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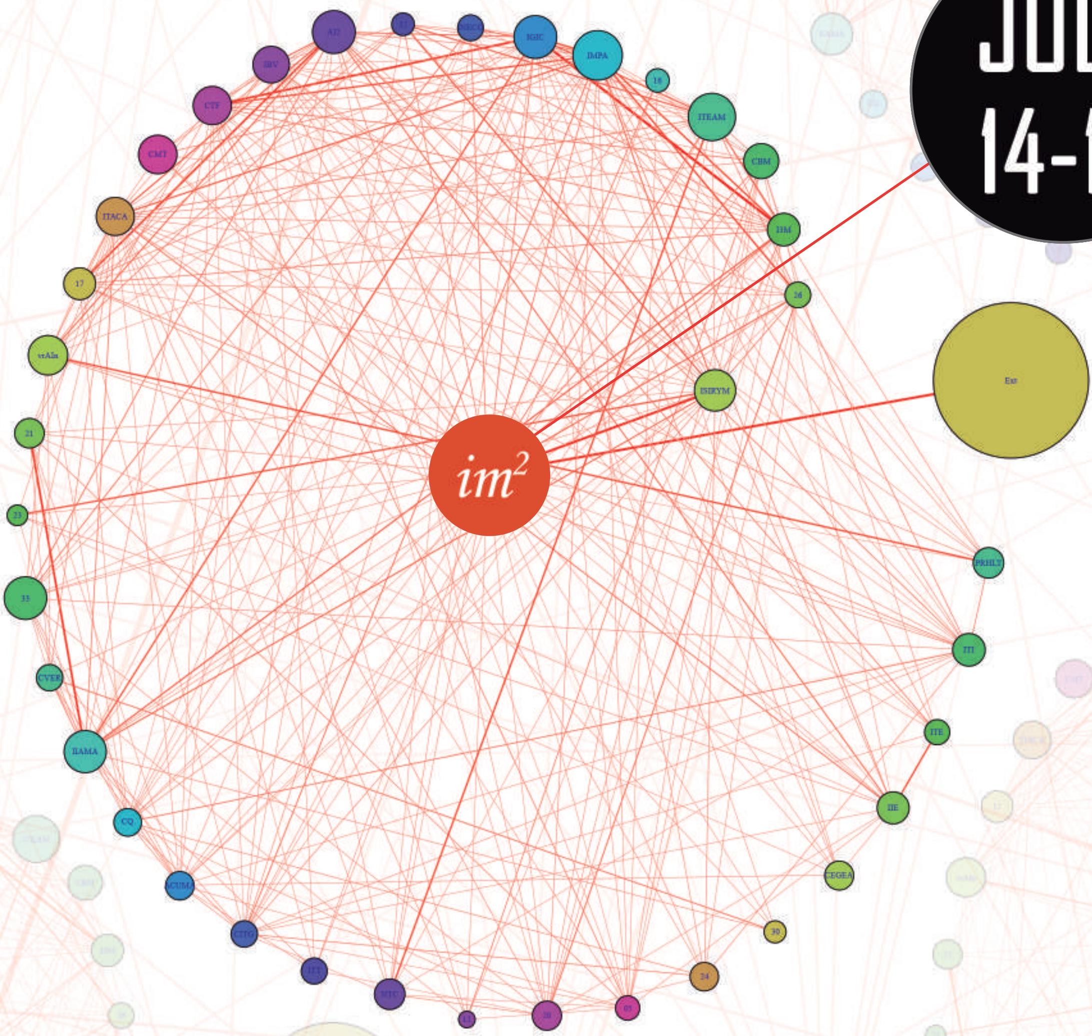

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Improved pairwise comparison transitivity using strategically selected reduced information

