Mediterr. J. Math. (2022) 19:96
https://doi.org/10.1007/s00009-021-01962-8

Mediterranean Journal of Mathematics

The L^r-Variational Integral

Francesco Tulone and Paul Musial

Abstract. We define the L^r-variational integral and we prove that it is equivalent to the HK_r-integral defined in 2004 by P. Musial and Y. Sagher in the Studia Mathematica paper *The L^r-Henstock–Kurzweil integral*. We prove also the continuity of L^r-variation function.

Mathematics Subject Classification. 26A39, 28C99.

Keywords. L^r-Variational Integral, HK_r Integral, Non-absolute integral.

1. History and Aim

At the beginning of the 1900s, Denjoy and Perron developed descriptive processes for recovering a function from its derivative that solved known problems of classical Riemann and Lebesgue integrals. Many years later, an equivalent constructive Riemann-type integral process was developed by Henstock and Kurzweil. Both integration processes were generalized quite recently for many different spaces (see [1,11] and [12]) solving the problem of recovering Fourier coefficients in Haar, Walsh and Vilenkin systems (see [9,10,14,15] and [16]). Many properties of these non-absolute integrals were investigated, for example, the Hake property was studied with an abstract differential basis in a topological spaces, in terms of variational measure and in Riesz spaces (see [13,17] and [2]).

To establish pointwise estimates for solutions of elliptic partial differential equations, in 1961 Calderon and Zygmund introduced the L^r-derivative (see [3]) and in 1968 L. Gordon described a Perron-type integral, the P_r-integral, that recovers a function from its L^r-derivative (see [4]). In 2004, Musial and Sagher extended the P_r-integral to the L^r-Henstock–Kurzweil integral, the HK_r-integral, that recovers also a function from its L^r-derivative (see [6]). Quite recently the integration by parts formula for the HK_r-integral was investigated by Musial and Tulone (see [7]) and the same authors described a norm on the space of HK_r-integrable functions and studied the dual and completion of this space (see [8]).

Birkhäuser

Published online: 27 March 2022

It is well known that the Henstock–Kurzweil integral is equivalent to the variational integral (see [5]). In this paper, we define the L^r-variational integral and we prove that it is equivalent to the HK_r-integral.

2. Introduction

We will assume that $r \geq 1$ and we will consider the case of the closed interval $[a, b]$.

Definition 2.1. A function $f : [a, b] \to \mathbb{R}$ is L^r-variational integrable on $[a, b]$ if there exists a function $F \in L^r[a, b]$ with the following property: for each $\varepsilon > 0$ there exist a non-decreasing function ϕ defined on $[a, b]$ and a gauge δ, i.e., a positive function, defined on $[a, b]$ such that $\phi(b) - \phi(a) < \varepsilon$ and for any δ-fine tagged interval $(x, [c, d])$, where $[c, d] \subseteq [a, b]$,

$$\left(\frac{1}{d-c} \int_c^d |F(y) - F(x) - f(x)(y-x)|^r \, dy \right)^{1/r} < \phi(d) - \phi(c). \quad (2.1)$$

We will use the following definition given in [6]

Definition 2.2. A function $f : [a, b] \to \mathbb{R}$ is L^r-Henstock–Kurzweil integrable on $[a, b]$ if there exists a function $F \in L^r[a, b]$ so that for any $\varepsilon > 0$ there exists a gauge δ so that for any finite collection of nonoverlapping δ-fine tagged intervals

$$\mathcal{Q} = \{(x_i, [c_i, d_i]), 1 \leq i \leq q\},$$

we have

$$\sum_{i=1}^{q} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r} < \varepsilon.$$

By Theorem 5 in [6], the function F in the Definition 2.2 is unique up to an additive constant, so we can state that for each $x \in (a, b]$

$$F(x) = (HK_r) \int_a^x f.$$

We need the following definition in a later theorem.

