

## REGULARITY AND $h$ -POLYNOMIALS OF TORIC IDEALS OF GRAPHS

GIUSEPPE FAVACCHIO, GRAHAM KEIPER, AND ADAM VAN TUYL

(Communicated by Claudia Polini)

**ABSTRACT.** For all integers  $4 \leq r \leq d$ , we show that there exists a finite simple graph  $G = G_{r,d}$  with toric ideal  $I_G \subset R$  such that  $R/I_G$  has (Castelnuovo–Mumford) regularity  $r$  and  $h$ -polynomial of degree  $d$ . To achieve this goal, we identify a family of graphs such that the graded Betti numbers of the associated toric ideal agree with its initial ideal, and, furthermore, that this initial ideal has linear quotients. As a corollary, we can recover a result of Hibi, Higashitani, Kimura, and O’Keefe that compares the depth and dimension of toric ideals of graphs.

### 1. INTRODUCTION

Let  $\mathbb{K}$  be an algebraically closed field of characteristic zero, and also let  $R = \mathbb{K}[x_1, \dots, x_n]$  be the standard graded polynomial ring over  $\mathbb{K}$ . Hibi and Matsuda [14] initiated a comparison of the (Castelnuovo–Mumford) regularity of  $R/I$  and the degree of the  $h$ -polynomial appearing in the Hilbert series of  $R/I$ . They showed that for any integers  $d, r \geq 1$ , there is a monomial ideal  $I$  such that the regularity of  $R/I$  is  $r$ , and the degree of the  $h$ -polynomial is  $d$ . Hibi and Matsuda later refined this result in [15] to show that  $I$  can be taken to be a lexsegment ideal, and later, with Van Tuyl [16], showed that  $I$  could be an edge ideal. Further comparisons of the regularity and degree have been carried out for the edge ideals of Cameron–Walker graphs [12] and binomial edge ideals [13, 17]. In this paper we compare these invariants for the toric ideals of finite simple graphs.

Given a finite simple graph  $G$  on the vertex set  $V = \{v_1, \dots, v_n\}$  with edge set  $E = \{e_1, \dots, e_q\}$ , the *toric ideal of  $G$* , denoted  $I_G$ , is the kernel of the map  $\varphi : \mathbb{K}[E] = \mathbb{K}[e_1, \dots, e_q] \rightarrow \mathbb{K}[v_1, \dots, v_n]$  given by  $\varphi(e_i) = v_{i_1}v_{i_2}$ , where  $e_i = \{v_{i_1}, v_{i_2}\} \in E$ . Some properties of the homological invariants of  $I_G$  can be found in [2–4, 6, 7, 11, 18, 21]. Our main result adds to this list of properties, and also contributes to Hibi and Matsuda’s program.

**Theorem 1.1.** *Let  $4 \leq r \leq d$  be integers. Then there is a connected finite simple graph  $G = G_{r,d}$  such that the toric ideal of  $G$  satisfies  $\text{reg}(\mathbb{K}[E]/I_G) = r$  and  $\deg h_{\mathbb{K}[E]/I_G}(x) = d$ .*

---

Received by the editors March 16, 2020.

2020 *Mathematics Subject Classification.* Primary 13D02, 13P10, 13D40, 14M25, 05E40.

*Key words and phrases.* Toric ideals, graphs, graded Betti numbers, regularity, Hilbert series.

The first author is grateful for the support of the Università degli Studi di Catania “Piano della Ricerca 2016/2018 Linea di intervento 2” and the “National Group for Algebraic and Geometric Structures, and their Applications” (GNSAGA-INdAM).

The second author acknowledges the support of NSERC RGPIN-2019-05412.

The proof of Theorem 1.1 has two components. First, we consider the family of graphs constructed from the complete bipartite graph  $K_{2,t}$  by adjoining a “triangle” to each vertex of degree two (see Figure 1). We prove that the toric ideals of the graphs in this family have a unique extremal graded Betti number. We use this fact to show that for any  $e \geq 5$ , we can construct a graph  $G$  such that  $\mathbb{K}[E]/I_G$  has regularity 4 and the degree of its  $h$ -polynomial is  $e$ . The second component is to leverage the splitting techniques of the authors and Hofscheier [3] to create the desired graphs of Theorem 1.1 from the graphs in this family. As a bonus corollary, we give a new proof for the main result of [11] which compared the depth and dimension of toric ideals of graphs (see Corollary 3.10).

Our paper is structured as follows. In Section 2, we give the relevant background, including the undefined terms from the introduction. We also recall some tools from [3]; they are used to show that if  $r \geq 1$ , there is a graph  $G$  with  $r = \text{reg}(\mathbb{K}[E]/I_G) = \text{deg } h_{\mathbb{K}[E]/I_G}(x)$ . In Section 3, we introduce a family of connected graphs, and we show that we can control the values of  $\text{reg}(\mathbb{K}[E]/I_G)$  and  $\text{deg } h_{\mathbb{K}[E]/I_G}(x)$ , where  $I_G$  is the toric ideal of a graph in this family. These graphs can then be used to prove Theorem 1.1. We conclude with remarks in Section 4 about pairs  $(r, d)$  not covered by Theorem 1.1.

## 2. PRELIMINARIES

We recall the relevant background on homological invariants and toric ideals of graphs.

**2.1. Homological invariants.** If  $I$  is a homogeneous ideal of  $R$ , then the minimal graded free resolution of  $R/I$  has the form

$$0 \rightarrow \bigoplus_{j \in \mathbb{N}} R(-j)^{\beta_{p,j}(R/I)} \rightarrow \dots \rightarrow \bigoplus_{j \in \mathbb{N}} R(-j)^{\beta_{1,j}(R/I)} \rightarrow R \rightarrow R/I \rightarrow 0,$$

where  $R(-j)$  is the ring  $R$  with its grading shifted by  $j$ , and where  $\beta_{i,j}(R/I) = \dim_{\mathbb{K}} \text{Tor}_i^R(R/I, \mathbb{K})_j$  is called the  $i, j$ th graded Betti number of  $R/I$ . The (Castelnuovo–Mumford) regularity of  $R/I$  is

$$\text{reg}(R/I) = \max\{j - i \mid \beta_{i,j}(R/I) \neq 0\}.$$

The projective dimension of  $R/I$  is the length of the minimal graded free resolution, that is,

$$\text{pdim}(R/I) = \max\{i \mid \beta_{i,j}(R/I) \neq 0\}.$$

The Hilbert series of a standard graded  $\mathbb{K}$ -algebra  $R/I$  is the formal power series

$$HS_{R/I}(x) = \sum_{i \geq 0} [\dim_{\mathbb{K}}(R/I)_i] x^i,$$

where  $\dim_{\mathbb{K}}(R/I)_i$  is the dimension of  $i$ th graded piece of  $R/I$ . The Hilbert series of  $R/I$  can be read from any resolution of  $R/I$  (e.g., see [8, p. 100]). In particular,

$$(2.1) \quad HS_{R/I}(x) = \frac{1 + \sum_{i,j} (-1)^i \beta_{i,j}(R/I) x^j}{(1-x)^n}.$$

By the Hilbert–Serre Theorem (e.g., see [23, Theorem 5.1.4]) there is a polynomial  $h_{R/I}(x) \in \mathbb{Z}[x]$ , called the  $h$ -polynomial of  $R/I$ , such that  $HS_{R/I}$  can be written as

$$(2.2) \quad HS_{R/I}(x) = \frac{h_{R/I}(x)}{(1-x)^{\dim(R/I)}} \text{ with } h_{R/I}(1) \neq 0,$$

where  $\dim(R/I)$  denotes the Krull dimension of  $R/I$ .

