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REGULARITY AND h-POLYNOMIALS OF TORIC IDEALS OF GRAPHS

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ABSTRACT. For all integers $4 \le r \le d$, we show that there exists a finite simple graph $G = G_{r,d}$ with toric ideal $I_G \subset R$ such that R/I_G has (Castelnuovo-Mumford) regularity r and h-polynomial of degree d. To achieve this goal, we identify a family of graphs such that the graded Betti numbers of the associated toric ideal agree with its initial ideal, and furthermore, this initial ideal has linear quotients. As a corollary, we can recover a result of Hibi, Higashitani, Kimura, and O'Keefe that compares the depth and dimension of toric ideals of graphs.

1. INTRODUCTION

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, and let $R = \mathbb{K}[x_1, \ldots, x_n]$ be the standard graded polynomial ring over $\mathbb{K}$. Hibi and Matsuda [14] initiated a comparison of the (Castelnuovo-Mumoford) regularity of R/I and the degree of the h-polynomial appearing in the Hilbert series of R/I. They showed that for any integers $d, r \ge 1$, there is a monomial ideal I such that the regularity of R/I is r, and the degree of the h-polynomial is d. Hibi and Matsuda later refined this result in [15] to show that I can be taken to be a lexsegment ideal, and later, with Van Tuyl [16], showed that I could be an edge ideal. Further comparisons of the regularity and degree have been carried out for the edge ideals of Cameron-Walker graphs [12] and binomial edge ideals [13, 17]. In this note we compare these invariants for the toric ideals of finite simple graphs.

Given a finite simple graph G on the vertex set $V = \{v_1, \ldots, v_n\}$ with edge set $E = \{e_1, \ldots, e_q\}$, the *toric ideal of* G, denoted I_G, is the kernel of the map $\varphi : \mathbb{K}[E] = \mathbb{K}[e_1, \ldots, e_q] \to \mathbb{K}[v_1, \ldots, v_n]$ given by $\varphi(e_i) = v_{i_1} v_{i_2}$ where $e_i = \{v_{i_1}, v_{i_2}\} \in E$. Some properties of the homological invariants of I_G can be found in [2, 3, 4, 6, 7, 11, 18, 21]. Our main result adds to this list of properties, and contributes to Hibi and Matsuda's program.

Theorem 1.1. *Let $4 \le r \le d$ be integers. Then there is a connected finite simple graph $G = G_{r,d}$ such that the toric ideal of G satisfies* $\operatorname{reg}(\mathbb{K}[E]/I_G) = r$ *and* $\deg h_{\mathbb{K}[E]/I_G}(x) = d$.

The proof of Theorem 1.1 has two components. First, we consider the family of graphs constructed from the complete bipartite graph $K_{2,t}$ by adjoining a "triangle" to each vertex of degree two (see Figure 1). We prove that the toric ideals of the graphs in this family have a unique extremal graded Betti number. We use this fact to show that for any $e \ge 5$, we can construct a graph G such that $\mathbb{K}[E]/I_G$ has regularity 4 and the degree of its h-polynomial is e. The second component is to leverage the splitting techniques of the authors and Hofscheier [3] to create the desired graphs of Theorem 1.1 from the graphs in this family. As a bonus corollary, we give a new proof for the main result of [11] which compared the depth and dimension of toric ideals of graphs (see Corollary 3.10).

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Our paper is structured as follows. In Section 2, we give the relevant background, including the undefined terms from the introduction. We also recall some tools from [3]; they are used to show that if $r \geq 1$, there is a graph G with $r = \operatorname{reg}(\mathbb{K}[E]/I_G) = \deg h_{\mathbb{K}[E]/I_G}(x)$. In Section 3, we introduce a family of connected graphs, and we show we can control the values of $\operatorname{reg}(\mathbb{K}[E]/I_G)$ and $\deg h_{\mathbb{K}[E]/I_G}(x)$, where I_G is the toric ideal of a graph in this family. These graphs can then be used to prove Theorem 1.1. We conclude with remarks in Section 4 about pairs (r,d) not covered by Theorem 1.1.

2. Preliminaries

We recall the relevant background on homological invariants and toric ideals of graphs.

2.1. **Homological invariants.** If I is a homogeneous ideal of R, then the minimal graded free resolution of R/I has the form

$$0 \to \bigoplus_{j\in\mathbb{N}} R(-j)^{\beta_{p,j}(R/I)} \to \cdots \to \bigoplus_{j\in\mathbb{N}} R(-j)^{\beta_{1,j}(R/I)} \to R \to R/I \to 0$$

where $R(-j)$ is the ring R with its grading shifted by j, and $\beta_{i,j}(R/I) = \dim_{\mathbb{K}} \operatorname{Tor}_i^R(R/I,\mathbb{K})_j$ is called the i,j-th *graded Betti number* of R/I. The *(Castelnuovo-Mumford) regularity* of R/I is

$$\operatorname{reg}(R/I) = \max\{j - i \mid \beta_{i,j}(R/I) \neq 0\}.$$

The *projective dimension* of R/I is the length of the minimal graded free resolution, that is

$$\operatorname{pdim}(R/I) = \max\{i \mid \beta_{i,j}(R/I) \neq 0\}.$$

The *Hilbert series* of a standard graded $\mathbb{K}$-algebra R/I is the formal power series

$$HS_{R/I}(x) = \sum_{i\geq 0} [\dim_{\mathbb{K}}(R/I)_i]\, x^i$$

where $\dim_{\mathbb{K}}(R/I)_i$ is the dimension of i-th graded piece of R/I. The Hilbert series of R/I can be read from any resolution of R/I (e.g., see [8, p. 100]). In particular,

$$HS_{R/I}(x) = \frac{1 + \sum_{i,j}(-1)^i \beta_{i,j}(R/I)x^j}{(1-x)^n}. \tag{2.1}$$

By the Hilbert-Serre Theorem (e.g., see [23, Theorem 5.1.4]) there is a polynomial $h_{R/I}(x) \in \mathbb{Z}[x]$, called the *h-polynomial* of R/I, such that $HS_{R/I}$ can be written as

$$HS_{R/I}(x) = \frac{h_{R/I}(x)}{(1-x)^{\dim(R/I)}} \quad \text{with } h_{R/I}(1) \neq 0, \tag{2.2}$$

where $\dim(R/I)$ denotes the Krull dimension of R/I.

