On Compound and Iterated Conditionals

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Abstract

We illustrate the notions of compound and iterated conditionals introduced, in recent papers, as suitable conditional random quantities, in the framework of coherence. We motivate our definitions by examining some concrete examples. Our logical operations among conditional events satisfy the basic probabilistic properties valid for unconditional events. We show that some, intuitively acceptable, compound sentences on conditionals can be analyzed in a rigorous way in terms of suitable iterated conditionals. We discuss the Import-Export principle, which is not valid in our approach, by also examining the inference from a material conditional to the associated conditional event. Then, we illustrate the characterization, in terms of iterated conditionals, of some well known p-valid and non p-valid inference rules.

Keywords: Coherence, Conditional events, Conditional random quantities, Conjunction, Disjunction, Iterated conditional, Inference rules, p-validity, p-entailment, Import-Export principle.

1. Introduction

Let us imagine an experiment where you flip a coin twice; then, let us consider the conjunction

"*the outcome of the 1st flip is head and the outcome of the 2nd flip is head*".

By defining the events A = "*the outcome of the 1st flip is head*" and B = "*the outcome of the 2nd flip is head*", we denote by $A \wedge B$, or simply by AB, the previous conjunction, which is true when both A and B are true, and false when A or B is false. If you judge $P(AB) = p$, then in a bet on AB you agree to pay, for instance, p by receiving 1, or 0, according to whether AB turns out to be *true*, or *false*, respectively.

What is the "logical value" of AB when the outcome of the first coin is head and the second coin stands up? We cannot answer because the event B is neither true nor false.

Moreover, what happens of the bet? Cases like this are not considered when assessing $P(AB)$ (they are assumed to be impossible, or to have zero probability).

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Usually, the bet is called off and you receive back the paid amount p. Actually, by introducing the events H =*the outcome of the 1*[st] *flip is head or tail* and K =*the outcome of the 2*[nd] *flip is head or tail*, we realize that when evaluating $P(AB)$, in fact we are evaluating $P(AB|HK)$, under the implicit assumption that $P(HK) = 1$. Indeed, by observing that $P(AB|\overline{HK}) = 0$, it follows that

$$P(AB) = P(AB|HK)P(HK) + P(AB|\overline{HK})P(\overline{HK}) = P(AB|HK)P(HK),$$

and when $P(HK) = 1$ it holds that $P(AB) = P(AB|HK)$. On the contrary, when $P(HK) < 1$, one has $P(AB) < P(AB|HK)$, in which case the paid amount $P(AB)$ is less than the amount (that should be paid) $P(AB|HK)$. Moreover,as $\Omega = HK \vee H\overline{K} \vee \overline{H}K \vee \overline{H}\overline{K} = HK \vee \overline{HK}$, the event $\overline{HK}$ is the disjunction of three logical cases, that is $\overline{HK} = H\overline{K} \vee \overline{H}K \vee \overline{H}\overline{K}$, and such cases could be judged to be not similar. In particular, $\overline{H}\overline{K}$ appears different from the other two cases. What should be a general approach to this kind of more complex situations? We observe that, in order to manage these cases, the two sentences

the outcome of the 1[st] *flip is head*,

the outcome of the 2[nd] *flip is head*

should be written, respectively, as the conditional sentences

the outcome of the 1[st] *flip is head, given that it is head or tail*,
the outcome of the 2[nd] *flip is head, given that it is head or tail*;

that is, the events A, B should be replaced by the conditional events $A|H$, $B|K$. Moreover, the conjunction AB should be written as a suitable conjoined conditional $(A|H) \wedge (B|K)$. Based on the theories of de Finetti (1936, 1980) and Ramsey (1990), we look at the conditional *if H then A* as the conditional event $A|H$, hence in our approach it is satisfied *the Equation* (Edgington 1995), or *Conditional Probability Hypothesis* (see, e.g., Sanfilippo, Pfeifer, Over et al. 2018; Sanfilippo, Gilio, Over et al. 2020; Cruz 2020; Over and Cruz 2021), which establishes that the probability of a conditional coincides with the probability of the associated conditional event: $P(\textit{if } H \textit{ then } A) = P(A|H)$. Then, the conditional events $A|H$ and $B|K$ are associated with the following two conditionals:

1) *if the outcome of the 1*[st] *flip is head or tail, then it is head,*
2) *if the outcome of the 2*[nd] *flip is head or tail, then it is head.*

Moreover, by defining *valid* the flip when "the coin does not stand, or similar things", that is when "the outcome of the flip is head or tail", the conjunction $(A|H) \wedge (B|K)$ can be interpreted as the **conjoined conditional**

(if the outcome of the 1[st] *flip is head or tail, then it is head)* and *(if the outcome of the 2*[nd] *flip is head or tail, then it is head*.

What are the possible values of this conjoined conditional $(A|H) \wedge (B|K)$? The same analysis can be done for the disjunction $(A|H) \vee (B|K)$.

The problem of suitably defining logical operations among conditional events has been largely discussed by many authors (see, e.g., Baratgin, Politzer, Over et al. 2018; Benferhat, Dubois and Prade 1997; Coletti, Scozzafava and Vantaggi

2013, 2015; Douven, Elqayam, Singmann et al. 2019; Flaminio, Godo and Hosni 2020; Goodman, Nguyen and Walker 1991; Kaufmann 2009; Mura 2011; McGee 1989; Nguyen and Walker 1994). In a pioneering paper, written in 1935, de Finetti (1936) proposed a three-valued logic for conditional events, also studied by Lukasiewicz. Moreover, several authors have given many contributions to research on three-valued logics and compounds of conditionals (see, e.g.,Edgington 1995; McGee 1989; Milne 1997). Coherence-based probability logic has been recently discussed in Pfeifer 2021.