We recall a fact about extremal Betti numbers; see [1] for more on their properties.

**Definition 2.1.** A graded Betti number of  $R/I$ , say  $\beta_{a,b}(R/I) \neq 0$ , is *extremal* if  $\beta_{i,j}(R/I) = 0$  for any pair  $(i, j)$  such that  $i \geq a$  and  $j > b$  and  $j - i \geq b - a$ .

**Lemma 2.2.** *Suppose  $\beta_{a,b}(R/I)$  is the only extremal Betti number of  $R/I$ . Then  $\text{reg}(R/I) = b - a$ ,  $\text{pdim}(R/I) = a$ , and  $\text{deg } h_{R/I}(x) = b - \dim R + \dim R/I$ .*

*Proof.* Since  $\beta_{a,b}(R/I)$  is an extremal Betti number, from the definition, we have  $\beta_{a,b}(R/I) \neq 0$  and  $\beta_{i,j}(R/I) = 0$  for any  $i \geq a$ ,  $j > b$  and  $j - i \geq b - a$ . Moreover, because it is the unique extremal Betti number,  $\beta_{i,j}(R/I) = 0$  if either  $i \geq a$  or  $j > b$  (otherwise there must be some other extremal Betti). Thus, the Betti table of  $R/I$  has a rectangular shape and the pair  $(a, b)$  determines the regularity and the projective dimension. Furthermore, from equation (2.1), the degree of the nonreduced numerator in the Hilbert series is  $b$ , so by (2.2), the degree of the  $h$ -polynomial is  $b - \dim R + \dim R/I$ .  $\square$

A monomial ideal  $I \subseteq R$  is said to have *linear quotients* if its minimal generators  $\{g_1, \dots, g_m\}$  can be ordered so that the quotient ideal  $\langle g_1, \dots, g_{j-1} \rangle : \langle g_j \rangle$  is generated by variables for every  $j = 2, \dots, m$ . Linear quotients were first defined in [10]. By [20, Corollary 2.7], a monomial ideal  $I \subseteq R$  with linear quotients with respect to the ordering  $g_1, \dots, g_m$  has graded Betti numbers given by the formula

$$(2.3) \quad \beta_{i+1,i+j}(R/I) = \sum_{1 \leq p \leq m, \text{deg}(g_p)=j} \binom{n_p}{i} \text{ for } i \geq 0,$$

where  $n_p$  denotes the number of different variables generating  $\langle g_1, \dots, g_{p-1} \rangle : \langle g_p \rangle$ .

For a fixed monomial ordering, we let  $\text{in}(I)$  denote the *initial ideal* of  $I$ . It is well known that  $\beta_{i,j}(R/I) \leq \beta_{i,j}(R/\text{in}(I))$  for all  $i, j \geq 0$  (e.g., see [19, Theorem 22.9]). The following result, found in [4, Lemma 2.6], gives a criterion for when we have equality for all  $i, j \geq 0$ .

**Lemma 2.3.** *Fix a monomial order. Suppose that  $I \subseteq R$  is a homogeneous ideal such that  $\beta_{i,i+j}(R/I) = \beta_{i,i+j}(R/\text{in}(I))$  for all  $i$  and all  $j \neq k$ . Then  $\beta_{i,i+k}(R/I) = \beta_{i,i+k}(R/\text{in}(I))$  for all  $i \geq 0$ .*

**2.2. Toric ideals of graphs.** We now turn to toric ideals of graphs, as defined in the introduction. Note that if  $G$  is a finite simple graph, then the toric ideal  $I_G$  is a prime homogeneous binomial ideal. Many of the algebraic and geometric invariants of  $I_G$  depend upon the combinatorics of  $G$ . In order to discuss these results, we briefly introduce some relevant terminology and results (see Villareal [23] and Herzog, Hibi, and Ohsgui [9] for details). Note that if  $G = (V, E)$  is a finite simple graph, we may sometimes write  $\mathbb{K}[E]$  for  $\mathbb{K}[e \mid e \in E]$  and  $\mathbb{K}[G]$  for the ring  $\mathbb{K}[E]/I_G$ .

If  $G$  is a finite simple graph, a *walk* in  $G$  is a sequence of edges  $w = (e_1, e_2, \dots, e_k)$  such that  $e_i \cap e_{i+1} \neq \emptyset$  for  $i = 1, \dots, k - 1$ . Equivalently, a walk is a sequence of vertices  $(x_1, \dots, x_k, x_{k+1})$  such that  $\{x_i, x_{i+1}\} \in E$  for  $i = 1, \dots, k$ . A walk is an *even walk* if  $k$  is even. A *closed walk* is a walk where  $x_{k+1} = x_1$ . Two closed even walks  $(e_0, \dots, e_{2k-1})$  and  $(e'_0, \dots, e'_{2k-1})$  are equivalent up to a *circular permutation* if there is an  $i$  such that  $e_j = e'_{j+i}$  for all  $j$  where  $j + i$  is taken modulo  $2k$  (or if the walk is in the reverse order, i.e.,  $e_j = e'_{(2k-j)+i}$  for all  $j$ ).

A finite graph  $G$  is *connected* if for every  $x, y \in V$  with  $x \neq y$ , there exists a walk having  $x$  as its first vertex and  $y$  as its last. A closed walk  $(e_1, \dots, e_k)$  where each vertex and edge is distinct is called a *cycle* of length  $k$ . A graph  $G$  is *bipartite* if there are no odd cycles in  $G$ . An  $n$ -*cycle*, denoted  $C_n$ , is the graph with vertex set  $V = \{x_1, \dots, x_n\}$  and edge set  $E = \{\{x_1, x_2\}, \{x_2, x_3\}, \dots, \{x_{n-1}, x_n\}, \{x_n, x_1\}\}$ .

The generators of the toric ideal  $I_G$  can be obtained from closed even walks in  $G$ ; we sketch out this connection. To each closed even walk  $w = (e_{i_1}, e_{i_2}, \dots, e_{i_{2n}})$  in  $G$ , we can associate the binomial  $f_w$  defined by

$$f_w = \prod_{2+j} e_{i_j} - \prod_{2|j} e_{i_j} \in I_G.$$

Note that it is straightforward to verify that  $\varphi(f_w) = 0$ , where  $\varphi : \mathbb{K}[e_1, \dots, e_q] \rightarrow \mathbb{K}[v_1, \dots, v_n]$  is the map defining  $I_G = \ker(\varphi)$ . Among all closed even walks, we identify a special subset.