We recall a fact about extremal Betti numbers; see [1] for more on their properties.

Definition 2.1. A graded Betti number of R/I, say $\beta_{a,b}(R/I) \neq 0$, is *extremal* if $\beta_{i,j}(R/I) = 0$ for any pair (i,j) such that $i \geq a$ and $j > b$ and $j - i \geq b - a$.

Lemma 2.2. *Suppose $\beta_{a,b}(R/I)$ is the only extremal Betti number of R/I. Then $\operatorname{reg}(R/I) = b - a$, $\operatorname{pdim}(R/I) = a$, and $\deg h_{R/I}(x) = b - \dim R + \dim R/I$.*

Proof. Since $\beta_{a,b}(R/I)$ is an extremal Betti number, from the definition, we have $\beta_{a,b}(R/I) \neq 0$ and $\beta_{i,j}(R/I) = 0$ for any $i \geq a$, $j > b$ and $j - i \geq b - a$. Moreover, because it is the unique extremal Betti number, $\beta_{i,j}(R/I) = 0$ if either $i \geq a$ or $j > b$ (otherwise there must be some other extremal Betti). Thus, the Betti table of R/I has a rectangular shape and the pair (a,b) determines the regularity and the projective dimension. Furthermore, from equation (2.1), the degree of the non-reduced numerator in the Hilbert series is b, so by (2.2), the degree of the h-polynomial is $b - \dim R + \dim R/I$. $\square$

A monomial ideal $I \subseteq R$ is said to have *linear quotients* if its minimal generators $\{g_1, \ldots, g_m\}$ can be ordered so that the quotient ideal $\langle g_1, \ldots, g_{j-1}\rangle : \langle g_j\rangle$ is generated by variables for every $j = 2, \ldots, m$. Linear quotients were first defined in [10]. By [20, Corollary 2.7], a monomial ideal $I \subseteq R$ with linear quotients with respect to the ordering $g_1, \ldots, g_m$, has graded Betti numbers given by the formula

$$\beta_{i+1,i+j}(R/I) = \sum_{1\leq p\leq m,\ \deg(g_p)=j} \binom{n_p}{i} \text{ for } i \geq 0 \tag{2.3}$$

where n_p denotes the number of different variables generating $\langle g_1, \ldots, g_{p-1}\rangle : \langle g_p\rangle$.

For a fixed monomial ordering, we let $\mathrm{in}(I)$ denote the *initial ideal of* I. It is well known that $\beta_{i,j}(R/I) \leq \beta_{i,j}(R/\mathrm{in}(I))$ for all $i, j \geq 0$ (e.g., see [19, Theorem 22.9]). The following result, found in [4, Lemma 2.6], gives a criterion for when we have equality for all $i, j \geq 0$.

Lemma 2.3. *Fix a monomial order. Suppose that $I \subseteq R$ is a homogeneous ideal such that $\beta_{i,i+j}(R/I) = \beta_{i,i+j}(R/\mathrm{in}(I))$ for all i and all $j \neq k$. Then $\beta_{i,i+k}(R/I) = \beta_{i,i+k}(R/\mathrm{in}(I))$ for all $i \geq 0$.*

2.2. **Toric ideals of graphs.** We now turn to toric ideals of graphs, as defined in the introduction. Note that if G is a finite simple graph, then the toric ideal I_G is a prime homogeneous binomial ideal. Many of the algebraic and geometric invariants of I_G depend upon the combinatorics of G. In order to discuss these results, we briefly introduce some relevant terminology and results (see Villareal [23] and Herzog, Hibi, and Ohsgui [9] for details). Note that if $G = (V, E)$ is a finite simple graph, we may sometimes write $\mathbb{K}[E]$ for $\mathbb{K}[e \mid e \in E]$ and $\mathbb{K}[G]$ for the ring $\mathbb{K}[E]/I_G$.

If G is a finite simple graph, a *walk* in G is a sequence of edges $w = (e_1, e_2, \ldots, e_k)$ such that $e_i \cap e_{i+1} \neq \varnothing$ for $i = 1, \ldots, k-1$. Equivalently, a walk is a sequence of vertices $(x_1, \ldots, x_k, x_{k+1})$ such that $\{x_i, x_{i+1}\} \in E$ for $i = 1, \ldots, k$. A walk is an *even walk* if k is even. A *closed walk* is a walk where $x_{k+1} = x_1$. Two closed even walks $(e_0, \ldots, e_{2k-1})$ and $(e'_0, \ldots, e'_{2k-1})$ are equivalent up to a *circular permutation* if there is an i such that $e_j = e'_{j+i}$ for all j where $j + i$ is taken modulo $2k$ (or if the walk is in the reverse order, i.e., $e_j = e'_{(2k-j)+i}$ for all j).

A finite graph G is *connected* if for every $x, y \in V$ with $x \neq y$, there exists a walk having x as its first vertex and y as its last. A closed walk $(e_1, \ldots, e_k)$ where each vertex and edge is distinct is called a *cycle* of length k. A graph G is *bipartite* if there are no odd cycles in G. An *n-cycle*, denoted C_n, is the graph with vertex set $V = \{x_1, \ldots, x_n\}$ and edge set $E = \{\{x_1, x_2\}, \{x_2, x_3\}, \ldots, \{x_{n-1}, x_n\}, \{x_n, x_1\}\}$.

