Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
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doi:10.3906/mat-2112-97

Capelli identities on algebras with involution or graded involution

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Received: 22.12.2021 • **Accepted/Published Online:** 25.03.2022 • **Final Version:** ..2022

Abstract: We present recent results about Capelli polynomials with involution or graded involution and their asymptotics. In the associative case, the asymptotic equality between the codimensions of the T-ideal generated by the Capelli polynomial of rank k^2+1 and the codimensions of the matrix algebra $M_k(F)$ was proved. This result was extended to superalgebras. Recently, similar results have been determined by the authors in the case of algebras with involution and superalgebras with graded involution.

Key words: Involution, graded involution, Capelli polynomials, codimension

1. Introduction

The Capelli polynomial plays a central role in the combinatorial PI-theory and in particular in the study of polynomial identities of the matrix algebra $M_k(F)$ in fact, a precise relation was determinated between the growth of the corresponding T-ideal and the growth of the T-ideal of polynomial identities of the matrix algebra (see [27]). From Kemer's theory (see [32]), the polynomial identities of $M_k(F)$ over a field F of characteristic zero are among the most intriguing topics in PI-theory. There are a lot of open problems and conjectures concerning the T-ideal of polynomial identities of $M_k(F)$. Similar problems are also considered for matrix algebras with additional structure as $\mathbb{Z}_2$-gradings, group gradings, involution, superinvolution or graded-involution.

Let us recall that, for any positive integer m, the m-th Capelli polynomial is the element of the free algebra $F\langle X\rangle$ defined as

$$Cap_m = Cap_m(t_1,\ldots,t_m;x_1,\ldots,x_{m-1}) =$$
$$= \sum_{\sigma\in S_m} (\mathrm{sgn}\sigma) t_{\sigma(1)}x_1t_{\sigma(2)}\cdots t_{\sigma(m-1)}x_{m-1}t_{\sigma(m)}$$

where S_m is the symmetric group on $\{1,\ldots,m\}$. It is a polynomial alternating on $t_1,\ldots,t_m$ and every polynomial which is alternating on $t_1,\ldots,t_m$ can be written as a linear combination of Capelli polynomials obtained by specializing the x_i's. These polynomials were first introduced by Razmyslov (see [34]) in his construction of central polynomials for $k\times k$ matrices. It is easy to show that if A is a finite dimensional algebra and $\dim A = m-1$ then A satisfies Cap_m. Hence, the matrix algebra $M_k(F)$ satisfies Cap_{k^2+1} and k^2+1 is actually the minimal degree of a Capelli polynomial satisfied by $M_k(F)$.

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2010 *AMS Mathematics Subject Classification:* Primary 16R10; Secondary 16P90, 16R50

The main purpose of this paper is to present a survey on recent and new results obtained by the authors in [12] and [13] concerning the Capelli polynomials on algebras with involution and on superalgebras with graded involution. Specifically, in Section 2, we recall shortly the results about the T-ideal generated by the m-th Capelli polynomial Cap_m and the results concerning the $T_{\mathbb{Z}_2}$-ideal generated by the $\mathbb{Z}_2$-graded Capelli polynomials Cap^0_M and Cap^1_L. We describe their relations with the T-ideal of the polynomial identities of $M_k(F)$ and, respectively, with the $T_{\mathbb{Z}_2}$-ideals of the $\mathbb{Z}_2$-graded identities of the simple finite dimensional superalgebras $M_k(F)$, $M_{k,l}(F)$, and $M_k(F \oplus cF)$. In Sections 3 and 4, we present analogous results in the case of algebras with involution and superalgebras with graded involution. In particular in Section 3, we analyze the involution case by showing that the $*$-codimensions of a finite dimensional $*$-simple algebra are asymptotically equal to the $*$-codimensions of the T^*-ideal generated by the $*$-Capelli polynomials Cap^+_M and Cap^-_L alternanting on M symmetric variables and L skew variables, respectively, for some fixed natural numbers M and L (see [11, 12]). Finally, the last section, Section 4, is dedicated to the study of $*$-graded Capelli polynomials and their relationship with superalgebras with graded-involution. In this section, we show that the $*$-graded codimensions of the finite dimensional simple $*$-superalgebras are asymptotically equal to the $*$-graded codimensions of $T^*_{\mathbb{Z}_2}$-ideal generated by the set of the $*$-graded Capelli polynomials.

2. Associative and $\mathbb{Z}_2$-graded cases

Let F be a field of characteristic zero and let $F\langle X\rangle = F\langle x_1, x_2, \ldots\rangle$ be the free associative algebra on a countable set X over F. Let A be an associative algebra over F, then an element $f = f(x_1, \ldots, x_n) \in F\langle X\rangle$ is a polynomial identity for A if $f(a_1, \ldots, a_n) = 0$ for any $a_1, \ldots, a_n \in A$. Let $Id(A) = \{f \in F\langle X\rangle \mid f \equiv 0 \text{ in A}\}$ be the set of polynomial identities of A. This is a T-ideal of $F\langle X\rangle$ i.e. an ideal invariant under all endomorphisms of $F\langle X\rangle$, and every T-ideal of $F\langle X\rangle$ is the ideal of identities of some F-algebra A. When $Id(A) \neq (0)$, we say that A is a PI-algebra. For $I = Id(A)$, we denote by $\mathrm{var}(I) = \mathrm{var}(A)$ the variety of all associative algebras having the elements of I as polynomial identities.

By the structure theory of T-ideals developed by Kemer in his solution of the Specht problem (see [31, 32]), the study of an arbitrary T-ideal can be reduced to the study of the verbally prime T-ideals. Recall that a T-ideal $I \subseteq F\langle X\rangle$ is verbally prime if for any T-ideals I_1, I_2 such $I_1 I_2 \subseteq I$ we must have $I_1 \subseteq I$ or $I_2 \subseteq I$. A PI-algebra A is called verbally prime if its T-ideal of identities $I = Id(A)$ is verbally prime. Also, the corresponding variety of associative algebras $\mathrm{var}(A)$ is called verbally prime. From the classification of the verbally prime T-ideals made by Kemer, it turns out that $Id(M_k(F))$ is a verbally prime T-ideal so to determine the polynomials identities of the matrix algebra is one of the main objectives of the PI-theory.

It is well known that in characteristic zero every T-ideal, $Id(A)$, is completely determined by its multilinear elements. Hence, if P_n is the space of multilinear polynomials of degree n in the variables $x_1, \ldots, x_n$, we can consider the sequence of spaces $\{P_n/(P_n \cap Id(A))\}_{n\geq 1}$. The integer $c_n(A) = \dim P_n/(P_n \cap Id(A))$ is called the n-th codimension of A and gives a quantitative estimate of the polynomial identities satisfied by A.

It is well known that A is a PI-algebra if and only if $c_n(A) < n!$ for some $n \geq 1$. Regev in [35] showed that if A is an associative PI-algebra, then $c_n(A)$ is exponentially bounded, i.e. there exist constants α, β which depend on A such that $c_n(A) \leq \alpha\beta^n$ for any $n \geq 1$. In the early 1980s, Amitsur conjectured that the exponential growth of the codimension sequence should be an integer. Giambruno and Zaicev in [25] and [26] gave a positive answer to this conjecture by proving that for a PI-algebra A

$$\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

exists and is an integer; $\exp(A)$ is called the PI-exponent of the algebra A. The PI-exponents of the verbally prime algebras are well known (see [15, 26, 36, 37]). In particular,

$$\exp(M_k(F)) = k^2.$$

In [36], Regev obtained the precise asymptotic behavior of the codimensions of the verbally prime algebra $M_k(F)$. It turns out that

Theorem 2.1 *[36, Theorem 1]*

$$c_n(M_k(F)) \simeq \left(\frac{1}{\sqrt{2\pi}}\right)^{k-1}\left(\frac{1}{2}\right)^{\frac{1}{2}(k^2-1)} 1!2!\cdots(k-1)!k^{\frac{1}{2}(k^2+4)}n^{\frac{1-k^2}{2}}(k^2)^n.$$

For the other verbally prime algebras, there are only some partial results (see [15] and [16]).

In [27], a relation was found between the asymptotics of codimensions of the verbally prime T-ideals and the T-ideals generated by Capelli polynomials.

Now, if $f \in F\langle X\rangle$, we denote by $\langle f\rangle_T$ the T-ideal generated by f. Also for $V \subset F\langle X\rangle$, we write $\langle V\rangle_T$ to indicate the T-ideal generated by V. Let C_m be the set of 2^{m-1} polynomials obtained from the m-th Capelli polynomial Cap_m by deleting any subset of variables x_i (by evaluating the variables x_i to 1 in all possible ways) and let $\langle C_m\rangle_T$ denote the T-ideal generated by C_m. Also, $\mathrm{var}(C_m)$ is the variety corresponding to $\langle C_m\rangle_T$. In case $m = k^2$, it follows from [36] and [33] that

$$\exp(C_{k^2+1}) = k^2 = \exp(M_k(F)).$$

In [27], Giambruno and Zaicev proved that the codimensions of $\mathrm{var}(C_{k^2+1})$ are asymptotically equal to the codimensions of the verbally prime algebra $M_k(F)$.

Theorem 2.2 *[27, Theorem 3, Corollary 4] Let $m = k^2$. Then $\mathrm{var}(C_{m+1}) = \mathrm{var}(M_k(F) \oplus B)$ for some finite dimensional algebra B such that $\exp(B) < k^2$. In particular*

$$c_n(C_{k^2+1}) \simeq c_n(M_k(F)).$$

This result has been extended to the other verbally prime algebras by the so called Amitsur's Capelli-type polynomials introduced in [5] (see [10]).

Kemer in [31] showed that any associative variety is generated by the Grassmann envelope of a suitable finite dimensional superalgebra (see also Section 3.7 of [28] for a concise treatment); moreover, he established that an associative variety is verbally prime if and only if it is generated by the Grassmann envelope of a simple finite dimensional superalgebra. Thus, by Kemer's theory (see [32]), superalgebras and their $\mathbb{Z}_2$-graded identities play a basic role in the study of the structure of varieties of associative algebras over a field of characteristic zero.

Recall that an algebra A is a superalgebra (or $\mathbb{Z}_2$-graded algebra) with grading $(A^{(0)}, A^{(1)})$ if $A = A^{(0)} \oplus A^{(1)}$, where $A^{(0)}, A^{(1)}$ are subspaces of A such that $A^{(0)}A^{(0)} + A^{(1)}A^{(1)} \subseteq A^{(0)}$ and $A^{(0)}A^{(1)} + A^{(1)}A^{(0)} \subseteq A^{(1)}$. The elements of $A^{(0)}$ and of $A^{(1)}$ are called homogeneous of degree zero (or even elements) and of degree one (or odd elements), respectively. If we write $X = Y \cup Z$ as the disjoint union of two countable sets, then the free associative algebra $F\langle X\rangle = F\langle Y \cup Z\rangle = \mathcal{F}^{(0)} \oplus \mathcal{F}^{(1)}$ has a natural structure of free superalgebra with grading $(\mathcal{F}^{(0)}, \mathcal{F}^{(1)})$, where $\mathcal{F}^{(0)}$ is the subspace generated by the monomials of even degree with respect to Z and $\mathcal{F}^{(1)}$ is the subspace generated by the monomials having odd degree in Z.