**Definition 2.4.** A binomial  $f_1 - f_2 \in I_G$  is *primitive* if there exists no binomial  $g_1 - g_2 \in I_G$  such that  $g_1 \mid f_1$  and  $g_2 \mid f_2$ . A closed even walk  $w$  in a graph  $G$  is said to be *primitive* if the corresponding binomial  $f_w$  is primitive in  $I_G$ .

The importance of primitive closed even walks lies in the next theorem.

**Theorem 2.5** ([23, Proposition 10.1.10]). *The set of binomials associated with primitive closed even walks is a universal Gröbner basis of  $I_G$ .*

We round out this section by specializing one of the results of [3] that will be a key ingredient in our proof of Theorem 1.1. Recall that given a graph  $G = (V, E)$  and  $W \subseteq V$ , the *induced subgraph* of  $G$  on  $W$  is the graph with vertex set  $W$  and edge set  $\{e \in E \mid e \subseteq W\}$ . Following [3, Construction 4.1], let  $G_1, G_2$  be two graphs and suppose that  $H_1 \subseteq G_1, H_2 \subseteq G_2$  are two induced subgraphs which are isomorphic with respect to some graph isomorphism  $\varphi : H_1 \rightarrow H_2$ . We define the *glued graph*  $G_1 \cup_\varphi G_2$  of  $G_1$  and  $G_2$  along  $\varphi$  as the disjoint union of  $G_1$  and  $G_2$ , and we use  $\varphi$  to identify vertices and edges in  $H_1$  with their images in  $H_2$ . At times, we may be more informal and say that  $G_1$  and  $G_2$  is *glued along  $H$*  if the induced subgraphs  $H \cong H_1$  and  $H \cong H_2$  and isomorphism  $\varphi$  are clear.

It was shown in [3] that under some hypotheses on  $G_1$  and  $G_2$ , if the  $G_1$  and  $G_2$  are glued along some induced subgraph  $H$ , then many of the homological invariants of  $G_1 \cup_\varphi G_2$  are related to those of  $G_1$  and  $G_2$ . In particular, if we specialize [3, Corollary 3.11], we have the following result.

**Theorem 2.6.** *Let  $G$  be any finite simple connected graph, and let  $C_{2s}$  be an even cycle of length  $2s \geq 4$ . Let  $e$  be any edge of  $G$ , and let  $e'$  be any edge of  $C_{2s}$ . If  $G' = (V', E')$  is the graph obtained by gluing  $G$  and  $C_{2s}$  along  $e \cong e'$ , then*

- (i)  $\text{reg}(\mathbb{K}[G']) = \text{reg}(\mathbb{K}[G]) + s - 1$  and
- (ii)  $\text{deg } h_{\mathbb{K}[G']}(x) = \text{deg } h_{\mathbb{K}[G]}(x) + s - 1$ .

**Corollary 2.7.** *Let  $G = (V, E)$  be a connected graph with  $\text{deg } h_{\mathbb{K}[G]}(x) = d$  and  $\text{reg}(\mathbb{K}[G]) = r$ . Then there exists a connected graph  $G' = (V', E')$  with  $\text{deg } h_{\mathbb{K}[G']}(x) = d + 1$  and  $\text{reg}(\mathbb{K}[G']) = r + 1$ .*

*Proof.* By Theorem 2.6, if we glue a  $C_4$  along any edge of  $G$ , we get the desired result. □

For all integers  $1 \leq r$ , there is a graph  $G$  satisfying  $\text{deg } h_{\mathbb{K}[G]}(x) = \text{reg}(\mathbb{K}[G]) = r$ .

**Example 2.8.** Consider the graph  $C_4^{(r)}$ , where  $r \geq 1$  is an integer, on the vertex set  $V^{(r)} = \{x_1, \dots, x_{2r+2}\}$  and edge set  $E^{(r)} = \{\{x_1, x_2\}\} \cup \{\{x_1, x_{2i+1}\}, \{x_2, x_{2i+2}\}, \{x_{2i+1}, x_{2i+2}\} \mid i = 1, \dots, r\}$ . So, the graph  $C_4^{(r)}$  consists of  $r$  squares glued along one edge. Since  $C_4 = C_4^{(1)}$  has  $\deg h_{\mathbb{K}[C_4]}(x) = \text{reg}(\mathbb{K}[C_4]) = 1$ , then, iteratively from Corollary 2.7, we get  $\deg h_{\mathbb{K}[C_4^{(r)}]}(x) = \text{reg}(\mathbb{K}[C_4^{(r)}]) = r$ .

3. SOME HOMOLOGICAL INVARIANTS OF THE TORIC IDEAL FOR A FIXED FAMILY OF GRAPHS

In this section we construct a family of simple graphs  $G_t$  with  $t \geq 2$  such that  $\text{reg}(\mathbb{K}[G_t]) = 4$  and  $\deg h_{\mathbb{K}[G_t]}(x) = t + 3$ . By combining this family with Corollary 2.7, we can prove Theorem 1.1.

To help the reader, we sketch out the broad strokes that we take in this section. We begin by defining a graph  $G_t$  on  $t + 6$  vertices and  $2t + 6$  edges, where  $t \geq 2$  is an integer. We then describe a set  $\mathcal{G}$  of binomials that form a universal Gröbner basis for  $I_{G_t}$  and a set  $\mathcal{M}$  of minimal generators of  $\text{in}(I_{G_t})$ , the initial ideal of  $I_{G_t}$  for a given monomial ordering. We show that  $\text{in}(I_{G_t})$  has linear quotients. Lastly, we prove that all the graded Betti numbers of  $\mathbb{K}[G_t]$  coincide with the ones of  $\mathbb{K}[E_t]/\text{in}(I_{G_t})$ , and that there exists a unique extremal Betti number. We derive Theorem 1.1 from these facts.

We begin by formally defining the graphs of interest.

**Definition 3.1.** Let  $t \geq 2$  be an integer. The graph  $G_t$  is defined having the vertex and edge sets:

$$V_t = \{x_1, x_2, y_1, \dots, y_t, z_1, z_2, w_1, w_2\} \text{ and}$$

$$E_t = \{\{x_i, y_j\} \mid 1 \leq i \leq 2, 1 \leq j \leq t\} \cup \{\{x_1, z_1\}, \{z_1, z_2\}, \{z_2, x_1\}\} \cup \{\{x_2, w_1\}, \{w_1, w_2\}, \{w_2, x_2\}\}.$$

We label the edges of  $G_t$  as follows:  $e_1 = \{x_1, z_1\}$ ,  $e_2 = \{z_1, z_2\}$ ,  $e_3 = \{z_2, x_1\}$ ,  $f_1 = \{x_2, w_1\}$ ,  $f_2 = \{w_1, w_2\}$ ,  $f_3 = \{w_2, x_2\}$ , and, for  $i \in \{1, \dots, t\}$ ,  $a_i = \{x_1, y_i\}$  and  $b_i = \{x_2, y_i\}$ .