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1 Introduction

To judge the mutual relationship among elements, pairwise comparisons (PC) are widely used in decision modelling. PC is especially useful when the involved elements are intangible. Frequently, the number of elements to be compared may be very large. When dealing with n elements, the number of PCs is, under the reciprocity hypothesis, $n(n-1)/2$. PCs are compiled in so-called pairwise comparison matrices (PCM). In the presence of missing entries due to uncertainty or lack of information, decision-making must be performed from the available incomplete information. Making all the comparisons in the complete case may be tedious, strenuous and time-consuming for the actors, may blur the body of judgment, and produce weak priorities and unreliable decisions, thus leading to wrong and harmful conclusions. We claim that a sample of PCs involving less than that number of comparisons may be suitable to develop sound decisions. As the problem has no general solution, we analyse and solve the case in which PCs focus on comparing the elements against only a reduced number of pivotal specific elements. This case include, among others, two practical cases: the actor is more familiar with those pivotal specific elements, and the Best-Worst method [1] has been used to identify the two extreme elements in the set. The approach, developed within the linearization theory [2], is supported with rigorous Mathematics, numerical tests and examples and may be implemented using straightforward and simple computational codes.

2 Methods

Let M_n denote the set of $n \times n$ matrices. A matrix $A = (a_{ij}) \in M_n$ is said to be *reciprocal* if $a_{ij} > 0$ and $a_{ij}a_{ji} = 1$ for all $1 \leq i, j \leq n$. A matrix $B = (b_{ij}) \in M_n$ is said to be *consistent* if $b_{ij} > 0$ and $b_{ij}b_{jk} = b_{ik}$ for all $1 \leq i, j, k \leq n$. Evidently, any consistent matrix is reciprocal.
Let A be an $n \times n$ reciprocal matrix and X_A be the consistent closest matrix to A in the sense of the following distance defined in the set of positive $n \times n$ matrices: $\mathrm{d}(X, Y) = \|L(X) - L(Y)\|_F$,

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where L is the entry-wise logarithm and $\|\cdot\|_F$ is the Frobenius norm. It is proved (see [3]) that

$$X_A = E\left(\frac{1}{n}\left[(L(A)U_n) - (L(A)U_n)^T\right]\right), \tag{1}$$

where E is the entry-wise exponential, $U_n = \mathbb{1}_n\mathbb{1}_n^T$, and $\mathbb{1}_n = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$ (we shall use columns for vectors in $\mathbb{R}^n$). Observe that the Frobenius norm derives from the following scalar product defined in M_n: if $A, B \in M_n$, then $\langle A, B\rangle = \mathrm{tr}(AB^T)$, where $\mathrm{tr}(\cdot)$ is the trace operator. The following is a useful linear mapping: $\phi_n : \mathbb{R}^n \to M_n$ given by $\phi_n(\mathbf{v}) = \mathbf{v}\mathbb{1}_n^T - \mathbb{1}_n\mathbf{v}^T$. This mapping satisfies $\ker\phi_n = \mathrm{span}\{\mathbb{1}_n\}$; $A \in M_n$ is consistent if and only if $L(A) \in \mathrm{im}\,\phi_n$; and

$$\langle\phi_n(\mathbf{v}), \phi_n(\mathbf{w})\rangle = 2n\mathbf{v}^T\mathbf{w} - 2\left(\mathbf{v}^T\mathbb{1}_n\right)\left(\mathbf{w}^T\mathbb{1}_n\right), \tag{2}$$

for any $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, see [4, Teorema 9].
If $X = (x_{ij}) \in M_n$ is consistent, there exists a positive vector $\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ such that $x_{ij} = x_i/x_j$ for all indexes i, j. This vector (or a positive scalar multiple) is said to be the *priority vector* of X, and is useful to rank the alternatives. The above condition can be written as $\log x_{ij} = \log x_i - \log x_j$, which can be restated as $L(X) = \phi_n(L(\mathbf{x}))$. In this contribution we address a way to reduce in a relevant and coherent way the number of pairwise comparisons for the survey to become more friendly. If there are n alternatives, the expert has to build an entire $n \times n$ reciprocal matrix, and therefore, assuming reciprocity, produce $n(n-1)/2$ numbers. If n is large, $n(n-1)/2$ is also large and the expert can be easily tired and lose the necessary concentration. For example, if $n = 10$ (which is not very large), then $n(n-1)/2 = 45$, and a survey consisting of 45 questions may be tedious, strenous and time-consuming. In contrast, if the expert is asked to fill fewer entries, the survey will become more friendly and, arguably, more reliable.
The main idea in this contribution is to use an incomplete reciprocal matrix, obtained after comparing the elements only with two pivotal ones, to build an explicit expression of the most suitable completion.
Let us assume that the expert is asked to compare two alternatives (without loss of generality, we assume that these alternatives are the first and the second) with the remaining ones. In this case, the problem is solved with $2n - 3$ PCs (plus their symmetric, reciprocal ones). While at a lower level than in the complete case, this case involves some redundancy.
With the next theorem, we can characterize when such an incomplete $n \times n$ reciprocal matrix B admits a consistent completion.