Recall that an element $f(y_1, \dots, y_n, z_1, \dots, z_m)$ of $F\langle Y \cup Z\rangle$ is a $\mathbb{Z}_2$-graded identity or a superidentity for A if $f(a_1, \dots, a_n, b_1, \dots, b_m) = 0$, for all $a_1, \dots, a_n \in A^{(0)}$ and $b_1, \dots, b_m \in A^{(1)}$. The set $Id_{\mathbb{Z}_2}(A)$ of all $\mathbb{Z}_2$-graded identities of A is a $T_{\mathbb{Z}_2}$-ideal of $F\langle Y \cup Z\rangle$, i.e. an ideal invariant under all endomorphisms of $F\langle Y \cup Z\rangle$ preserving the grading. Moreover, every $T_{\mathbb{Z}_2}$-ideal Γ of $F\langle Y \cup Z\rangle$ is the ideal of $\mathbb{Z}_2$-graded identities of some superalgebra $A = A^{(0)} \oplus A^{(1)}$, $\Gamma = Id_{\mathbb{Z}_2}(A)$. For a $T_{\mathbb{Z}_2}$-ideal $\Gamma = Id_{\mathbb{Z}_2}(A)$ of $F\langle Y \cup Z\rangle$, we denote by $\mathrm{var}_{\mathbb{Z}_2}(\Gamma)$ the supervariety of superalgebras having the elements of Γ as $\mathbb{Z}_2$-graded identities.

In the case of algebraically closed field of characteristic zero, the following classification of the simple finite dimensional superalgebras is well known (see [28], Section 3.5).

Theorem 2.3 *If F is an algebraically closed field of characteristic zero, then a simple finite dimensional superalgebra over F is isomorphic to one of the following algebras:*

1. *$M_k(F)$ with trivial grading $(M_k(F), 0)$;*
2. *$M_{k,l}(F)$ with grading $\left(\begin{pmatrix} F_{11} & 0 \\ 0 & F_{22} \end{pmatrix}, \begin{pmatrix} 0 & F_{12} \\ F_{21} & 0 \end{pmatrix}\right)$, where F_{11}, F_{12}, F_{21}, F_{22} are $k \times k$, $k \times l$, $l \times k$, and $l \times l$ matrices respectively, $k \geq 1$ and $l \geq 1$;*
3. *$M_k(F \oplus cF)$ with grading $(M_k(F), cM_k(F))$, where $c^2 = 1$.*

An interesting problem in the theory of PI-algebras is to describe the following $T_{\mathbb{Z}_2}$-ideals: $Id_{\mathbb{Z}_2}(M_k(F))$, $Id_{\mathbb{Z}_2}(M_{k,l}(F))$, $Id_{\mathbb{Z}_2}(M_k(F \oplus cF))$.

In case $\mathrm{char}\, F = 0$, it is well known that $Id_{\mathbb{Z}_2}(A)$ is completely determined by its multilinear polynomials and an approach to the description of the $\mathbb{Z}_2$-graded identities of A is based on the study of the $\mathbb{Z}_2$-graded codimensions sequence of this superalgebra. If $P_n^{\mathbb{Z}_2}$ denotes the space of multilinear polynomials of degree n in the variables $y_1, z_1, \dots, y_n, z_n$ (i.e. y_i or z_i appears in each monomial at degree 1), then the sequence of subspaces $\{P_n^{\mathbb{Z}_2} \cap Id_{\mathbb{Z}_2}(A)\}_{n\geq 1}$ determines $Id_{\mathbb{Z}_2}(A)$ and

$$c_n^{\mathbb{Z}_2}(A) = \dim \frac{P_n^{\mathbb{Z}_2}}{P_n^{\mathbb{Z}_2} \cap Id_{\mathbb{Z}_2}(A)}$$

is called the n-th $\mathbb{Z}_2$-graded codimension or supercodimension of A. In 1985, Giambruno and Regev (see [24]) proved that the sequence $\{c_n^{\mathbb{Z}_2}(A)\}_{n\geq 1}$ is exponentially bounded if and only if A satisfies an ordinary polynomial identity. In [9], it was proved that if A is a finitely generated superalgebra satisfying a polynomial identity, then $\lim_{n\to\infty} \sqrt[n]{c_n^{\mathbb{Z}_2}(A)}$ exists and is a nonnegative integer. It is called superexponent (or $\mathbb{Z}_2$-exponent) of A and it is denoted by

$$\exp_{\mathbb{Z}_2}(A) = \lim_{n\to\infty} \sqrt[n]{c_n^{\mathbb{Z}_2}(A)}.$$

We remark that in [1] the existence of the G-exponent has been proved for an arbitrary PI-algebras A graded by a finite group G. Partial results were obtained in [2, 18, 21].

For the simple finite dimensional superalgebras we have that (see [9])

$$\exp_{\mathbb{Z}_2}(M_k(F)) = k^2, \; \exp_{\mathbb{Z}_2}(M_{k,l}(F)) = (k+l)^2, \; \exp_{\mathbb{Z}_2}(M_k(F \oplus cF)) = 2k^2.$$

From a result of Karasik and Shpigelman, the asymptotic behavior of the supercodimensions of the simple finite dimensional superalgebra follows. More precisely

Theorem 2.4 *[30, Theorem (A), Theorem (B)]*

$$c_n^{\mathbb{Z}_2}(M_{k,l}(F)) \simeq \alpha n^{\frac{1-(k^2+l^2)}{2}} (k+l)^{2n}$$

and

$$c_n^{\mathbb{Z}_2}(M_k(F \oplus cF)) \simeq \beta n^{\frac{1-k^2}{2}} (2k^2)^n,$$

for some costants α and β.

As in the ordinary case, a close relation was found among the asymptotics of the simple finite dimensional superalgebras and the $\mathbb{Z}_2$-graded Capelli polynomials.

Now, if $f \in F\langle Y \cup Z\rangle$ we denote by $\langle f\rangle_{T_{\mathbb{Z}_2}}$ the $T_{\mathbb{Z}_2}$-ideal generated by f. Also for a set of polynomials $V \subset F\langle Y \cup Z\rangle$, we write $\langle V\rangle_{T_{\mathbb{Z}_2}}$ to indicate the $T_{\mathbb{Z}_2}$-ideal generated by V. Let us denote by $Cap_m[Y,X] = Cap_m(y_1,\dots,y_m;x_1,\dots,x_{m-1})$ and $Cap_m[Z,X] = Cap_m(z_1,\dots,z_m;x_1,\dots,x_{m-1})$ the m-th $\mathbb{Z}_2$-graded Capelli polynomials alternanting in the variables of homogeneous degree zero $y_1,\dots,y_m$ and of homogeneous degree one $z_1,\dots,z_m$, respectively. Then Cap_m^0 indicates the set of 2^{m-1} polynomials obtained from $Cap_m[Y,X]$ by deleting any subset of variables x_i. Similarly, we define by Cap_m^1 the set of 2^{m-1} polynomials obtained from $Cap_m[Z,X]$ by deleting any subset of variables x_i. If L and M are two natural numbers, let $\Gamma_{M,L} = \langle Cap_M^0, Cap_L^1\rangle_{T_{\mathbb{Z}_2}}$ be the $T_{\mathbb{Z}_2}$-ideal generated by Cap_M^0, Cap_L^1. The following relations between the superexponent of the $\mathbb{Z}_2$-graded Capelli polynomials and the superexponent of the simple finite dimensional superalgebras are well known (see [7, 9])

$$\exp_{\mathbb{Z}_2}(\Gamma_{k^2+1,1}) = k^2 = \exp_{\mathbb{Z}_2}(M_k(F))$$

$$\exp_{\mathbb{Z}_2}(\Gamma_{k^2+l^2+1,2kl+1}) = (k+l)^2 = \exp_{\mathbb{Z}_2}(M_{k,l}(F))$$

$$\exp_{\mathbb{Z}_2}(\Gamma_{k^2+1,k^2+1}) = 2k^2 = \exp_{\mathbb{Z}_2}(M_k(F \oplus cF)).$$

Moreover, in [8], a relation was found among the asymptotics of the $\mathbb{Z}_2$-graded Capelli polynomials and the simple finite dimensional superalgebras. More precisely, we have the following:

Theorem 2.5 *[8, Theorem 9] Let $M = k^2+l^2$ and $L = 2kl$ with $k, l \in \mathbb{N}$, $k > l > 0$. Then $var_{\mathbb{Z}_2}(\Gamma_{M+1,L+1}) = var_{\mathbb{Z}_2}(M_{k,l}(F) \oplus D')$, where D' is a finite dimensional superalgebra such that $exp_{\mathbb{Z}_2}(D') < M + L$. In particular*

$$c_n^{\mathbb{Z}_2}(\Gamma_{M+1,L+1}) \simeq c_n^{\mathbb{Z}_2}(M_{k,l}(F)).$$

Theorem 2.6 *[8, Theorem 14] Let $M = L = k^2$ with $k \in \mathbb{N}$, $k > 0$. Then $var_{\mathbb{Z}_2}(\Gamma_{M+1,L+1}) = var_{\mathbb{Z}_2}(M_k(F \oplus cF) \oplus D'')$, where D'' is a finite dimensional superalgebra such that $exp_{\mathbb{Z}_2}(D'') < M + L$. In particular,*

$$c_n^{\mathbb{Z}_2}(\Gamma_{M+1,L+1}) \simeq c_n^{\mathbb{Z}_2}(M_k(F \oplus cF)).$$

3. Involution case

Let $F\langle X, *\rangle = F\langle x_1, x_1^*, x_2, x_2^*, \ldots\rangle$ denote the free associative algebra with involution $*$ generated by the countable set of variables $X = \{x_1, x_1^*, x_2, x_2^*, \ldots\}$ over a field F of characteristic zero. If $(A, *)$ is any algebra with involution $*$, let $A^+ = \{a \in A \,|\, a^* = a\}$ and $A^- = \{a \in A \,|\, a^* = -a\}$ denote the subspaces of symmetric and skew elements of A, respectively. We can regard the free associative algebra with involution $F\langle X, *\rangle$ as generated by symmetric and skew variables. In particular, for $i = 1, 2, \ldots$, we let $x_i^+ = x_i + x_i^*$ and $x_i^- = x_i - x_i^*$, then we write $X = X^+ \cup X^-$ as the disjoint union of the set X^+ of symmetric variables and the set X^- of skew variables and $F\langle X, *\rangle = F\langle X^+ \cup X^-\rangle$. A polynomial $f = f(x_1^+, \ldots, x_m^+, x_1^-, \ldots, x_n^-) \in F\langle X^+ \cup X^-\rangle$ is a $*$-polynomial identity of A if and only if $f(a_1, \ldots, a_m, b_1, \ldots, b_n) = 0$ for all $a_i \in A^+$, $b_i \in A^-$. We denote by $Id^*(A)$ the set of all $*$-polynomial identities satisfied by A. $Id^*(A)$ is a T^*-ideal of $F\langle X, *\rangle$, i.e. an ideal invariant under all endomorphisms of $F\langle X, *\rangle$ commuting with the involution of the free algebra. For $\Gamma = Id^*(A)$ we denote by $\mathrm{var}^*(\Gamma) = \mathrm{var}^*(A)$ the variety of $*$-algebras having the elements of Γ as $*$-identities.

