{g^{p,a}})^{(s)}_{p,a}(t) = \begin{cases} \sum_{i=0}^{m-s} \frac{1}{i!}\, g^{(s+i)}\left(1 - \frac{1}{\sqrt[p]{a}}\right)\left(t - 1 + \frac{1}{\sqrt[p]{a}}\right)^i & \text{if } t \in \left[0, 1 - \frac{1}{\sqrt[p]{a}}\right] \\ g^{(s)}(t) & \text{if } t \in \left(1 - \frac{1}{\sqrt[p]{a}}, +\infty\right), \end{cases}$$

which, in particular, for $s = m$ reduces to

$$(\widetilde{g^{p,a}})^{(m)}_{p,a}(t) = \begin{cases} g^{(m)}\left(1 - \frac{1}{\sqrt[p]{a}}\right) & \text{if } \quad t \in \left[0, 1 - \frac{1}{\sqrt[p]{a}}\right] \\ g^{(m)}(t) & \text{if } \quad t \in \left(1 - \frac{1}{\sqrt[p]{a}}, +\infty\right). \end{cases}$$

To prove the thesis we will show that, for $s = 0, 1, \cdots, m$,

$$\left\| \tilde{f}^{(s)}_{p,a} - (\widetilde{g^{p,a}})^{(s)}_{p,a} \right\|_\infty \le (m+1) \left\| \tilde{f}_{p,a} - g \right\|_m .$$

??? Since, for each s,

$$\left\| \tilde{f}^{(s)}_{p,a} - (\widetilde{g^{p,a}})^{(s)}_{p,a} \right\|_\infty = \Big\{ \max_{t \in \left[0, 1 - \frac{1}{\sqrt[p]{a}}\right]} \left| \tilde{f}^{(s)}_{p,a}(t) - (\widetilde{g^{p,a}})^{(s)}_{p,a}(t) \right|, \sup_{t \in \left(1 - \frac{1}{\sqrt[p]{a}}, +\infty\right)} \left| \tilde{f}^{(s)}_{p,a}(t) - g^{(s)}(t) \right| \Big\},$$

it suffices to prove

$$\max_{t \in \left[0, 1 - \frac{1}{\sqrt[p]{a}}\right]} \left| \tilde{f}^{(s)}_{p,a}(t) - (\widetilde{g^{p,a}})^{(s)}_{p,a}(t) \right| \le (m+1) \left\| \tilde{f}_{p,a} - g \right\|_m . \tag{4.2}$$

Let us consider first the case $s = m$. Since, for $t \in \left[0, 1 - \frac{1}{\sqrt[p]{a}}\right]$, $\tilde{f}^{(m)}_{p,a}(t) = f^{(m)}(0) = \tilde{f}^{(m)}_{p,a}\left(1 - \frac{1}{\sqrt[p]{a}}\right)$, we have

$$\max_{t \in \left[0, 1 - \frac{1}{\sqrt[p]{a}}\right]} \left| \tilde{f}^{(m)}_{p,a}(t) - (\widetilde{g^{p,a}})^{(m)}_{p,a}(t) \right| = \left| \tilde{f}^{(m)}_{p,a}\left(1 - \frac{1}{\sqrt[p]{a}}\right) - g^{(m)}\left(1 - \frac{1}{\sqrt[p]{a}}\right) \right|$$
$$\leq \left\| \tilde{f}^{(m)}_{p,a} - g^{(m)} \right\|_\infty \leq \left\| \tilde{f}_{p,a} - g \right\|_m,$$

hence (4.2) holds. Next let $s \in \{0, 1, \cdots, m-1\}$. Let $t \in \left[0, 1 - \frac{1}{\sqrt[p]{a}}\right]$. Then

$$\tilde{f}^{(s)}_{p,a}(t) = \frac{1}{(m-s)!} f^{(m)}(0) \left(t - 1 + \frac{1}{\sqrt[p]{a}}\right)^{m-s}$$
$$+ \sum_{j=1}^{m-s} \frac{1}{(m-s-j)!} f^{(m-j)}_{p,a}\left(1 - \frac{1}{\sqrt[p]{a}}\right)\left(t - 1 + \frac{1}{\sqrt[p]{a}}\right)^{m-s-j}.$$