Note that the subgraph of  $G_t$  on the vertices  $\{x_1, x_2, y_1, \dots, y_t\}$  is a complete bipartite graph  $K_{2,t}$  consisting of only the edges  $\{a_1, \dots, a_t, b_1, \dots, b_t\}$ . Thus, less formally, the graph  $G_t$  is obtained from the complete bipartite graph  $K_{2,t}$  by joining a 3-cycle to each of the two vertices of degree  $t$ . See Figure 1 for the case  $t = 5$ . Note that the toric ideals of these graphs were also considered in [11].

Going forward, we work in the standard graded polynomial ring

$$\mathbb{K}[E_t] = \mathbb{K}[a_1, \dots, a_t, f_1, f_2, f_3, e_1, e_2, e_3, b_1, \dots, b_t].$$

Let  $>$  denote the graded reverse lexicographic monomial ordering on  $\mathbb{K}[E_t]$  satisfying

$$(3.1) \quad a_1 > \dots > a_t > f_1 > f_2 > f_3 > e_1 > e_2 > e_3 > b_1 > \dots > b_t.$$

We denote the *initial ideal* of an ideal  $I$  with respect to this ordering by  $\text{in}(I)$ .

Before focusing on  $I_{G_t}$ , we summarize some known results about the toric ideal of  $I_{K_{2,t}}$ . Here, we see  $K_{2,t}$ , the complete bipartite graph, as the induced subgraph of  $G_t$  on  $V' := \{x_1, x_2, y_1, \dots, y_t\}$ .

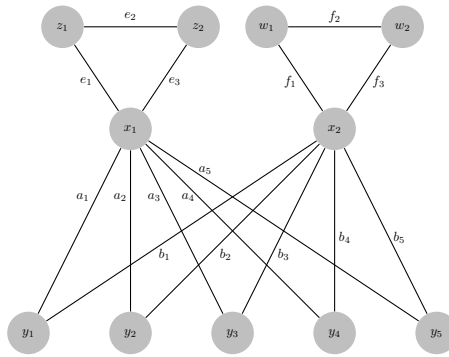


FIGURE 1. The graph  $G_5$

**Lemma 3.2.** Fix some integer  $t \geq 2$ . Using the same labelling as in Definition 3.1, let  $I_{K_{2,t}}$  be the toric ideal of the graph  $K_{2,t} = (V', E')$  in the polynomial ring  $\mathbb{K}[E'] = \mathbb{K}[a_1, \dots, a_t, b_1, \dots, b_t]$ . Then:

- (i)  $I_{K_{2,t}} = \langle a_i b_j - a_j b_i \mid 1 \leq i < j \leq t \rangle$ ;
- (ii)  $\text{in}(I_{K_{2,t}}) = \langle a_i b_j \mid 1 \leq j < i \leq t \rangle$  with respect to the graded reverse lexicographical order where  $a_1 > a_2 > \dots > a_t > b_1 > \dots > b_t$ ;
- (iii)  $\text{in}(I_{K_{2,t}})$  has linear quotients if one orders the generators with respect to the graded reverse lexicographical order; and
- (iv) if  $\{g_1, \dots, g_k\}$  are the generators of  $\text{in}(I_{K_{2,t}})$  ordered with respect to the graded reverse lexicographical order, then  $n_p \leq t - 1$  for all  $p$ , where  $n_p$  is the number of generators of  $\langle g_1, \dots, g_{p-1} \rangle : \langle g_p \rangle$  for  $p = 1, \dots, k$ ;
- (v)  $\beta_{i,j}(\mathbb{K}[K_{2,t}]) = \beta_{i,j}(\mathbb{K}[E']/\text{in}(I_{K_{2,t}}))$  for all  $i, j \geq 0$ .

*Proof.* Statements (i) and (ii) follow from [4, Remark 3.4] which shows that the given generators are a universal Gröbner basis of  $I_{K_{2,t}}$ . Statements (iii), (iv), and (v) follow from [4, Corollary 2.8]. □

The next result describes the set of primitive binomials of  $I_{G_t}$  which we denote by  $\mathcal{G}$ .

**Theorem 3.3.** For any integer  $t \geq 2$ , the ideal  $I_{G_t}$  is generated by the primitive binomials in  $\mathcal{G} = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3$  where

- (i)  $\mathcal{G}_1 = \{a_i b_j - a_j b_i \mid 1 \leq i < j \leq t\}$ ,
- (ii)  $\mathcal{G}_2 = \{a_i a_j f_1 f_3 e_2 - f_2 e_1 e_3 b_i b_j \mid 1 \leq i < j \leq t\}$ , and
- (iii)  $\mathcal{G}_3 = \{a_i^2 f_1 f_3 e_2 - f_2 e_1 e_3 b_i^2 \mid 1 \leq i \leq t\}$ .

In particular,  $\mathcal{G}$  is a universal Gröbner basis for  $I_{G_t}$ .

*Proof.* By Theorem 2.5, it suffices to show that the binomials in  $\mathcal{G}$  correspond to the primitive closed even walks in  $G_t$ . We only need to identify these even walks up to a circular permutation, since the associated binomials will be equal up to a sign.

Note that the elements of  $\mathcal{G}$  correspond to the following closed even walks in the graph  $G_t$ :

- $(a_i, b_i, b_j, a_j)$ , where  $1 \leq i < j \leq t$ ,
- $(a_i, b_i, f_1, f_2, f_3, b_j, a_j, e_1, e_2, e_3)$ , where  $1 \leq i < j \leq t$ , and
- $(a_i, b_i, f_1, f_2, f_3, b_i, a_i, e_1, e_2, e_3)$ , where  $1 \leq i \leq t$ .

However, as noted in [11] (prior to Lemma 2.1), these closed even walks form a complete set of primitive closed even walks.  $\square$

**Corollary 3.4.** *Using the graded reverse lexicographic order that satisfies (3.1), we have that  $\text{in}(I_{G_t})$  is generated by the monomials in  $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3$  where:*

- (i)  $\mathcal{M}_1 = \{a_i b_j \mid 1 \leq j < i \leq t\}$ ,
- (ii)  $\mathcal{M}_2 = \{a_i a_j f_1 f_3 e_2 \mid 1 \leq i < j \leq t\}$ , and
- (iii)  $\mathcal{M}_3 = \{a_i^2 f_1 f_3 e_2 \mid 1 \leq i \leq t\}$ .

Furthermore,  $\mathcal{M}$  is a minimal set of generators for  $\text{in}(I_{G_t})$ .

*Proof.* That  $\mathcal{M}$  is a generating set with respect to the given order follows from Theorem 3.3. That it is minimal follows from the fact that none of the monomials are divided by any of the others.  $\square$

We will show that  $\text{in}(I_{G_t})$  has linear quotients with respect to an order of its generators.