Definition 2.3. Let $F \in L^r[a, b]$. For $x \in [a, b]$ we say that F is L^r-continuous at x if

$$\lim_{h \to 0} \left(\frac{1}{2h} \int_{x-h}^{x+h} |F(y) - F(x)|^r \, dy \right)^{1/r} = 0.$$

If F is L^r-continuous for all $x \in E$, we say that F is L^r-continuous on E.

The Henstock–Kurzweil integral primitive is continuous in the usual sense. In [6] is proved an equivalent result for L^r-Henstock–Kurzweil indefinite integral.

Theorem 2.4. *The function F in the definition of the L^r-Henstock–Kurzweil is L^r-continuous on $[a, b]$.*

Definition 2.5. Let Φ be a function defined on the subintervals of $[a,b]$. The function Φ is superadditive if

$$\Phi([u,v]) + \Phi([v,w]) \le \Phi([u,w]),$$

whenever $a \le u < v < w \le b$. The function Φ is continuous if for each $c \in (a,b)$,

$$\lim_{x\to c^-} \Phi([x,c]) = 0 = \lim_{x\to c^+} \Phi([c,x])$$

and

$$\lim_{x\to b^-} \Phi([x,b]) = 0 = \lim_{x\to a^+} \Phi([a,x]).$$

Remark 2.6. Throughout this paper, if an interval function is said to be continuous, it is to be considered continuous in the sense of Definition 2.5.

Definition 2.7. Let δ be a gauge and let

$$\mathcal{P} = \{(x_i, [c_i, d_i]), 1 \le i \le n\}$$

be a δ-fine partition of $[a,b]$. Let

$$W(\mathcal{P}) = \sum_{i=1}^{q} \left(\frac{1}{d_i - c_i} \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \right)^{1/r}. \quad (2.2)$$

The main tool we need to get the L^r-variational integral is the following definition of L^r-variation function.

Definition 2.8. For each subinterval $[c,d] \subseteq [a,b]$ define

$$\Phi([c,d]) = \Phi(F, \delta, [c,d]) = \sup\{W(\mathcal{P})\}, \quad (2.3)$$

where the supremum is taken over all δ-fine partitions $\mathcal{P}$ of $[c,d]$.

Theorem 2.9. *The function Φ is superadditive.*

Proof. Let u, v and w be such that $a \le u < v < w \le b$ and let $\varepsilon > 0$. If either $\Phi([u,v]) = \infty$ or $\Phi([v,w]) = \infty$ then surely $\Phi([u,w]) = \infty$ and the assertion holds. Otherwise let $\mathcal{P}_1$ be a partition of $[u,v]$ such that $W(\mathcal{P}_1) > \Phi([u,v]) - \varepsilon$ and let $\mathcal{P}_2$ be a partition of $[v,w]$ such that $W(\mathcal{P}_2) > \Phi([v,w]) - \varepsilon$. Let $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, and clearly $W(\mathcal{P}) = W(\mathcal{P}_1) + W(\mathcal{P}_2)$. But $W(\mathcal{P}) \le \Phi([u,w])$. Therefore,

$$\Phi([u,v]) + \Phi([v,w]) - 2\varepsilon < W(\mathcal{P}_1) + W(\mathcal{P}_2) \le \Phi([u,w]).$$

□

Now we can prove the following theorem that extends Theorem 11.9 in [5]

3. Main Results

Theorem 3.1. *A function $f : [a, b] \to \mathbb{R}$ is L^r-Henstock–Kurzweil integrable on $[a, b]$ if and only if there exists a function $F : [a, b] \to \mathbb{R}$ with the following property: for each $\varepsilon > 0$ there exists a superadditive interval function Φ defined on the subintervals of $[a, b]$ and a gauge δ defined on $[a, b]$ such that $\Phi([a, b]) < \varepsilon$ and for any δ-fine tagged interval $(x, [c, d])$, where $[c, d] \subseteq [a, b]$,*

$$\left(\frac{1}{d-c}\int_c^d |F(y) - F(x) - f(x)(y-x)|^r \, dy\right)^{1/r} < \Phi([c, d]).$$