Since $f^{(m)}(0) = f^{(m)}_{p,a}\left(1 - \frac{1}{\sqrt[p]{a}}\right)$ and $f^{(s)}_{p,a}\left(1 - \frac{1}{\sqrt[p]{a}}\right) = \tilde{f}^{(s)}_{p,a}\left(1 - \frac{1}{\sqrt[p]{a}}\right)$, for all s, we can write

$$\tilde{f}^{(s)}_{p,a}(t) = \sum_{j=0}^{m-s} \frac{1}{(m-s-j)!} \tilde{f}^{(m-j)}_{p,a}\left(1 - \frac{1}{\sqrt[p]{a}}\right)\left(t - 1 + \frac{1}{\sqrt[p]{a}}\right)^{m-s-j}.$$

Moreover, changing the summation index (letting $j = m - s - i$) in (4.1) we can write

$$(\widetilde{g^{p,a}})^{(s)}_{p,a}(t) = \sum_{j=0}^{m-s} \frac{1}{(m-s-j)!} g^{(m-j)}\left(1 - \frac{1}{\sqrt[p]{a}}\right)\left(t - 1 + \frac{1}{\sqrt[p]{a}}\right)^{m-s-j}.$$

Therefore we obtain

$$\max_{t \in \left[0, 1 - \frac{1}{\sqrt[p]{a}}\right]} \left| \tilde{f}^{(s)}_{p,a}(t) - (\widetilde{g^{p,a}})^{(s)}_{p,a}(t) \right| \leq \sum_{j=0}^{m-s} \left| \tilde{f}^{(m-j)}_{p,a}\left(1 - \frac{1}{\sqrt[p]{a}}\right) - g^{(m-j)}\left(1 - \frac{1}{\sqrt[p]{a}}\right) \right|$$
$$\leq \sum_{j=0}^{m-s} \left\| \tilde{f}^{(m-j)}_{p,a} - g^{(m-j)} \right\|_\infty \leq (m+1) \left\| \tilde{f}_{p,a} - g \right\|_m.$$

Hence (4.2) is proved and the proof is complete. $\square$

Now given $p \in \mathbb{N}$, let D_p as given in (3.2), then we have

PROPOSITION 4.3. *Let $p \in \mathbb{N}$. Given $M \subseteq B(C^m_b)$ we have*

$$\frac{D_p}{m+1}\gamma(M) \leq \gamma(Q_p M). \tag{4.3}$$

In particular, the following estimate of the lower Hausdorff measure of noncompactness $\omega(Q_p)$ of Q_p holds:

$$\omega(Q_p) \geq \frac{D_p}{m+1}.$$

Proof. Let $p \in \mathbb{N}$, $M \subseteq B(C_b^m)$ and $\eta > \gamma(Q_p M)$. Fix an η-net $\{\lambda_1, \cdots, \lambda_q\}$ for $Q_p M$ in C_b^m. Similarly as in Lemma 3.8 it can be proved that given $\lambda \in C_b^m$ and a sequence $\{a_n\}$ in $[1,2]$ such that $a_n \to a$, then $\|\lambda^{p,a_n} - \lambda^{p,a}\|_m \to 0$. Then we have that $A^{p,\lambda_i} = \{\lambda_i^{p,a} : a \in [1,2]\}$ is a compact subset of C_b^m and therefore $A^p = \bigcup_{i=1}^q A^{p,\lambda_i}$ is a compact set in C_b^m. Hence, given $\delta > 0$ we choose a δ-net $\{\xi_1, \cdots, \xi_r\}$ for A^p in C_b^m.
Now let $f \in M$. Fix $i \in \{1, \cdots, q\}$ such that $\|Q_p f - \lambda_i\|_m \le \eta$. Since $(\lambda_i)^{p,\frac{2}{1+\|f\|_m}}$ is in A^p, we can choose $j \in \{1, \cdots, r\}$ such that $\|(\lambda_i)^{\frac{2}{1+\|f\|_m}} - \xi_j\|_m \le \delta$. Then, also in view of Lemma 3.1