**Theorem 3.5.** *Let  $\mathcal{M}_1, \mathcal{M}_2$  and  $\mathcal{M}_3$  be as in Corollary 3.4, and order each set from smallest to largest with respect to the graded reverse lexicographical order. Then the initial ideal of  $I_{G_t}$ ,*

$$\begin{aligned} \text{in}(I_{G_t}) = \langle & a_t b_{t-1}, a_t b_{t-2}, \dots, a_{t-1} b_{t-2}, \dots, a_2 b_1, \\ & a_t a_{t-1} f_1 f_3 e_2, a_t a_{t-2} f_1 f_3 e_2, \dots, a_2 a_1 f_1 f_3 e_2, \\ & a_t^2 f_1 f_3 e_2, a_{t-1}^2 f_1 f_3 e_2, \dots, a_1^2 f_1 f_3 e_2 \rangle, \end{aligned}$$

has linear quotients with respect to this order of the generators. Furthermore, if  $\text{in}(I_{G_t}) = \langle g_1, \dots, g_{t^2} \rangle$ , and  $n_p$  is the number of generator of  $\langle g_1, \dots, g_{p-1} \rangle : \langle g_p \rangle$ , then

$$\max\{n_p \mid 2 \leq p \leq t^2\} = 2t - 2.$$

*Proof.* It follows from Corollary 3.4 that  $\text{in}(I_{G_t})$  has  $t^2$  generators. Let  $g_1, \dots, g_{t^2}$  be these generators, ordered as in the statement of the theorem. For each  $p \in \{2, \dots, t^2\}$ , let  $I(p) = \langle g_1, \dots, g_{p-1} \rangle : \langle g_p \rangle$ . A generating set of  $I(p)$  is given by

$$(3.2) \quad I(p) = \left\langle \frac{LCM(g_1, g_p)}{g_p}, \frac{LCM(g_2, g_p)}{g_p}, \dots, \frac{LCM(g_{p-1}, g_p)}{g_p} \right\rangle.$$

We initially observe that the first  $\frac{t(t-1)}{2}$  generators of  $\text{in}(I_{G_t})$  with respect to our ordering are the exact same as the generators of  $\text{in}(I_{K_{2,t}})$  by Lemma 3.2(ii). So, by Lemma 3.2(iii), since this order has linear quotients,  $I(p)$  is generated by variables for  $p = 2, \dots, \frac{t(t-1)}{2}$ .

It suffices to show that  $I(p)$  is generated by variables for  $p \in \left\{ \frac{t(t-1)}{2} + 1, \dots, t^2 \right\}$ . We consider the following two cases.

*Case 1.* Suppose that  $g_p = a_i a_j f_1 f_3 e_2$  with  $t \geq i > j \geq 1$ . Then the ideal  $I(p)$  is

$$= \begin{cases} \langle a_t b_{t-1}, a_t b_{t-2}, \dots, a_2 b_1 \rangle : \langle a_t a_{t-1} f_1 f_3 e_2 \rangle & \text{if } i = t \text{ and } j = t - 1, \\ \langle a_t b_{t-1}, a_t b_{t-2}, \dots, a_2 b_1, a_t a_{t-1} f_1 f_3 e_2, \dots, a_{j+2} a_{j+1} f_1 f_3 e_2 \rangle \\ \quad : \langle a_t a_j f_1 f_3 e_2 \rangle & \text{if } i = t \text{ and } 1 \leq j < t - 1, \\ \langle a_t b_{t-1}, a_t b_{t-2}, \dots, a_2 b_1, a_t a_{t-1} f_1 f_3 e_2, \dots, a_{i+1} a_j f_1 f_3 e_2 \rangle \\ \quad : \langle a_i a_j f_1 f_3 e_2 \rangle & \text{if } i < t. \end{cases}$$

If we calculate each ideal using (3.2), we get

$$I(p) = \begin{cases} \langle b_1, \dots, b_{t-1} \rangle & \text{if } i = t \text{ and } j = t - 1, \\ \langle b_1, \dots, b_{t-1}, a_{j+1}, \dots, a_{t-1} \rangle & \text{if } i = t \text{ and } 1 \leq j < t - 1, \\ \langle b_1, \dots, b_{i-1}, a_{j+1}, \dots, a_{i-1}, a_{i+1}, \dots, a_t \rangle & \text{if } i < t. \end{cases}$$

Case 2. If  $g_p = a_i^2 f_1 f_3 e_2$ , then

$$I(p) = \begin{cases} \langle a_t b_{t-1}, \dots, a_2 a_1 f_1 f_3 e_2 \rangle : \langle a_t^2 f_1 f_3 e_2 \rangle & \text{if } i = t, \\ \langle a_t b_{t-1}, \dots, a_2 a_1 f_1 f_3 e_2, a_1^2 f_1 f_3 e_2, \dots, a_{i+1}^2 f_1 f_3 e_2 \rangle : \langle a_i^2 f_1 f_3 e_2 \rangle & \text{if } 1 \leq i < t. \end{cases}$$

Computing each colon ideal gives

$$I(p) = \begin{cases} \langle b_1, \dots, b_{t-1}, a_1, \dots, a_{t-1} \rangle & \text{if } i = t, \\ \langle b_1, \dots, b_{i-1}, a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_t \rangle & \text{if } 1 \leq i < t. \end{cases}$$

It thus follows that  $\text{in}(I_{G_t})$  has linear quotients with respect to the given order.

To prove the final statement, it follows that  $n_p \leq t - 1$  for  $p = 2, \dots, \frac{t(t-1)}{2}$  by Lemma 3.2(iv). On the other hand, from our above computations, we saw that

$$\langle a_t b_{t-1}, \dots, a_2 a_1 f_1 f_3 e_2 \rangle : \langle a_t^2 f_1 f_3 e_2 \rangle = \langle b_1, \dots, b_{t-1}, a_1, \dots, a_{t-1} \rangle$$

has  $2t - 2$  generators, and every ideal  $I(p)$  with  $\frac{t(t-1)}{2} + 1 \leq p \leq t^2$  has  $n_p \leq 2t - 2$ .  $\square$

**Corollary 3.6.** *For any integer  $t \geq 2$ , we have  $\beta_{i,i+j}(\mathbb{K}[G_t]) = \beta_{i,i+j}(\mathbb{K}[E_t]/\text{in}(I_{G_t}))$  for all  $i, j \geq 0$ .*

*Proof.* Recall that we have  $\beta_{i,i+j}(\mathbb{K}[G_t]) \leq \beta_{i,i+j}(\mathbb{K}[E_t]/\text{in}(I_{G_t}))$  for all  $i, j \geq 0$ . Because  $\text{in}(I_{G_t})$  has linear quotients and is only generated in degrees 2 and 5, formula (2.3) thus gives

$$\beta_{i,i+j}(\mathbb{K}[G_t]) = \beta_{i,i+j}(\mathbb{K}[E_t]/\text{in}(I_{G_t})) = 0 \text{ for all } i \geq 0 \text{ and all } j \neq 1, 4.$$