Usually, conjunction and disjunction of conditionals have been defined as suitable conditionals (see, e.g., Adams 1975; Calabrese 1987, 2017; Ciucci and Dubois 2012, 2013; Goodman, Nguyen and Walker 1991). However, in this way many classical probabilistic properties are lost. In particular, differently from the case of unconditional events, the lower and upper probability bounds for the conjunction of two conditional events are no more the Fréchet-Hoeffding bounds. This aspect has been studied in Sanfilippo 2018.

A more general approach where the result of conjunction or disjunction is no longer a three-valued object has been given in Kaufmann 2009; McGee 1989. In recent years, a related theory (Gilio and Sanfilippo 2013a, 2013b, 2014, 2017, 2019, 2020), with some applications (Gilio, Over, Pfeifer et al. 2017; Gilio, Pfeifer and Sanfilippo 2020; Sanfilippo, Gilio, Over et al. 2020; Sanfilippo, Pfeifer and Gilio 2017; Sanfilippo, Pfeifer, Over et al. 2018), has been developed in the setting of coherence, where conditioning events with zero probability are properly managed. In these papers the notions of compound and iterated conditionals are defined as suitable *conditional random quantities* with a finite number of possible values in the interval $[0,1]$. Within our approach the basic properties, valid for unconditional events, are preserved. In particular:

- the inequality $AB \leq \min\{A,B\}$ becomes $(A|H) \wedge (B|K) \leq \min\{A|H, B|K\}$ (the inequality $A \vee B \geq \max\{A,B\}$ becomes $(A|H) \vee (B|K) \geq \max\{A|H, B|K\}$);
- the Fréchet-Hoeffding lower and upper prevision bounds for the conjunction of conditional events still hold (Gilio and Sanfilippo 2021a);
- De Morgan's Laws are satisfied (Gilio and Sanfilippo 2019);
- the inclusion-exclusion formula for the disjunction of conditional events is valid (Gilio and Sanfilippo 2020); for instance, the formula $P(E_1 \vee E_2) = P(E_1) + P(E_2) - P(E_1E_2)$ becomes $\mathbb{P}[(E_1|H_1) \vee (E_2|H_2)] = P(E_1|H_1) + P(E_2|H_2) - \mathbb{P}[(E_1|H_1) \wedge (E_2|H_2)]$ (Gilio and Sanfilippo 2014);
- we can introduce the set of (conditional) constituents, with properties analogous to the case of unconditional events (Gilio and Sanfilippo 2020);
- by exploiting conjunction we obtain a characterization of the probabilistic entailment of Adams (Adams 1975) for conditionals (Gilio and Sanfilippo 2019); moreover, by exploiting iterated conditionals, the p-entailment of $E_3|H_3$ from a p-consistent family $\{E_1|H_1, E_2|H_2\}$ is characterized by the property that the iterated conditional $(E_3|H_3)|((E_1|H_1)\wedge(E_2|H_2))$ is constant and coincides with 1 (Gilio, Pfeifer and Sanfilippo 2020).

In our theory of compound and iterated conditionals, as in Adams 1975, Kaufmann 2009 and differently from McGee 1989, the Import-Export principle is not valid. As a consequence, as shown in Gilio and Sanfilippo 2014 (see also Sanfilippo, Gilio, Over et al. 2020; Sanfilippo, Pfeifer, Over et al. 2018), we avoid Lewis' triviality results (Lewis 1976). Probabilistic modus ponens has been generalized to conditional events (Sanfilippo, Pfeifer and Gilio 2017); moreover, one-premise and two-premise centering inferences has been studied in Gilio, Over, Pfeifer et al. 2017; Sanfilippo, Pfeifer, Over et al. 2018. In Sanfilippo, Gilio, Over et al. 2020 some intuitive probabilistic assessments discussed in Douven and Dietz 2011 have been explained, by making some implicit background information explicit.

The paper is organized as follows: In Section 2. we recall some basic notions and results on coherence, conditional events, and conditional random quantities. Moreover, we recall the definitions of p-consistency and p-entailment in the setting of coherence. Then we illustrate the notions of conjoined, disjoined and iterated conditional. In Section 3. we recall some well known probabilistic properties valid for unconditional events. Then, we show that these properties continue to hold when replacing events by conditional events. In Section 4. we show that some compound sentences on conditionals, which seem intuitively acceptable, can be analyzed in a rigorous way in terms of iterated conditionals. Moreover, we discuss the Import-Export principle, by also examining the iterated conditional $(A|H)|(\overline{H} \vee A)$. Then we illustrate, in terms of suitable iterated conditionals, several well known, p-valid and non p-valid, inference rules. In Section 5. we give some conclusions.