Theorem 12. *Let $B \in M_n$ be a reciprocal incomplete matrix with known entries b_{1i}, b_{2i} for $i = 1, \ldots, n$. The matrix B has a consistent completion if and only if its first and second columns are proportional. In this case, the completion is unique and its (i, j) entry is $b_{i1}b_{1j}$ for all i, j.*

We can establish the main result in this contribution:

Theorem 13. *Let $B \in M_n$ be a reciprocal incomplete matrix with known entries b_{1i}, b_{2i} for $i = 1, \ldots, n$.*

1. *There is a unique reciprocal completion of B, say D, such that $\mathrm{d}(D, \mathcal{C}_n) \leq \mathrm{d}(D', \mathcal{C}_n)$ for all $D' \in M_n$ reciprocal completion of B.*
2. *There is a unique $Z \in \mathcal{C}_n$ such that $\mathrm{d}(D, Z) = \mathrm{d}(D, \mathcal{C}_n)$.*
3. *$Z = E[\phi_n(\mathcal{L}^\dagger\mathcal{Q}\boldsymbol{\rho})]$, where $\boldsymbol{\rho} = (\log b_{12}, \ldots, \log b_{1n}, \log b_{21}, \log b_{23}, \ldots, \log b_{2n})^T$, matrix $\mathcal{Q}$ is the incidence matrix associated to the graph of B, and $\mathcal{L} = \mathcal{Q}\mathcal{Q}^T$.*
4. *If (i, j) is an unknown entry of B, then the (i, j) entry of D and Z are equal.*

It is not difficult to give a simple expression for $\mathcal{L}^{\dagger}\mathcal{Q}$ (omitted here), what makes all the calculations straightforward.
With the above considerations we may prove the following calculation practical result.

Theorem 14. *Let $B \in M_n$ be a reciprocal incomplete matrix with known entries b_{1i}, b_{2i} for $i = 1, \ldots, n$. Under the notation of Theorem 13, one has $Z = E[\phi_n(\mathbf{w})]$, where*

$$\mathbf{w} = \frac{1}{n+2}\begin{bmatrix} 2\log b_{12} + (s_1 - s_2)/2 \\ 2\log b_{21} + (s_2 - s_1)/2 \\ -(\log b_{13} + \log b_{23})/2 \\ \vdots \\ -(\log b_{1n} + \log b_{2n})/2 \end{bmatrix},$$

where $s_1 = \sum_{i=3}^{n} \log b_{1i}$ and $s_2 = \sum_{i=3}^{n} \log b_{2i}$.

We can see that $n(n-1)/2$ comparisons are required to build a complete $n \times n$ comparison matrix and only $2n - 3$ comparisons if we use Theorem 14.
The involved numerical operations to find vector $\mathbf{w}$ in Theorem 14 (and $Z = E[\phi_n(\mathbf{w})]$) are elemental (for instance, neither linear systems are solved, nor pseudoinverses computed).

3 Results

The BW method is a multi-criteria decision-making method proposed by J. Rezaei, [1]. In this method, two elements are selected by the expert: the best (the more influential) and the worst (the less influential). The expert gives the preferences of the best element over all the other elements using numbers between 1 to 9, which can be stored in the vector $\mathbf{m} = (m_1, \ldots, m_n)^T$. Analogously for the worst element, producing $\mathbf{p} = (p_1, \ldots, p_n)^T$, being $p_i \in \{1, \ldots, 9\}$ the preference of the i-th element over the worst. The last step is to find the optimal weights $\mathbf{w} = (w_1, \ldots, w_n)^T$ that are the solution of the following problem