On the other hand, the generators of  $I_{G_t}$  of degree two are exactly the same as the generators of  $I_{K_{2,t}}$  by Theorem 3.3 and Lemma 3.2. So  $\beta_{i,i+1}(\mathbb{K}[G_t]) = \beta_{i,i+1}(\mathbb{K}[K_{2,t}])$  for all  $i \geq 0$ . The minimal generators of  $\text{in}(I_{G_t})$  of degree 2 are also the minimal generators of  $\text{in}(I_{K_{2,t}})$ . Therefore,

$$\begin{aligned} \beta_{i,i+1}(\mathbb{K}[K_{2,t}]) &= \beta_{i,i+1}(\mathbb{K}[G_t]) \leq \beta_{i,i+1}(\mathbb{K}[E_t]/\text{in}(I_{G_t})) = \beta_{i,i+1}(\mathbb{K}[E_t]/\text{in}(I_{K_{2,t}})) \\ &= \beta_{i,i+1}(\mathbb{K}[K_{2,t}]), \end{aligned}$$

where the last inequality is Lemma 3.2(v). So we have shown that  $\beta_{i,i+j}(\mathbb{K}[G_t]) = \beta_{i,i+j}(\mathbb{K}[E_t]/\text{in}(I_{G_t}))$  for all  $i, j \geq 0$  except  $j = 4$ . To complete the proof, we now apply Lemma 2.3.  $\square$

*Remark 3.7.* It is possible to find an explicit formula for  $\beta_{i,i+j}(\mathbb{K}[E_t]/\text{in}(I_{G_t}))$  using the formula (2.3), and to determine the exact values of  $n_p$  for each  $p$ . These values can be extracted from the proof of Theorem 3.5 and [4, Theorem 3.6].

**Corollary 3.8.** *For any integer  $t \geq 2$ ,  $\beta_{2t-1,2t+3}(\mathbb{K}[G_t])$  is the unique extremal Betti number of  $\mathbb{K}[G_t]$ .*

*Proof.* By Corollary 3.6, it suffices to show that  $\beta_{2t-1,2t+3}(\mathbb{K}[E_t]/\text{in}(I_{G_t}))$  is the unique extremal graded Betti number of  $\mathbb{K}[E_t]/\text{in}(I_{G_t})$ . Since the ideal  $\text{in}(I_{G_t})$  is generated in degrees two and five, and because this ideal has linear quotients, any extremal Betti number will have the form  $\beta_{i,i+1}(\mathbb{K}[E_t]/\text{in}(I_{G_t}))$  or



$\beta_{i,i+4}(\mathbb{K}[E_t]/\text{in}(I_{G_t}))$ , where  $i \geq 1$ . By Lemma 3.2(iv) and formula (2.3)  $\beta_{i,i+1}(\mathbb{K}[E_t]/\text{in}(I_{G_t})) = 0$  if  $i \geq t$  since  $n_p \leq t - 1$  in this range. On the other hand, since  $n_p = 2t - 2$  for a generator of degree five (by Theorem 3.5), and because this is the maximal such value for  $n_p$ , we have  $\beta_{2t-1,2t+3}(\mathbb{K}[E_t]/\text{in}(I_{G_t})) \neq 0$ , but  $\beta_{i,i+4}(\mathbb{K}[E_t]/\text{in}(I_{G_t})) = 0$  for all  $i \geq 2t - 1$ . Since  $2t - 1 \geq t - 1$  because  $t \geq 2$ ,  $\beta_{2t-1,2t+3}(\mathbb{K}[E_t]/\text{in}(I_{G_t}))$  is the unique extremal graded Betti number.  $\square$

We can now compute the regularity and the degree of the  $h$ -polynomial of  $\mathbb{K}[G_t]$ .

**Theorem 3.9.** *For any integer  $t \geq 2$ , the graph  $G_t$  has  $\text{reg}(\mathbb{K}[G_t]) = 4$  and  $\text{deg } h_{\mathbb{K}[G_t]}(x) = t + 3$ .*

*Proof.* This results follows by combining Corollary 3.8 and Lemma 2.2, and using the fact that  $\dim \mathbb{K}[E_t] = 2t+6$  and  $\dim(\mathbb{K}[G_t]) = |V(G_t)| = 2t+4$ ; see [23, Corollary 10.1.21] for the latter assertion.  $\square$

We now have all the pieces to prove the main theorem of this paper.

*Proof of Theorem 1.1.* Let  $(r, d)$  be a pair of integers such that  $4 \leq r \leq d$ . If  $r = d$ , then the graph  $C_4^{(r)}$  introduced in Example 2.8 has the required invariants. Assume now that  $r < d$ . Set  $q = d-r+1 \geq 2$ . By Theorem 3.9 the graph  $G_q$  has  $\text{reg}(\mathbb{K}[G_q]) = 4$  and  $\text{deg } h_{\mathbb{K}[G_q]}(x) = q + 3 = d - r + 4$ . Thus, applying Corollary 2.7 ( $r - 4$ ) times, we get the existence of a graph  $G_{r,d}$  with  $\text{reg}(\mathbb{K}[G_{r,d}]) = r$  and  $\text{deg } h_{\mathbb{K}[G_{r,d}]}(x) = d$ . (The graph  $G_{r,d}$  is obtained by gluing  $G_q$  with  $r - 4$  squares  $C_4$  along one edge, no matter which one.)  $\square$

As another consequence of Corollary 3.8 we derive a new proof of the main result of [11].

**Corollary 3.10** ([11, Theorem 0.2]). *Fix integers  $7 \leq f \leq d$ . Then there exists a graph  $G$  whose toric ring satisfies  $\text{depth}(\mathbb{K}[G]) = f$  and  $\dim(\mathbb{K}[G]) = d$ .*

*Proof.* As described in [11], the proof of the above result hinges upon finding a graph on  $k + 6$  vertices with  $k \geq 1$  whose toric ideal  $I_G$  has the property that  $\text{depth}(\mathbb{K}[G]) = 7$ . Using our notation, [11] shows that the graphs  $G_t$  with  $t \geq 1$  (where  $G_1$  is the graph of two triangles joined by a path of length two) satisfy  $\text{depth}(\mathbb{K}[G_t]) = 7$ . But, for  $t \geq 2$  this also follows from Lemma 2.2, Corollary 3.8, and the Auslander–Buchsbaum formula since  $\text{depth}(\mathbb{K}[G_t]) = \dim(\mathbb{K}[E_t]) - \text{pdim}(\mathbb{K}[G_t]) = 2t + 6 - (2t - 1) = 7$ . When  $t = 1$ , then  $I_{G_t}$  has a single generator, so  $\text{pdim}(\mathbb{K}[G_t]) = 1$  and  $\text{depth}(\mathbb{K}[G_1]) = 8 - 1 = 7$ . The proof now runs as in the introduction of [11].  $\square$

#### 4. FURTHER COMMENTS AND OBSERVATIONS

We now turn our attention to integers  $d, r \geq 1$  not covered by Theorem 1.1.

While Hibi and Matsuda [14] showed that for all  $d, r \geq 1$ , there is a monomial ideal  $I$  with  $(r, d) = (\text{reg}(R/I), \text{deg } h_{R/I}(x))$ , this behaviour will not hold for toric ideals of graphs. In particular, if  $r = 1$ , then  $d$  must also equal 1.