$$\begin{aligned}\|f - \xi_j\|_m &\le \|f - (\lambda_i)^{p,\frac{2}{1+\|f\|_m}}\|_m + \|(\lambda_i)^{p,\frac{2}{1+\|f\|_m}} - \xi_j\|_m \\ &\le \frac{1}{D_p}\left\|\tilde{f}_{p,\frac{2}{1+\|f\|_m}} - (\widetilde{(\lambda_i)^{p,a}})_{p,\frac{2}{1+\|f\|_m}}\right\|_m + \delta.\end{aligned}$$

By Lemma 4.2 we have

$$\left\|\tilde{f}_{p,\frac{2}{1+\|f\|_m}} - (\widetilde{(\lambda_i)^{p,\frac{2}{1+\|f\|_m}}})_{p,\frac{2}{1+\|f\|_m}}\right\|_m \le (m+1)\|\tilde{f}_{p,\frac{2}{1+\|f\|_m}} - \lambda_i\|_m,$$

hence we obtain

$$\begin{aligned}\|f - \xi_j\|_m &\le \frac{m+1}{D_p}\left\|\tilde{f}_{p,\frac{2}{1+\|f\|_m}} - \lambda_i\right\|_m + \delta \\ &= \frac{m+1}{D_p}\|Q_p f - \lambda_i\|_m + \delta \le \frac{m+1}{D_p}\,\eta + \delta.\end{aligned}$$

Therefore $\gamma(M) \le ((m+1)/D_p)\,\eta + \delta$, so that

$$\frac{D_p}{m+1}\gamma(M) \le \eta + \frac{D_p}{m+1}\delta,$$

which by the arbitrariness of δ gives (4.3). Thus the proof is complete. □

5. The mapping $P_{u,p}$

For $p \in \mathbb{N}$ and $u > 0$, we define $P_{u,p} : B(C_b^m) \to C_b^m$ by setting

$$(P_{u,p}f)(t) = \begin{cases} -\frac{u}{(m+1)!}\left(t - 1 + \sqrt[p]{\frac{1+\|f\|_m}{2}}\right)^{m+1} & \text{if} \quad t \in \left[0, 1 - \sqrt[p]{\frac{1+\|f\|_m}{2}}\right] \\ 0 & \text{if} \quad t \in \left(1 - \sqrt[p]{\frac{1+\|f\|_m}{2}}, +\infty\right). \end{cases}$$

We observe that if f and $g \in B(C_b^m)$ and $\|f\|_m = \|g\|_m$ we have $P_{u,p}f = P_{u,p}g$, in particular $P_{u,p}f$ coincides with the null function if $\|f\|_m = 1$.
Clearly $P_{u,p}f \in C_b^m$, and for $s = 0, 1, \cdots, m$ we have

$$(P_{u,p}f)^{(s)}(t) = \begin{cases} -\frac{u}{(m+1-s)!}\left(t - 1 + \sqrt[p]{\frac{1+\|f\|_m}{2}}\right)^{m+1-s} & \text{if} \quad t \in \left[0, 1 - \sqrt[p]{\frac{1+\|f\|_m}{2}}\right] \\ 0 & \text{if} \quad t \in \left(1 - \sqrt[p]{\frac{1+\|f\|_m}{2}}, +\infty\right). \end{cases}$$