2. Preliminary Notions and Results

In our approach events represent uncertain facts described by (non ambiguous) logical propositions. An event A is a two-valued logical entity which is either *true*, or *false*. The indicator of an event A is a two-valued numerical quantity which is 1, or 0, according to whether A is true, or false, respectively. We use the same symbol to refer to an event and its indicator. We denote by Ω the sure event and by $\emptyset$ the impossible one (notice that, when necessary, the symbol $\emptyset$ will denote the empty set). Given two events A and B, we denote by $A \wedge B$, or simply by AB, the intersection, or conjunction, of A and B, as defined in propositional logic; likewise, we denote by $A \vee B$ the union, or disjunction, of A and B. We denote by $\overline{A}$ the negation of A. Of course, the truth values for conjunctions, disjunctions and negations are defined as usual. Given any events A and B, we simply write $A \subseteq B$ to denote that A logically implies B, that is $A\overline{B} = \emptyset$, which means that A and $\overline{B}$ cannot both be true.

2.1 Conditional Events, Coherence, and Conditional Random Quantities

Given two events E, H, with $H \neq \emptyset$, the conditional event $E|H$ is defined as a three-valued logical entity which is *true*, or *false*, or *void*, according to whether EH is true, or $\overline{E}H$ is true, or $\overline{H}$ is true, respectively. In the betting framework, to assess $P(E|H) = x$ amounts to say that, for every real number s, you are willing to pay an amount sx and to receive s, or 0, or sx, according to whether EH is true, or $\overline{E}H$ is true, or $\overline{H}$ is true (bet called off), respectively. Then for the random gain $G = sH(E - x)$, the possible values are $s(1 - x)$, or $-sx$, or 0, according to whether EH is true, or $\overline{E}H$ is true, or $\overline{H}$ is true, respectively. More generally speaking, consider a real-valued function $P : \mathcal{K} \to \mathbb{R}$, where $\mathcal{K}$ is an arbitrary (possibly not finite) family of conditional events. Let $\mathcal{F} = \{E_1|H_1, \ldots, E_n|H_n\}$ be a family of conditional events, where $E_i|H_i \in \mathcal{K}$, $i = 1,\dots,n$, and let $\mathcal{P} = (p_1,\dots,p_n)$ be the vector of values $p_i = P(E_i|H_i)$, where $i = 1,\dots,n$. We denote by $\mathcal{H}_n$ the disjunction $H_1 \vee \cdots \vee H_n$. With the pair $(\mathcal{F},\mathcal{P})$ we associate the random gain $G = \sum_{i=1}^{n} s_i H_i(E_i - p_i)$, where $s_1,\dots,s_n$ are n arbitrary real numbers. G represents the net gain of n transactions. Let $\mathcal{G}_{\mathcal{H}_n}$ denote the set of possible values of G restricted to $\mathcal{H}_n$, that is, the values of G when at least one conditioning event is true.

Definition 1

The function P defined on $\mathcal{K}$ is coherent *if and only if, for every integer n, for every finite subfamily $\mathcal{F} = \{E_1|H_1,\dots,E_n|H_n\}$ of $\mathcal{K}$ and for every real numbers $s_1,\dots,s_n$, it holds that:* $\min \mathcal{G}_{\mathcal{H}_n} \leq 0 \leq \max \mathcal{G}_{\mathcal{H}_n}$.

Intuitively, Definition 1 means in betting terms that a probability assessment is coherent if and only if, in any finite combination of n bets, it cannot happen that the values in $\mathcal{G}_{\mathcal{H}_n}$ are all positive, or all negative (*no Dutch Book*).

We denote by X a *random quantity*, that is an uncertain real quantity, which has a well determined but unknown value. We assume that X has a finite set of possible values. Given any event $H \neq \emptyset$, agreeing to the betting metaphor, if you assess that the prevision of "*X conditional on H*" (or short: "*X given H*"), $\mathbb{P}(X|H)$, is equal to μ, this means that for any given real number s you are willing to pay an amount μs and to receive sX, or μs, according to whether H is true, or false (bet called off), respectively. In particular, when X is (the indicator of) an event A, then $\mathbb{P}(X|H) = P(A|H)$. Definition 1 can be generalized to the case of prevision assessments on a family of conditional random quantities (see, e.g., Gilio and Sanfilippo 2020). Given a conditional event $A|H$ with $P(A|H) = x$, the indicator of $A|H$, denoted by the same symbol, is

$$A|H = AH + x\overline{H} = AH + x(1-H) = \begin{cases} 1, & \text{if } AH \text{ is true,} \\ 0, & \text{if } \overline{A}H \text{ is true,} \\ x, & \text{if } \overline{H} \text{ is true.} \end{cases} \tag{1}$$

The third value of the random quantity $A|H$ (subjectively) depends on the assessed probability $P(A|H) = x$. When $H \subseteq A$ (i.e., $AH = H$), it holds that $P(A|H) = 1$; then, for the indicator $A|H$ it holds that

$$A|H = AH + x\overline{H} = H + \overline{H} = 1, \text{ (when } H \subseteq A). \tag{2}$$

Likewise, if $AH = \emptyset$, it holds that $P(A|H) = 0$; then

$$A|H = 0 + 0\overline{H} = 0, \text{ (when } AH = \emptyset).$$

Given a random quantity X and an event $H \neq \emptyset$, with $P(X|H) = \mu$, in our approach, likewise formula (1), the conditional random quantity $X|H$ is defined as

$$X|H = XH + \mu\overline{H}.$$

(For a discussion on this extended notion of a conditional random quantity see, e.g., Gilio and Sanfilippo 2014; Sanfilippo, Gilio, Over et al. 2020.)

Remark 1

Given a conditional random quantity $X|H$ and a prevision assessment $\mathbb{P}(X|H) = \mu$, if conditionally on H being true X is constant, say $X = c$, then by coherence $\mu = c$.

