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UNDERSTANDING STAR-FUNDAMENTAL ALGEBRAS

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Abstract. Star-fundamental algebras are special finite dimensional algebras with involution $*$ over an algebraically closed field of characteristic zero defined in terms of multialternating $*$-polynomials.

We prove that the upper-block matrix algebras with involution introduced in Di Vincenzo and La Scala [J. Algebra 317 (2007), pp. 642–657] are star-fundamental. Moreover, any finite dimensional algebra with involution contains a subalgebra mapping homomorphically onto one of such algebras.

We also give a characterization of star-fundamental algebras through the representation theory of the symmetric group.

1. Introduction

Let F be an algebraically closed field of characteristic zero. This paper is devoted to the study of a class of finite dimensional algebras with involution or $*$-algebras over F called $*$-fundamental.

The theory of polynomial identities for $*$-algebras has been developed mainly following the pattern of the ordinary theory of polynomial identities ([8], [16], [17], [18], [19]). By rediscovering the theory of varieties developed by Kemer ([15]), in recent years much interest has been devoted to the so-called "fundamental algebras" ([2], [3], [20]). In this framework in ([9]) we developed a theory of $*$-fundamental algebras. These are algebras defined in terms of multialternating $*$-polynomials non-vanishing in them; one of their basic properties is that any finite dimensional $*$-algebra over F has the same $*$-identities as a finite direct sum of $*$-fundamental algebras. By exploiting the methods of [2] we were able to prove that if A is any finitely generated PI-algebra with involution, then $\lim_{n\to\infty} \log_n \frac{c_n^*(A)}{exp^*(A)^n}$ exists and is an integer or a half integer, where $c_n^*(A)$ is the nth $*$-codimension of A and $exp^*(A)$ is its $*$-exponent. This result agrees with an extension to rings with involution of a conjecture of Regev implying that the polynomial growth rate of the ordinary codimensions of a PI-algebra is an integer or a half integer (see [4], [5]).

For a better understanding of $*$-fundamental algebras we feel that the theory still needs two ingredients: a class of examples of $*$-fundamental algebras (that are not fundamental), and an explicit definition via the representation theory of the symmetric group S_n. We achieve both objectives in this paper. In fact, on one hand

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we give a family of matrix algebras that are examples of $*$-fundamental algebras and on the other we give a characterization of $*$-fundamental algebras in terms of S_n-characters.

We achieve the first objective by considering the algebras $UT_*(A_1, \ldots, A_n)$ of upper-block triangular matrices with involution introduced in [6]. These algebras play a special role in the theory of $*$-polynomial identities since they generate the only varieties of algebras with involution of finite basic rank which are minimal with respect to the $*$-exponent, as shown in [7]. We prove that any finite dimensional $*$-algebra contains a $*$-subalgebra which maps homomorphically onto some $UT_*(A_1, \ldots, A_n)$ (Theorem 4). Moreover we show that such algebras are $*$-fundamental, for any choice of the $*$-simple components $A_1, \ldots, A_n$ (Theorem 3). If we allow at least one $*$-simple component that is not simple, then we find out that the algebras $UT_*(A_1, \ldots, A_n)$ are examples of algebras that are $*$-fundamental but not fundamental.

Along the proof of these results we find a connection between the Kemer $*$-index and the $*$-exponent of any finite dimensional $*$-algebra. The first index is defined in terms of multialternating $*$-polynomials non-vanishing in the algebra (see [9]), while the $*$-exponent measures the exponential growth of the $*$-codimensions of the algebra (see [12], [13]).

As for the second result, one considers the free algebra with involution $F\langle X, *\rangle$ on a countable set X and if A is a $*$-algebra, we let $\mathrm{Id}^*(A)$ be the ideal of $F\langle X, *\rangle$ of $*$-polynomial identities satisfied by A. We write $A = \bar{A} + J$, where $\bar{A}$ is a $*$-semisimple subalgebra and J is the Jacobson radical. Let $t = \dim \bar{A}$ and $s \geq 0$ the least integer such that $J^{s+1} = 0$.

As in the ordinary case ([11]) one considers $P_n^*(A)$, the space of multilinear $*$-polynomials in n fixed variables modulo $\mathrm{Id}^*(A)$. Through the permutation action of the symmetric group S_n, the space $P_n^*(A)$ becomes an S_n-module and we call $\chi_n^*(A)$ its character. Such character decomposes into a sum of irreducible S_n-characters χ_λ indexed by partitions λ of n.

We prove that A is $*$-fundamental if and only if for all n large enough, there is a partition λ of n with s boxes below the first t rows such that χ_λ appears in $\chi_n^*(A)$ with non-zero multiplicity (Theorem 2).

As a final remark we mention that some of the theory of codimensions of finite dimensional algebras has been generalized to the context of H-codimensions, with H a finite dimensional semisimple algebra having a generalized action on A ([14]). Unfortunately the methods of this paper cannot work in that generality due to the structure of finite dimensional $*$-algebras.

2. $*$-FUNDAMENTAL ALGEBRAS AND THEIR $*$-COCHARACTERS

Throughout this paper, we shall denote by F a field of characteristic zero and by A an associative F-algebra with involution $*$ (also called a $*$-algebra).

Let $X = \{x_1, x_2, \ldots\}$ be a countable set and let $F\langle X, *\rangle = F\langle x_1, x_1^*, x_2, x_2^*, \ldots\rangle$ be the free associative algebra with involution on X over F. In order to simplify the notation we shall write $f(x_1, \ldots, x_n)$ to indicate a $*$-polynomial of $F\langle X, *\rangle$ in which the variables $x_1, \ldots, x_n$ or their star appear.

Recall that $f(x_1, \ldots, x_n) \in F\langle X, *\rangle$ is a $*$-polynomial identity (or simply a $*$-identity) of A, and we write $f \equiv 0$, if $f(a_1, \ldots, a_n) = 0$, for all $a_1, \ldots, a_n \in A$.