Also in the theory of PI-algebras with involution it is interesting to describe the T^*-ideals of $*$-polynomial identities of $*$-simple finite dimensional algebras. Let us recall that

Theorem 3.1 *[38, Proposition 2.2.12] If F is an algebraically closed field of characteristic zero, then, up to isomorphisms, all finite dimensional $*$-simple algebras are the following ones:*

- *$(M_k(F), t)$ the algebra of $k \times k$ matrices with the transpose involution;*
- *$(M_{2m}(F), s)$ the algebra of $2m \times 2m$ matrices with the symplectic involution;*
- *$(M_h(F) \oplus M_h(F)^{op}, exc)$ the direct sum of the algebra of $h \times h$ matrices and the opposite algebra with the exchange involution.*

Similar to the case of ordinary identities, in characteristic zero $Id^*(A)$ is completely determinated by the multilinear $*$-polynomials it contains. To the T^*-ideal $\Gamma = Id^*(A)$ one associates a numerical sequence called the sequence of $*$-codimensions $c_n^*(\Gamma) = c_n^*(A)$ which is the main tool for the quantitative investigation of the $*$-polynomial identities of A. Thus, if we denote by P_n^* the space of all multilinear polynomials of degree n in $x_1, x_1^*, \cdots, x_n, x_n^*$ then

$$c_n^*(A) = \dim \frac{P_n^*}{P_n^* \cap Id^*(A)}.$$

A celebrated theorem of Amitsur ([3, 4]) states that if an algebra with involution satisfies a $*$-polynomial identity then it satisfies an ordinary polynomial identity. In light of this result in [24], it was proved that, as in the ordinary case, if A satisfies a nontrivial $*$-polynomial identity then $c_n^*(A)$ is exponentially bounded, i.e. there exist constants a and b such that $c_n^*(A) \leq ab^n$, for all $n \geq 1$. Later, an explicit exponential bound for $c_n^*(A)$ was exhibited (see [6]).

The asymptotic behavior of the $*$-codimensions was determined in case of matrices with involution in [14]. More precisely,

Theorem 3.2

$$c_n^*((M_k(F),t)) \simeq \left[\sqrt{k}^{\frac{k(k-1)}{2}} \frac{1}{k!} \Gamma\left(\frac{3}{2}\right)^{-k} \prod_{j=1}^{k} \Gamma\left(1+\frac{1}{2}j\right) \left(\frac{1}{2}\right)^{k-1} \right] \left(\frac{1}{\sqrt{2n}}\right)^{\frac{k(k-1)}{2}} k^{2n},$$

$$c_n^*((M_{2m}(F),s)) \simeq \left[\left(\frac{1}{\sqrt{2\pi}}\right)^{m} 2^{\frac{m^2+m+2}{4}} m^{\frac{m(7m-1)}{4}} \frac{1}{m!} \prod_{j=1}^{m} \Gamma(2j+1) \right] \left(\frac{1}{2n}\right)^{\frac{m(2m+1)}{2}} (2m)^{2n},$$

where $\Gamma(x)$ is the Euler's gamma function.

Recently, the authors found the asymptotic behavior of the $*$-codimensions of the $*$-simple algebra $(M_h(F) \oplus M_h(F)^{op}, exc)$ (see also [22]).

Theorem 3.3 *[12, Theorem 6]*

$$c_n^*((M_h(F) \oplus M_h(F)^{op}, exc)) \simeq \left(\frac{1}{\sqrt{2\pi}}\right)^{h-1} \left(\frac{1}{2}\right)^{\frac{1}{2}(h^2+1)} 1!2!\cdots(h-1)!h^{\frac{1}{2}(h^2+4)} n^{\frac{1-h^2}{2}} (2h^2)^n.$$

In [23], for any algebra with involution, the exponential behavior of $c_n^*(A)$ was studied, and it was shown that the $*$-exponent of A, $\exp^*(A) = \lim_{n\to\infty} \sqrt[n]{c_n^*(A)}$, exists and is a nonnegative integer. Moreover, an explicit way to calculate the $*$-exponent of A was described. As consequences, it follows that if A is a $*$-simple algebra then $\exp^*(A) = \dim_F A$ and so

$$\exp^*((M_k(F),t)) = k^2, \ \exp^*((M_{2m}(F),s)) = 4m^2, \ \exp^*((M_h(F) \oplus M_h(F)^{op}, exc)) = 2h^2.$$

Newly, in [12], we characterized the T^*-ideal of $*$-identities of any $*$-simple finite dimensional algebra and as a consequence we found a relation among the asymptotics of their $*$-codimensions and the T^*-ideals generated by the $*$-Capelli polynomials alternanting on $M+1$ symmetric variables and $L+1$ skew variables, respectively. More precisely, let

$$Cap_m^*[X^+,U] = Cap_m(x_1^+,\ldots,x_m^+;u_1,\ldots,u_{m-1})$$

and

$$Cap_m^*[X^-,U] = Cap_m(x_1^-,\ldots,x_m^-;u_1,\ldots,u_{m-1})$$

be the m-th $*$-Capelli polynomial alternating in the symmetric variables $x_1^+,\dots,x_m^+$ and in the skew variables $x_1^-,\dots,x_m^-$ respectively, ($u_1,\dots,u_{m-1}$ are arbitrary variables).
Let Cap_m^+ be the set of 2^{m-1} polynomials obtained from $Cap_m^*[X^+,U]$ by deleting any subset of variables u_i. Similarly, we define by Cap_m^- the set of 2^{m-1} polynomials obtained from $Cap_m^*[X^-,U]$ by deleting any subset of variables u_i. If L and M are two natural numbers, we denote by $\Gamma^*_{M,L}=\langle Cap_M^+, Cap_L^-\rangle$ the T^*-ideal generated by the polynomials Cap_M^+, Cap_L^-. We have the following:

Theorem 3.4 *[12, Theorem 4] Let $M=k(k+1)/2$ and $L=k(k-1)/2$ with $k\in\mathbb{N}$, $k>0$. Then*

$$\mathrm{var}^*(\Gamma^*_{M+1,L+1})=\mathrm{var}^*(M_k(F)\oplus D'),$$

where D' is a finite dimensional $*$-algebra such that $\exp^*(D')<M+L$. In particular,

$$c_n^*(\Gamma^*_{M+1,L+1})\simeq c_n^*((M_k(F),t))\simeq$$

$$\left[\sqrt{k}^{\frac{k(k-1)}{2}}\frac{1}{k!}\Gamma\left(\frac{3}{2}\right)^{-k}\prod_{j=1}^{k}\Gamma\left(1+\frac{1}{2}j\right)\left(\frac{1}{2}\right)^{k-1}\right]\left(\frac{1}{\sqrt{2n}}\right)^{\frac{k(k-1)}{2}}k^{2n},$$

where $\Gamma(x)$ is the Euler's gamma function.

Theorem 3.5 *[12, Theorem 5] Let $M=m(2m-1)$ and $L=m(2m+1)$ with $m\in\mathbb{N}$, $m>0$. Then*

$$\mathrm{var}^*(\Gamma^*_{M+1,L+1})=\mathrm{var}^*(M_{2m}(F)\oplus D''),$$

where D'' is a finite dimensional $*$-algebra such that $\exp^*(D'')<M+L$. In particular,

$$c_n^*(\Gamma^*_{M+1,L+1})\simeq c_n^*((M_{2m}(F),s))\simeq$$

$$\left[\left(\frac{1}{\sqrt{2\pi}}\right)^m 2^{\frac{m^2+m+2}{4}}m^{\frac{m(7m-1)}{4}}\frac{1}{m!}\prod_{j=1}^{m}\Gamma(2j+1)\right]\left(\frac{1}{2n}\right)^{\frac{m(2m+1)}{2}}(2m)^{2n},$$

where $\Gamma(x)$ is the Euler's gamma function.

Theorem 3.6 *[12, Theorem 6] Let $M=L=h^2$ with $h\in\mathbb{N}$, $h>0$. Then*

$$\mathrm{var}^*(\Gamma^*_{M+1,L+1})=\mathrm{var}^*((M_h(F)\oplus M_h(F)^{op})\oplus D'''),$$

where D''' is a finite dimensional $*$-algebra such that $\exp^*(D''')<M+L$. In particular,

$$c_n^*(\Gamma^*_{M+1,L+1})\simeq c_n^*((M_h(F)\oplus M_h(F)^{op},exc))\simeq$$

$$\left(\frac{1}{\sqrt{2\pi}}\right)^{h-1}\left(\frac{1}{2}\right)^{\frac{1}{2}(h^2+1)} 1!2!\cdots(h-1)!h^{\frac{1}{2}(h^2+4)}n^{\frac{1-h^2}{2}}(2h^2)^n.$$

4. Graded-involution case

Given a superalgebra $A = A_0 \oplus A_1$ endowed with an involution $*$, we say that $*$ is a graded involution if it preserves the homogeneous components of A, i.e. if $A_i^* \subseteq A_i$, $i = 0,1$. A superalgebra endowed with a graded involution is called $*$-superalgebra. It is clear that a superalgebra A is a $*$-superalgebra if and only if the subspaces A^+ and A^- are graded subspaces. Thus, since char $F = 0$, the $*$-superalgebra A can be written as

$$A = A_0^+ \oplus A_0^- \oplus A_1^+ \oplus A_1^-$$

where, for $i = 0,1$, $A_i^+ = \{a \in A_i \mid a^* = a\}$ and $A_i^- = \{a \in A_i \mid a^* = -a\}$ denote the sets of homogeneous symmetric and skew elements of A_i, respectively.

Let A be a $*$-superalgebra and let I be an ideal of A, we say that I is a $*$-graded ideal of A if it is homogeneous in the $\mathbb{Z}_2$-grading and invariant under $*$. Moreover, A is called simple $*$-superalgebra if $A^2 \neq \{0\}$ and it has no nonzero $*$-graded ideals.