Lemma 5.1. *Let $p \in \mathbb{N}$ and $u > 0$. Let $\{f_n\}$ be a sequence in $B(C_b^m)$ and $f \in B(C_b^m)$ such that $\|f_n\|_m \to \|f\|_m$, then*

$$\|P_{u,p}f_n - P_{u,p}f\|_m \to 0.$$

Proof. We will show that for each $s = 0, 1, \cdots, m$ we have

$$\|(P_{u,p}f_n)^{(s)} - (P_{u,p}f)^{(s)}\|_\infty \to 0, \tag{5.1}$$

To this end, fix $s \in \{0, 1, \cdots, m\}$ and $\varepsilon > 0$. Find $\overline{n}$ such that for all $n \geq \overline{n}$ we have

$$\left| \sqrt[p]{\frac{1 + \|f\|_m}{2}} - \sqrt[p]{\frac{1 + \|f_n\|_m}{2}} \right| \leq \frac{\varepsilon}{u}.$$

Let $n \geq \overline{n}$. We will prove

$$\left|(P_{u,p}f_n)^{(s)}(t) - (P_{u,p}f)^{(s)}(t)\right| \leq \varepsilon, \quad \text{for all } t \in [0, +\infty). \tag{5.2}$$

If $t \in \left[0, 1 - \sqrt[p]{\frac{1+\|f\|_m}{2}}\right] \cap \left[0, 1 - \sqrt[p]{\frac{1+\|f_n\|_m}{2}}\right]$, then

$$\begin{aligned} &\left|(P_{u,p}f_n)^{(s)}(t) - (P_{u,p}f)^{(s)}(t)\right| \\ \leq\ & u \left| \left(t - 1 + \sqrt[p]{\tfrac{1+\|f_n\|_m}{2}}\right)^{m+1-s} - \left(t - 1 + \sqrt[p]{\tfrac{1+\|f\|_m}{2}}\right)^{m+1-s} \right| \\ \leq\ & u \left| \sqrt[p]{\tfrac{1+\|f_n\|_m}{2}} - \sqrt[p]{\tfrac{1+\|f\|_m}{2}} \right| (m + 1 - s) \leq u\tfrac{\varepsilon}{u} = \varepsilon. \end{aligned}$$

Assume now $\|f\|_m \leq \|f_n\|_m$ and $t \in \left[1 - \sqrt[p]{\frac{1+\|f_n\|_m}{2}}, 1 - \sqrt[p]{\frac{1+\|f\|_m}{2}}\right]$, then

$$\begin{aligned} \left|(P_{u,p}f_n)^{(s)}(t) - (P_{u,p}f)^{(s)}(t)\right| &\leq u \left| t - 1 + \sqrt[p]{\tfrac{1+\|f\|_m}{2}} \right|^{m+1-s} \\ &\leq u \left| \sqrt[p]{\tfrac{1+\|f_n\|_m}{2}} - \sqrt[p]{\tfrac{1+\|f\|_m}{2}} \right|^{m+1-s} \\ &\leq u \left| \sqrt[p]{\tfrac{1+\|f_n\|_m}{2}} - \sqrt[p]{\tfrac{1+\|f\|_m}{2}} \right| \leq u\tfrac{\varepsilon}{u} = \varepsilon. \end{aligned}$$

If we assume $\|f_n\|_m \leq \|f\|_m$ and $t \in \left[1 - \sqrt[p]{\frac{1+\|f\|_m}{2}}, 1 - \sqrt[p]{\frac{1+\|f_n\|_m}{2}}\right]$, then in a similarly way we find

$$\left|(P_{u,p}f_n)^{(s)}(t) - (P_{u,p}f)^{(s)}(t)\right| \leq \varepsilon.$$

Since for $t \in \left(\max\left\{1 - \sqrt[p]{\frac{1+\|f\|_m}{2}},\ 1 - \sqrt[p]{\frac{1+\|f_n\|_m}{2}}\right\},\ +\infty\right)$ we have

$$(P_{u,p}f_n)^{(s)}(t) = (P_{u,p}f)^{(s)}(t) = 0,$$

the proof is complete. ∎

Proposition 5.2. *Let $u > 0$. The mapping $P_{u,p}$ is compact.*

Proof. Let $\{f_n\}$ be a sequence in $B(C_b^m)$ and $f \in B(C_b^m)$ such that $\|f_n - f\|_m \to 0$. Then $\|f_n\|_m \to \|f\|_m$, and Lemma 5.1 implies that $P_{u,p}$ is continuous.