The generators of the toric ideal I_G can be obtained from closed even walks in G; we sketch out this connection. To each closed even walk $w = (e_{i_1}, e_{i_2}, \ldots, e_{i_{2n}})$ in G, we can associate the binomial f_w defined by

$$f_w = \prod_{2\nmid j} e_{i_j} - \prod_{2\mid j} e_{i_j} \in I_G.$$

Note that it is straightforward to verify that $\varphi(f_w) = 0$ where $\varphi : \mathbb{K}[e_1, \ldots, e_q] \to \mathbb{K}[v_1, \ldots, v_n]$ is the map defining $I_G = \ker(\varphi)$. Among all closed even walks, we identify a special subset.

Definition 2.4. A binomial $f_1 - f_2 \in I_G$ is *primitive* if there exists no binomial $g_1 - g_2 \in I_G$ such that $g_1 \mid f_1$ and $g_2 \mid f_2$. A closed even walk w in a graph G is said to be *primitive* if the corresponding binomial f_w is primitive in I_G.

The importance of primitive closed even walks lies in the next theorem.

Theorem 2.5 ([23, Proposition 10.1.10])**.** *The set of binomials associated with primitive closed even walks is a universal Gröbner basis of I_G.*

We round out this section by specializing one of the results of [3] that will be a key ingredient in our proof of Theorem 1.1. Recall that given a graph $G = (V, E)$ and $W \subseteq V$, the *induced subgraph* of G on W is the graph with vertex set W and edge set $\{e \in E \mid e \subseteq W\}$. Following [3, Construction

4.1], let G_1, G_2 be two graphs and suppose that $H_1 \subseteq G_1, H_2 \subseteq G_2$ are two induced subgraphs which are isomorphic with respect to some graph isomorphism $\varphi : H_1 \to H_2$. We define the *glued graph* $G_1 \cup_\varphi G_2$ of G_1 and G_2 along φ as the disjoint union of G_1 and G_2, and we use φ to identify vertices and edges in H_1 with their images in H_2. At times, we may be more informal and say that G_1 *and* G_2 *is glued along* H if the induced subgraphs $H \cong H_1$ and $H \cong H_2$ and isomorphism φ are clear.

It was shown in [3] that under some hypotheses on G_1 and G_2, if the G_1 and G_2 are glued along some induced subgraph H, then many of the homological invariants of $G_1 \cup_\varphi G_2$ are related to those of G_1 and G_2. In particular, if we specialize [3, Corollary 3.11], we have the following result.

Theorem 2.6. *Let G be any finite simple connected graph, and let C_{2s} be an even cycle of length $2s \geq 4$. Let e be any edge of G and let e' be any edge of C_{2s}. If $G' = (V', E')$ is the graph obtained by gluing G and C_{2s} along $e \cong e'$, then*

(*i*) $\operatorname{reg}(\mathbb{K}[G']) = \operatorname{reg}(\mathbb{K}[G]) + s - 1$, *and*
(*ii*) $\deg h_{\mathbb{K}[G']}(x) = \deg h_{\mathbb{K}[G]}(x) + s - 1$.

Corollary 2.7. *Let $G = (V, E)$ be a connected graph with $\deg h_{\mathbb{K}[G]}(x) = d$ and $\operatorname{reg}(\mathbb{K}[G]) = r$. Then there exists a connected graph $G' = (V', E')$ with $\deg h_{\mathbb{K}[G']}(x) = d + 1$ and $\operatorname{reg}(\mathbb{K}[G']) = r + 1$.*

Proof. By Theorem 2.6, if we glue a C_4 along any edge of G, we get the desired result. □

For all integers $1 \leq r$, there is a graph G satisfying $\deg h_{\mathbb{K}[G]}(x) = \operatorname{reg}(\mathbb{K}[G]) = r$.

Example 2.8. Consider the graph $C_4^{(r)}$, where $r \geq 1$ is an integer, on the vertex set $V^{(r)} = \{x_1, \ldots, x_{2r+2}\}$ and edge set $E^{(r)} = \{\{x_1, x_2\}\} \cup \{\{x_1, x_{2i+1}\}, \{x_2, x_{2i+2}\}, \{x_{2i+1}, x_{2i+2}\} \mid iv = 1, \ldots, r\}$. So, the graph $C_4^{(r)}$ consists of r squares glued along one edge. Since, $C_4 = C_4^{(1)}$ has $\deg h_{\mathbb{K}[C_4]}(x) = \operatorname{reg}(\mathbb{K}[C_4]) = 1$ then, iteratively from Corollary 2.7, we get $\deg h_{\mathbb{K}[C_4^{(r)}]}(x) = \operatorname{reg}(\mathbb{K}[C_4^{(r)}]) = r$.

3. Some homological invariants of the toric ideal for a fixed family of graphs

In this section we construct a family of simple graphs G_t with $t \geq 2$ such that $\operatorname{reg}(\mathbb{K}[G_t]) = 4$ and $\deg h_{\mathbb{K}[G_t]}(x) = t + 3$. By combining this family with Corollary 2.7, we can prove Theorem 1.1.

To help the reader, we sketch out the broad strokes that we take in this section. We begin by defining a graph G_t on $t + 6$ vertices and $2t + 6$ edges, where $t \geq 2$ is an integer. We then describe a set $\mathcal{G}$ of binomials that form a universal Gröbner basis for I_{G_t} and a set $\mathcal{M}$ of minimal generators of $\operatorname{in}(I_{G_t})$, the initial ideal of I_{G_t} for a given monomial ordering. We show that $\operatorname{in}(I_{G_t})$ has linear quotients. Lastly, we prove that all the graded Betti numbers of $\mathbb{K}[G_t]$ coincide with the ones of $\mathbb{K}[E_t]/\operatorname{in}(I_{G_t})$, and that there exists a unique extremal Betti number. We derive Theorem 1.1 from these facts.

We begin by formally defining the graphs of interest.