Let $X = \{x_1, x_2, \ldots\}$ be a countable set of noncommutative variables and $F\langle X\rangle$ the free associative algebra on X over F. We write $X = Y \cup Z$ as the disjoint union of two countable sets of variables $Y = \{y_1, y_2, \ldots\}$ and $Z = \{z_1, z_2, \ldots\}$, then $F\langle X\rangle = F\langle Y \cup Z\rangle = F\langle y_1, z_1, y_2, z_2, \ldots\rangle$ is the free superalgebra over F. Moreover, if we write each set as the disjoint union of two other infinite sets of symmetric and skew elements, respectively, then we obtain the free $*$-superalgebra

$$F\langle Y \cup Z, *\rangle = F\langle y_1^+, y_1^-, z_1^+, z_1^-, \ldots\rangle$$

where $y_i^+ = y_i + y_i^*$ denotes a symmetric variable of even degree, $y_i^- = y_i - y_i^*$ a skew variable of even degree, $z_i^+ = z_i + z_i^*$ a symmetric variable of odd degree and $z_i^- = z_i - z_i^*$ a skew variable of odd degree.

An element $f = f(y_1^+, \ldots, y_n^+, y_1^-, \ldots, y_m^-, z_1^+, \ldots, z_p^+, z_1^-, \ldots, z_q^-)$ of $F\langle Y \cup Z, *\rangle$ is a $*$-graded polynomial identity for the $*$-superalgebra A if

$$f(a_{1,0}^+, \ldots, a_{n,0}^+, a_{1,0}^-, \ldots, a_{m,0}^-, a_{1,1}^+, \ldots, a_{p,1}^+, a_{1,1}^-, \ldots, a_{q,1}^-) = 0_A$$

for every $a_{1,0}^+, \ldots, a_{n,0}^+ \in A_0^+$, $a_{1,0}^-, \ldots, a_{m,0}^- \in A_0^-$, $a_{1,1}^+, \ldots, a_{p,1}^+ \in A_1^+$, $a_{1,1}^-, \ldots, a_{q,1}^- \in A_1^-$ and we write $f \equiv 0$. The set of all $*$-graded polynomial identities satisfied by A

$$Id_{\mathbb{Z}_2}^*(A) = \{f \in F\langle Y \cup Z, *\rangle \mid f \equiv 0 \text{ on } A\}$$

is an ideal of $F\langle Y \cup Z, *\rangle$ called the ideal of $*$-graded identities of A. It is easy to show that $Id_{\mathbb{Z}_2}^*(A)$ is a $T_{\mathbb{Z}_2}^*$-ideal of $F\langle Y \cup Z, *\rangle$, i.e. a two-sided ideal invariant under all endomorphisms of the free $*$-superalgebra that preserve the superstructure and commute with the graded involution $*$. We denote by $\mathrm{var}_{\mathbb{Z}_2}^*(\Gamma) = \mathrm{var}_{\mathbb{Z}_2}^*(A)$ the variety of $*$-superalgebras having the elements of $\Gamma = Id_{\mathbb{Z}_2}^*(A)$ as $*$-graded identities.

As in the previuos cases, we want to describe the $T_{\mathbb{Z}_2}^*$-ideals of $*$-graded polynomial identities of finite dimensional simple $*$-superalgebras. Recently, for these $*$-superalgebras, the following classification was proven:

Theorem 4.1 *[19, Theorem 7.6] If F is an algebraically closed field of characteristic zero, then, up to graded isomorphisms, the only finite dimensional simple $*$-superalgebras are the following:*

- $(M_{k,l}(F),\diamond)$, *with* $k\geq l\geq 0$, $k\neq 0$;
- $(M_{k,l}(F)\oplus M_{k,l}(F)^{op}, exc)$, *with* $k\geq l\geq 0$, $k\neq 0$, *and induced grading;*
- $(M_k(F+cF),\star)$, *with involution given by* $(a+cb)^\star = a^\diamond - cb^\diamond$;
- $(M_k(F+cF),\dagger)$, *with involution given by* $(a+cb)^\dagger = a^\diamond + cb^\diamond$;
- $(M_k(F+cF)\oplus M_k(F+cF)^{op}, exc)$, *with grading* $(M_k(F)\oplus M_k(F)^{op}, c(M_k(F)\oplus M_k(F)^{op}))$;

where $\diamond = t, s$ denotes the transpose or symplectic involution and exc is the exchange involution. The symplectic involution can occur only when $k=l$.

Since char $F=0$, it is well known that $Id^*_{\mathbb{Z}_2}(A)$ is completely determined by its multilinear polynomials. Now, let

$$P_n^{(\mathbb{Z}_2,*)} = \{w_{\sigma(1)},\dots,w_{\sigma(n)} \mid \sigma\in S_n,\, w_i\in\{y_i^+, y_i^-, z_i^+, z_i^-\},\, i=1,\dots,n\}$$

be the space of multilinear polynomials of degree n in the variables y_1^+, y_1^-, z_1^+, $z_1^-,\dots,$ y_n^+, y_n^-, z_n^+, z_n^-, (i.e. y_i^+, y_i^-, z_i^+ or z_i^- appears in each monomial at degree 1). Then, for all $n\geq 1$, one defines the n-th $*$-graded codimension $c_n^{(\mathbb{Z}_2,*)}(A)$ of the $*$-superalgebra A as

$$c_n^{(\mathbb{Z}_2,*)}(A) = \dim \frac{P_n^{(\mathbb{Z}_2,*)}}{P_n^{(\mathbb{Z}_2,*)}\cap Id^*_{\mathbb{Z}_2}(A)}.$$

If A satisfies an ordinary polynomial identity, then the sequence $\{c_n^{(\mathbb{Z}_2,*)}(A)\}_{n\geq 1}$ is exponentially bounded (see [19, Lemma 3.1]).

Now it is easy to determine the asymptotic behavior of the $*$-graded codimension of the finite dimensional simple $*$-superalgebras $(M_{k,l}(F)\oplus M_{k,l}(F)^{op}, exc)$ and $(M_k(F+cF)\oplus M_k(F+cF)^{op}, exc)$. For the other finite dimensional simple $*$-superalgebras these asymptotic behavior are unknown. We have the following

Theorem 4.2 *Let $A=A_0\oplus A_1$ be a finite dimensional simple superalgebra. Then*

$$c_n^{(\mathbb{Z}_2,*)}((A\oplus A^{op}, exc)) = 2^n c_n^{\mathbb{Z}_2}(A)$$

Proof. For any $k=0,\dots,n$, we denote by $P^{\mathbb{Z}_2}_{k,n-k}$ the space of multilinear polynomials where the first k variables are homogeneous of degree zero and the remaning $n-k$ variables are homogeneous of degree one. Now, for any $\bar{k}$ and $\bar{h}$ such that $0\leq\bar{k}\leq k$, $0\leq\bar{h}\leq n-k$, we define

$$P^{(\mathbb{Z}_2,*)}_{\bar{k},k-\bar{k},\bar{h},n-(k+\bar{h})} = \{w_{\sigma(1)},\dots,w_{\sigma(n)} \mid \sigma\in S_n,\, w_i=y_i^+,\, 1\leq i\leq\bar{k},\, w_i=y_i^-,\, \bar{k}+1\leq i\leq k,$$

$$w_i=z_i^+,\, k+1\leq i\leq k+\bar{h},\ \text{and}\ w_i=z_i^-,\, k+\bar{h}+1\leq i\leq n\}.$$

If $f(y_1,\dots,y_k;z_{k+1},\dots,z_n)\in P^{\mathbb{Z}_2}_{k,n-k}$, then we can consider the $*$-graded polynomial $\bar f(y_1^+,\dots,y_{\bar k}^+,y_{\bar k+1}^-,\dots,y_k^-;z_{k+1}^+,\dots,z_{k+\bar h}^+,z_{k+\bar h+1}^-,\dots,z_n^-)\in P^{(\mathbb{Z}_2,*)}_{\bar k,k-\bar k,\bar h,n-(k+\bar h)}$ obtained from f by substituting the variables $y_1,\dots,y_k,z_{k+1},\dots,z_n$ with $y_1^+,\dots,y_{\bar k}^+,y_{\bar k+1}^-,\dots,y_k^-,z_{k+1}^+,\dots,z_{k+\bar h}^+,z_{k+\bar h+1}^-,\dots,z_n^-$, respectively.

Notice that if $A=A_0\oplus A_1$ is a superalgebra with graded involution $*$ then the map $\varphi: A\to A^{op}$ such that $\varphi(a)=a^*$ is an isomorphism of superalgebras. Hence, $Id_{\mathbb{Z}_2}(A)=Id_{\mathbb{Z}_2}(A^{op})$ and so it easily follows that $f\in P^{\mathbb{Z}_2}_{k,n-k}\cap Id_{\mathbb{Z}_2}(A)$ if and only if $\bar f\in P^{(\mathbb{Z}_2,*)}_{\bar k,k-\bar k,\bar h,n-(k+\bar h)}\cap Id^*_{\mathbb{Z}_2}(A\oplus A^{op})$. In fact, if $f\in P^{\mathbb{Z}_2}_{k,n-k}\cap Id_{\mathbb{Z}_2}(A)$ then $f(a_1,\dots,a_k;a_{k+1},\dots,a_n)=0$ for all $a_1,\dots,a_k\in A_0$ and $a_{k+1},\dots,a_n\in A_1$. Thus,

$$\bar f((a_1,a_1),\dots,(a_{\bar k},a_{\bar k}),(a_{\bar k+1},-a_{\bar k+1}),\dots,(a_k,-a_k),(a_{k+1},a_{k+1}),\dots$$

$$\dots,(a_{k+\bar h},a_{k+\bar h}),(a_{k+\bar h+1},-a_{k+\bar h+1}),\dots,(a_n,-a_n))=$$

$$(f(a_1,\dots,a_k;a_{k+1},\dots,a_n),f(a_1,\dots,a_{\bar k},-a_{\bar k+1},\dots,-a_k;a_{k+1},\dots,a_{k+\bar h},-a_{k+\bar h+1},\dots,-a_n))=$$

$$=(0,0).$$

Viceversa, let $\bar f\in P^{(\mathbb{Z}_2,*)}_{\bar k,k-\bar k,\bar h,n-(k+\bar h)}\cap Id^*_{\mathbb{Z}_2}(A\oplus A^{op})$. If there exist $a_1,\dots,a_k\in A_0$ and $a_{k+1},\dots,a_n\in A_1$ such that $f(a_1,\dots,a_k;a_{k+1},\dots,a_n)\neq 0$, then for any $\bar k$ and $\bar h$ with $0\le\bar k\le k$, $0\le\bar h\le n-k$, we obtain

$$0=\bar f((a_1,a_1),\dots,(a_{\bar k},a_{\bar k}),(a_{\bar k+1},-a_{\bar k+1}),\dots,(a_k,-a_k),(a_{k+1},a_{k+1}),\dots$$

$$\dots,(a_{k+\bar h},a_{k+\bar h}),(a_{k+\bar h+1},-a_{k+\bar h+1}),\dots,(a_n,-a_n))=$$

$$(f(a_1,\dots,a_k;a_{k+1},\dots,a_n),f(a_1,\dots,a_{\bar k},-a_{\bar k+1},\dots,-a_k;a_{k+1},\dots,a_{k+\bar h},-a_{k+\bar h+1},\dots,-a_n))\neq$$

$$\neq(0,0)$$

and this is a contradiction. As a consequence, we obtain that

$$c^{\mathbb{Z}_2}_{k,n-k}(A)=\dim\frac{P^{\mathbb{Z}_2}_{k,n-k}}{P^{\mathbb{Z}_2}_{k,n-k}\cap Id_{\mathbb{Z}_2}(A)}=$$

$$\dim\frac{P^{(\mathbb{Z}_2,*)}_{\bar k,k-\bar k,\bar h,n-(k+\bar h)}}{P^{(\mathbb{Z}_2,*)}_{\bar k,k-\bar k,\bar h,n-(k+\bar h)}\cap Id^*_{\mathbb{Z}_2}(A\oplus A^{op})}=c^{(\mathbb{Z}_2,*)}_{\bar k,k-\bar k,\bar h,n-(k+\bar h)}(A\oplus A^{op}).$$