**Theorem 4.1.** *Let  $G$  be a graph such that  $\text{reg}(\mathbb{K}[G]) = 1$ . Then  $\text{deg } h_{\mathbb{K}[G]}(x) = 1$ .*

*Proof.* It can be assumed that the graph  $G = (V, E)$  is connected. Since  $\text{reg}(\mathbb{K}[G]) = 1$ , then  $\mathbb{K}[G]$  has a linear resolution (hence it has a unique extremal Betti number  $\beta_{a,a+1}(\mathbb{K}[G])$ ) and, in particular,  $I_G$  is only generated by quadratic binomials.

Thus, from [9, Corollary 5.26], the ring  $\mathbb{K}[G]$  is Cohen–Macaulay ( $\text{depth}(\mathbb{K}[G]) = \dim \mathbb{K}[G]$ ). So, the Auslander–Buchsbaum formula implies that  $|E| - \dim \mathbb{K}[G] = \text{pdim}(\mathbb{K}[G]) = a$  and, from Lemma 2.2, we get  $\deg h_{\mathbb{K}[G]}(x) = \text{reg}(\mathbb{K}[G])$ .  $\square$

*Remark 4.2.* In the proof of Theorem 4.1, we saw that  $\mathbb{K}[G]$  was Cohen–Macaulay, from which we deduced that  $\deg h_{\mathbb{K}[G]}(x) = \text{reg}(\mathbb{K}[G])$ . As shown in [22, Corollary B.4.1], this holds in general, i.e., if  $\mathbb{K}[G]$  is Cohen–Macaulay, then the regularity and the degree of the  $h$ -polynomial are equal. We know of no example of a graph  $G$  such that  $I_G$  is generated in degrees  $\leq 3$  and  $\mathbb{K}[G]$  is not a Cohen–Macaulay ring, thus suggesting that if  $\text{reg}(\mathbb{K}[G]) \leq 3$ , there may be restrictions for  $\deg h_{\mathbb{K}[G]}(x)$ . On the other hand, the graph  $G$  with vertex set  $V = \{x_1, \dots, x_8\}$  and edge set

$$E = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_4, x_6\}, \{x_4, x_7\}, \{x_5, x_6\}, \{x_5, x_7\}, \{x_6, x_7\}\} \cup \{\{x_i, x_8\} \mid i = 1, \dots, 7\}$$

is generated in degrees  $\leq 4$ , including a generator of degree four, and  $\mathbb{K}[G]$  is not Cohen–Macaulay. (We thank Kazunori Matsuda for pointing us towards this example.)

Now we make an observation about the graphs having  $\deg h_{\mathbb{K}[G]}(x) = 1$ .

*Remark 4.3.* Let  $G = (V, E)$  be a connected and nonbipartite graph such that  $h_{\mathbb{K}[G]}(x) = 1 + ax$ ,  $a \neq 0$ . From equations (2.1) and (2.2) in Section 2 we have

$$HS_{\mathbb{K}[G]}(x) = \frac{1 + ax}{(1 - x)^{|V|}} = \frac{1 + \sum_j B_j x^j}{(1 - x)^{|E|}}, \text{ where } B_j = \sum_i (-1)^i \beta_{i,j}(\mathbb{K}[G]).$$

Note that, in particular,  $B_1 = 0$  and  $B_2 = -\beta_{1,2}(\mathbb{K}[G])$ . Thus, we get  $(1 + ax)(1 - x)^{|E|-|V|} = 1 + \sum_j B_j x^j$ . So, by comparing coefficients,  $a = |E| - |V|$  and  $\beta_{1,2}(I_G) = a^2 - \binom{a}{2} = \binom{a+1}{2} = \binom{|E|-|V|+1}{2}$ . So, if there is a nonbipartite graph  $G$  with  $\deg h_{\mathbb{K}[G]}(x) = 1$ , then it must have  $\binom{|E|-|V|+1}{2}$  quadratic generators.

*Remark 4.4.* We know of no example of a graph  $G$  such that  $\beta_{1,2}(\mathbb{K}[G]) = \binom{|E|-|V|+1}{2}$  and  $\mathbb{K}[G]$  is not a Cohen–Macaulay ring.

Finally, note that the strategy of Theorem 1.1 is to find graphs where we can control the regularity and the degree of the  $h$ -polynomial, and use it as a “seed” to repeatedly apply Corollary 2.7. Thus, to extend Theorem 1.1 for integers  $d < r$ , we need an appropriate initial graph. As the next example shows, we can extend Theorem 1.1 slightly to include all integers  $(r, d)$  with  $5 \leq r = d + 1$ .

**Example 4.5.** Let  $Z$  be the graph in Figure 2 on the vertex set  $V = \{x_1, \dots, x_{10}\}$  and edges  $E = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_1, x_7\}, \{x_1, x_8\}, \{x_1, x_9\}, \{x_4, x_5\}, \{x_4, x_6\}, \{x_4, x_7\}, \{x_4, x_8\}, \{x_4, x_9\}, \{x_2, x_3\}, \{x_5, x_6\}, \{x_7, x_8\}, \{x_8, x_{10}\}, \{x_9, x_{10}\}\}$ . One can compute the Betti diagram and the Hilbert series of  $\mathbb{K}[Z]$  by using

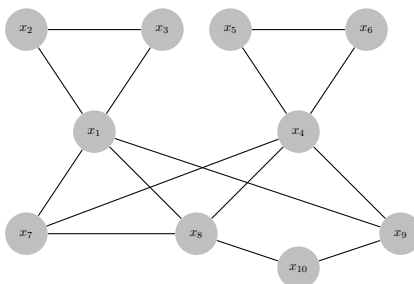


FIGURE 2. The graph  $Z$

Macaulay2 [5]:

	0	1	2	3	4	5	6
total:	1	12	40	56	37	11	1
$\beta(\mathbb{K}[Z]) =$	0:	1	.	.	.	.	.
	1:	.	5	5	.	.	.
	2:	.	2	.	1	.	.
	3:	.	.	10	10	.	.
	4:	.	5	25	45	37	10
	5:	.	.	.	.	1	.

and

$$HS_{\mathbb{K}[Z]}(x) = \frac{1 + 5x + 10x^2 + 13x^3 + 10x^4}{(1 - x)^{10}}.$$

Thus, the graph  $Z$  covers the new case  $\text{reg}(\mathbb{K}[Z]) = 5$  and  $\text{deg } h_{\mathbb{K}[Z]}(x) = 4$ . As a consequence of Corollary 2.7, for any pair  $(r, d)$  of positive integers such that  $d \geq 4$  and  $r - d = 1$ , there is a graph  $G$  with  $\text{reg}(\mathbb{K}[G]) = r$  and  $\text{deg } h_{\mathbb{K}[G]}(x) = d$ .

Table 1 summarizes all the results from this paper. In the table, a filled-in circle  $\bullet$  denotes a pair  $(r, d)$  for which there is a graph  $G$  with  $(\text{reg}(\mathbb{K}[G]), \text{deg } h_{\mathbb{K}[G]}(x)) = (r, d)$ , the empty circle  $\circ$  denotes a pair  $(r, d)$  for which there is no such graph, and the unfilled spots denote pairs for which we currently do not know of a graph that satisfies  $(\text{reg}(\mathbb{K}[G]), \text{deg } h_{\mathbb{K}[G]}(x)) = (r, d)$ .