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Let $\mathrm{Id}^*(A) = \{f \in F\langle X, *\rangle \mid f \equiv 0 \text{ on } A\}$ be the ideal of $F\langle X, *\rangle$ of $*$-polynomial identities of A. Clearly $\mathrm{Id}^*(A)$ is a T^*-ideal of $F\langle X, *\rangle$, i.e., an ideal invariant under all endomorphisms of the free algebra commuting with the involution. Let

$$P_n^* = \mathrm{span}_F\{w_{\sigma(1)} \cdots w_{\sigma(n)} | \ \sigma \in S_n, w_i = x_i \text{ or } w_i = x_i^*, 1 \leq i \leq n\}$$

be the space of multilinear $*$-polynomials of degree n in $x_1, \dots, x_n$, i.e., for every $i = 1, \dots, n$, either x_i or x_i^* appears in every monomial of P_n^* at degree 1 (but not both). Since in characteristic zero $\mathrm{Id}^*(A)$ is generated as T^*-ideal by its multilinear polynomials one studies its intersection with P_n^*, for all $n \geq 1$.

There is a natural action of the symmetric group S_n on P_n^* : if $\sigma \in S_n$ and $f = f(x_1, \dots, x_n) \in P_n^*$ then

$$\sigma f = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

It is easily seen that the space

$$P_n^*(A) = \frac{P_n^*}{P_n^* \cap \mathrm{Id}^*(A)}$$

inherits a structure of S_n-module and its dimension, $c_n^*(A)$, is called the nth $*$-codimension of A.

In order to capture the exponential rate of growth of the sequence of $*$-codimensions, in [10] the authors proved that for any associative $*$-algebra A, satisfying an ordinary identity, the limit

$$\exp^*(A) = \lim_{n\to\infty} \sqrt[n]{c_n^*(A)}$$

exists and is an integer. It is called the $*$-exponent of A. Moreover $\exp^*(A)$ can be explicitly computed; it turns out to be the dimension of a suitable finite dimensional semisimple $*$-algebra when the base field F is algebraically closed.

Now the character $\chi_n^*(A)$ of $P_n^*(A)$ is called the nth $*$-cocharacter of A, and by complete reducibility, we can write

$$\chi_n^*(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda, \tag{1}$$

where χ_λ is the irreducible S_n-character associated to the partition λ and $m_\lambda \geq 0$ is the corresponding multiplicity.

If $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$ is such that $\lambda_1 \geq \cdots \geq \lambda_r > 0$, we call $r = ht(\lambda)$ the height of λ.

We recall that any irreducible left S_n-module M wih character χ_λ can be generated as an S_n-module by an element of the form $e_{T_\lambda} f$, for some $f \in M$ and some Young tableau T_λ of shape λ. Here $e_{T_\lambda} = R_{T_\lambda}^+ C_{T_\lambda}^-$ is an essential idempotent, where $R_{T_\lambda}^+ = \sum_{\sigma \in R_{T_\lambda}} \sigma$, $C_{T_\lambda}^- = \sum_{\tau \in C_{T_\lambda}} (\mathrm{sign}\tau)\tau$, and R_{T_λ} and C_{T_λ} are the row and column stabilizers of T_λ, respectively. Notice that the $*$-polynomial $f_{T_\lambda} = C_{T_\lambda}^-(e_{T_\lambda} f)$ generates also M, as an S_n-module, and it is alternating in each set of variables indexed by the columns of T_λ.

We recall that a $*$-polynomial $f(x_1, \dots, x_n, Y)$ linear in the variables $x_1, \dots, x_n$ (and in some other set of variables Y) is alternating in $x_1, \dots, x_n$ if f vanishes whenever we identify any two of these variables. This is equivalent to say that the polynomial changes sign whenever we exchange any two of these variables (here we exchange the indices of the two variables). For instance the polynomial $x_1 x_2^* - x_2 x_1^*$ is alternating in x_1 and x_2 whereas $x_1^* x_2 - x_2 x_1^*$ is not alternating in x_1 and x_2.

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From now on we assume that $A = \bar{A} + J$ is a finite dimensional $*$-algebra, where $\bar{A}$ is a semisimple subalgebra of A and $J = J(A)$ is the Jacobson radical. It is well-known that there is a $*$-Wedderburn-Malcev decomposition of A, i.e., $J = J^*$ and we may take $\bar{A}$ to be stable under $*$ ([13, Theorem 3.4.4]). We recall that the (t,s)-index of A is $Ind_{t,s}(A) = (\dim \bar{A}, s_A)$ where $s_A \geq 0$ is the smallest integer such that $J^{s_A+1} = 0$.

Next we define the Kemer $*$-index of A.

Let $\Gamma \subseteq F\langle X, *\rangle$ be the T^*-ideal of $*$-identities of A. Then $\beta(\Gamma)$ is defined as the greatest integer t such that for every $\mu \geq 1$, there exists a multilinear $*$-polynomial $f(X_1, \dots, X_\mu, Y) \notin \Gamma$ alternating in the μ sets X_i with $|X_i| = t$. Moreover $\gamma(\Gamma)$ is defined as the greatest integer s for which there exists for all $\mu \geq 1$, a multilinear $*$-polynomial $f(X_1, \dots, X_\mu, Z_1, \dots, Z_s, Y) \notin \Gamma$ alternating in the μ sets X_i with $|X_i| = \beta(\Gamma)$ and in the s sets Z_j with $|Z_j| = \beta(\Gamma) + 1$.

Then $Ind_K^*(\Gamma) = (\beta(\Gamma), \gamma(\Gamma))$ is called the *Kemer $*$-index* of Γ.

Since $\Gamma = \mathrm{Id}^*(A)$, we also say that $(\beta(\Gamma), \gamma(\Gamma)) = (\beta(A), \gamma(A)) = Ind_K^*(A)$ is the Kemer $*$-index of A.

We remark that by the definition of $\gamma(\Gamma)$ there exists a smallest integer μ_0 such that every $*$-polynomial $f(X_1, \dots, X_\mu, Z_1, \dots, Z_{\gamma(\Gamma)+1}, Y)$, alternating in $\mu \geq \mu_0$ sets X_i with $\beta(\Gamma)$ elements and in $\gamma(\Gamma) + 1$ sets Z_j with $\beta(\Gamma) + 1$ elements lies in Γ.

A multilinear $*$-polynomial $f(X_1, \dots, X_\mu, Z_1, \dots, Z_{\gamma(\Gamma)}, Y) \notin \Gamma$ which is alternating in $\mu > \mu_0$ sets X_i with $|X_i| = \beta(\Gamma)$ and in $\gamma(\Gamma)$ sets Z_i with $|Z_i| = \beta(\Gamma) + 1$ is called a *Kemer $*$-polynomial* related to Γ.