Definition 3.1. Let $t \geq 2$ be an integer. The graph G_t is defined having the vertex and edge sets:

$$V_t = \{x_1, x_2, y_1, \ldots, y_t, z_1, z_2, w_1, w_2\}, \text{ and}$$

$$E_t = \{\{x_i, y_j\} \mid 1 \leq i \leq 2, 1 \leq j \leq t\} \cup \{\{x_1, z_1\}, \{z_1, z_2\}, \{z_2, x_1\}\} \cup \{\{x_2, w_1\}, \{w_1, w_2\}, \{w_2, x_2\}\}.$$

We label the edges of G_t as follows: $e_1 = \{x_1, z_1\}$, $e_2 = \{z_1, z_2\}$, $e_3 = \{z_2, x_1\}$, $f_1 = \{x_2, w_1\}$, $f_2 = \{w_1, w_2\}$, $f_3 = \{w_2, x_2\}$ and, for $i \in \{1, \ldots, t\}$, $a_i = \{x_1, y_i\}$ and $b_i = \{x_2, y_i\}$.

Note that the subgraph of G_t on the vertices $\{x_1, x_2, y_1, \ldots, y_t\}$ is a complete bipartite graph $K_{2,t}$ consisting of only the edges $\{a_1, \ldots, a_t, b_1, \ldots, b_t\}$. Thus, less formally, the graph G_t is obtained from the complete bipartite graph $K_{2,t}$ by joining a 3-cycle to each of the two vertices of degree t. See Figure 1 for the case $t = 5$. Note that the toric ideals of these graphs were also considered in [11].

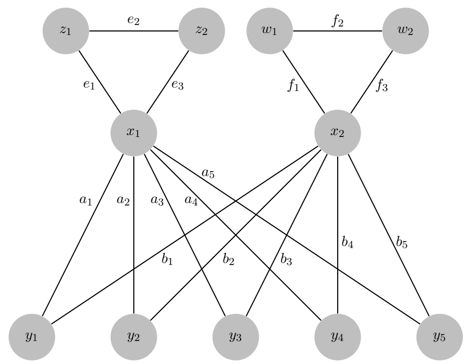

FIGURE 1. The graph G_5.

Going forward, we work in the standard graded polynomial ring

$$\mathbb{K}[E_t] = \mathbb{K}[a_1, \ldots, a_t, f_1, f_2, f_3, e_1, e_2, e_3, b_1, \ldots, b_t].$$

Let $>$ denote the graded reverse lexicographic monomial ordering on $\mathbb{K}[E_t]$ satisfying

$$a_1 > \cdots > a_t > f_1 > f_2 > f_3 > e_1 > e_2 > e_3 > b_1 > \cdots > b_t. \tag{3.1}$$

We denote the *initial ideal* of an ideal I with respect to this ordering by $\mathrm{in}(I)$.

Before focusing on I_{G_t}, we summarize some known results about the toric ideal of $I_{K_{2,t}}$. Here, we see $K_{2,t}$, the complete bipartite graph, as the induced subgraph of G_t on $V' := \{x_1, x_2, y_1, \ldots, y_t\}$.

Lemma 3.2. *Fix some integer $t \geq 2$. Using the same labelling as in Definition 3.1, let $I_{K_{2,t}}$ be the toric ideal of the graph $K_{2,t} = (V', E')$ in the polynomial ring $\mathbb{K}[E'] = \mathbb{K}[a_1, \ldots, a_t, b_1, \ldots, b_t]$. Then*

- (i) $I_{K_{2,t}} = \langle a_i b_j - a_j b_i \mid 1 \leq i < j \leq t \rangle$;
- (ii) $\mathrm{in}(I_{K_{2,t}}) = \langle a_i b_j \mid 1 \leq j < i \leq t \rangle$ *with respect to the graded reverse lexicographical order where* $a_1 > a_2 > \cdots > a_t > b_1 > \cdots > b_t$;
- (iii) $\mathrm{in}(I_{K_{2,t}})$ *has linear quotients if one orders the generators with respect to the graded reverse lexicographical order; and*
- (iv) *if* $\{g_1, \ldots, g_k\}$ *are the generators of* $\mathrm{in}(I_{K_{2,t}})$ *ordered with respect to the graded reverse lexicographical order, then* $n_p \leq t - 1$ *for all* p*, where* n_p *is the number of generators of* $\langle g_1, \ldots, g_{p-1} \rangle : \langle g_p \rangle$ *for* $p = 1, \ldots, k$;
- (v) $\beta_{i,j}(\mathbb{K}[K_{2,t}]) = \beta_{i,j}(\mathbb{K}[E']/\mathrm{in}(I_{K_{2,t}}))$ *for all* $i, j \geq 0$.

Proof. Statements (i) and (ii) follow from [4, Remark 3.4] which shows that the given generators are a universal Gröbner basis of $I_{K_{2,t}}$. Statements $(iii), (iv)$, and (v) follow from [4, Corollary 2.8]. □

The next result describes the set of primitive binomials of I_{G_t} which we denote by $\mathcal{G}$.

Theorem 3.3. *For any integer $t \geq 2$, the ideal I_{G_t} is generated by the primitive binomials in $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$ where*

- (i) $\mathcal{G}_1 = \{a_i b_j - a_j b_i \mid 1 \leq i < j \leq t\}$,
- (ii) $\mathcal{G}_2 = \{a_i a_j f_1 f_3 e_2 - f_2 e_1 e_3 b_i b_j \mid 1 \leq i < j \leq t\}$, *and*
- (iii) $\mathcal{G}_3 = \{a_i^2 f_1 f_3 e_2 - f_2 e_1 e_3 b_i^2 \mid 1 \leq i \leq t\}$.

In particular, $\mathcal{G}$ is a universal Gröbner basis for I_{G_t}.