Proof. Suppose there exists a function F with the property stated in the theorem. Let $\varepsilon > 0$ and choose Φ and δ according to the hypotheses. If $\mathcal{P} := \{(x_i, [c_i, d_i]), 1 \le i \le n\}$ is a δ-fine tagged partition of $[a, b]$, then

$$\sum_{i=1}^{n}\left(\frac{1}{d_i-c_i}\int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y-x_i)|^r \, dy\right)^{1/r}$$
$$\le \sum_{i=1}^{n} \Phi([c_i, d_i]) \le \Phi([a, b]), < \varepsilon$$

and so f is L^r-Henstock–Kurzweil integrable on $[a, b]$.

Now suppose that f is L^r-Henstock–Kurzweil integrable on $[a, b]$ and let

$$F(x) = (HK_r)\int_a^x f,$$

for each $x \in (a, b]$. Let $\varepsilon > 0$. By hypothesis, there exists a gauge δ on $[a, b]$ such that

$$\sum_{i=1}^{n}\left(\frac{1}{d_i-c_i}\int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y-x_i)|^r \, dy\right)^{1/r} < \varepsilon/2,$$

whenever $\mathcal{P}$ is a δ-fine tagged partition of $[a, b]$. Let

$$\mathcal{P} = \{(x_i, [c_i, d_i]), 1 \le i \le n\}$$

and let $W(\mathcal{P})$ be defined as in (2.2) and let Φ be defined on the subintervals of $[a, b]$ as in (2.3). By Theorem 2.9, Φ is superadditive. Also,

$$\Phi([a, b]) \le \varepsilon/2 < \varepsilon.$$

Finally, by the definition of Φ, if $(x, [c, d])$ is a δ-fine tagged interval such that $[c, d] \subseteq [a, b]$,

$$\left(\frac{1}{d-c}\int_c^d |F(y) - F(x) - f(x)(y-x)|^r \, dy\right)^{1/r} < \Phi([c, d]).$$

This completes the proof. □

Theorem 3.2. *A function $f : [a, b] \to$ is L^r-Henstock–Kurzweil integrable on $[a, b]$ if and only if f is L^r-variational integrable on $[a, b]$.*

Proof. Suppose first that f is L^r-variational integrable on $[a,b]$. Let $\varepsilon > 0$ and let F, δ and ϕ satisfy the conditions in Definition 2.1. If $\mathcal{P} = \{(x_i,[c_i,d_i]), 1 \le i \le n\}$ is a δ-fine tagged partition of $[a,b]$, then

$$\sum_{i=1}^{n}\left(\frac{1}{d_i-c_i}\int_{c_i}^{d_i}|F(y)-F(x_i)-f(x_i)(y-x_i)|^r\,dy\right)^{1/r} \le \sum_{i=1}^{n}(\phi(d_i)-\phi(c_i)) = \phi(b)-\phi(a) < \varepsilon$$

and so f is L^r-Henstock–Kurzweil integrable on $[a,b]$ and

$$(HK_r)\int_a^b f = F(b)-F(a).$$

Now suppose that f is L^r-Henstock–Kurzweil integrable on $[a,b]$ and that for each $x \in (a,b]$,

$$F(x) = (HK_r)\int_a^x f.$$

Let $\varepsilon > 0$. By Theorem 3.1 there exists a superadditive interval function Φ defined on $[a,b]$ such that $\Phi([a,b]) < \varepsilon$ and

$$\left(\frac{1}{d-c}\int_c^d|F(y)-F(x)-f(x)(y-x)|^r\,dy\right)^{1/r} < \Phi([c,d]),$$

whenever $(x,[c,d])$ is a δ-fine tagged interval such that $[c,d] \subseteq [a,b]$. Define $\phi : [a,b] \to \mathbb{R}$ by $\phi(a) = 0$ and $\phi(x) = \Phi([a,x])$ for all $x \in (a,b]$. If $a \le c < d \le b$, then