Remark 1 ([9, Remark 5.1]). $Ind_K^*(A) \leq Ind_{t,s}(A)$ in the left lexicographic order.

In what follows, by abuse of notation, we shall write $s_A = s$.

Next we give the definition of $*$-fundamental algebra (see [9, Section 6] for more details). We start with the following construction. Let $A = \bar{A} + J$ be a finite dimensional algebra with involution over an algebraically closed field F, $J^s \neq 0$, $J^{s+1} = 0$ and let $n = \dim J$.

If $A' = \bar{A} * F\langle x_1, \dots, x_n, *\rangle$ is the free product of $\bar{A}$ and the free algebra $F\langle x_1, \dots, x_n, *\rangle$ then we can write $A' = \bar{A} \oplus I$, where I is the $*$-ideal of A' generated by $x_1, \dots, x_n$. Let I_1 be the $*$-ideal generated by $\{f(A') \mid f \in \mathrm{Id}^*(A)\}$ and let

$$\mathcal{A}_s = A'/(I^{s+1} + I_1).$$

Notice that $\mathcal{A}_s$ is a finite dimensional algebra with $\mathrm{Id}^*(\mathcal{A}_s) = \mathrm{Id}^*(A)$ ([9, Lemma 6.1]). Moreover $Ind_{t,s}(\mathcal{A}_s) = Ind_{t,s}(A)$.

Now consider the ideal $I' = I/(I^{s+1} + I_1)$ of $\mathcal{A}_s$ and let

$$\mathcal{B}_0 = \mathcal{A}_s/(I')^s.$$

Hence $\mathrm{Id}^*(A) = \mathrm{Id}^*(\mathcal{A}_s) \subseteq \mathrm{Id}^*(\mathcal{B}_0)$, and $Ind_{t,s}(\mathcal{B}_0) = (\dim \bar{A}, s - 1)$.

Let $\bar{A} = A_1 \oplus \cdots \oplus A_q$, where the A_i's are $*$-simple algebras and let $s \geq 0$ be the smallest integer such that $J^{s+1} = 0$.

Now, for any $1 \leq i \leq q$, we define

$$B_i = A_1 \oplus \cdots \oplus \hat{A}_i \oplus \cdots \oplus A_q + J,$$

where the symbol $\hat{A}_i$ means that the algebra A_i is omitted in the direct sum.

Definition 1. The algebra A is $*$-fundamental if either A is $*$-simple or $s > 0$ and

$$\mathrm{Id}^*(A) \subsetneq \cap_{i=1}^q \mathrm{Id}^*(B_i) \cap \mathrm{Id}^*(\mathcal{B}_0).$$

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In this case a multilinear $*$-polynomial f is $*$-fundamental if $f \in \cap_{i=1}^{q} \mathrm{Id}^*(B_i) \cap \mathrm{Id}^*(\mathcal{B}_0)$ and $f \notin \mathrm{Id}^*(A)$.

It is not difficult to see that any finite dimensional algebra with involution has the same $*$-identities as a finite direct sum of $*$-fundamental algebras.

The $*$-fundamental algebras can also be characterized through the Kemer $*$-index, in fact we have.

Theorem 1 ([9, Theorem 6.1]). *A finite dimensional $*$-algebra A is $*$-fundamental if and only if $Ind_K^*(A) = Ind_{t,s}(A)$.*

In Theorem 2 we give a characterization of $*$-fundamental algebras through the representation theory of the symmetric group.

Theorem 2. *Let A be a finite dimensional $*$-algebra with $Ind_{t,s}(A) = (t, s)$ and let $\chi_n^*(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$ be its nth $*$-cocharacter. Then A is $*$-fundamental if and only if for all n large enough, there is a partition λ of n of height $ht(\lambda) \geq t$ with precisely s boxes under the first t rows such that $m_\lambda \neq 0$.*

Proof. Suppose first that for all n large enough there is a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ with $\lambda_{t+1} + \cdots + \lambda_r = s$ such that $m_\lambda \neq 0$.

If $s = 0$, A is semisimple and by hypothesis there is an irreducible S_n-module $M \nsubseteq \mathrm{Id}^*(A)$ generated by a $*$-polynomial $f_{T_\lambda} \in P_n^*$ alternating in t variables, say, $x_1, \ldots, x_t$. Since $f_{T_\lambda} \notin \mathrm{Id}^*(A)$ is multilinear, there is a non-zero evaluation of the variables x_i on a basis of A. Being $t = \dim A$, it is clear that A must be $*$-simple, hence $*$-fundamental.

If $s > 0$, there exists an irreducible S_n-module $M \nsubseteq \mathrm{Id}^*(A)$ that can be generated by a multilinear $*$-polynomial $f_{T_\lambda}(x_1, \ldots, x_n) \in P_n^*$, alternating on λ_{t+1} sets of variables each of size greater or equal than $t + 1$.

In order to prove that A is $*$-fundamental, we shall show that $f_{T_\lambda} \in \cap_{i=1}^{q} \mathrm{Id}^*(B_i) \cap \mathrm{Id}^*(\mathcal{B}_0)$. Now, since $Ind_{t,s}(\mathcal{B}_0) = (t, s-1)$, then $f_{T_\lambda} \in \mathrm{Id}^*(\mathcal{B}_0)$. In fact, f_{T_λ} is alternating on λ_{t+1} sets of variables each of size greater or equal than $t+1$; these alternating sets correspond to the first λ_{t+1} columns of the diagram of λ. Hence if we want to get a non-zero evaluation, in each of the alternating sets we have to substitute at most t linearly independent elements from the maximal semisimple subalgebra of $\mathcal{B}_0$. Hence we need to evaluate the remaining variables in the Jacobson radical $J(\mathcal{B}_0)$ of $\mathcal{B}_0$. Now, since $\lambda_{t+1} + \cdots + \lambda_r = s$ and $J(\mathcal{B}_0)^s = 0$ we get that any evaluation of f_{T_λ} into $\mathcal{B}_0$ gives zero.

Since for any $i = 1, \ldots, q$, we have that $\dim \bar{B}_i < t$, if we evaluate f_{T_λ} in B_i, in order to get a non-zero value we have to evaluate more than s variables in J. This says that $f_{T_\lambda} \in \mathrm{Id}^*(B_i)$. So we have proved that $f_{T_\lambda} \in \cap_{i=1}^{q} \mathrm{Id}^*(B_i) \cap \mathrm{Id}^*(\mathcal{B}_0)$ and, since $f_{T_\lambda} \notin \mathrm{Id}^*(A)$, we get that A is $*$-fundamental.