Now, since

$$P_n^{(\mathbb{Z}_2,*)}=\bigoplus_{(\bar k,\,k-\bar k,\,\bar h,\,(n-k)-\bar h)}\binom{n}{\bar k,\ k-\bar k,\ \bar h,\ (n-k)-\bar h}P^{(\mathbb{Z}_2,*)}_{(\bar k,\,k-\bar k,\,\bar h,\,(n-k)-\bar h)}$$

where $\binom{n}{\bar k,\ k-\bar k,\ \bar h,\ (n-k)-\bar h}$ denotes the multimonomial coefficient, we obtain

$$c_n^{(\mathbb{Z}_2,*)}(A\oplus A^{op})=\sum_{k=0}^{n}\sum_{\bar k=0}^{k}\sum_{\bar h=0}^{n-k}\binom{n}{\bar k,\ k-\bar k,\ \bar h,\ (n-k)-\bar h}c^{(\mathbb{Z}_2,*)}_{\bar k,k-\bar k,\bar h,n-(k+\bar h)}(A\oplus A^{op})=$$

$$= \sum_{k=0}^{n} \left(\sum_{\bar{k}=0}^{k} \sum_{\bar{h}=0}^{n-k} \frac{n!}{\bar{k}!(k-\bar{k})!\bar{h}![n-(k+\bar{h})]!} \right) c_{k,n-k}^{\mathbb{Z}_2}(A) =$$

$$= \sum_{k=0}^{n} \left(\sum_{\bar{k}=0}^{k} \frac{[(n-k)+1]\cdots n}{\bar{k}!(k-\bar{k})!} \sum_{\bar{h}=0}^{n-k} \frac{(n-k)!}{\bar{h}![n-(k+\bar{h})]!} \right) c_{k,n-k}^{\mathbb{Z}_2}(A) =$$

$$= \sum_{k=0}^{n} \left(\sum_{\bar{k}=0}^{k} \frac{[(n-k)+1]\cdots n}{\bar{k}!(k-\bar{k})!} \sum_{\bar{h}=0}^{n-k} \binom{n-k}{\bar{h}} \right) c_{k,n-k}^{\mathbb{Z}_2}(A) =$$

$$= \sum_{k=0}^{n} \left(\sum_{\bar{k}=0}^{k} \frac{[(n-k)+1]\cdots n}{\bar{k}!(k-\bar{k})!} \right) 2^{n-k} c_{k,n-k}^{\mathbb{Z}_2}(A) =$$

$$= \sum_{k=0}^{n} \left(\sum_{\bar{k}=0}^{k} \frac{n!}{(n-k)!\bar{k}!(k-\bar{k})!} \right) 2^{n-k} c_{k,n-k}^{\mathbb{Z}_2}(A) =$$

$$= \sum_{k=0}^{n} \left(\sum_{\bar{k}=0}^{k} \binom{k}{\bar{k}} \right) 2^{n-k} \binom{n}{k} c_{k,n-k}^{\mathbb{Z}_2}(A) =$$

$$= 2^n \sum_{k=0}^{n} \binom{n}{k} c_{k,n-k}^{\mathbb{Z}_2}(A) = 2^n c_n^{\mathbb{Z}_2}(A).$$

So the theorem is proven.

By Theorem 2.4, we obtain immediately

Lemma 4.3

$$c_n^{(\mathbb{Z}_2,*)}((M_{k,l}(F) \oplus M_{k,l}(F)^{op}, exc)) \simeq \alpha n^{\frac{1-(k^2+l^2)}{2}} (2(k+l)^2)^n$$

and

$$c_n^{(\mathbb{Z}_2,*)}((M_k(F+cF)) \oplus (M_k(F+cF))^{op}, exc) \simeq \beta n^{\frac{1-k^2}{2}} (2k)^{2n},$$

for some costant α and β.

If A is a finite dimensional PI-algebra, Gordienko in [29] proved that

$$\exp_{\mathbb{Z}_2}^*(A) = \lim_{n\to\infty} \sqrt[n]{c_n^{(\mathbb{Z}_2,*)}(A)}$$

exists and is a nonnegative integer. It is called the $*$-graded exponent of the $*$-superalgebra A. Moreover, he showed that $\exp_{\mathbb{Z}_2}^*(A) = d$ where d is the maximal dimension of an admissible subalgebra of A. By the generalization of the Wedderburn-Malcev Theorem (see [19, Theorem 7.3]), we can write $A = A_1 \oplus \cdots \oplus A_r + J$, where $A_1, \ldots, A_r$ are simple $*$-superalgebras and $J = J(A)$ is the Jacobson radical of A which is a $*$-graded ideal.

We say that a subalgebra $A_{i_1} \oplus \cdots \oplus A_{i_k}$ of A, where $A_{i_1}, \ldots, A_{i_k}$ are distinct simple components, is admissible if for some permutation $(l_1, \ldots, l_k)$ of $(i_1, \ldots, i_k)$ we have that $A_{l_1} J \cdots J A_{l_k} \neq 0$. Moreover, if $A_{i_1} \oplus \cdots \oplus A_{i_k}$ is an admissible subalgebra of A then $A' = A_{i_1} \oplus \cdots \oplus A_{i_k} + J$ is called a reduced algebra. It follows immediately that

Remark 4.4 *If A is a simple $*$-superalgebra then $\exp^*_{\mathbb{Z}_2}(A) = \dim_F A$.*

By [20, Theorem 5.3], the Gordienko's result on the existence of the $*$-graded exponent can be actually extended to any finitely generated PI-$*$-superalgebra since it satisfies the same $*$-graded polynomial identities of a finite-dimensional $*$-superalgebra.

In [27], it was shown that reduced superalgebras are building blocks of any proper variety. We obtained the analogous result for varieties of $*$-superalgebras in [13].

Lemma 4.5 *Let A and B be $*$-superalgebras satisfying an ordinary polynomial identity. Then*

$$c_n^{(\mathbb{Z}_2,*)}(A), c_n^{(\mathbb{Z}_2,*)}(B) \leq c_n^{(\mathbb{Z}_2,*)}(A \oplus B) \leq c_n^{(\mathbb{Z}_2,*)}(A) + c_n^{(\mathbb{Z}_2,*)}(B).$$

If A and B are finitely generated $*$-superalgebras, then

$$\exp^*_{\mathbb{Z}_2}(A \oplus B) = max\{\exp^*_{\mathbb{Z}_2}(A), \exp^*_{\mathbb{Z}_2}(B)\}.$$

Theorem 4.6 *Let A be a finitely generated $*$-superalgebra satisfying an ordinary polynomial identity. Then there exists a finite number of reduced $*$-superalgebras $B_1, \ldots, B_r$ and a finite dimensional $*$-superalgebra D such that*

$$\mathrm{var}^*_{\mathbb{Z}_2}(A) = \mathrm{var}^*_{\mathbb{Z}_2}(B_1 \oplus \cdots \oplus B_r \oplus D)$$

with $\exp^*_{\mathbb{Z}_2}(A) = \exp^*_{\mathbb{Z}_2}(B_1) = \cdots = \exp^*_{\mathbb{Z}_2}(B_r)$ and $\exp^*_{\mathbb{Z}_2}(D) < \exp^*_{\mathbb{Z}_2}(A)$.

An application of Theorem 4.6 is given in terms of $*$-graded codimensions.

Corollary 4.7 *Let A be a finitely generated PI-$*$-superalgebra. Then there exists a finite number of reduced $*$-superalgebras $B_1, \ldots, B_r$ such that*

$$c_n^{(\mathbb{Z}_2,*)}(A) \simeq c_n^{(\mathbb{Z}_2,*)}(B_1 \oplus \cdots \oplus B_r).$$

Now we shall recall the construction of the $*$-superalgebra $UT^*_{\mathbb{Z}_2}(A_1, \ldots, A_m)$ given in Section 3 of [17] and we shall investigate the relation among the asymptotics of the $*$-supercodimensions of the finite dimensional simple $*$-superalgebras and the $T^*_{\mathbb{Z}_2}$-ideals generated by the $*$-graded Capelli polynomials recentely proved by the authors.