$$\phi(d)-\phi(c) = \Phi([a,d])-\Phi([a,c]) \ge \Phi([c,d]) \ge 0$$

and so ϕ is non-decreasing. In addition,

$$\phi(b)-\phi(a) = \Phi([a,b]) < \varepsilon.$$

Suppose that $(x,[c,d])$ is a δ-fine tagged interval such that $[c,d] \subseteq [a,b]$. Then,

$$\left(\frac{1}{d-c}\int_c^d|F(y)-F(x)-f(x)(y-x)|^r\,dy\right)^{1/r} \le \Phi([c,d]) \le \phi(d)-\phi(c).$$

Hence, the function f is L^r-variational integrable on $[a,b]$. This completes the proof. □

Corollary 3.3. *If f is L^r-variational integrable on $[a,b]$, then the function F which satisfies the conditions of Definition 2.1 is unique up to an additive constant.*

We now prove the continuity of the interval function Φ.

Proposition 3.4. *Let f be L^r-variational integrable on $[a,b]$ and let F be a function that satisfies (2.1). Let δ be a gauge, $\Phi = \Phi(\delta, F)$ be as in (2.3), and assume that $\Phi([a,b])$ is finite. Then, Φ is continuous.*

Proof. We will prove that $\lim_{x\to c^-} \Phi([x,c]) = 0$ for each $c \in (a,b]$; the proof for right-handed limits is similar. Suppose by way of contradiction that $\lim_{x\to c^-} \Phi([x,c])$ either fails to exist or exists and is not equal to zero. Since Φ is nonnegative, there exists $\eta > 0$ such that $\limsup_{x\to c^-} \Phi([x,c]) > \eta$. Let us see that for every $\xi \in [a,c)$, $\Phi([\xi,c]) > \eta$. Fix ξ, there exists $\xi < \zeta < c$ such that $\Phi([\zeta,c]) > \eta$. Since Φ is superadditive, we have that

$$\Phi([\xi,c]) \geq \Phi([\xi,\zeta]) + \Phi([\zeta,c]) \geq \Phi([\zeta,c]) > \eta.$$

Consequently, for each $\xi \in [a,c)$, there exists $\mathcal{P}_\xi$, a δ-fine tagged partition of $[\xi,c]$ such that $W(\mathcal{P}_\xi) > \eta$.

We now prove that we can make the following three assumptions about $\mathcal{P}_x$:

1. $\mathcal{P}_x$ contains at least two tagged intervals,
2. c is a tag of $\mathcal{P}_x$, and
3. the interval containing c is arbitrarily small.

Fix x and $\varepsilon > 0$. Choose $y \in (\max(x, c-\varepsilon), c)$. By Cousin's Lemma there exists $\mathcal{Q}$, a δ-fine tagged partition of $[x,y]$. Define $\mathcal{P}_x = \mathcal{Q} \cup \mathcal{P}_y$. We then have

$$W(\mathcal{P}_x) = W(\mathcal{Q}) + W(\mathcal{P}_y) \geq W(\mathcal{P}_y) > \eta.$$

If c is the tag of its interval, then $\mathcal{P}_x$ has the desired properties.

Now suppose that c is not the tag of its interval. Let s and t be such that $(t,[s,c])$ is the tagged interval which contains c. It is possible that $s=t$ but we assume that $t<c$.