Conversely, suppose that A is $*$-fundamental.

If $s = 0$, by definition A is $*$-simple and by [9, Proposition 3.1] there exists a multilinear $*$-polynomial $f(x_1, \ldots, x_t, Y) \notin \mathrm{Id}^*(A)$ alternating in $x_1, \ldots, x_t$, $|Y| < \infty$.

Now, for any $n > \deg f$, by multiplying on the right by new variables we may assume that $f \in P_n^*$. Consider the permutation action of the symmetric group S_t on $x_1, \ldots, x_t$, so that we can regard P_n^* and, so, $P_n^*(A)$ as an S_t-module. Since $f \notin \mathrm{Id}^*(A)$, for any tableau T of shape (1^t) we have that $f_T = e_T f \notin \mathrm{Id}^*(A)$. Thus the character $\chi_{(1^t)}$ appears with non-zero multiplicity in the S_t-character of $P_n^*(A)$.

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Now, by the branching rule ([13, Theorem 2.3.1]) the induced character $\chi_{(1^t)} \uparrow S_n$ has an irreducible component $\chi_{\lambda'}$ with non-zero multiplicity and $ht(\lambda') = t$ since $\dim A = t$.

Suppose now that $s > 0$ and consider a Kemer $*$-polynomial $f \in P_n^*$. Since $Ind_K^*(A) = Ind_{t,s}(A)$ f can be written as

$$f = f(X_1, \dots, X_s, z_1, \dots, z_s, Y) \notin \mathrm{Id}^*(A),$$

where $|X_1| = \cdots = |X_s| = t$ and f is alternating on each set $\{X_i, z_i\}$, for $i = 1, \dots, s$.

We can consider P_n^* as an S_k-module, where $k = (t+1)s$, by letting S_k act on the set $\{X_1, \dots, X_s, z_1, \dots, z_s\}$. Then the quotient space $P_n^*(A)$ inherits a structure of S_k-module.

Since $f \notin \mathrm{Id}^*(A)$, there exists a partition $\lambda = (\lambda_1, \dots, \lambda_r) \vdash k$ and a tableau T_λ such that $f_{T_\lambda} = C_{T_\lambda}^- e_{T_\lambda} f \notin \mathrm{Id}^*(A)$. Let χ be the S_k-character of $P_n^*(A)$. Then, if we decompose $\chi = \sum_{\mu \vdash k} m'_\mu \chi'_\mu$, we have that $m'_\lambda \neq 0$. Since f_{T_λ} is not a $*$-identity for A, we get that also $f' = R_{T_\lambda}^+ f_{T_\lambda} \notin \mathrm{Id}^*(A)$. Notice that f' is symmetric in λ_1 variables and f is alternating on s disjoint sets of variables; hence $m'_\lambda \neq 0$ implies that $\lambda_1 \leq s$ and, also, $\lambda_1 + \cdots + \lambda_t \leq ts$.

If $\lambda_1 + \cdots + \lambda_t < ts$, then we would have $\lambda_{t+1} + \cdots + \lambda_r \geq s+1$ and as above we get $m'_\lambda = 0$, a contradiction. Thus $\lambda_1 + \cdots + \lambda_t = ts$ and $\lambda_{t+1} + \cdots + \lambda_r = s$. Now consider the induced character $\chi'_\lambda \uparrow S_n$. By the branching rule we have $\chi'_\lambda \uparrow S_n = \sum_{\mu \vdash n} m_\mu \chi_\mu$, a decomposition into irreducible S_n-characters, where $\mu = (\mu_1, \dots, \mu_{t+1}, \dots, \mu_r, \dots)$ runs over the set of partitions of n with $\mu_{t+1} + \cdots + \mu_r + \cdots \geq \lambda_{t+1} + \cdots + \lambda_r = s$. Since $m'_\lambda \neq 0$, there exists at least one χ_μ which appears with non-zero multiplicity. Then as in the first part of the proof we get that $\mu_{t+1} + \cdots + \mu_r + \cdots = s$ and we are done. $\square$

3. Star-fundamental algebras

We start by recalling the definition of the algebra $UT_*(A_1, \dots, A_n)$ given in [6].

Let $(A_1, \dots, A_n)$ be an n-tuple of $*$-simple algebras. We recall that any $*$-simple algebra A_i is isomorphic to $(M_{d_i}(F), *)$, the algebra of $d_i \times d_i$ matrices over F where $*$ is the transpose or the symplectic involution or to $M_{d_i}(F) \oplus M_{d_i}(F)^{op}$, the direct sum of $M_{d_i}(F)$ and its opposite, with exchange involution ([13, Theorem 3.4.4, Theorem 3.6.8]). In order to simplify the notation we shall identify $A_i \equiv M_{d_i}(F)$ in the first case, and $A_i \equiv M_{d_i}(F) \oplus M_{d_i}(F)^{op}$, in the second case.

For any $i = 1, \dots, n$, let γ_i denote the reflection involution along the secondary diagonal of $M_{d_i}(F)$. Recall that γ_i acts on matrix units as $e_{ij}^{\gamma_i} = e_{d_i-j+1, d_i-i+1}$.

Let $d = d_1 + \cdots + d_n$ and let $UT(d_1, \dots, d_n)$ denote the algebra of upper block triangular matrices of size $d_1, \dots, d_n$. We set $s_1 = 0$, and for any $2 \leq i \leq n+1$ $s_i = \sum_{k=1}^{i-1} d_k$.

For $1 \leq i, j \leq n$ define the map

$$\pi_{ij} : M_d(F) \to M_{d_i \times d_j}$$

such that

$$Y = (a_{\alpha,\beta}) \to \begin{pmatrix} a_{s_i+1, s_j+1} & \cdots & a_{s_i+1, s_{j+1}} \\ \vdots & & \vdots \\ a_{s_{i+1}, s_j+1} & \cdots & a_{s_{i+1}, s_{j+1}} \end{pmatrix}.$$

Notice that $\pi_{ij}(Y)$ is the projection of Y on its $d_i \times d_j$-block.