Now we prove that the mapping $P_{u,p}$ is sequentially-compact. To this end let $\{g_n\}$ be a sequence in $P_{u,p}(B(C_b^m))$. For each $n \in \mathbb{N}$ fix $h_n \in B(C_b^m)$ such that $g_n = P_{u,p}h_n$. Passing, if necessary, to a subsequence, we may assume without loss of generality that $\|h_n\|_m \to c \in [0,1]$. Now we choose $h \in B(C_b^m)$ such that $\|h\|_m = c$ so that $\|h_n\|_m \to \|h\|_m$. Set $g := P_{u,p}h$. Since $\|g_n - g\|_m = \|P_{u,p}h_n - P_{u,p}h\|_m$, Lemma 5.1 implies $\|g_n - g\|_m \to 0$, as desired. □

6. The retraction $R_{u,p}$

Let $p \in \mathbb{N}$. Let $u > 0$ be arbitrarily fixed. We define $T_{u,p} : B(C_b^m) \to C_b^m$, by setting

$$T_{u,p} = Q_p + P_{u,p}.$$

The mapping $T_{u,p}$, being a compact perturbation of Q_p, is C_p-ball-contractive. Our first step is that of proving that $\inf_{f\in B(C_b^m)} \|T_{u,p}f\|_m > 0$ (next Proposition 6.2). To this end, preliminarily let us consider the function $h_{u,p} : [0,1] \to \mathbb{R}$, defined by

$$h_{u,p}(c) = \frac{u}{2}\left(1 - \sqrt[p]{\frac{1+c}{2}}\right) - c, \quad \text{for } c \in [0,1].$$

Since $h_{u,p}(0)h_{u,p}(1) < 0$ and $h_{u,p}$ is strictly decreasing on $[0,1]$, there exists a unique solution $c_{u,p} \in (0,1)$ of the equation

$$c = \frac{u}{2}\left(1 - \sqrt[p]{\frac{1+c}{2}}\right). \tag{6.1}$$

Observe that, for any fixed p, we have

$$\lim_{u\to+\infty} c_{u,p} = 1. \tag{6.2}$$

Moreover, the following lemma holds true.

Lemma 6.1. *Let $p \in \mathbb{N}$ and $u > 0$. Given $f \in B(C_b^m)$, if*

$$\|f\|_m \le c_{u,p}$$

where $c_{u,p} \in (0,1)$ is the unique solution of the equation

$$c = \frac{u}{2}\left(1 - \sqrt[p]{\frac{1+c}{2}}\right),$$

then we have

$$\max\left\{-\|f^{(m)}\|_\infty + u\left(1 - \sqrt[p]{\frac{1+\|f\|_m}{2}}\right),\ \|f^{(m)}\|_\infty\right\} \ge c_{u,p}.$$

PROOF. Let $p \in \mathbb{N}$. Let $u > 0$. Then, for every $c \in [0,1]$, we define the auxiliary function $\varphi_{c,p} : [0,c] \to \mathbb{R}$ by setting

$$\varphi_{c,p}(x) = -x + u\left(1 - \sqrt[p]{\frac{1+c}{2}}\right), \quad \text{for } x \in [0,c].$$

Further, we set

$$\xi_{u,p} = \max\{c : \ c \in [0,1] \text{ and } \varphi_{c,p}(x) \geq x \quad \text{for } x \in [0,c]\}.$$

Since, for every $c \in [0,1]$, the function $\varphi_{c,p}$ is decreasing on $[0,c]$, we have that $\xi_{u,p} = c_{u,p}$. Then, for every $c \in [0, c_{u,p}]$ the function $\psi_{c,p} : [0,1] \to \mathbb{R}$ defined by

$$\psi_{c,p}(x) = \max\{x, \varphi_{c,p}(x)\} = \max\left\{x, \ -x + u\left(1 - \sqrt[p]{\frac{1+c}{2}}\right)\right\}$$

satisfies

$$\min_{x\in[0,c]} \psi_{c,p}(x) \geq c_{u,p}. \tag{6.3}$$

Now let $f \in B(C_b^m)$ with $\|f\|_m \leq c_{u,p}$. Then the result follows by (6.3) considering $c = \|f\|_m$ and setting $x = \|f^{(m)}\|_\infty$. $\square$

Having in mind the constant D_p given in (3.2), without loss of generality we may assume $D_p > 0$. We prove the following result.