It suffices to show that

$$\lim_{u\to c^-} W(\{(t,[s,u]),(c,[u,c])\}) = W(\{(t,[s,c])\}).$$

Note that

$$\begin{aligned} &W(\{(t,[s,u]),(c,[u,c])\}) \\ &\quad= \left(\frac{1}{u-s}\int_s^u |F(y)-F(t)-f(t)(y-t)|^r\,dy\right)^{1/r} \\ &\qquad+ \left(\frac{1}{c-u}\int_u^c |F(y)-F(c)-f(c)(y-c)|^r\,dy\right)^{1/r}. \end{aligned}$$

Using Minkowski's inequality, we have

$$\begin{aligned}
&\left(\frac{1}{c-u}\int_u^c |F(y)-F(c)-f(c)(y-c)|^r\,dy\right)^{1/r}\\
&\quad=\left(\frac{1}{c-u}\right)^{1/r}\left(\int_u^c |F(y)-F(c)-f(c)(y-c)|^r\,dy\right)^{1/r}\\
&\quad\le\left(\frac{1}{c-u}\right)^{1/r}\left(\int_u^c |F(y)-F(c)|^r\,dy\right)^{1/r}\\
&\qquad+\left(\frac{1}{c-u}\right)^{1/r}\left(\int_u^c |f(c)(y-c)|^r\,dy\right)^{1/r}\\
&\quad\le\left(\frac{1}{c-u}\int_u^c |F(y)-F(c)|^r\,dy\right)^{1/r}\\
&\qquad+|f(c)|\left(\frac{1}{c-u}\right)^{1/r}\left(\int_u^c |(c-u)|^r\,dy\right)^{1/r}\\
&\quad=\left(\frac{1}{c-u}\int_u^c |F(y)-F(c)|^r\,dy\right)^{1/r}+|f(c)|(c-u).
\end{aligned}$$

By Theorem 2.4 the function F is L^r-continuous at each point of $[a,b]$, and so we have that

$$\begin{aligned}
&\lim_{u\to c^-}\left(\frac{1}{c-u}\int_u^c |F(y)-F(c)-f(c)(y-c)|^r\,dy\right)^{1/r}\\
&\quad\le\lim_{u\to c^-}\left[\left(\frac{1}{c-u}\int_u^c |F(y)-F(c)|^r\,dy\right)^{1/r}+|f(c)|(c-u)\right]=0
\end{aligned}\tag{3.1}$$

We also have that

$$\begin{aligned}
&\lim_{u\to c^-}\left(\frac{1}{u-s}\int_s^u |F(y)-F(t)-f(t)(y-t)|^r\,dy\right)^{1/r}\\
&\quad=\left(\frac{1}{c-s}\int_s^c |F(y)-F(t)-f(t)(y-t)|^r\,dy\right)^{1/r}.
\end{aligned}$$

It follows that

$$\begin{aligned}
&\lim_{u\to c^-} W(\{(t,[s,u]),(c,[u,c])\})\\
&\quad=\lim_{u\to c^-} W(\{(t,[s,u])\})=W(\{(t,[s,c])\}).
\end{aligned}$$

We now prove the proposition. Set $x_1=a$ and write

$$\begin{aligned}
\mathcal{P}_{x_1}&=\mathcal{Q}_1\cup(c,[x_2,c])\\
\mathcal{P}_{x_2}&=\mathcal{Q}_2\cup(c,[x_3,c])\\
&\ \vdots\\
\mathcal{P}_{x_k}&=\mathcal{Q}_k\cup(c,[x_{k+1},c]).
\end{aligned}$$

By the result proved above, we may assume that for each k, $c-x_k<1/k$ and, therefore, that $x_k\to c$.

For each n, the collection

$$\mathcal{P}'_n = \bigcup_{k=1}^{n} \mathcal{Q}_k$$

is a δ-fine tagged partition of $[a, x_{n+1}]$. Hence,

$$W(\mathcal{P}'_n) = \sum_{k=1}^{n} W(\mathcal{Q}_k) \leq \Phi([a, x_{n+1}]) \leq \Phi([a, b]) < \infty.$$

This shows that the series

$$\sum_{k=1}^{\infty} W(\mathcal{Q}_k)$$

converges and hence

$$\lim_{k \to \infty} W(\mathcal{Q}_k) = 0.$$

We then have for each k,

$$\begin{aligned} \eta &< W(\mathcal{P}_{x_k}) \\ &= W(\mathcal{Q}_k) + \left(\frac{1}{c - x_{k+1}} \int_{x_{k+1}}^{c} |F(y) - F(c) - f(c)(y - c)|^r \, dy \right)^{1/r}. \end{aligned}$$