Let $(A_1, \ldots, A_m)$ be a m-tuple of finite dimensional simple $*$-superalgebras. For every $r = 1, \ldots, m$, the size of A_r is given by

$$s_r = \begin{cases} k_r + l_r & \text{if } A_r = M_{k_r,l_r}(F) \text{ or } A_r = M_{k_r,l_r}(F) \oplus M_{k_r,l_r}(F)^{op}; \\ 2k_r & \text{if } A_r = M_{k_r}(F+cF) \text{ or } A_k = M_{k_r}(F+cF) \oplus M_{k_r}(F+cF)^{op} \end{cases}$$

and, set $\eta_0 = 0$, let $\eta_r = \Sigma_{i=1}^r s_i$ and $\mathrm{B}l_r = \{\eta_{r-1}+1, \dots, \eta_r\}$. Let γ_d be the orthogonal involution defined on the matrix algebra $M_d(F)$ by sending each $a \in M_d(F)$ into the element $a^{\gamma_d} \in M_d(F)$ obtained reflecting a along its secondary diagonal. In particular for any matrix unit $e_{i,j}$ of $M_d(F)$, $e_{i,j}^{\gamma_d} = e_{d-j+1,d-i+1}$. Then, we have a monomorphism of $*$-algebra

$$\Delta : \bigoplus_{r=1}^{m} A_r \to (M_{2\eta_m}(F), \gamma_{2\eta_m})$$

defined by

$$(c_1, \dots, c_m) \to \begin{pmatrix} \bar{a}_1 & & & & & \\ & \ddots & & & & \\ & & \bar{a}_m & & & \\ & & & \bar{b}_m & & \\ & & & & \ddots & \\ & & & & & \bar{b}_1 \end{pmatrix}$$

where the elements $\bar{a}_i$ and $\bar{b}_i$ are defined as follows:

- if $c_i \in (M_{k,l}(F); \diamond)$, then $\bar{a}_i = c_i$ and $\bar{b}_i = (c_i^{\diamond})^{\gamma_{k+l}}$;
- if $c_i = (a_i, b_i) \in (M_{k,l}(F) \oplus M_{k,l}(F)^{op}, exc)$, then $\bar{a}_i = a_i$ and $\bar{b}_i = b_i^{\gamma_{k+l}}$;
- if $c_i = a_i + cb_i \in (M_k(F+cF), \star)$, then $\bar{a}_i = \begin{pmatrix} a_i & b_i \\ b_i & a_i \end{pmatrix}$ and $\bar{b}_i = (\bar{a}_i^{\perp})^{\gamma_{2k}}$ where $\begin{pmatrix} x & y \\ y & x \end{pmatrix}^{\perp} = \begin{pmatrix} x^{\diamond} & -y^{\diamond} \\ -y^{\diamond} & x^{\diamond} \end{pmatrix}$;
- if $c_i = a_i + cb_i \in (M_k(F+cF), \dagger)$, then $\bar{a}_i = \begin{pmatrix} a_i & b_i \\ b_i & a_i \end{pmatrix}$ and $\bar{b}_i = (\bar{a}_i^{\top})^{\gamma_{2k}}$ where $\begin{pmatrix} x & y \\ y & x \end{pmatrix}^{\top} = \begin{pmatrix} x^{\diamond} & y^{\diamond} \\ y^{\diamond} & x^{\diamond} \end{pmatrix}$;
- if $c_i = (a_i+cb_i, u_i+cv_i) \in (M_k(F+cF) \oplus M_k(F+cF)^{op}, exc)$, then $\bar{a}_i = \begin{pmatrix} a_i & b_i \\ b_i & a_i \end{pmatrix}$ and $\bar{b}_i = \begin{pmatrix} u_i & v_i \\ v_i & u_i \end{pmatrix}^{\gamma_{2k}}$.

Let denote by $D \subseteq (M_{2\eta_m}(F), \gamma_{2\eta_m})$ the $*$-algebra image of $\bigoplus_{i=1}^m A_i$ by Δ and set

$$V = \begin{pmatrix} 0 & V_{12} & \cdots & V_{1m} & & & & \\ & \ddots & \ddots & \vdots & & & & \\ & & 0 & V_{m-1m} & & & & \\ & & & 0 & & & & \\ & & & & 0 & V_{mm-1} & \cdots & V_{m1} \\ & & & & & \ddots & \ddots & \vdots \\ & & & & & & 0 & V_{21} \\ & & & & & & & 0 \end{pmatrix} \subseteq M_{2\eta_m}(F)$$

where, for $1 \leq i, j \leq m$, $i \neq j$, $V_{ij} = M_{s_i \times s_j}(F)$ is the vector space of $s_i \times s_j$ matrices of F. Let define

$$UT^*(A_1, \ldots, A_m) = D \oplus V \subseteq M_{2\eta_m}(F).$$

It is easy to see that $UT^*(A_1, \ldots, A_m)$ is a subalgebra with involution of $(M_{2\eta_m}(F), \gamma_{2\eta_m})$ whose Jacobson radical coincides with V.

Now, for any m-tuple $\tilde{g} = (g_1, \ldots, g_m) \in \mathbb{Z}_2^m$, we consider the map

$$\alpha_{\tilde{g}} : \{1, \ldots, 2\eta_m\} \to \mathbb{Z}_2, \;\; i \to \begin{cases} \alpha_r(i - \eta_{r-1}) + g_r & 1 \leq i \leq \eta_m; \\ \alpha_r(2\eta_m - i + 1 - \eta_{r-1}) + g_r & \eta_m + 1 \leq i \leq 2\eta_m. \end{cases}$$

where $r \in \{1, \ldots, m\}$ is the (unique) integer such that $i \in \mathrm{B}l_r$ and α_r's are maps so defined:

· if $A_r \simeq M_{k,l}(F)$ or $A_r \simeq M_{k,l}(F) \oplus M_{k,l}(F)^{op}$, then

$$\alpha_r : \{1, \ldots, k+l\} \to \mathbb{Z}_2, \;\; \alpha_r(i) = \begin{cases} 0 & 1 \leq i \leq k; \\ 1 & k+1 \leq i \leq k+l. \end{cases}$$

· if $A_r \simeq M_k(F + cF)$ or $A_r \simeq M_k(F + cF) \oplus M_k(F + cF)^{op}$, then

$$\alpha_r : \{1, \ldots, 2k\} \to \mathbb{Z}_2, \;\; \alpha_r(i) = \begin{cases} 0 & 1 \leq i \leq k; \\ 1 & k+1 \leq i \leq 2k. \end{cases}$$

The map $\alpha_{\tilde{g}}$ induces an elementary grading on $UT^*(A_1, \ldots, A_m)$ with respect to which $\gamma_{2\eta_m}$ is a graded involution. We shall use the symbol

$$UT^*_{\mathbb{Z}_2, \tilde{g}}(A_1, \ldots, A_m)$$

to indicate the $*$-superalgebra defined by the m-tuple $\tilde{g}$.

In the next lemma we establish the link between the degrees of the $*$-graded Capelli polynomials and the $*$-graded polynomial identities of $UT^*_{\mathbb{Z}_2, \tilde{g}}(A_1, \ldots, A_m)$. We write

$$Cap_m^{(\mathbb{Z}_2,*)}[Y^+, X], \;\; Cap_m^{(\mathbb{Z}_2,*)}[Y^-, X], \;\; Cap_m^{(\mathbb{Z}_2,*)}[Z^+, X] \;\; \text{and} \;\; Cap_m^{(\mathbb{Z}_2,*)}[Z^-, X]$$

to indicate the m-th $*$-graded Capelli polynomial alternating in the symmetric variables of degree zero $y_1^+, \ldots, y_m^+$, in the skew variables of degree zero $y_1^-, \ldots, y_m^-$, in the symmetric variables of degree one $z_1^+, \ldots, z_m^+$

and in the skew variables of degree one $z_1^-, \ldots, z_m^-$, respectively ($x_1, \ldots, x_{m-1}$ are arbitrary variables). Let $Cap_m^{(0,+)}$ denote the set of 2^{m-1} polynomials obtained from $Cap_m^{(\mathbb{Z}_2,*)}[Y^+, X]$ by deleting any subset of variables x_i (by evaluating the variables x_i to 1 in all possible way). In a similar way we define $Cap_m^{(0,-)}$, $Cap_m^{(1,+)}$ and $Cap_m^{(1,-)}$. If M^+, M^-, L^+ and L^- are natural numbers, we denote by

$$\Gamma_{M^\pm, L^\pm}^{(\mathbb{Z}_2,*)} = \langle Cap_{M^+}^{(0,+)}, Cap_{M^-}^{(0,-)}, Cap_{L^+}^{(1,+)}, Cap_{L^-}^{(1,-)} \rangle_{\mathbb{Z}_2}^*$$

the $T_{\mathbb{Z}_2}^*$-ideal generated by $Cap_{M^+}^{(0,+)}$, $Cap_{M^-}^{(0,-)}$, $Cap_{L^+}^{(1,+)}$ and $Cap_{L^-}^{(1,-)}$.

For all $i = 1, \ldots, m$, let's write $A_i = A_{i,0}^+ \oplus A_{i,0}^- \oplus A_{i,1}^+ \oplus A_{i,1}^-$ and let $(d_0^\pm)_i = \dim_F A_{i,0}^\pm$ and $(d_1^\pm)_i = \dim_F A_{i,1}^\pm$. If we set $d_0^\pm := \sum_{i=1}^m (d_0^\pm)_i$ and $d_1^\pm := \sum_{i=1}^m (d_1^\pm)_i$, then in [13] we proved the following

Lemma 4.8 *Let $\tilde{g} = (g_1, \ldots, g_m)$ be a fixed element of $\mathbb{Z}_2^m$ and $A = UT_{\mathbb{Z}_2,\tilde{g}}^*(A_1, \ldots, A_m)$, with A_i finite dimensional simple $*$-superalgebra. Let $0 \le \bar{m} \le m$ denote the number of the finite dimensional simple $*$-superalgebras with trivial grading.*

1. *If $\bar{m} = 0$, $Cap_{q^+}^{(\mathbb{Z}_2,*)}[Y^+, X]$, $Cap_{q^-}^{(\mathbb{Z}_2,*)}[Y^-, X]$, $Cap_{k^+}^{(\mathbb{Z}_2,*)}[Z^+, X]$ and $Cap_{k^-}^{(\mathbb{Z}_2,*)}[Z^-, X]$ are in $Id_{\mathbb{Z}_2}^*(A)$ if and only if $q^+ \ge d_0^+ + m$, $q^- \ge d_0^- + m$, $k^+ \ge d_1^+ + m$ and $k^- \ge d_1^- + m$;*
2. *If $0 < \bar{m} \le m$, let $\tilde{m}$ be the number of blocks of consecutive $*$-superalgebras with trivial grading that appear in $(A_1, \ldots, A_m)$. Then $Cap_{q^+}^{(\mathbb{Z}_2,*)}[Y^+, X], Cap_{q^-}^{(\mathbb{Z}_2,*)}[Y^-, X]$, $Cap_{k^+}^{(\mathbb{Z}_2,*)}[Z^+, X]$ and $Cap_{k^-}^{(\mathbb{Z}_2,*)}[Z^-, X]$ are in $Id_{\mathbb{Z}_2}^*(A)$ if and only if $q^+ > d_0^+ + (m - \bar{m}) + (\tilde{m} - 1) + r_0$, $q^- > d_0^- + (m - \bar{m}) + (\tilde{m} - 1) + r_0$, $k^+ > d_1^+ + (m - \bar{m}) + (\tilde{m} - 1) + r_1$ and $k^- > d_1^- + (m - \bar{m}) + (\tilde{m} - 1) + r_1$, where r_0, r_1 are two non negative integers depending on the grading $\tilde{g}$, with $r_0 + r_1 = \bar{m} - \tilde{m}$.*

The next results give us a characterization of the varieties of $*$-superalgebras satisfying a Capelli identity.