The result below establishes some conditions under which two conditional random quantities $X|H$ and $Y|K$ coincide (Gilio and Sanfilippo 2014, Theorem 4).

Theorem 1

Given any events $H \neq \emptyset$ and $K \neq \emptyset$, and any random quantities X and Y, let Π be the set of the coherent prevision assessments $\mathbb{P}(X|H) = \mu$ and $\mathbb{P}(Y|K) = \nu$.

(i) Assume that, for every $(\mu, \nu) \in \Pi$, the values of $X|H$ and $Y|K$ always coincide when $H \vee K$ is true; then $\mu = \nu$ for every $(\mu, \nu) \in \Pi$.

(ii) For every $(\mu, \nu) \in \Pi$, the values of $X|H$ and $Y|K$ always coincide when $H \vee K$ is true if and only if $X|H = Y|K$.

2.2 Probabilistic Consistency and Entailment

We recall below the notion of logical implication of Goodman and Nguyen 1988 for conditional events (see also Gilio and Sanfilippo 2013d).

Definition 2

Given two conditional events $A|H$ and $B|K$ we define that $A|H$ logically implies *$B|K$ (denoted by $A|H \subseteq B|K$) if and only if AH logically implies BK and $\bar{B}K$ logically implies $\bar{A}H$; i.e., $AH \subseteq BK$ and $\bar{B}K \subseteq \bar{A}H$.*

A generalization of the Goodman and Nguyen logical implication to conditional random quantities has been given in Pelessoni and Vicig 2014.

The notions of p-consistency and p-entailment of Adams 1975 were formulated for conditional events in the setting of coherence in Gilio and Sanfilippo 2013d (see also Biazzo, Gilio, Lukasiewicz et al. 2005; Gilio 2002; Gilio and Sanfilippo 2013c).

Definition 3

Let $\mathcal{F}_n = \{E_i|H_i\,,\ i = 1, \ldots, n\}$ be a family of n conditional events. Then, $\mathcal{F}_n$ is p-consistent *if and only if the probability assessment $(p_1, p_2, \ldots, p_n) = (1, 1, \ldots, 1)$ on $\mathcal{F}_n$ is coherent.*

Definition 4

A p-consistent family $\mathcal{F}_n = \{E_i|H_i\,,\ i = 1, \ldots, n\}$ p-entails *a conditional event $E|H$ (denoted by $\mathcal{F}_n \Rightarrow_p E|H$) if and only if for any coherent probability assessment $(p_1, \ldots, p_n, z)$ on $\mathcal{F}_n \cup \{E|H\}$ it holds that: if $p_1 = \cdots = p_n = 1$, then $z = 1$.*

Of course, when $\mathcal{F}_n$ p-entails $E|H$, there may be coherent assessments $(p_1, \dots, p_n, z)$ with $z \neq 1$, but in such cases $p_i \neq 1$ for at least one index i. We say that the inference from a p-consistent family $\mathcal{F}_n$ to $E|H$ is *p-valid* if and only if $\mathcal{F}_n$ p-entails $E|H$.

We also recall the characterization of the p-entailment for two conditional events (Gilio and Sanfilippo 2013d, Theorem 7):

Theorem 2

Given two conditional events $A|H$, $B|K$, with $AH \neq \emptyset$. It holds that

$$A|H \Rightarrow_p B|K \Longleftrightarrow A|H \subseteq B|K, \text{ or } K \subseteq B \Longleftrightarrow \Pi \subseteq \{(x,y) \in [0,1]^2 : x \leq y\},$$

where Π is the set of coherent assessments (x,y) on $\{A|H, B|K\}$.

2.3 Conjunction, Disjunction, and Iterated Conditioning

Given two conditional events $A|H$ and $B|K$, the associated constituents, denoted $C_1, \dots, C_8, C_0$ in Table 1, are the conjunctions of the logical disjunction in the formula below.

$$\Omega = (AH \vee \bar{A}H \vee \bar{H}) \wedge (BK \vee \bar{B}K \vee \bar{K}) = AHBK \vee AH\bar{B}K \vee \cdots \vee \bar{H}\bar{K}.$$

	C_h	$A\|H$	$B\|K$	$\max\{A\|H + B\|K - 1, 0\}$	$(A\|H) \wedge (B\|K)$	$\min\{A\|H, B\|K\}$
C_1	$AHBK$	1	1	1	1	1
C_2	$AH\bar{B}K$	1	0	0	0	0
C_3	$AH\bar{K}$	1	y	y	y	y
C_4	$\bar{A}HBK$	0	1	0	0	0
C_5	$\bar{A}H\bar{B}K$	0	0	0	0	0
C_6	$\bar{A}H\bar{K}$	0	y	0	0	0
C_7	$\bar{H}BK$	x	1	x	x	x
C_8	$\bar{H}\bar{B}K$	x	0	0	0	0
C_0	$\bar{H}\bar{K}$	x	y	$\max\{x+y-1, 0\}$	z	$\min\{x, y\}$

Table 1: *Possible values of* $\max\{A|H + B|K - 1, 0\}$, $(A|H) \wedge (B|K)$, *and* $\min\{A|H, B|K\}$, *associated with the constituents* $C_1, \dots, C_8, C_0$, *where* $x = P(A|H)$, $y = P(B|K)$, *and* $z = \mathbb{P}[(A|H) \wedge (B|K)]$.