By (3.1), the term on the right goes to zero; therefore, the entire right side of the equality goes to zero. This contradiction completes the proof. □

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

References

[1] Boccuto, A., Skvortsov, V.A., Tulone, F.: Integration of functions ranging in complex Riesz space and some applications in harmonic analysis. Math. Notes **98**(1), 25–37 (2015)

[2] Boccuto, A., Skvortsov, V.A., Tulone, F.: A hake-type theorem for integrals with respect to abstract derivation bases in the Riesz space setting. Math. Slovaca **65**(6), 1319–1336 (2015)

[3] Calderon, A.P., Zygmund, A.: Local properties of solutions of elliptic partial differential equations. Studia Math. **20**, 171–225 (1961)

[4] Gordon, L.: Perron's integral for derivatives in L^r. Studia Math. **28**, 295–316 (1967)

[5] Gordon, R.A.: The Integrals of Lebesgue, Denjoy, Perron, and Henstock, Graduate Studies in Mathematics, vol. 4. American Mathematical Society, Washington (1994)

[6] Musial, P., Sagher, Y.: The L^r-Henstock-Kurzweil integral. Studia Math. **160**(1), 53–81 (2004)

[7] Musial, P., Tulone, F.: Integration by parts for the L^r-Henstock-Kurzweil integral. Electron. J. Differ. Equ. **2015**(44), 1–7 (2015)

[8] Musial, P., Tulone, F.: Dual of the class of HKr integrable functions. Minim. Theory Appl. **4**(1), 151–160 (2019)

[9] Oniani, G., Tulone, F.: On the possible values of upper and lower derivatives with respect to differentiation bases of product structure. Bull. Ga. Natl. Acad. Sci. **12**(1), 12–15 (2018)

[10] Oniani, G., Tulone, F.: On the almost everywhere convergence of multiple Fourier-Haar series. J. Contemp. Math. **54**(5), 288–295 (2019)

[11] Skvortsov, V.A., Tulone, F.: Generality of Henstock-Kurzweil type integral on a compact zero-dimensional metric space. Tatra Mt. Math. Publ. **49**(1), 81–88 (2011)

[12] Skvortsov, V.A., Tulone, F.: Henstock type integral in compact zero-dimensional metric space and quasi-measures representations. Mosc. Univ. Math. Bull. **67**(2), 55–60 (2012)

[13] Skvortsov, V.A., Tulone, F.: Generalized Hake property for integrals of Henstock type. Mosc. Univ. Math. Bull. **68**(6), 270–274 (2013)

[14] Skvortsov, V.A., Tulone, F.: Multidimensional dyadic Kurzweil-Henstock- and Perron-type integrals in the theory of Haar and Walsh series. J. Math. Anal. Appl. **421**(2), 1502–1518 (2015)

[15] Skvortsov, V.A., Tulone, F.: On the Coefficients of Multiple Series with Respect to Vilenkin System. Tatra Mt. Math. Publ. **68**(1), 81–92 (2017)

[16] Skvortsov, V.A., Tulone, F.: Multidimensional P-adic integrals in some problems of harmonic analysis. Minim. Theory Appl. **2**(1), 153–174 (2017)

[17] Skvortsov, V.A., Tulone, F.: A version of Hake's theorem for Kurzweil-Henstock integral in terms of variational measure. Ga. Math. J. **5**, 1–6 (2019). https://doi.org/10.1515/gmj-2019-2074

Francesco Tulone
University of Palermo
Via Archirafi 34
90123 Palermo
Italy
e-mail: `francesco.tulone@unipa.it`

Paul Musial
Chicago State University
9501 South King Drive
Chicago IL60628
USA
e-mail: pmusial@csu.edu

Received: January 12, 2021.
Revised: April 24, 2021.
Accepted: December 7, 2021.