Proof. By Theorem 2.5, it suffices to show that the binomials in $\mathcal{G}$ correspond to the primitive closed even walks in G_t. We only need to identify these even walks up to a circular permutation, since the associated binomials will be equal up to a sign.

Note that the elements of $\mathcal{G}$ correspond to the following closed even walks in the graph G_t:

- (a_i, b_i, b_j, a_j), where $1 \leq i < j \leq t$,
- $(a_i, b_i, f_1, f_2, f_3, b_j, a_j, e_1, e_2, e_3)$, where $1 \leq i < j \leq t$, and
- $(a_i, b_i, f_1, f_2, f_3, b_i, a_i, e_1, e_2, e_3)$ where $1 \leq i \leq t$.

However, as noted in [11] (prior to Lemma 2.1), these closed even walks form a complete set of primitive closed even walks. □

Corollary 3.4. *Using the graded reverse lexicographic order that satisfies (3.1), we have that* $\mathrm{in}(I_{G_t})$ *is generated by the monomials in* $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$ *where:*

(i) $\mathcal{M}_1 = \{a_i b_j \mid 1 \leq j < i \leq t\}$,
(ii) $\mathcal{M}_2 = \{a_i a_j f_1 f_3 e_2 \mid 1 \leq i < j \leq t\}$, *and*
(iii) $\mathcal{M}_3 = \left\{a_i^2 f_1 f_3 e_2 \mid 1 \leq i \leq t\right\}$.

Furthermore, $\mathcal{M}$ *is a minimal set of generators for* $\mathrm{in}(I_{G_t})$.

Proof. That $\mathcal{M}$ is a generating set with respect to the given order follows from Theorem 3.3. That it is minimal follows from the fact that none of the monomials are divided by any of the others. □

We will show $\mathrm{in}(I_{G_t})$ has linear quotients with respect to an order of its generators.

Theorem 3.5. *Let* $\mathcal{M}_1, \mathcal{M}_2$ *and* $\mathcal{M}_3$ *be as in Corollary 3.4, and order each set from smallest to largest with respect to the graded reverse lexicographical order. Then the initial ideal of* I_{G_t}

$$\begin{aligned}\mathrm{in}(I_{G_t}) = \langle\, & a_t b_{t-1},\ a_t b_{t-2}, \ldots, a_{t-1} b_{t-2}, \ldots, a_2 b_1, \\ & a_t a_{t-1} f_1 f_3 e_2,\ a_t a_{t-2} f_1 f_3 e_2, \ldots, a_2 a_1 f_1 f_3 e_2, \\ & a_t^2 f_1 f_3 e_2,\ a_{t-1}^2 f_1 f_3 e_2, \ldots, a_1^2 f_1 f_3 e_2 \,\rangle\end{aligned}$$

has linear quotients with respect to this order of the generators. Furthermore, if $\mathrm{in}(I_{G_t}) = \langle g_1, \ldots, g_{t^2}\rangle$, *and* n_p *is the number of generator of* $\langle g_1, \ldots, g_{p-1}\rangle : \langle g_p\rangle$, *then*

$$\max\{n_p \mid 2 \leq p \leq t^2\} = 2t - 2.$$

Proof. It follows from Corollary 3.4 that $\mathrm{in}(I_{G_t})$ has t^2 generators. Let $g_1, \ldots, g_{t^2}$ be these generators, ordered as in the statement of the theorem. For each $p \in \{2, \ldots, t^2\}$, let $I(p) = \langle g_1, \ldots, g_{p-1}\rangle : \langle g_p\rangle$. A generating set of $I(p)$ is given by:

$$I(p) = \left\langle \frac{LCM(g_1, g_p)}{g_p}, \frac{LCM(g_2, g_p)}{g_p}, \ldots, \frac{LCM(g_{p-1}, g_p)}{g_p} \right\rangle. \tag{3.2}$$

We first observe that the first $\frac{t(t-1)}{2}$ generators of $\mathrm{in}(I_{G_t})$ with respect to our ordering are the exact same as the generators of $\mathrm{in}(I_{K_{2,t}})$ by Lemma 3.2 *(ii)*. So, by Lemma 3.2 *(iii)*, since this order has linear quotients, $I(p)$ is generated by variables for $p = 2, \ldots, \frac{t(t-1)}{2}$.

It suffices to show that $I(p)$ is generated by variables for $p \in \left\{\frac{t(t-1)}{2} + 1, \ldots, t^2\right\}$. We consider two cases.

Case 1. Suppose that $g_p = a_i a_j f_1 f_3 e_2$ with $t \geq i > j \geq 1$. Then the ideal $I(p)$ is

$$= \begin{cases} \langle a_t b_{t-1}, a_t b_{t-2}, \ldots, a_2 b_1\rangle : \langle a_t a_{t-1} f_1 f_3 e_2\rangle & \text{if } i = t \text{ and } j = t-1 \\ \langle a_t b_{t-1}, a_t b_{t-2}, \ldots, a_2 b_1, a_t a_{t-1} f_1 f_3 e_2, \ldots, a_{j+2} a_{j+1} f_1 f_3 e_2\rangle : \langle a_t a_j f_1 f_3 e_2\rangle & \text{if } i = t \text{ and } 1 \leq j < t-1 \\ \langle a_t b_{t-1}, a_t b_{t-2}, \ldots, a_2 b_1, a_t a_{t-1} f_1 f_3 e_2, \ldots, a_{i+1} a_j f_1 f_3 e_2\rangle : \langle a_i a_j f_1 f_3 e_2\rangle & \text{if } i < t. \end{cases}$$

If we calculate each ideal using (3.2), we get

$$I(p)=\begin{cases}\langle b_1,\dots,b_{t-1}\rangle & \text{if } i=t \text{ and } j=t-1\\ \langle b_1,\dots,b_{t-1},a_{j+1},\dots,a_{t-1}\rangle & \text{if } i=t \text{ and } 1\le j<t-1\\ \langle b_1,\dots,b_{i-1},a_{j+1},\dots,a_{i-1},a_{i+1},\dots,a_t\rangle & \text{if } i<t.\end{cases}$$