We recall now the notion of conjoined conditionals which was introduced in the framework of conditional random quantities (Gilio and Sanfilippo 2013b; Gilio and Sanfilippo 2013a; Gilio and Sanfilippo 2014; Gilio and Sanfilippo 2019). Given a coherent probability assessment (x,y) on $\{A|H, B|K\}$, we consider the random quantity $AHBK + x\bar{H}BK + y\bar{K}AH$ and we set $\mathbb{P}[(AHBK + x\bar{H}BK + y\bar{K}AH)|(H \vee K)] = z$. Then we define the conjunction $(A|H) \wedge (B|K)$ as follows:

Definition 5

Given a coherent prevision assessment $P(A|H) = x$, $P(B|K) = y$, and $\mathbb{P}[(AHBK + x\bar{H}BK + y\bar{K}AH)|(H \vee K)] = z$, the conjunction $(A|H) \wedge (B|K)$

is the conditional random quantity defined as

$$(A|H)\wedge(B|K) = (AHBK + x\overline{H}BK + y\overline{K}AH)|(H\vee K) = \\ = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } \overline{A}H \vee \overline{B}K \text{ is true,} \\ x, & \text{if } \overline{H}BK \text{ is true,} \\ y, & \text{if } AH\overline{K} \text{ is true,} \\ z, & \text{if } \overline{H}\,\overline{K} \text{ is true.} \end{cases} \tag{3}$$

Of course, $\mathbb{P}[(A|H)\wedge(B|K)] = z$. Notice that, once the (coherent) assessment (x,y,z) is given, the conjunction $(A|H)\wedge(B|K)$ is (subjectively) determined. We recall that, in betting terms, z represents the amount you agree to pay, with the proviso that you will receive the quantity

$$(A|H)\wedge(B|K) = AHBK + x\overline{H}BK + y\overline{K}AH + z\overline{H}\,\overline{K}, \tag{4}$$

which assumes one of the following values:

- 1, if both conditional events are true;
- 0, if at least one of the conditional events is false;
- the probability of the conditional event that is void, if one conditional event is void and the other one is true;
- z (the amount that you paid), if both conditional events are void.

We observe that $(A|H)\wedge(A|H) = A|H$ and $(A|H)\wedge(B|K) = (B|K)\wedge(A|H)$. Moreover, if $H = K$, then

$$(A|H)\wedge(B|H) = AB|H. \tag{5}$$

Indeed, in this case $\overline{H}BK = AH\overline{K} = \emptyset$, so that by Definition 5 it holds that $z = \mathbb{P}(ABH|H) = P(AB|H)$ and $(A|H)\wedge(B|H) = ABH|H = AB|H$. The result below shows that Fréchet-Hoeffding bounds still hold for the conjunction of two conditional events (Gilio and Sanfilippo 2014, Theorem 7).

Theorem 3

Given any coherent assessment (x,y) on $\{A|H, B|K\}$, with A,H,B,K logically independent, and with $H\neq\emptyset, K\neq\emptyset$, the extension $z = \mathbb{P}[(A|H)\wedge(B|K)]$ is coherent if and only if the following Fréchet-Hoeffding bounds are satisfied:

$$\max\{x+y-1,0\} = z' \leq z \leq z'' = \min\{x,y\}. \tag{6}$$

Remark 2

Notice that, from (3) and (6), it holds that (see Table 1)

$$\max\{A|H + B|K - 1, 0\} \leq (A|H)\wedge(B|K) \leq \min\{A|H, B|K\}. \tag{7}$$

Then, when $AH = \emptyset$, it holds that $A|H = 0$ and $(A|H)\wedge(B|K) = 0\wedge(B|K) = 0$. Moreover, when $K \subseteq B$, it holds that $B|K = 1$ and $(A|H)\wedge(B|K) = (A|H)\wedge 1 = A|H$.

We recall now the notion of disjoined conditional. Given a coherent probability assessment (x,y) on $\{A|H,B|K\}$ we consider the random quantity $(AH\vee BK)+x\overline{H}\overline{B}K+y\overline{K}\overline{A}H$ and we set $\mathbb{P}[((AH\vee BK)+x\overline{H}\overline{B}K+y\overline{K}\overline{A}H)|(H\vee K)] = w$. Then we define the disjunction $(A|H)\vee(B|K)$ as follows:

Definition 6

Given a coherent prevision assessment $P(A|H)=x$, $P(B|K)=y$, and $\mathbb{P}[((AH\vee BK)+x\overline{H}\overline{B}K+y\overline{K}\overline{A}H)|(H\vee K)]=w$, the disjunction $(A|H)\vee(B|K)$ is the conditional random quantity defined as

$$(A|H)\vee(B|K) = ((AH\vee BK)+x\overline{H}\overline{B}K+y\overline{K}\overline{A}H)|(H\vee K) = \\ = \begin{cases} 1, & \text{if } AH\vee BK \text{ is true,} \\ 0, & \text{if } \overline{A}H\overline{B}K \text{ is true,} \\ x, & \text{if } \overline{H}\overline{B}K \text{ is true,} \\ y, & \text{if } \overline{A}H\overline{K} \text{ is true,} \\ w, & \text{if } \overline{H}\overline{K} \text{ is true.} \end{cases} \tag{8}$$

Remark 3

Given any coherent assessment (x,y) on $\{A|H,B|K\}$, with A,H,B,K logically independent, and with $H\neq\emptyset, K\neq\emptyset$, the extension $w=\mathbb{P}[(A|H)\vee(B|K)]$ is coherent if and only if (Gilio and Sanfilippo 2014, Section 6)

$$\max\{x,y\} = w' \leq w \leq w'' = \min\{1,x+y\}. \tag{9}$$

Notice that, from (8) and (9), it holds that

$$\max\{A|H,B|K\} \leq (A|H)\vee(B|K) \leq \min\{1, A|H+B|K\}. \tag{10}$$

Then, when $AH=\emptyset$, it holds that $A|H=0$ and $(A|H)\vee(B|K)=0\vee(B|K)=B|K$. Moreover, when $K\subseteq B$, it holds that $B|K=1$ and $(A|H)\vee(B|K)=(A|H)\vee 1=1$.