PROPOSITION 6.2. *Let $p \in \mathbb{N}$, $u > 0$ and $f \in B(C_b^m)$. Then*

$$\|T_{u,p}f\|_m \geq D_p \ c_{u,p}.$$

PROOF. Fix $p \in \mathbb{N}$ and $u > 0$. Let $f \in B(C_b^m)$. Assume first $\|f\|_m \leq c_{u,p}$. We have

$$\begin{aligned}
&\|T_{u,p}f\|_m \geq \|(T_{u,p}f)^{(m)}\|_\infty = \sup\nolimits_{t\in[0,+\infty)} \left|(T_{u,p}f)^{(m)}(t)\right| \\
&= \max\Bigg\{ \max\nolimits_{t\in\left[0,1-\sqrt[p]{\frac{1+\|f\|_m}{2}}\right]} \left| f^{(m)}(0) - u\left(t - 1 + \sqrt[p]{\tfrac{1+\|f\|_m}{2}}\right)\right|, \\
&\qquad \max\nolimits_{t\in\left[1-\sqrt[p]{\frac{1+\|f\|_m}{2}},1\right]} \left| f^{(m)}\left(1 + \sqrt[p]{\tfrac{2}{1+\|f\|_m}}(t-1)\right)\right|, \ \sup\nolimits_{t\in(1,+\infty)} |f^{(m)}(t)| \Bigg\} \\
&\geq \max\left\{ f^{(m)}(0) + u\left(1 - \sqrt[p]{\tfrac{1+\|f\|_m}{2}}\right), \ \max\nolimits_{t\in[0,1]} |f^{(m)}(t)|, \ \sup\nolimits_{t\in(1,+\infty)} |f^{(m)}(t)| \right\} \\
&\geq \max\left\{ -\|f^{(m)}\|_\infty + u\left(1 - \sqrt[p]{\tfrac{1+\|f\|_m}{2}}\right), \ \|f^{(m)}\|_\infty \right\}.
\end{aligned}$$

Thus, in view of Lemma 6.1 we obtain $\|T_{u,p}f\|_m \geq c_{u,p}$.
Now assume $c_{u,p} \leq \|f\|_m \leq 1$, and let $s \in \{0, \cdots, m\}$ such that $\|f\|_m = \|f^{(s)}\|_\infty$. We distinguish two cases, that is, whether or not $s = m$. In the first case, $s = m$,

we have

$$\begin{aligned}&\|T_{u,p}f\|_m \geq \|(T_{u,p}f)^{(m)}\|_\infty\\ &\geq \max\left\{\max_{t\in\left[1-\sqrt[p]{\frac{1+\|f\|_m}{2}},1\right]}\left|f^{(m)}\left(1+\sqrt[p]{\tfrac{2}{1+\|f\|_m}}(t-1)\right)\right|,\ \sup_{t\in(1,+\infty)}|f^{(m)}(t)|\right\}\\ &= \|f^{(m)}\|_\infty = \|f\|_m \geq c_{u,p}.\end{aligned}$$

In the case in which $s\in\{0,\cdots,m-1\}$, if $\|f^{(s)}\|_\infty = \sup_{t\in(1,+\infty)}|f^{(s)}(t)|$ we have

$$\begin{aligned}\|T_{u,p}f\|_m &\geq \|(T_{u,p}f)^{(s)}\|_\infty\\ &\geq \sup_{t\in(1,+\infty)}|f^{(s)}(t)| = \|f^{(s)}\|_\infty = \|f\|_m \geq c_{u,p}.\end{aligned}$$