Lemma 4.9 *Let M^+, M^-, L^+ and L^- be natural numbers. If A is a $*$-superalgebra satisfying the $*$-graded Capelli polynomials $Cap_{M^+}^{(\mathbb{Z}_2,*)}[Y^+, X]$, $Cap_{M^-}^{(\mathbb{Z}_2,*)}[Y^-, X]$, $Cap_{L^+}^{(\mathbb{Z}_2,*)}[Z^+, X]$ and $Cap_{L^-}^{(\mathbb{Z}_2,*)}[Z^-, X]$, then A satisfies the Capelli identity $Cap_k(x_1, \ldots, x_k; \bar{x}_1, \ldots, \bar{x}_{k-1})$, where $k = M^+ + M^- + L^+ + L^-$.*

Theorem 4.10 *Let $\mathcal{V}_{\mathbb{Z}_2}^*$ be a variety of $*$-superalgebras. If $\mathcal{V}_{\mathbb{Z}_2}^*$ satisfies the Capelli identity of some rank, then $\mathcal{V}_{\mathbb{Z}_2}^* = \mathrm{var}_{\mathbb{Z}_2}^*(A)$, for some finitely generated $*$-superalgebra A.*

Let

$$A = \bar{A} \oplus J$$

where $\bar{A}$ is a finite dimensional simple $*$-superalgebra and $J = J(A)$ is its Jacobson radical. From now on we put $M^\pm = \dim_F \bar{A}_0^\pm$ and $L^\pm = \dim_F \bar{A}_1^\pm$. Then

Lemma 4.11 $\exp^*_{\mathbb{Z}_2}(\Gamma^{(\mathbb{Z}_2,*)}_{M^\pm+1,L^\pm+1}) = M^+ + M^- + L^+ + L^- = M + L = \exp^*_{\mathbb{Z}_2}(\bar{A})$.

Moreover we have the following two technical lemmas

Lemma 4.12 *The Jacobson radical J can be decomposed into the direct sum of four $\bar{A}$-bimodules*

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11}$$

where, for $p, q \in \{0,1\}$, J_{pq} is a left faithful module or a 0-left module according to $p = 1$, or $p = 0$, respectively. Similarly, J_{pq} is a right faithful module or a 0-right module according to $q = 1$ or $q = 0$, respectively. Moreover, for $p, q, i, l \in \{0,1\}$, $J_{pq}J_{ql} \subseteq J_{pl}$, $J_{pq}J_{il} = 0$ for $q \neq i$ and there exists a finite dimensional nilpotent $*$-superalgebra N such that N commutes with $\bar{A}$ and $J_{11} \cong \bar{A} \otimes_F N$ (isomorphism of $\bar{A}$-bimodules and of $*$-superalgebras).

Notice that J_{00} and J_{11} are stable under the involution whereas $J_{01}^* = J_{10}$.

Lemma 4.13 *If $\Gamma^{(\mathbb{Z}_2,*)}_{M^\pm+1,L^\pm+1} \subseteq Id^*_{\mathbb{Z}_2}(A)$, then $J_{10} = J_{01} = (0)$ and N is commutative.*

Now we are able to state the main result of [13].

Theorem 4.14 *For suitable natural numbers M^+, M^-, L^+, L^- there exists a finite dimensional simple $*$-superalgebra A such that*

$$\mathcal{U} = \mathrm{var}^*_{\mathbb{Z}_2}(\Gamma^{(\mathbb{Z}_2,*)}_{M^\pm+1,L^\pm+1}) = \mathrm{var}^*_{\mathbb{Z}_2}(A \oplus D),$$

where D is a finite dimensional $*$-superalgebra such that $\exp^*_{\mathbb{Z}_2}(D) < M + L$, with $M = M^+ + M^-$ and $L = L^+ + L^-$. In particular

1) *If $M^\pm = \frac{k(k\pm1)}{2} + \frac{l(l\pm1)}{2}$ and $L^\pm = kl$, with $k \geq l > 0$, then $A = (M_{k,l}(F), t)$;*

2) *If $M^\pm = k^2$ and $L^\pm = k(k \mp 1)$, with $k > 0$, then $A = (M_{k,k}(F), s)$;*

3) *If $M^\pm = k^2 + l^2$ and $L^\pm = 2kl$, with $k \geq l > 0$, then $A = (M_{k,l}(F) \oplus M_{k,l}(F)^{op}, exc)$;*

4) *If $M^+ = L^\pm = \frac{k(k+1)}{2}$, $M^- = L^\mp = \frac{k(k-1)}{2}$, with $k > 0$, then $A = (M_k(F + cF), *)$, where $(a + cb)^* = a^t \pm cb^t$;*

5) *If $M^+ = L^\pm = \frac{k(k-1)}{2}$, $M^- = L^\mp = \frac{k(k+1)}{2}$, with $k > 0$, then $A = (M_k(F + cF), *)$, where $(a + cb)^* = a^s \pm cb^s$;*

6) *If $M^\pm = L^\pm = k^2$, with $k > 0$, then $A = (M_k(F + cF) \oplus M_k(F + cF)^{op}, exc)$.*

Sketch of the proof. By Lemma 4.11 we have that $\exp^*_{\mathbb{Z}_2}(\mathcal{U}) = M+L$. Let B be a generating $*$-superalgebra of $\mathcal{U}$. From Lemma 4.9 and Theorem 4.10, since any finitely generated $*$-superalgebra satisfies the same $*$-graded polynomial identities of a finite-dimensional $*$-superalgebra (see [20]), we can assume that B is finite dimensional. Thus, by Theorem 4.6, there exists a finite number of reduced $*$-superalgebras $B_1, \ldots, B_r$ and a finite dimensional $*$-superalgebra D such that

$$\mathcal{U} = \mathrm{var}^*_{\mathbb{Z}_2}(B) = \mathrm{var}^*_{\mathbb{Z}_2}(B_1 \oplus \cdots \oplus B_r \oplus D). \tag{1}$$

Moreover

$$\exp^*_{\mathbb{Z}_2}(B_1) = \cdots = \exp^*_{\mathbb{Z}_2}(B_r) = \exp^*_{\mathbb{Z}_2}(\mathcal{U}) = M+L$$

and

$$\exp^*_{\mathbb{Z}_2}(D) < \exp^*_{\mathbb{Z}_2}(\mathcal{U}) = M+L.$$

Let's now analyze the structure of a finite dimensional reduced $*$-superalgebra R such that $\exp^*_{\mathbb{Z}_2}(R) = M+L = \exp^*_{\mathbb{Z}_2}(\mathcal{U})$ and $\Gamma^*_{M^\pm+1, L^\pm+1} \subseteq Id^*_{\mathbb{Z}_2}(R)$. We have that

$$R = R_1 \oplus \cdots \oplus R_m + J, \tag{2}$$

where R_i are simple $*$-graded subalgebras of R, $J = J(R)$ is the Jacobson radical of R and $R_1 J \cdots J R_m \neq 0$. By [17, Theorem 4.3] there exists a $*$-superalgebra $\overline{R}$ isomorphic to the $*$-superalgebra $UT^*_{\mathbb{Z}_2,\tilde{g}}(R_1, \ldots, R_m)$, for some $\tilde{g} = (g_1, \ldots, g_m) \in \mathbb{Z}_2^m$, such that $Id(R) \subseteq Id(\overline{R})$ and

$$\exp^*_{\mathbb{Z}_2}(R) = \exp^*_{\mathbb{Z}_2}(\overline{R}) = \exp^*_{\mathbb{Z}_2}(UT^*_{\mathbb{Z}_2,\tilde{g}}(R_1, \ldots, R_m)).$$

It follows that

$$M+L = \exp^*_{\mathbb{Z}_2}(R) = \exp^*_{\mathbb{Z}_2}(\overline{R}) =$$

$$\exp^*_{\mathbb{Z}_2}(UT^*_{\mathbb{Z}_2,\tilde{g}}(R_1, \ldots, R_m)) = \dim_F R_1 + \cdots + \dim_F R_m = d_0^+ + d_0^- + d_1^+ + d_1^-$$

where $d_i^\pm = \dim_F (R_1 \oplus \cdots \oplus R_m)^\pm_{(i)}$, for $i = 0, 1$.

Let $0 \leq \bar{m} \leq m$ denote the number of the $*$-superalgebras R_i with trivial grading appearing in (2). We want to prove that $\bar{m} = 0$.

Let's suppose $\bar{m} > 0$. By Lemma 4.8, $\overline{R}$ does not satisfy the $*$-graded Capelli polynomials

$$Cap^{(\mathbb{Z}_2,*)}_{d_0^+ + (m-\bar{m}) + (\tilde{m}-1) + r_0}[Y^+, X], \quad Cap^{(\mathbb{Z}_2,*)}_{d_0^- + (m-\bar{m}) + (\tilde{m}-1) + r_0}[Y^-, X],$$

$$Cap^{(\mathbb{Z}_2,*)}_{d_1^+ + (m-\bar{m}) + (\tilde{m}-1) + r_1}[Z^+, X], \quad Cap^{(\mathbb{Z}_2,*)}_{d_1^- + (m-\bar{m}) + (\tilde{m}-1) + r_1}[Z^-, X],$$

where r_0, r_1 are two non negative integers dependent on the grading $\tilde{g}$ with $r_0 + r_1 = \bar{m} - \tilde{m}$.
But $\overline{R}$ satisfies $Cap^{(\mathbb{Z}_2,*)}_{M^+ + 1}[Y^+, X]$, $Cap^{(\mathbb{Z}_2,*)}_{M^- + 1}[Y^-, X]$, $Cap^{(\mathbb{Z}_2,*)}_{L^+ + 1}[Z^+, X]$ and $Cap^{(\mathbb{Z}_2,*)}_{L^- + 1}[Z^-, X]$, then

$$d_0^+ + (m-\bar{m}) + (\tilde{m}-1) + r_0 + d_0^- + (m-\bar{m}) + (\tilde{m}-1) + r_0 +$$

$$d_1^+ + (m-\bar{m}) + (\tilde{m}-1) + r_1 + d_1^- + (m-\bar{m}) + (\tilde{m}-1) + r_1 \leq M+L.$$

Since $d_0^+ + d_0^- + d_1^+ + d_1^- = M + L$ we obtain that $4(m - \bar{m}) + 4(\tilde{m} - 1) + 2(r_0 + r_1) = 0$ and so $2(m - 1) + \tilde{m} - \bar{m} = 0$ and this implies that $m \leq 2$. If $m = 2$ then we easily obtain a contradiction. Thus $m = \bar{m} = \tilde{m} = 1$.

Hence we can assume that $R = R_1 \oplus J$ where $R_1 \simeq (M_{h_1}(F), t)$ or $R_1 \simeq (M_{2h_1}(F), s)$ or $R_1 \simeq (M_{h_1}(F) \oplus M_{h_1}(F)^{op}, exc)$ with $h_1 > 0$.