We recall that, as defined in (1), the indicator of a conditional event $A|H$, with $P(A|H)=x$, is

$$A|H = A\wedge H + x\overline{H}.$$

Likewise, we define the notion of an iterated conditional based on the same structure, i.e. $\Box|\bigcirc = \Box\wedge\bigcirc+\mathbb{P}(\Box|\bigcirc)\overline{\bigcirc}$, where $\Box$ denotes $B|K$ and $\bigcirc$ denotes $A|H$, and where we set $\mathbb{P}(\Box|\bigcirc)=\mu$. In the framework of subjective probability $\mu=\mathbb{P}(\Box|\bigcirc)$ is the amount that you agree to pay, by knowing that you will receive the random quantity $\Box\wedge\bigcirc+\mu\overline{\bigcirc}$. The negation $\overline{A|H}$ of $A|H$ is defined as $1-A|H=\overline{A}|H$. Then, the iterated conditional $(B|K)|(A|H)$ is defined (see, e.g., Gilio and Sanfilippo 2013b; Gilio and Sanfilippo 2013a; Gilio and Sanfilippo 2014) as follows:

Definition 7

Given any pair of conditional events $A|H$ and $B|K$, with $AH\neq\emptyset$, let (x,y,z) be a coherent assessment on $\{A|H,B|K,(A|H)\wedge(B|K)\}$. The iterated conditional

$(B|K)|(A|H)$ *is defined as*

$$(B|K)|(A|H) = (B|K) \wedge (A|H) + \mu \bar{A}|H = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ y, & \text{if } AH\bar{K} \text{ is true,} \\ x + \mu(1-x), & \text{if } \bar{H}BK \text{ is true,} \\ \mu(1-x), & \text{if } \bar{H}\bar{B}K \text{ is true,} \\ z + \mu(1-x), & \text{if } \bar{H}\bar{K} \text{ is true,} \\ \mu, & \text{if } \bar{A}H \text{ is true,} \end{cases} \qquad (11)$$

where $\mu = \mathbb{P}[(B|K)|(A|H)] = \mathbb{P}[(B|K) \wedge (A|H) + \mu \bar{A}|H]$.

Notice that we assume $AH \neq \emptyset$ to avoid trivial cases of iterated conditionals. By the linearity of prevision, it holds that

$$\begin{aligned} \mu &= \mathbb{P}((B|K)|(A|H)) = \mathbb{P}((B|K) \wedge (A|H) + \mu \bar{A}|H) = \\ &= \mathbb{P}((B|K) \wedge (A|H)) + \mathbb{P}(\mu \bar{A}|H) = \\ &= \mathbb{P}((B|K) \wedge (A|H)) + \mu P(\bar{A}|H) = z + \mu(1-x), \end{aligned}$$

from which it follows that (Gilio and Sanfilippo 2013a)

$$z = \mathbb{P}((B|K) \wedge (A|H)) = \mu x = \mathbb{P}((B|K)|(A|H))P(A|H), \qquad (12)$$

and $\mu = \mathbb{P}((B|K)|(A|H)) = \frac{\mathbb{P}((B|K)\wedge(A|H))}{P(A|H)} = \frac{z}{x} \in [0,1]$, when $x > 0$. We observe that, when $x = 0$, one has

$$(B|K)|(A|H) = \begin{cases} 1, & \text{if } AHBK \text{ is true,} \\ 0, & \text{if } AH\bar{B}K \text{ is true,} \\ y, & \text{if } AH\bar{K} \text{ is true,} \\ \mu, & \text{if } \bar{A}H \vee \bar{H} \text{ is true.} \end{cases}$$

Then, in order that the prevision assessment μ on $(B|K)|(A|H)$ be coherent, μ must belong to the convex hull of the values $0, y, 1$; that is, (also when $x = 0$) it must be that $\mu \in [0,1]$. Therefore in all cases $(B|K)|(A|H) \in [0,1]$.

The notions of conjunction $\mathcal{C}_{1\cdots n} = (E_1|H_1) \wedge \cdots \wedge (E_n|H_n)$ and disjunction $\mathcal{D}_{1\cdots n} = (E_1|H_1) \vee \cdots \vee (E_n|H_n)$ of n conditional events have been defined as (Gilio and Sanfilippo 2019, see also Gilio and Sanfilippo 2020)

$$\mathcal{C}_{1\cdots n} = \begin{cases} 1, & \text{if } \bigwedge_{i=1}^{n} E_i H_i, \text{ is true} \\ 0, & \text{if } \bigvee_{i=1}^{n} \bar{E}_i H_i, \text{ is true,} \\ x_S, & \text{if } \bigwedge_{i \in S} \bar{H}_i \bigwedge_{i \notin S} E_i H_i \text{ is true,} \end{cases} \qquad (13)$$