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Let $\Gamma = \{i \mid A_i = (M_{d_i}(F), *)\}$, where $*$ is the transpose or the symplectic involution. The algebra

$$UT_*(A_1, \dots, A_n) \subseteq UT(d_1, \dots, d_n, d_n, \dots, d_1) \subseteq M_{2d}(F)$$

is defined by the following conditions: given $Y \in UT(d_1, \dots, d_n, d_n, \dots, d_1)$, then $Y \in UT_*(A_1, \dots, A_n)$ if and only if

(1) $\pi_{ij}(Y) = 0$ if $i \leq n$ and $j > n$;
(2) $\pi_{2n-i+1,2n-i+1}(Y) = (\pi_{ii}(Y)^*)^{\gamma_i}$, when $i \in \Gamma$.

The involution on $UT_*(A_1, \dots, A_n)$ is the reflection involution γ.

The relevance of the algebras $UT_*(A_1, \dots, A_n)$ is due to a result proved in [7]. We recall that a variety $\mathcal{V}$ of algebras with involution of finite basic rank (i.e., the variety is generated by a finitely generated algebra with involution) are minimal with respect to the $*$-exponent $e \geq 2$ if for any proper subvariety $\mathcal{U}$ of finite basic rank, we have that $e = \exp^*(\mathcal{V}) > \exp^*(\mathcal{U})$. Here the $*$-exponent of a variety is the $*$-exponent of a generating algebra.

In [7] the authors proved that the variety $\mathcal{V}$ is minimal with respect to the $*$-exponent if and only if it is generated by $UT_*(A_1, \dots, A_n)$, for some $*$-simple algebras $A_1, \dots, A_n$. We remark that the definition of minimal variety given in [7] refers to varieties generated by finite dimensional algebras, but by the result in [21] this is equivalent to our definition.

Next we shall prove that the algebras $UT_*(A_1, \dots, A_n)$ are $*$-fundamental.

To this end we need first to prove two lemmas of independent interest.

Let $A = \bar{A} + J$ be a finite dimensional algebra with involution $*$, with $\bar{A} = A_1 \oplus \cdots \oplus A_q$, where the A_i's are $*$-simple algebras and $s \geq 0$ is the smallest integer such that $J^{s+1} = 0$.

In what follows we shall be dealing with multilinear $*$-polynomials. In order to check that a multilinear $*$-polynomial is a $*$-identity of A, we shall evaluate the variables only on elements of a basis of A. Then we choose a basis of A as the union of a basis of J and a basis of $\bar{A}$, which is the union of bases of the $*$-simple components.

Recall that given any $*$-polynomial $f(x_1, \dots, x_n, Y)$ linear in each of the variables in $X = \{x_1, \dots, x_n\}$, the operator of alternation Alt_X on X is defined as

$$Alt_X f(x_1, \dots, x_n, Y) = \sum_{\sigma \in S_n} (\mathrm{sgn}\sigma) f(x_{\sigma(1)}, \dots, x_{\sigma(n)}, Y).$$

The new polynomial $Alt_X f(x_1, \dots, x_n, Y)$ is multilinear and alternating in $x_1, \dots, x_n$.

Let the $e_{l,j}$'s be the matrix units of $M_{d_i}(F)$ and, by abuse of notation, in case $A_i = M_{d_i}(F) \oplus M_{d_i}(F)^{op}$, let $e_{l,j}$ denote $(e_{l,j}, 0)$ or $(0, e_{l,j})$.

In what follows we shall make use of the following.

Proposition 1 ([9, Proposition 3.1]). *Let A be a $*$-simple algebra. For every $\mu \geq 1$ there exists a multilinear $*$-polynomial*

$$f(X_1, \dots, X_\mu, Y) \notin \mathrm{Id}^*(A) \tag{2}$$

alternating on each of the disjoint sets $X_1, \dots, X_\mu$, where $|X_1| = \cdots = |X_\mu| = \dim A$ and $|Y| < \infty$. Such a polynomial has the property that it can take any value of the type $e_{i,i}$, when evaluated in A.

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Recall that a $*$-algebra $A = \bar{A} + J$, with $\bar{A} = A_1 \oplus \cdots \oplus A_q$ is $*$-reduced ([9]) if up to a rearrangement of the $*$-simple components we have that $A_1 J A_2 J \cdots J A_q \neq 0$.

Let $f(X_1, \ldots, X_r, Y)$ be a multilinear $*$-polynomial in disjoint sets of variables $X_1, \ldots, X_r, Y$. Then f is said to be r-fold t-alternating if f is alternating on each set X_i and $|X_i| = t$, $1 \leq i \leq r$.

Lemma 1. *Let $A = \bar{A} + J$ be a finite dimensional $*$-reduced algebra, and let $\dim \bar{A} = \dim(A_1 \oplus \cdots \oplus A_q) = t$. Then, for any $\mu \geq 1$, there exists a multilinear $*$-polynomial*

$$f(X_1, \ldots, X_\mu, Z_1, \ldots, Z_{q-1}, Y) \notin \mathrm{Id}^*(A)$$

with $|X_j| = t$, $1 \leq j \leq \mu$, $|Z_i| = t+1$, $1 \leq i \leq q-1$ such that

(1) *f is $(q-1)$-fold $(t+1)$-alternating,*
(2) *f is μ-fold t-alternating.*

Proof. We assume as we may that $A_1 J A_2 J \cdots J A_q \neq 0$. Now let $\mu \geq 1$ and for every $1 \leq i \leq q$, let $f_i(X_{1,i}, \ldots, X_{\mu+q-1,i}, Y_i) \notin \mathrm{Id}^*(A_i)$ be the multilinear $*$-polynomial constructed in Proposition 1. Recall that f_i is alternating on each of the disjoint sets $X_{1,i}, \ldots, X_{\mu+q-1,i}$, where $|X_{1,i}| = \cdots = |X_{\mu+q-1,i}| = \dim A_i$. Also such $*$-polynomial has the property that it can take any value of the type $e_{j_i \cdot j_i} \in A_i$.

Next we define a new polynomial

$$f = f_1 z_1 f_2 z_2 \cdots z_{q-1} f_q,$$

where $z_1, \ldots, z_{q-1}$ are new variables distinct from the ones appearing in each f_i. Then we let

$$\tilde{f} = Alt_{X,Z} f,$$

where $Alt_{X,Z}$ is the operator of alternation on each set

$$X_j = X_{j,1} \cup \cdots \cup X_{j,q}, \quad q \leq j \leq \mu + q - 1,$$

and on each set

$$Z_j = X_{j,1} \cup \cdots \cup X_{j,q} \cup \{z_j\}, \quad 1 \leq j \leq q-1.$$

Since $\dim A_1 + \cdots + \dim A_q = t$, $\tilde{f}$ is $(q-1)$-fold $(t+1)$-alternating, and μ-fold t-alternating.