Let consider the case when $M^{\pm} = \frac{h(h\pm1)}{2} + \frac{l(l\pm1)}{2}$ and $L^{\pm} = hl$, with $h \geq l > 0$.
If $R \simeq (M_{h_1}(F), t) + J$ then $\exp^*_{\mathbb{Z}_2}(R) = h_1^2$. Since $\exp^*_{\mathbb{Z}_2}(R) = M + L = (h + l)^2$ we obtain that $h_1 = h + l$. By hypotesis, R satisfies $Cap^{(\mathbb{Z}_2,*)}_{M^++1}[Y^+; X]$ but, since $Id^*_{\mathbb{Z}_2}(R) \subseteq Id^*_{\mathbb{Z}_2}(UT^*_{\mathbb{Z}_2,\tilde{g}}(R_1, \dots, R_q))$, R does not satisfy $Cap^{(\mathbb{Z}_2,*)}_{d_0^+}[Y^+; X]$. Hence, for $h \geq l > 0$, we have

$$M^+ + 1 = \frac{h(h+1)}{2} + \frac{l(l+1)}{2} + 1 = \frac{h^2 + l^2 + (h+l) + 2}{2} \leq$$

$$\frac{h^2 + l^2 + (h+l) + 2hl}{2} = \frac{(h+l)(h+l+1)}{2} = \frac{h_1(h_1+1)}{2} = d_0^+$$

and this is impossible.
If $R \simeq (M_{2h_1}(F), s) + J$ then $\exp^*_{\mathbb{Z}_2}(R) = 4h_1^2$. Since $\exp^*_{\mathbb{Z}_2}(R) = M + L = (h + l)^2$ we have that $2h_1 = h + l$. Moreover R satisfies $Cap^{(\mathbb{Z}_2,*)}_{M^-+1}[Y^-; X]$ but does not satisfy $Cap^{(\mathbb{Z}_2,*)}_{d_0^-}[Y^-; X]$ and so we get a contradiction since

$$M^- + 1 = \frac{h(h-1)}{2} + \frac{l(l-1)}{2} + 1 = \frac{h^2 + l^2 - (h+l) + 2}{2} <$$

$$\frac{h^2 + l^2 + (h+l) + 2hl}{2} = \frac{(h+l)^2 + (h+l)}{2} = \frac{4h_1^2 + 2h_1}{2} = 2h_1^2 + h_1 = d_0^-.$$

Finally, let $R \simeq (M_{h_1}(F) \oplus M_{h_1}(F)^{op}, exc) + J$, with $h_1 > 0$. Then $(h + l)^2 = M + L = \exp^*_{\mathbb{Z}_2}(R) = 2h_1^2$, a contradiction.

In the same way, by analyzing all possible cases as M and L vary, we obtain a contradiction and so we can assume that $\bar{m} = 0$.

Let $R = R_1 \oplus \cdots \oplus R_m + J$, where R_i are simple $*$-superalgebras with non trivial grading. Let's prove that $m = 1$. By Lemma 4.8, $\overline{R}$ does not satisfy the $*$-graded Capelli polynomials $Cap^{(\mathbb{Z}_2,*)}_{d_0^++m-1}[Y^+, X]$, $Cap^{(\mathbb{Z}_2,*)}_{d_0^-+m-1}[Y^-, X]$, $Cap^{(\mathbb{Z}_2,*)}_{d_1^++m-1}[Z^+, X]$ and $Cap^{(\mathbb{Z}_2,*)}_{d_1^-+m-1}[Z^-, X]$ but satisfies $Cap^{(\mathbb{Z}_2,*)}_{M^++1}[Y^+, X]$, $Cap^{(\mathbb{Z}_2,*)}_{M^-+1}[Y^-, X]$, $Cap^{(\mathbb{Z}_2,*)}_{L^++1}[Z^+, X]$ and $Cap^{(\mathbb{Z}_2,*)}_{L^-+1}[Z^-, X]$ thus $d_0^+ + m - 1 \leq M^+$, $d_0^- + m - 1 \leq M^-$, $d_1^+ + m - 1 \leq L^+$ and $d_1^- + m - 1 \leq L^-$. Hence we have that

$$d_0^+ + (m-1) + d_0^- + (m-1) + d_1^+ + (m-1) + d_1^- + (m-1) \leq M^+ + M^- + L^+ + L^- = M + L.$$

Since $d_0^+ + d_0^- + d_1^- + d_1^- = M + L$ we obtain that $4(m - 1) = 0$ and so $m = 1$.

It follows that $R = R_1 \oplus J$ where R_1 is a simple $*$-superalgebra with non trivial grading. Now let's analyze the cases corresponding to the different values of M and L.

Let $M^{\pm} = \frac{h(h\pm 1)}{2} + \frac{l(l\pm 1)}{2}$ and $L^{\pm} = hl$, with $h \geq l > 0$.
If $R \simeq (M_{h_1,h_1}(F), s) + J$ then $(h+l)^2 = M + L = \exp^*_{\mathbb{Z}_2}(R) = 4h_1^2$ so we have $2h_1 = h + l$. By hypothesis R satisfies $Cap^{(\mathbb{Z}_2,*)}_{L^-+1}[Z^-; X]$ but does not satisfy $Cap^{(\mathbb{Z}_2,*)}_{d_1^-}[Z^+; X]$, where $d_1^- = h_1(h_1+1)$. Since $h + l = 2h_1$ and $h \geq l > 0$ we have that $h_1^2 \geq hl$ and so

$$L^- + 1 = hl + 1 \leq h_1^2 + 1 \leq h_1(h_1+1) = d_1^-$$

a contradiction.
If $R \simeq (M_{h_1,l_1}(F) \oplus M_{h_1,l_1}(F)^{op}, exc) + J$, with $h_1 \geq l_1 > 0$, then $(h+l)^2 = M + L = \exp^*_{\mathbb{Z}_2}(R) = 2(h_1+l_1)^2$ and so we have again a contradiction.
If $R \simeq (M_n(F + cF), *) + J$, where $(a + cb)^* = a^\diamond \pm cb^\diamond$ and $\diamond = t, s$, then we obtain the contradiction $(h+l)^2 = 2n^2$.
If $R \simeq (M_n(F + cF) \oplus M_n(F + cF)^{op}, exc) + J$ with $n > 0$, then $(h+l)^2 = M + L = \exp^*_{\mathbb{Z}_2}(R) = 4n^2$ and so $2n = h + l$. As before we can easily obtain a contradiction. It follows that $R \simeq (M_{h,l}(F), t) + J$.

In the same way by analyzing all other possible cases corresponding to the different values of M and L we obtain that $R \simeq A + J$ where A is a simple $*$-superalgebra with non trivial grading satisfying the thesis of the theorem. Then, from Lemmas 4.12 and 4.13 we obtain that

$$R \cong (A + J_{11}) \oplus J_{00} \cong (A \otimes N^\sharp) \oplus J_{00}$$

where $N^\sharp$ is the algebra obtained from N by adjoining a unit element. Since $N^\sharp$ is commutative, it follows that $A + J_{11}$ and A satisfy the same $*$-graded identities. Thus $\mathrm{var}^*_{\mathbb{Z}_2}(R) = \mathrm{var}^*_{\mathbb{Z}_2}(A \oplus J_{00})$ with J_{00} finite dimensional nilpotent $*$-superalgebra. Hence, from the decomposition (1), we get

$$\mathcal{U} = \mathrm{var}^*_{\mathbb{Z}_2}(\Gamma^*_{M^\pm+1, L^\pm+1}) = \mathrm{var}^*_{\mathbb{Z}_2}(A \oplus D),$$

where D is a finite dimensional $*$-superalgebra with $\exp^*_{\mathbb{Z}_2}(D) < M + L$ and the theorem is proven.

From Corollary 4.7 and Lemma 4.3, we easily obtain the following:

Corollary 4.15 *1) If $M^{\pm} = \frac{k(k\pm 1)}{2} + \frac{l(l\pm 1)}{2}$ and $L^{\pm} = kl$, with $k \geq l > 0$, then*

$$c_n^{(\mathbb{Z}_2,*)}(\Gamma^*_{M^\pm+1, L^\pm+1}) \simeq c_n^{(\mathbb{Z}_2,*)}((M_{k,l}(F), t));$$

2) If $M^{\pm} = k^2$ and $L^{\pm} = k(k \mp 1)$, with $k > 0$, then

$$c_n^{(\mathbb{Z}_2,*)}(\Gamma^*_{M^\pm+1, L^\pm+1}) \simeq c_n^{(\mathbb{Z}_2,*)}((M_{k,k}(F), s));$$

3) If $M^{\pm} = k^2 + l^2$ *and* $L^{\pm} = 2kl$, *with* $k \geq l > 0$, *then*

$$c_n^{(\mathbb{Z}_2,*)}(\Gamma^*_{M^{\pm}+1,L^{\pm}+1}) \simeq c_n^{(\mathbb{Z}_2,*)}((M_{k,l}(F) \oplus M_{k,l}(F)^{op}, exc)) \simeq \alpha n^{\frac{1-(k^2+l^2)}{2}}(2(k+l)^2)^n,$$

for some costant α*;*

4) If $M^+ = L^{\pm} = \frac{k(k+1)}{2}$, $M^- = L^{\mp} = \frac{k(k-1)}{2}$, *with* $k > 0$, *then*

$$c_n^{(\mathbb{Z}_2,*)}(\Gamma^*_{M^{\pm}+1,L^{\pm}+1}) \simeq c_n^{(\mathbb{Z}_2,*)}((M_k(F+cF),*)),$$

where $(a+cb)^* = a^t \pm cb^t$;

5) If $M^+ = L^{\pm} = \frac{k(k-1)}{2}$, $M^- = L^{\mp} = \frac{k(k+1)}{2}$, *with* $k > 0$, *then*

$$c_n^{(\mathbb{Z}_2,*)}(\Gamma^*_{M^{\pm}+1,L^{\pm}+1}) \simeq c_n^{(\mathbb{Z}_2,*)}((M_k(F+cF),*)),$$

where $(a+cb)^* = a^s \pm cb^s$*;*

6) If $M^{\pm} = L^{\pm} = k^2$, *with* $k > 0$, *then*

$$c_n^{(\mathbb{Z}_2,*)}(\Gamma^*_{M^{\pm}+1,L^{\pm}+1}) \simeq c_n^{(\mathbb{Z}_2,*)}((M_k(F+cF) \oplus M_k(F+cF)^{op}, exc)) \simeq \beta n^{\frac{1-k^2}{2}}(2k)^{2n},$$

for some costant β.

Acknowledgement

This paper is dedicated to our dear friend and colleague Vesselin Drensky on his 70th birthday.
The authors thank the referee for his useful suggestions.

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