$$\begin{aligned} &\min \max_j \left\{ \left| \frac{w_B}{w_j} - m_j \right|, \left| \frac{w_j}{w_W} - p_j \right| \right\} \\ &\text{such that} \\ &\qquad w_i > 0 \text{ for all } j \text{ and } w_1 + \cdots + w_n = 1, \end{aligned} \tag{3}$$

where w_B is the B-th component of $\mathbf{w}$ corresponding to the position of the best element, and similarly for w_W. From now on, without loss of generality, we will assume that the best element is the first one and the worst element is the second one. With this assumption, it is evident that $m_1 = p_2 = 1$ and $m_2 = p_1$.
As we can easily see, in the BW method and in the method described in Theorem 14, there are two outstanding elements. In [1, Definition 3] it is defined when a comparison given in the BW method is *fully consistent*: it must satisfy $m_j p_j = m_2$ for all j. We can show that this concept is equivalent to the characterization given in Theorem 12.

Theorem 15. *Let $\mathbf{m}, \mathbf{p} \in \mathbb{R}^n$ be the two vectors given by the BW method. Let*

$$B = \begin{bmatrix} m_1 & m_2 & m_3 & \cdots & m_n \\ 1/p_1 & 1/p_2 & 1/p_3 & \cdots & 1/p_n \\ 1/m_3 & p_3 & \star & \cdots & \star \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1/m_n & p_n & \star & \cdots & \star \end{bmatrix}, \tag{4}$$

an incomplete reciprocal matrix. Then the criterion given in Theorem 12 holds if and only if the BW method is fully consistent.

We can also show that if we change the optimization function of the problem established in (3), then the BW method and the method provided in Theorem 14 are the same.
Finally, we can empirically test the main result given in the Method's Section. To this end, we employ the following pseudocode

1. We generate r random reciprocal matrices $n \times n$. Let $A_1, \ldots, A_r$ be these matrices.

2. We compute the consistent matrix closest to each matrix A_i using (1). Let $X_1, \ldots, X_r$ be these matrices.

3. Since each X_i is consistent, its priority vector is any normalised column. Let b_i and w_i the position of the largest and smallest, respectively, component of the priority vector of X_i.

4. We apply the main result of Section 4 to matrix A_i considering the pivotal elements b_i and w_i obtaining Z_i.

5. We calculate the (arithmetic) mean of $\mathrm{d}(X_i, Z_i)$.

By executing this pseudocode for $r = 1000$ and $n = 10$ we get numbers around 0.01. By selecting in a concrete way the "best" and the "worst" criteria we get very small numbers.

4 Conclusions

The large number of alternatives and multiple conflicting goals to be considered make decision-making increasingly complex. Many studies in the literature address comparing alternatives in large databases. It is reasonable to be inconsistent When considering a large number of options. Several MCDM methods rely on pairwise comparisons between elements. However, if the number n of elements for pairwise comparison is large, the number of comparisons, $n(n-1)/2$ is large too, and an individual making the comparisons may become easily bored, tired, and eventually lose concentration, thus leading to wrong decisions. Insisting in issuing every single comparison will be of no help. If the actor is instead asked to perform fewer comparisons, the survey will be friendlier, and, most importantly, the issued opinions more reliable.
In this contribution, we have addressed the provision of fewer PCs in a decision-making problem, while still producing sound decision-making. We think there is not a general solution to the problem of finding an optimal sample of PCs to be issued so that card(sample) $< n(n-1)/2$ and still produce a sound DM. This seems to be especially true if we focus the problem from too general a perspective. There is a trivial solution, providing a lower bound for the sample size: just produce $n-1$ PCs, since it is equivalent to directly giving the priority vector.
Among many other possibilities, we have concentrated on considering a reduced number of elements that the expert is familiar with, and issue comparisons only on those elements with all the others. This case is very frequent in the applications, since experts sometimes do not have the same degree of acquaintance with all the elements under comparison. Only $2n-3$ comparisons, using Theorem 14, would be asked, instead of the $n(n-1)/2$ comparisons required to build a complete $n \times n$ comparison matrix. Additionally, we have considered the BW method [1] and shown the close relationship with the approach herein developed.
Further studies could consider other efficient ways of selecting representative samples of PCs and mechanisms to prove that efficiency.

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