The proof of the lemma will be completed once we prove that $\tilde{f}$ is not a $*$-identity of A.

In fact, since $A_1 J A_2 J \cdots J A_q \neq 0$, there exist $a_1 \in A_1, \ldots, a_q \in A_q$, $u_1, \ldots, u_{q-1} \in J$, such that

$$a_1 u_1 a_2 u_2 \cdots u_{q-1} a_q \neq 0.$$

Let e_i be the unit element of A_i. It follows that

$$e_{i_1,i_1} v_1 e_{i_2,i_2} v_2 \cdots v_{q-1} e_{i_q,i_q} \neq 0,$$

where e_{i_j,i_j} are suitable elements of A_j, $1 \leq j \leq q$, and $v_k = e_k a_k u_k e_{k+1}$, $1 \leq k \leq q-2$, $v_{q-1} = e_{q-1} a_{q-1} u_{q-1} a_q e_q$.

Let φ be the evaluation such that $\varphi(f_j) = e_{i_j,i_j}$, $1 \leq j \leq q$, and $\varphi(z_j) = v_j$ for $1 \leq j \leq q-1$. Then, since for $i \neq j$, $A_i A_j = 0$, we get

$$\varphi(\tilde{f}) = C e_{i_1,i_1} v_1 e_{i_2,i_2} v_2 \cdots v_{q-1} e_{i_q i_q}$$

where $C = \prod_{i=1}^{q} (\dim A_i)!^{\mu+q-1}$. $\square$

The connection between the $*$-exponent and the Kemer $*$-index of a finite dimensional algebra with involution is given in the following.

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Remark 2. If $A = \bar{A} + J$ is a finite dimensional algebra with involution and $Ind_K^*(A) = (\alpha, \beta)$, then $\alpha = exp^*(A)$.

Proof. Recall (see [12] or [13, Corollary 10.8.5]) that $exp^*(A)$ is the largest dimension of a $*$-semisimple subalgebra of $\bar{A}$, say, $A_1 \oplus \cdots \oplus A_n$, such that

$$A_1 J A_2 J \cdots J A_n \neq 0.$$

Then the algebra $B = A_1 \oplus \cdots \oplus A_n + J$ is $*$-reduced and by Lemma 1 for every $\mu \geq 1$ there exists a multilinear $*$-polynomial $f \notin \mathrm{Id}^*(B)$ which is μ-fold t-alternating, where $t = \dim(A_1 \oplus \cdots \oplus A_n) = exp^*(A)$. Since $f \notin \mathrm{Id}^*(A)$, then for the first Kemer $*$-index we have $\alpha \geq exp^*(A)$.

On the other hand let $J^k = 0$ and let g be a multilinear $*$-polynomial k-fold $(exp^*(A) + 1)$-alternating. We shall prove that g is a $*$-identity of A.

Suppose to the contrary that there is a non-zero evaluation φ of g in A.

If we evaluate all variables of g in $\bar{A}$, since $A_i A_j = 0$, for $i \neq j$, in order for $\varphi(g)$ to be non-zero, all variables must be evaluated in one $*$-simple component, say A_i. Since g is alternating on $exp^*(A) + 1 > \dim A_i$ variables we get $\varphi(g) = 0$, a contradiction. It follows that some variables of g must be evaluated in J.

Suppose now that we evaluate all variables of an alternating set X_u in $\bar{A}$, say in $A_{i_1}, \ldots, A_{i_s}$. But then $0 \neq \varphi(g) \in B = A_{i_1} + \cdots + A_{i_s} + J$ and, by definition of $*$-exponent, $\dim(A_{i_1} + \cdots + A_{i_s}) \leq exp^*(A)$. Since $|X_u| = exp^*(A) + 1$, we get that $\varphi(g) = 0$, a contradiction.

It follows that at least one variable of each alternating set must be evaluated under φ in J. Since there are k alternating sets, we get that $\varphi(g) \in J^k = 0$.

Since for any $l \geq k$, any multilinear $*$-polynomial l-fold $(exp^*(A)+1)$-alternating is a $*$-identity of A, it follows that α, the first Kemer $*$-index of A, cannot be strictly greater than $\exp^*(A)$. □

We can now prove the following result.

Theorem 3. *The algebra $UT_*(A_1, \ldots, A_n)$ is $*$-fundamental, for any $*$-simple algebras $A_1, \ldots, A_n$.*

Proof. If J is the Jacobson radical of $A = UT_*(A_1, \ldots, A_n)$, we have that $J^n = 0$ and $J^{n-1} \neq 0$. It is not difficult to see that the algebra A is $*$-reduced. Hence the (t, s)-index of A is $Ind_{t,s}(A) = (\dim(A_1 + \cdots + A_n), n-1) = (exp^*(A), n-1)$. Since $Ind_K^*(A) \leq Ind_{t,s}(A)$, by Lemma 1 and Remark 2 we get that $Ind_K^*(A) = Ind_{t,s}(A)$ and A is $*$-fundamental. □

By applying Theorem 10.1 in [9] one easily gets the following.

Corollary 1. *Given $A = UT_*(A_1, \ldots, A_n)$ let r be the number of $*$-simple algebras A_i which are not simple algebras. Then*

$$C_1 m^{-\frac{1}{2}(\dim(\bar{A})^- - r) + n - 1} (\dim \bar{A})^m \leq c_m^*(A) \leq C_2 m^{-\frac{1}{2}(\dim(\bar{A})^- - r) + n - 1} (\dim \bar{A})^m,$$

for some constants $C_1 > 0, C_2$ where $\bar{A} = A_1 \oplus \cdots \oplus A_n$ and $(\bar{A})^-$ is the Lie algebra of skew elements of $\bar{A}$. Hence

$$\lim_{m\to\infty} \log_m \frac{c_m^*(A)}{\exp^*(A)^m} = -\frac{1}{2}(\dim(\bar{A})^- - r) + n - 1.$$

Final remarks are in order. Let $A = \bar{A} + J$ be a finite dimensional algebra over F and let $\bar{A} = A_1 \oplus \cdots \oplus A_r$, with the A_i's simple algebras (no involution is

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involved). Then A is reduced if up to a rearrangement of the simple components we have that $A_1JA_2J\cdots A_r \neq 0$. It is well-known that in the ordinary case any fundamental algebra ([3], see [20] where the term fundamental is used for the first time) is reduced and the analogous of Theorem 1 holds, i.e, a finite dimensional algebra A is fundamental if and only if $Ind_K(A) = Ind_{t,s}(A)$.