TABLE 1. Summary of comparison of the regularity and the degree of the  $h$ -polynomial.

	$r = 1$	2	3	4	5	6	7	...
$d = 1$	$\bullet$							
2	$\circ$	$\bullet$						
3	$\circ$		$\bullet$					
4	$\circ$			$\bullet$	$\bullet$			
5	$\circ$			$\bullet$	$\bullet$	$\bullet$		
6	$\circ$			$\bullet$	$\bullet$	$\bullet$	$\bullet$	
$\vdots$	$\vdots$			$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\ddots$

## ACKNOWLEDGMENTS

The authors thank Johannes Hofscheier for his feedback. Our results were inspired by calculations using *Macaulay2* [5].

## REFERENCES

- [1] Dave Bayer, Hara Charalambous, and Sorin Popescu, *Extremal Betti numbers and applications to monomial ideals*, J. Algebra **221** (1999), no. 2, 497–512, DOI 10.1006/jabr.1999.7970. MR1726711
- [2] Jennifer Biermann, Augustine O’Keefe, and Adam Van Tuyl, *Bounds on the regularity of toric ideals of graphs*, Adv. in Appl. Math. **85** (2017), 84–102, DOI 10.1016/j.aam.2016.11.003. MR3595300
- [3] Giuseppe Favacchio, Johannes Hofscheier, Graham Keiper, and Adam Van Tuyl, *Splittings of toric ideals*, Preprint, [arXiv:1909.12820](https://arxiv.org/abs/1909.12820) (2019).
- [4] Federico Galetto, Johannes Hofscheier, Graham Keiper, Craig Kohne, Adam Van Tuyl, and Miguel Eduardo Uribe Paczka, *Betti numbers of toric ideals of graphs: a case study*, J. Algebra Appl. **18** (2019), no. 12, 1950226, 14, DOI 10.1142/S0219498819502268. MR4026786
- [5] Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at <https://faculty.math.illinois.edu/Macaulay2/>.
- [6] Zachary Greif and Jason McCullough, *Green-Lazarsfeld Condition for Toric Edge Ideals of Bipartite Graphs*, Preprint, [arXiv:1908.02744](https://arxiv.org/abs/1908.02744) (2019).
- [7] Huy Tài Hà, Selvi Kara Beyarslan, and Augustine O’Keefe, *Algebraic properties of toric rings of graphs*, Comm. Algebra **47** (2019), no. 1, 1–16, DOI 10.1080/00927872.2018.1439047. MR3924764
- [8] Jürgen Herzog and Takayuki Hibi, *Monomial ideals*, Graduate Texts in Mathematics, vol. 260, Springer-Verlag London, Ltd., London, 2011. MR2724673
- [9] Jürgen Herzog, Takayuki Hibi, and Hidefumi Ohsugi, *Binomial ideals*, Graduate Texts in Mathematics, vol. 279, Springer, Cham, 2018. MR3838370
- [10] Jürgen Herzog and Yukihide Takayama, *Resolutions by mapping cones*, The Roos Festschrift volume (2), Homology Homotopy Appl. **4** (2002), no. 2, 277–294, DOI 10.4310/hha.2002.v4.n2.a13. MR1918513
- [11] Takayuki Hibi, Akihiro Higashitani, Kyouko Kimura, and Augustine B. O’Keefe, *Depth of edge rings arising from finite graphs*, Proc. Amer. Math. Soc. **139** (2011), no. 11, 3807–3813, DOI 10.1090/S0002-9939-2011-11083-9. MR2823027
- [12] Takayuki Hibi, Kyouko Kimura, Kazunori Matsuda, and Adam Van Tuyl, *The regularity and  $h$ -polynomial of Cameron-Walker graphs*, In progress, 2020.
- [13] Takayuki Hibi and Kazunori Matsuda, *Regularity and  $h$ -polynomials of binomial edge ideals*. Preprint, [arXiv:1808.06984](https://arxiv.org/abs/1808.06984) (2018).
- [14] Takayuki Hibi and Kazunori Matsuda, *Regularity and  $h$ -polynomials of monomial ideals*, Math. Nachr. **291** (2018), no. 16, 2427–2434, DOI 10.1002/mana.201700476. MR3884268
- [15] Takayuki Hibi and Kazunori Matsuda, *Lexsegment ideals and their  $h$ -polynomials*, Acta Math. Vietnam. **44** (2019), no. 1, 83–86, DOI 10.1007/s40306-018-0297-5. MR3935291
- [16] Takayuki Hibi, Kazunori Matsuda, and Adam Van Tuyl, *Regularity and  $h$ -polynomials of edge ideals*, Electron. J. Combin. **26** (2019), no. 1, Paper 1.22, 11, DOI 10.37236/8247. MR3919621
- [17] Thomas Kahle and Jonas Krüsemann, *Binomial edge ideals of cographs*, Preprint, [arXiv:1906.05510](https://arxiv.org/abs/1906.05510) (2019).
- [18] Hidefumi Ohsugi and Takayuki Hibi, *Toric ideals generated by quadratic binomials*, J. Algebra **218** (1999), no. 2, 509–527, DOI 10.1006/jabr.1999.7918. MR1705794
- [19] Irena Peeva, *Graded syzygies*, Algebra and Applications, vol. 14, Springer-Verlag London, Ltd., London, 2011. MR2560561
- [20] Leila Sharifan and Matteo Varbaro, *Graded Betti numbers of ideals with linear quotients*, Matematiche (Catania) **63** (2008), no. 2, 257–265 (2009). MR2531666
- [21] Christos Tatakis and Apostolos Thoma, *On the universal Gröbner bases of toric ideals of graphs*, J. Combin. Theory Ser. A **118** (2011), no. 5, 1540–1548, DOI 10.1016/j.jcta.2011.01.005. MR2771599
- [22] Wolmer V. Vasconcelos, *Computational methods in commutative algebra and algebraic geometry*, Algorithms and Computation in Mathematics, vol. 2, Springer-Verlag, Berlin, 1998.

With chapters by David Eisenbud, Daniel R. Grayson, Jürgen Herzog and Michael Stillman.  
MR1484973

- [23] Rafael H. Villarreal, *Monomial algebras*, 2nd ed., Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 2015. MR3362802

DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DEGLI STUDI DI CATANIA, VIALE  
A. DORIA, 6, 95100 - CATANIA, ITALY  
*Email address: favacchio@dmf.unict.it*

DEPARTMENT OF MATHEMATICS AND STATISTICS, MCMASTER UNIVERSITY, HAMILTON, ON-  
TARIO, L8S 4L8, CANADA  
*Email address: keipergt@mcmaster.ca*

DEPARTMENT OF MATHEMATICS AND STATISTICS, MCMASTER UNIVERSITY, HAMILTON, ON-  
TARIO, L8S 4L8, CANADA  
*Email address: vantuy1@math.mcmaster.ca*