Finally, always in the case $s\in\{0,\cdots,m-1\}$, if $\|f^{(s)}\|_\infty = \max_{t\in[0,1]}\left|f^{(s)}(t)\right|$ we have

$$\begin{aligned}\|T_{u,p}f\|_m &\geq \|(T_{u,p}f)^{(s)}\|_\infty\\ &\geq \max_{t\in\left[1-\sqrt[p]{\frac{1+\|f\|_m}{2}},1\right]}\left|f^{(s)}_{p,\frac{2}{1+\|f\|_m}}(t)\right|.\end{aligned}$$

Therefore using Lemma 3.1

$$\|T_{u,p}f\|_m \geq D_p\ \|f\|_m \geq D_p\ c_{u,p},$$

and this completes the proof. □

We are now in a position to prove our main result.

Theorem 6.3. *For any $\varepsilon > 0$ there exists a proper k-ball-contractive retraction of the closed unit ball $B(C^m_b)$ onto $S(C^m_b)$ with $k < 1+\varepsilon$, so that $W_\gamma(C^m_b) = 1$.*

Proof. Given $u > 0$, in view of Proposition 6.2, we have $\|T_{u,p}f\|_m > 0$ so we can define a retraction $R_{u,p} : B(C^m_b) \to S(C^m_b)$ by setting

$$R_{u,p}f = \frac{1}{\|T_{u,p}f\|_m}T_{u,p}f.$$

Let now $M \subseteq B(C^m_b)$. Since $P_{u,p}$ is a compact mapping, from Proposition 4.1 and Proposition 4.3 it follows that

$$\frac{D_p}{m+1}\ \gamma(M) \leq \gamma(T_{u,p}M) \leq C_p\ \gamma(M). \tag{6.4}$$

Moreover by the definition of $R_{u,p}$ and by Proposition 6.2 we get

$$R_{u,p}M \subseteq \left[0,\frac{1}{D_p c_{u,p}}\right]\cdot T_{u,p}M.$$

Therefore using the property of absorption invariance of γ and the right hand side of (6.4) we infer

$$\gamma(R_{u,p}M) \leq \frac{C_p}{D_p c_{u,p}}\gamma(M),$$

this means that the retraction $R_{u,p}$ is $k_{u,p}$- ball-contractive with $k_{u,p} = C_p/(D_p c_{u,p})$. On the other hand from Lemma 3.1 and the definition of $P_{u,p}$ we get

$$\|T_{u,p}f\|_m \leq \|Q_p f\|_m \; + \|P_{u,p}f\|_m \; \leq C_p + \frac{u}{2}$$

for all $f \in B(C_b^m)$, and so we have

$$T_{u,p}M \subseteq \left[0, \; C_p + \frac{u}{2}\right] \cdot R_{u,p}M.$$

Therefore we get

$$\gamma(T_{u,p}M) \leq \left(C_p + \frac{u}{2}\right)\gamma(R_{u,p}M),$$

and from the left hand side of (6.4)

$$\frac{D_p}{m+1}\left(C_p + \frac{u}{2}\right)^{-1}\gamma(M) \leq \gamma(R_{u,p}M).$$

The latter inequality implies

$$\omega(R_{u,p}) \geq \frac{D_p}{m+1}\left(C_p + \frac{u}{2}\right)^{-1},$$

consequently $\omega(R_{u,p}) > 0$ for every $u > 0$, so that $R_{u,p}$ is a proper retraction. Now given $\varepsilon > 0$, since

$$\lim_{u\to\infty}\frac{C_p}{D_p c_{u,p}} = 1,$$

we can find $\bar{u} > 0$ such that $k_{\bar{u}} < 1 + \varepsilon$. Then letting $k = k_{\bar{u}}$ we see that $R_{\bar{u},k}$ is the desired proper k-ball-contractive retraction. □