Clearly if A is a fundamental algebra then A is also a $*$-fundamental algebra for any involution $*$ defined on A. In fact, if A is fundamental we have that $Ind_{t,s}(A) = Ind_K(A)$ but $Ind_K(A) \leq Ind_K^*(A) \leq Ind_{t,s}(A)$. Hence $Ind_K^*(A) = Ind_{t,s}(A)$ and by Theorem 1 A is $*$-fundamental.

Next we shall give examples of algebras which are $*$-fundamental but not fundamental.

Consider the algebra $A = UT_*(A_1, \ldots, A_n)$. If at least one of the components A_i is a $*$-simple but not simple algebra, then A is not reduced, so, cannot be fundamental. On the other hand if all the components A_i are simple algebras, then $UT_*(A_1, \ldots, A_n)$ is isomorphic to $UT(A_1, \ldots, A_n)$, an algebra of upper block triangular matrices, and it is well-known that such an algebra is fundamental ([1] or [2, Section 2.2.1]).

Remark 3. The algebra $UT_*(A_1, \ldots, A_n)$ is $*$-fundamental but not fundamental if and only if at least one of the components A_i is a $*$-simple but not simple algebra.

4. Connecting finite dimensional algebras with $*$-minimal algebras

As in the previous section we let A be a finite dimensional algebra with involution $*$ over an algebraically closed field F of characteristic zero. We write $A = \bar{A} + J$, where $\bar{A}$ is a maximal semisimple subalgebra with involution and J is the Jacobson radical. Also $\bar{A} = A_1 \oplus \cdots \oplus A_q$, where $A_i = M_{d_i}(F)$ with transpose or symplectic involution or $A_i = M_{d_i}(F) \oplus M_{d_i}(F)^{op}$ with exchange involution.

A connection between finite dimensional $*$-algebras and $*$-fundamental algebras is given by the following.

Theorem 4. *Let $A = \bar{A} + J$ be a finite dimensional algebra with involution $*$ over an algebraically closed field F of characteristic zero. If $A_1, \ldots, A_n$ are distinct $*$-simple subalgebras of $\bar{A}$ such that $A_1JA_2J\cdots JA_n \neq 0$, then the $*$-algebra $UT_*(A_1, \ldots, A_n)$ is isomorphic to a quotient of a $*$-subalgebra of A.*

Proof. For any $i = 1, \ldots, n$, write $A_i = (M_{d_i}(F), *)$ or $A_i = M_{d_i}(F) \oplus M_{d_i}(F)^{op}$. As in the previous section let $d = d_1 + \cdots + d_n$, $s_1 = 0$, and for any $2 \leq i \leq n+1$, $s_i = \sum_{k=1}^{i-1} d_k$. Also $\Gamma = \{i \mid A_i = (M_{d_i}(F), *)\}$.

It is convenient to identify $\tilde{A} = A_1 \oplus \cdots \oplus A_n$ with the algebra

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{pmatrix},$$

where if $i \in \Gamma$, $A_i = (M_{d_i}(F), *)$ is identified with the $d_i \times d_i$ matrices with basis $\{e_{i,j} \mid s_i + 1 \leq i, j \leq s_{i+1}\}$ and if $i \notin \Gamma$, $A_i = M_{d_i}(F) \oplus M_{d_i}(F)^{op}$ is identified with the pairs of matrices with basis $\{(e_{i,j}, 0), (0, e_{k,l}) \mid s_i + 1 \leq i, j, k, l \leq s_{i+1}\}$.

Now, define a map $\varphi : \tilde{A} \to (M_{2d}(F), \gamma)$, where γ is the reflection involution, such that if $i \in \Gamma$, then for any $e_{\alpha,\beta} \in A_i$,

$$\varphi(e_{\alpha,\beta}) = e_{\alpha,\beta} + (e^*_{\alpha,\beta})^\gamma,$$

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whereas if $i \notin \Gamma$, for any $(e_{\alpha,\beta}, 0), (0, e_{\alpha,\beta}) \in A_i$,

$$\varphi((e_{\alpha,\beta}, 0)) = e_{\alpha,\beta} \quad \text{and} \quad \varphi((0, e_{\alpha,\beta})) = (e_{\alpha,\beta})^{\gamma}.$$

Then φ induces an isomorphism of algebras with involution

$$(\tilde{A}, *) \to (\varphi(\tilde{A}), \gamma) \subseteq UT_*(A_1, \dots, A_n).$$

Now by the hypothesis of the theorem, there exist $a_1 \in A_1, \dots, a_n \in A_n$ and $u_1, \dots, u_{n-1} \in J$ such that

$$a_1 u_1 a_2 \cdots a_{n-1} u_{n-1} a_n \neq 0.$$

If $e_1, \dots, e_n$ are the unit elements of $A_1, \dots, A_n$, respectively, we have

$$e_1 v_1' e_2 \cdots e_{n-1} v_{n-1}' e_n \neq 0,$$

where $v_i' = e_i a_i u_i e_{i+1}$, $1 \leq i \leq n-2$ and $v_{n-1}' = e_{n-1} a_{n-1} u_{n-1} a_n e_n$. Thus, recalling that we write e_{α_i,α_i} for $(e_{\alpha_i,\alpha_i}, 0)$ or $(0, e_{\alpha_i,\alpha_i})$, we can say that there exist matrix units $e_{\alpha_i,\alpha_i} \in A_i = M_{d_i}(F)$, if $i \in \Gamma$, or $e_{\alpha_i,\alpha_i} \in A_i = M_{d_i}(F) \oplus M_{d_i}(F)^{op}$, if $i \notin \Gamma$, such that

$$e_{\alpha_1,\alpha_1} v_1' e_{\alpha_2,\alpha_2} v_2' \cdots v_{n-1}' e_{\alpha_n,\alpha_n} \neq 0. \tag{3}$$

Now define $v_i = e_{\alpha_i,\alpha_i} v_1' e_{\alpha_{i+1},\alpha_{i+1}}$, $i = 1, \dots, n-1$ and we have

$$e_{\alpha_1,\alpha_1} v_1 e_{\alpha_2,\alpha_2} v_2 \cdots v_{n-1} e_{\alpha_n,\alpha_n} \neq 0. \tag{4}$$

Notice that $M_{d_i}(F) \oplus M_{d_i}(F)^{op}$ and $M_{d_i}(F)^{op} \oplus M_{d_i}(F)$ are isomorphic as algebras with involution. Hence, if $i \notin \Gamma$, by eventually replacing A_i with an isomorphic copy, we may assume that $e_{\alpha_i,\alpha_i} = (e_{\alpha_i,\alpha_i}, 0)$.