Case 2. If $g_p=a_i^2f_1f_3e_2$, then

$$I(p)=\begin{cases}\langle a_tb_{t-1},\dots,a_2a_1f_1f_3e_2\rangle:\langle a_t^2f_1f_3e_2\rangle & \text{if } i=t\\ \langle a_tb_{t-1},\dots,a_2a_1f_1f_3e_2,a_1^2f_1f_3e_2,\dots,a_{i+1}^2f_1f_3e_2\rangle:\langle a_i^2f_1f_3e_2\rangle & \text{if } 1\le i<t.\end{cases}$$

Computing each colon ideal gives

$$I(p)=\begin{cases}\langle b_1,\dots,b_{t-1},a_1,\dots,a_{t-1}\rangle & \text{if } i=t\\ \langle b_1,\dots,b_{i-1},a_1,\dots,a_{i-1},a_{i+1},\dots,a_t\rangle & \text{if } 1\le i<t.\end{cases}$$

It thus follows that $\mathrm{in}(I_{G_t})$ has linear quotients with respect to the given order.

To prove the final statement, it follows that $n_p\le t-1$ for $p=2,\dots,\frac{t(t-1)}{2}$ by Lemma 3.2 (iv). On the other hand, from our above computations, we saw that

$$\langle a_tb_{t-1},\dots,a_2a_1f_1f_3e_2\rangle:\langle a_t^2f_1f_3e_2\rangle=\langle b_1,\dots,b_{t-1},a_1,\dots,a_{t-1}\rangle$$

has $2t-2$ generators, and every ideal $I(p)$ with $\frac{t(t-1)}{2}+1\le p\le t^2$ has $n_p\le 2t-2$. □

Corollary 3.6. *For any integer $t\ge 2$, we have $\beta_{i,i+j}(\mathbb{K}[G_t])=\beta_{i,i+j}(\mathbb{K}[E_t]/\mathrm{in}(I_{G_t}))$ for all $i,j\ge 0$.*

Proof. Recall that we have $\beta_{i,i+j}(\mathbb{K}[G_t])\le\beta_{i,i+j}(\mathbb{K}[E_t]/\mathrm{in}(I_{G_t}))$ for all $i,j\ge 0$. Because $\mathrm{in}(I_{G_t})$ has linear quotients and is only generated in degrees 2 and 5, formula (2.3) thus gives

$$\beta_{i,i+j}(\mathbb{K}[G_t])=\beta_{i,i+j}(\mathbb{K}[E_t]/\mathrm{in}(I_{G_t}))=0 \text{ for all } i\ge 0 \text{ and all } j\ne 1,4.$$

On the other hand, the generators of I_{G_t} of degree two are the exact same as the generators of $I_{K_{2,t}}$ by Theorem 3.3 and Lemma 3.2. So $\beta_{i,i+1}(\mathbb{K}[G_t])=\beta_{i,i+1}(\mathbb{K}[K_{2,t}])$ for all $i\ge 0$. The minimal generators of $\mathrm{in}(I_{G_t})$ of degree 2 are also the minimal generators of $\mathrm{in}(I_{K_{2,t}})$. So

$$\beta_{i,i+1}(\mathbb{K}[K_{2,t}])=\beta_{i,i+1}(\mathbb{K}[G_t])\le\beta_{i,i+1}(\mathbb{K}[E_t]/\mathrm{in}(I_{G_t}))=\beta_{i,i+1}(\mathbb{K}[E_t]/\mathrm{in}(I_{K_{2,t}}))=\beta_{i,i+1}(\mathbb{K}[K_{2,t}])$$

where the last inequality is Lemma 3.2 (v). So we have shown that $\beta_{i,i+j}(\mathbb{K}[G_t])=\beta_{i,i+j}(R/\mathrm{in}(I_{G_t}))$ for all $i,j\ge 0$ except $j=4$. To complete the proof, we now apply Lemma 2.3. □

Remark 3.7. It is possible to find an explicit formula for $\beta_{i,i+j}(\mathbb{K}[E_t]/\mathrm{in}(I_{G_i}))$ using the formula (2.3), and determining the exact values of n_p for each p. These values can be extracted from the proof of Theorem 3.5 and [4, Theorem 3.6].

Corollary 3.8. *For any integer $t\ge 2$, $\beta_{2t-1,2t+3}(\mathbb{K}[G_t])$ is the unique extremal Betti number of $\mathbb{K}[G_t]$.*

Proof. By Corollary 3.6, it suffices to show that $\beta_{2t-1,2t+3}(\mathbb{K}[E_t]/\mathrm{in}(I_{G_t}))$ is the unique extremal graded Betti number of $\mathbb{K}[E_t]/\mathrm{in}(I_{G_t})$. Since the ideal $\mathrm{in}(I_{G_t})$ is generated in degrees two and five, and because this ideal has linear quotients, any extremal Betti number will have the form $\beta_{i,i+1}(\mathbb{K}[E_t]/\mathrm{in}(I_{G_t}))$ or $\beta_{i,i+4}(\mathbb{K}[E_t]/\mathrm{in}(I_{G_t}))$, where $i\ge 1$. By Lemma 3.2 (iv) and formula (2.3) $\beta_{i,i+1}(\mathbb{K}[E_t]/\mathrm{in}(I_{G_t}))=0$ if $i\ge t$ since $n_p\le t-1$ in this range. On the other hand, since $n_p=2t-2$ for a generator of degree five (by Theorem 3.5), and because this is the maximal such value for n_p, we have $\beta_{2t-1,2t+3}(\mathbb{K}[E_t]/\mathrm{in}(I_{G_t}))\ne 0$ but $\beta_{i,i+4}(\mathbb{K}[E_t]/\mathrm{in}(I_{G_t}))=0$ for all $i\ge 2t-1$. Since $2t-1\ge t-1$ because $t\ge 2$, $\beta_{2t-1,2t+3}(\mathbb{K}[E_t]/\mathrm{in}(I_{G_t}))$ is the unique extremal graded Betti number. □

We can now compute the regularity and the degree of the h-polynomial of $\mathbb{K}[G_t]$.