and

$$\mathcal{D}_{1\cdots n} = \begin{cases} 1, & \text{if } \bigvee_{i=1}^{n} E_i H_i, \text{ is true} \\ 0, & \text{if } \bigwedge_{i=1}^{n} \bar{E}_i H_i, \text{ is true,} \\ y_S, & \text{if } \bigwedge_{i \in S} \bar{H}_i \bigwedge_{i \notin S} \bar{E}_i H_i \text{ is true,} \end{cases} \qquad (14)$$

where, for each non-empty subset S of $\{1,\ldots,n\}$, x_S is the prevision of $\bigwedge_{i\in S}(E_i|H_i)$ and y_S is the prevision of $\bigvee_{i\in S}(E_i|H_i)$. Notice that $\mathcal{C}_{1\cdots n}$ and $\mathcal{D}_{1\cdots n}$ are conditional random quantities, with conditioning event $\bigvee_{i=1}^{n} H_i$ and with a finite set of possible values in $[0,1]$.

3. A Survey of Basic Properties, from Unconditional to Conditional Events

In this section we recall some well known logical and probabilistic properties for the case of unconditional events. Then, we illustrate analogous properties for the case of conditional events.

3.1 Some Basic Properties of Unconditional Events

We recall below some basic properties which concern unconditional events. The indicator of an event E is a random quantity, denoted by the same symbol, which is 1, or 0, according to whether E is true, or false, respectively.

1. Given two events A and B, denoting by the same symbols their indicators, it holds that
$$A \leq B \Longleftrightarrow A \subseteq B \iff AB = A,\ A \vee B = B\,.$$
2. Under the hypothesis $\emptyset \neq A \subseteq B$, one has $P(B|A) = 1$ and $B|A = AB + P(B|A)\overline{A} = A + \overline{A} = 1$.
3. Logical and probabilistic relations between disjunction and conjunction:
$$A \vee B = A + B - AB,\quad P(A \vee B) = P(A) + P(B) - P(AB).$$
4. De Morgan's laws
$$\overline{AB} = \overline{A} \vee \overline{B},\quad \overline{A \vee B} = \overline{A}\,\overline{B}.$$
5. Inclusion-exclusion principle
$$E_1 \vee \cdots \vee E_n = \sum_{i=1}^{n} E_i - \sum_{i<j} E_iE_j + \cdots + (-1)^{n+1}E_1\cdots E_n.$$
6. Fréchet-Hoeffding bounds
$$\max\{\sum_{i=1}^{n} P(E_i) - n + 1, 0\} \leq P(E_1\cdots E_n) \leq \min\{P(E_1),\ldots,P(E_n)\}.$$
7. Probability consistency. A family $\mathcal{F} = \{E_1,\ldots,E_n\}$ of n events is p-consistent if the assessment $P(E_i) = 1$, $i = 1,\ldots,n$, is coherent. We observe that
$$P(E_i) = 1, i = 1,\ldots,n \iff P(E_1\cdots E_n) = 1\,.$$
Indeed, defining $P(E_i) = x_i, i = 1,\ldots,n$, and $P(E_1\cdots E_n) = z$, by the Fréchet-Hoeffding bounds it holds that
$$\max\{x_1 + \cdots + x_n - n + 1, 0\} \leq z \leq \min\{x_1,\ldots,x_n\},$$
from which it follows that $x_i = 1, i = 1,\ldots,n,$ if and only if $z = 1$. Then, p-consistency amounts to the coherence of the assessment $P(E_1\cdots E_n) = 1$.

8. Probabilistic entailment. Given a p-consistent family $\mathcal{F} = \{E_1, \dots, E_n\}$ and a further event E_{n+1}, the family $\mathcal{F}$ p-entails the event E_{n+1} if and only if $P(E_i) = 1, i = 1, \dots, n$, implies that $P(E_{n+1}) = 1$. The p-entailment of E_{n+1} from $\mathcal{F}$ is equivalent to each one of the following properties

   (i) $E_1 \cdots E_n \subseteq E_{n+1}$, that is $E_1 \cdots E_n E_{n+1} = E_1 \cdots E_n$

   (ii) the (indicator of the) conditional event $E_{n+1}|E_1 \cdots E_n$ is constant and coincides with 1.

   We first observe that, given two events A and B, it holds that

   $$A \text{ p-entails } B \iff A \subseteq B, \text{ that is } A \leq B.$$

   Indeed, if $A \subseteq B$, then $P(A) = 1$ implies $P(B) = 1$. Conversely, by assuming that A p-entails B, if it were $A \nsubseteq B$, i.e. $A\bar{B} \neq \emptyset$, then given any assessment

   $$P(AB) = p, \; P(A\bar{B}) = 1 - p, \text{ with } p < 1,$$

   it would follow $P(A) = 1$, $P(\bar{A}B) = 0$, and $P(B) = P(AB) + P(\bar{A}B) = p < 1$, which contradicts the hypothesis. Then, the p-entailment of the event E_{n+1} from the event $E_1 \cdots E_n$ amounts to $E_1 \cdots E_n \subseteq E_{n+1}$, that is $E_1 \cdots E_n \leq E_{n+1}$. Then, it can be verified that

   $$\mathcal{F} = \{E_1, \dots, E_n\} \text{ p-entails } E_{n+1} \iff E_1 \cdots E_n \text{ p-entails } E_{n+1}.$$