Let B be the subalgebra with involution of A generated by $\tilde{A}$ and $v_1, \dots, v_{n-1}$. Let also I be the ideal with involution of B generated by the elements $v_i B v_i^*$, $v_i^* B v_i$, $1 \leq i \leq n-1$. The quotient algebra B/I has an induced involution, and we shall prove that it is isomorphic to $UT_*(A_1, \dots, A_n)$. To this end we shall induce the isomorphism $(\tilde{A}, *) \to (\varphi(\tilde{A}), \gamma)$ to

$$\varphi' : B/I \to UT_*(A_1, \dots, A_n).$$

We start by determining a basis of B/I. By abuse of notation we shall identify the elements of the quotient algebra with the elements of B keeping in mind that any word in elements of $\bar{A}$ and v_i's containing v_i and v_i^* must be zero.

Let $1 \leq i, j \leq d$. If $s_k + 1 \leq i, j \leq s_{k+1}$, for some $1 \leq k \leq n$, then we define $x_{i,j} = e_{i,j} \in A_k$.

Let now $1 \leq i < j \leq d$. For any $1 \leq k \leq n-1$ and $t \geq 1$ such that $k + t \leq n$, if

$$s_k + 1 \leq i \leq s_{k+1}, \quad s_{k+t} + 1 \leq j \leq s_{k+1+t},$$

define

$$x_{i,j} = e_{i,\alpha_k} e_{\alpha_k,\alpha_k} v_k e_{\alpha_{k+1},\alpha_{k+1}} v_{k+1} \cdots v_{k+t-1} e_{\alpha_{k+t},\alpha_{k+t}} e_{\alpha_{k+t},j}.$$

Since $e_{\alpha_k,i} x_{i,j} e_{j,\alpha_{k+t}}$ is a factor of the non-zero product in (4), we have that $x_{ij} \neq 0$.

Recalling also the definition of the ideal I, it can be checked that for $x_{i,j}, x_{k,l} \in J(B/I)$, we have that

$$x_{i,j} x_{k,l}^* = x_{i,j}^* x_{k,l} = 0. \tag{5}$$

In fact, let $x_{i,j} \in A_r v_r \cdots v_{s-1} A_s$ and $x_{k,l} \in A_t v_t \cdots v_{c-1} A_c$. Then $x_{i,j} x_{k,l}^* \in A_r v_r \cdots A_s A_c v_{c-1}^* \cdots v_t^* A_t$ and either $s \neq c$ and, so, $x_{i,j} x_{k,l}^* = 0$ or $s = c$ and the above product must contain v_{s-1} and v_{s-1}^* and, by definition of I, $x_{i,j} x_{k,l}^* = 0$. Similarly $x_{i,j}^* x_{k,l} = 0$.

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By direct computation it follows that the $x_{i,j}$'s and the $x^*_{i,j}$'s multiply as matrix units taking into account the relations in (5). Thus they are linearly independent over F. Since these are the only non-zero words of B/I, it follows that they form a basis of the algebra with involution B/I.

Next we define a map $\varphi' : B/I \to UT_*(A_1, \dots, A_n)$ by setting $\varphi'(\tilde{A}) = \varphi(\tilde{A})$. Moreover for $1 \leq k \leq n-1$, $k+t \leq n$, $t \geq 1$ if $s_k + 1 \leq i \leq s_{k+1}$, $\quad s_{k+t}+1 \leq j \leq s_{k+1+t}$, we let

$$\varphi'(x_{i,j}) = e_{i,j} \quad \text{and} \quad \varphi'(x^*_{i,j}) = e^{\gamma}_{i,j}.$$

The map φ' is a multiplicative linear isomorphism since $\dim B/I = \dim UT^*(A_1, \dots, A_n)$ and the x_{ij}'s multiply like matrix units. Since it is also an isomorphism of algebras with involution, the theorem is proved. □

Given an algebra with involution A let us denote by $\mathrm{var}^*(A)$ the $*$-variety generated by A.

The connection between finitely generated $*$-algebras and $*$-fundamental algebras with respect to the $*$-exponent is given in the following.

Corollary 2. *Let A be a finitely generated PI-algebra with involution $*$ over an algebraically closed field of characteristic zero. Then there exist $*$-simple algebras $A_1, \dots, A_n$ such that $UT_*(A_1, \dots, A_n) \in \mathrm{var}^*(A)$ and $exp^*(A) = exp^*(UT_*(A_1, \dots, A_n))$.*

Proof. By [21], we may assume that A is a finite dimensional algebra. Hence the $*$-exponent of A is equal to the largest dimension of a $*$-semisimple subalgebra of $\bar{A}$, say, $A_1 \oplus \cdots \oplus A_n$, such that

$$A_1 J A_2 J \cdots J A_n \neq 0.$$

Hence $\exp^*(A) = \dim(A_1 \oplus \cdots \oplus A_n)$ and, by Theorem 4, there exists a $*$-subalgebra of A mapping homomorphically onto $UT_*(A_1, \dots, A_n)$. Hence $UT_*(A_1, \dots, A_n) \in \mathrm{var}^*(A)$ and we are done since $\exp^*(UT_*(A_1, \dots, A_n)) = \dim(A_1 \oplus \cdots \oplus A_n)$. □

Now recalling the definition of $*$-minimal variety, by [21] this implies the following result proved in [7].

Theorem 5. *Let $\mathcal{V}$ be a $*$-variety of finite basic rank minimal with respect to the $*$-exponent. Then $\mathcal{V}$ is generated by an algebra of the type $UT_*(A_1, \dots, A_n)$ for some $*$-simple algebras $A_1, \dots, A_n$.*

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