Theorem 3.9. *For any integer $t \geq 2$, the graph G_t has* $\operatorname{reg}(\mathbb{K}[G_t]) = 4$ *and* $\deg h_{\mathbb{K}[G_t]}(x) = t+3$.

Proof. This results follows by combining Corollary 3.8 and Lemma 2.2, and using the fact that $\dim \mathbb{K}[E_t] = 2t+6$ and $\dim(\mathbb{K}[G_t]) = |V(G_t)| = 2t+4$; see [23, Corollary 10.1.21] for the latter assertion. □

We now have all the pieces to prove the main theorem of this paper.

Proof of Theorem 1.1. Let (r,d) be a pair of integers such that $4 \leq r \leq d$. If $r = d$, then the graph $C_4^{(r)}$ introduced in Example 2.8 has the required invariants. Assume now $r < d$. Set $q = d - r + 1 \geq 2$. By Theorem 3.9 the graph G_q has $\operatorname{reg}(\mathbb{K}[G_q]) = 4$ and $\deg h_{\mathbb{K}[G_q]}(x) = q+3 = d-r+4$. Thus, applying Corollary 2.7 $(r-4)$ times, we get the existence of a graph $G_{r,d}$ with $\operatorname{reg}(\mathbb{K}[G_{r,d}]) = r$ and $\deg h_{\mathbb{K}[G_{r,d}]}(x) = d$. (The graph $G_{r,d}$ is obtained by gluing G_q with $r-4$ squares C_4 along one edge, no matter which one.) □

As another consequence of Corollary 3.8 we derive a new proof of the main result of [11].

Corollary 3.10 ([11, Theorem 0.2])**.** *Fix integers $7 \leq f \leq d$. Then there exists a graph G whose toric ring satisfies* $\operatorname{depth}(\mathbb{K}[G]) = f$ *and* $\dim(\mathbb{K}[G]) = d$.

Proof. As described in [11], the proof of the above result hinges upon finding a graph on $k+6$ vertices with $k \geq 1$ whose toric ideal I_G has the property that $\operatorname{depth}(\mathbb{K}[G]) = 7$. Using our notation, [11] show that the graphs G_t with $t \geq 1$ (where G_1 is the graph of two triangles joined by a path of length two) satisfy $\operatorname{depth}(\mathbb{K}[G_t]) = 7$. But, for $t \geq 2$ this also follows from Lemma 2.2, Corollary 3.8, and the Auslander-Buchsbaum formula since $\operatorname{depth}(\mathbb{K}[G_t]) = \dim(\mathbb{K}[E_t]) - \operatorname{pdim}(\mathbb{K}[G_t]) = 2t+6-(2t-1) = 7$. When $t = 1$, then I_{G_t} has a single generator, so $\operatorname{pdim}(\mathbb{K}[G_t]) = 1$ and $\operatorname{depth}(\mathbb{K}[G_1]) = 8 - 1 = 7$. The proof now runs as in the introduction of [11]. □

4. Further comments and observations

We now turn our attention to integers $d, r \geq 1$ not covered by Theorem 1.1.

While Hibi and Matsuda [14] showed that for all $d, r \geq 1$, there is a monomial ideal I with $(r,d) = (\operatorname{reg}(R/I), \deg h_{R/I}(x))$, this behaviour will not hold for toric ideals of graphs. In particular, if $r = 1$, then d must also equal 1.

Theorem 4.1. *Let G be a graph such that* $\operatorname{reg}(\mathbb{K}[G]) = 1$. *Then* $\deg h_{\mathbb{K}[G]}(x) = 1$.

Proof. It can be assumed that the graph $G = (V,E)$ is connected. Since $\operatorname{reg}(\mathbb{K}[G]) = 1$, then $\mathbb{K}[G]$ has a linear resolution (hence it has a unique extremal Betti number $\beta_{a,a+1}(\mathbb{K}[G])$) and, in particular, I_G is only generated by quadratic binomials. Thus, from [9, Corollary 5.26], the ring $\mathbb{K}[G]$ is Cohen-Macaulay ($\operatorname{depth}(\mathbb{K}[G]) = \dim \mathbb{K}[G]$). So, the Auslander-Buchsbaum formula implies $|E| - \dim \mathbb{K}[G] = \operatorname{pdim}(\mathbb{K}[G]) = a$ and, from Lemma 2.2, we get $\deg h_{\mathbb{K}[G]}(x) = \operatorname{reg}(\mathbb{K}[G])$. □

Remark 4.2. In the proof of Lemma 4.1, we saw that $\mathbb{K}[G]$ was Cohen-Macaualay, from which we deduced that $\deg h_{\mathbb{K}[G]}(x) = \operatorname{reg}(\mathbb{K}[G])$. As shown in [22, Corollary B.4.1], this holds in general, i.e., if $\mathbb{K}[G]$ is Cohen-Macaulay, then the regularity and the degree of the h-polynomial are equal. We know of no example of a graph G such that I_G is generated in degrees ≤ 3 and $\mathbb{K}[G]$ is not a Cohen-Macaulay ring, thus suggesting that if $\operatorname{reg}(\mathbb{K}[G]) \leq 3$, there may be restrictions for $\deg h_{\mathbb{K}[G]}(x)$. On the other hand, the graph G with vertex set $V = \{x_1, \ldots, x_8\}$ and edge set

$$\begin{aligned} E &= \{\{x_1,x_2\},\{x_1,x_3\},\{x_2,x_3\},\{x_3,x_4\},\{x_4,x_5\},\{x_4,x_6\},\{x_4,x_7\},\{x_5,x_6\},\{x_5,x_7\},\{x_6,x_7\}\} \\ &\quad \cup \{\{x_i,x_8\} \mid i = 1,\ldots,7\} \end{aligned}$$

is generated in degrees ≤ 4, including a generator of degree four, and $\mathbb{K}[G]$ is not Cohen-Macaulay. (We thank Kazunori Matsuda for pointing us towards this example.)