Finally, we apply the result of this paper to consider the formulation of Birkhoff-Kellogg type theorems in C_b^m. Let us recall that Guo in [18, Lemma 1], proved that if a completely continuous operator $A : \overline{\Omega} \to X$, defined on the closure $\overline{\Omega}$ of a bounded open subset Ω of an infinite-dimensional Banach space $(X, \|\cdot\|)$, satisfies the Birkhoff-Kellogg condition $\inf_{f\in\partial\Omega}\|Af\| > 0$ and $Af \neq \lambda f$ for $f \in \partial\Omega$ and $0 < \lambda \leq 1$, then the Leray-Schauder degree $deg(I - A, \Omega, 0) = 0$. In [11] the result of Guo has been extended to k-ball-contractive operators, under a condition, $\inf_{f\in\partial\Omega}\|Af\| > kW_\gamma(X)\sup_{f\in\partial\Omega}\|f\|$, depending on the Wośko constant of the space, and considering the Nussbaum fixed point index $\mathrm{ind}(A, \Omega)$ of A on Ω (see [1]), which in the case of completely continuous operator agrees with Leray-Schauder degree. In particular, from [11, Theorem 3.2], being $W_\gamma(C_b^m) = 1$, we have the following result in C_b^m.

Theorem 6.4. *Let Ω be a bounded open set in C_b^m, with $0 \in \Omega$, and let $A : \overline{\Omega} \to C_b^m$ be a k-ball-contractive operator with $k < 1$, satisfying*

$$\inf_{f\in\partial\Omega}\|Af\| > k\sup_{f\in\partial\Omega}\|f\| \tag{6.5}$$

and $Af \neq \lambda f$ for $f \in \partial\Omega$ and $k < \lambda \leq 1$, then $ind(A, \Omega) = 0$.

Notice that condition (6.5) in C_b^m, as well in any space in which the Wośko constant is 1, is optimal, indeed if $A : B(C_b^m) \to C_b^m$ is defined by $Af = -kf$ with $k < 1$, then $\inf_{f \in \partial B(C_b^m)} ||Af|| > 0$, and $f \in \partial B(C_b^m)$ implies $Af \neq \lambda f$ for $\lambda > 0$, but $\operatorname{ind}(A, B(C_b^m) \setminus \partial B(C_b^m)) = 1$. Now we state the results on the existence of eigenvalues and eigenvectors and on the extension of Guo's domain compression and expansion fixed point theorem ([19]) to k-ball-contractions (cf. [11, Corollary 3.5] and [11, Corollary 3.7], respectively).

THEOREM 6.5. *Let Ω be a bounded open set in C_b^m, with $0 \in \Omega$, and let $A : \overline{\Omega} \to C_b^m$ be a k-ball-contractive operator (for any $k > 0$), satisfying*

$$\inf_{f \in \partial\Omega} ||Af|| > k \sup_{f \in \partial\Omega} ||f||.$$

Then there exist $\lambda > k$ and $f_\lambda \in \partial\Omega$ such that $\lambda f_\lambda = Af_\lambda$, and also there exist $\mu < -k$ and $f_\mu \in \partial\Omega$ such that $\mu f_\mu = Af_\mu$.

THEOREM 6.6. *Let Ω_1 and Ω_2 bounded open sets in C_b^m, such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $A : \overline{\Omega}_2 \to C_b^m$ be a k-ball-contractive operator, with $k < 1$. Suppose that one of the following groups of conditions holds*

$$\begin{cases} \inf_{f \in \partial\Omega_1} ||Af|| > & k \sup_{x \in \partial\Omega_1} ||f|| \\ \|Af\| \geq \|f\| & f \in \partial\Omega_1 \\ \|Af\| \leq \|f\| & f \in \partial\Omega_2 \end{cases},$$

or

$$\begin{cases} \inf_{f \in \partial\Omega_2} ||Af|| > & k \sup_{f \in \partial\Omega_2} ||f|| \\ \|Af\| \geq \|f\| & f \in \partial\Omega_2 \\ \|Af\| \leq \|f\| & f \in \partial\Omega_1. \end{cases}$$

Then A has at least a fixed point in $\overline{\Omega}_2 \setminus \Omega_1$.

For details and analogous results for condensing operators, we refer to [11].

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