Now we make an observation about the graphs having $\deg h_{\mathbb{K}[G]}(x) = 1$.

Remark 4.3. Let $G = (V, E)$ be a connected and non-bipartite graph such that $h_{\mathbb{K}[G]}(x) = 1 + ax$, $a \neq 0$. From equations (2.1) and (2.2) in Section 2 we have

$$HS_{\mathbb{K}[G]}(x) = \frac{1 + ax}{(1-x)^{|V|}} = \frac{1 + \sum_j B_j x^j}{(1-x)^{|E|}} \quad \text{where} \quad B_j = \sum_i (-1)^i \beta_{i,j}(\mathbb{K}[G]).$$

Note that in particular $B_1 = 0$ and $B_2 = -\beta_{1,2}(\mathbb{K}[G])$. Thus, we get $(1+ax)(1-x)^{|E|-|V|} = 1+\sum_j B_j x^j$. So, by comparing coefficients, $a = |E| - |V|$ and $\beta_{1,2}(I_G) = a^2 - \binom{a}{2} = \binom{a+1}{2} = \binom{|E|-|V|+1}{2}$. So, if there is a non-bipartite graph G with $\deg h_{\mathbb{K}[G]}(x) = 1$, then it must have $\binom{|E|-|V|+1}{2}$ quadratic generators.

Remark 4.4. We know of no example of a graph G such that $\beta_{1,2}(\mathbb{K}[G]) = \binom{|E|-|V|+1}{2}$ and $\mathbb{K}[G]$ is not a Cohen-Macaulay ring.

Finally, note that the strategy of Theorem 1.1 is to find graphs where we can control the regularity and the degree of the h-polynomial, and use it as a "seed" to repeatedly apply Corollary 2.7. Thus, to extend Theorem 1.1 for integers $d < r$, we need an appropriate initial graph. As the next example shows, we can extend Theorem 1.1 slightly to include all integers (r, d) with $5 \leq r = d + 1$.

Example 4.5. Let Z be the graph in Figure 2 on the vertex set $V = \{x_1, \ldots, x_{10}\}$ and edges $E = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_7\}, \{x_1, x_8\}, \{x_1, x_9\}, \{x_4, x_5\}, \{x_4, x_6\}, \{x_4, x_7\}, \{x_4, x_8\}, \{x_4, x_9\}, \{x_2, x_3\}, \{x_5, x_6\}, \{x_7, x_8\}, \{x_8, x_{10}\}, \{x_9, x_{10}\}\}$. One can compute the Betti diagram and the Hilbert

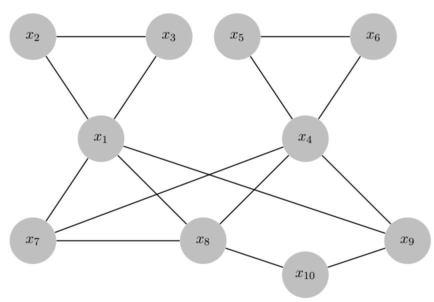

FIGURE 2. The graph Z.

series of $\mathbb{K}[Z]$ by using *Macaulay2* [5]:

$$\beta(\mathbb{K}[Z]) = \begin{array}{rccccccc} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \text{total:} & 1 & 12 & 40 & 56 & 37 & 11 & 1 \\ 0: & 1 & . & . & . & . & . & . \\ 1: & . & 5 & 5 & . & . & . & . \\ 2: & . & 2 & . & 1 & . & . & . \\ 3: & . & . & 10 & 10 & . & . & . \\ 4: & . & 5 & 25 & 45 & 37 & 10 & 1 \\ 5: & . & . & . & . & . & 1 & . \end{array} \quad \text{and} \quad HS_{\mathbb{K}[Z]}(x) = \frac{1 + 5x + 10x^2 + 13x^3 + 10x^4}{(1-x)^{10}}.$$

Thus, the graph Z covers the new case $\operatorname{reg}(\mathbb{K}[Z]) = 5$ and $\deg h_{\mathbb{K}[Z]}(x) = 4$. As a consequence of Corollary 2.7, for any pair (r, d) of positive integers such that $d \geq 4$ and $r - d = 1$, there is a graph G with $\operatorname{reg}(\mathbb{K}[G]) = r$ and $\deg h_{\mathbb{K}[G]}(x) = d$.

Table 1 summarizes all the results from this paper. In the table, a filled in circle $\bullet$ denotes a pair (r,d) for which there is a graph G with $(\operatorname{reg}(\mathbb{K}[G]), \deg h_{\mathbb{K}[G]}(x)) = (r,d)$, the empty circle $\circ$ denotes a pair (r,d) for which there is no such graph, and the unfilled spots denote pairs for which we currently do not know of a graph that satisfies $(\operatorname{reg}(\mathbb{K}[G]), \deg h_{\mathbb{K}[G]}(x)) = (r,d)$.

	$r=1$	2	3	4	5	6	7	$\cdots$
$d=1$	$\bullet$							
2	$\circ$	$\bullet$						
3	$\circ$		$\bullet$					
4	$\circ$			$\bullet$	$\bullet$			
5	$\circ$			$\bullet$	$\bullet$	$\bullet$		
6	$\circ$			$\bullet$	$\bullet$	$\bullet$	$\bullet$	
$\vdots$	$\vdots$			$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\ddots$

Table 1. Summary of comparison of the regularity and the degree of the h-polynomial.

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