   Indeed, as $P(E_1 \cdots E_n) = 1$ is equivalent to $P(E_i) = 1, i = 1, \dots, n$, if $\mathcal{F}$ p-entails E_{n+1}, then $P(E_1 \cdots E_n) = 1$ implies $P(E_{n+1}) = 1$, that is $E_1 \cdots E_n$ p-entails E_{n+1}. Conversely, as $P(E_i) = 1, i = 1, \dots, n$, is equivalent to $P(E_1 \cdots E_n) = 1$, if $E_1 \cdots E_n$ p-entails E_{n+1}, then $P(E_i) = 1, i = 1, \dots, n$, implies $P(E_{n+1}) = 1$, that is $\mathcal{F}$ p-entails E_{n+1}. Therefore

   $$\mathcal{F} \text{ p-entails } E_{n+1} \iff E_1 \cdots E_n \text{ p-entails } E_{n+1} \iff E_1 \cdots E_n \leq E_{n+1},$$

   that is, the p-entailment of E_{n+1} from $\mathcal{F}$ is equivalent to the property (i).

   Concerning the property (ii), if $\mathcal{F}$ p-entails E_{n+1}, then $E_1 \cdots E_n \subseteq E_{n+1}$ and hence, from (2), $E_{n+1}|E_1 \cdots E_n$ is constant and coincides with 1.

   Conversely, if $E_{n+1}|E_1 \cdots E_n$ coincides with 1, then $P(E_{n+1}|E_1 \cdots E_n) = 1$ and $E_1 \cdots E_n \overline{E_{n+1}} = \emptyset$, that is $E_1 \cdots E_n \subseteq E_{n+1}$, from which it follows that $E_1 \cdots E_n$ p-entails E_{n+1} and hence $\mathcal{F}$ p-entails E_{n+1}. Therefore, the p-entailment of E_{n+1} from $\mathcal{F}$ is equivalent to the property (ii).

3.2 Basic Properties of Conditional Events

In this section we show that the basic properties considered in Section 3.1 continue to hold when replacing events by conditional events. Here we denote by h^* the property h in the previous section.

1.* Given two conditional events $A|H$ and $B|K$, it holds that

   $$A|H \leq B|K \iff A|H \subseteq B|K, \text{ or } AH = \emptyset, \text{ or } K \subseteq B, \tag{15}$$

   and

   $$A|H \leq B|K \iff (A|H) \wedge (B|K) = A|H, \; (A|H) \vee (B|K) = B|K. \tag{16}$$

   Indeed, concerning (15), if $AH = \emptyset$, or $K \subseteq B$, then $A|H = 0$, or $B|K = 1$, and trivially it holds that $A|H \leq B|K$. If $A|H \subseteq B|K$, then by coherence

$P(A|H) \leq P(B|K)$ and hence $A|H \leq B|K$. Conversely, if $A|H \leq B|K$, then $P(A|H) \leq P(B|K)$, and the inequality may be satisfied, in three different ways, because: $AH = \emptyset$ (in which case $A|H = 0 \leq B|K$), or $K \subseteq B$ (in which case $B|K = 1 \geq A|H$), or $A|H \subseteq B|K$. For more details, see Gilio and Sanfilippo 2013d, Theorem 6. Moreover, concerning (16), if $A|H \leq B|K$, then by (15), we have three cases: (a) $AH = \emptyset$; (b) $K \subseteq B$; (c) $A|H \subseteq B|K$.

Case (a). It holds that $A|H = 0$ and by Remarks 2 and 3 it follows that

$$(A|H) \wedge (B|K) = 0 \wedge (B|K) = 0 = A|H, \; (A|H) \vee (B|K) = 0 \vee (B|K) = B|K.$$

Case (b). It holds that $B|K = 1$ and by Remarks 2 and 3 it follows that

$$(A|H) \wedge (B|K) = (A|H) \wedge 1 = A|H, \; (A|H) \vee (B|K) = (A|H) \vee 1 = 1 = B|K.$$

Case (c). If $A|H \subseteq B|K$, that is $AH \subseteq BK$ and $\bar{B}K \subseteq \bar{A}H$, it holds that $AH\bar{B}K = AH\bar{K} = \bar{H}\bar{B}K = \emptyset$ and the constituents are $C_1 = AHBK, C_2 = \bar{A}HBK, C_3 = \bar{A}H\bar{B}K, C_4 = \bar{A}H\bar{K}, C_5 = \bar{H}BK, C_0 = \bar{H}\bar{K}$. By defining $P(A|H) = x$, $P(B|K) = y$, $z = \mathbb{P}[(A|H) \wedge (B|K)]$, and $w = \mathbb{P}[(A|H) \vee (B|K)]$, the possible values of $A|H$, $B|K$, $(A|H) \wedge (B|K)$ and $(A|H) \vee (B|K)$ are illustrated in Table 2.

From Table 2, we observe that $(A|H) \wedge (B|K) = A|H$ when $H \vee K$ is true. Then, by Theorem 1 it follows that $z = x$; therefore $(A|H) \wedge (B|K) = A|H$ in all cases (see also Gilio and Sanfilippo 2013a, Section 3). Likewise, we observe that $(A|H) \vee (B|K) = B|K$ when $H \vee K$ is true. Then, by Theorem 1 it follows that $w = y$; therefore $(A|H) \vee (B|K) = B|K$ in all cases.

	C_h	$A\|H$	$B\|K$	$(A\|H) \wedge (B\|K)$	$(A\|H) \vee (B\|K)$
C_1	$AHBK$	1	1	1	1
C_2	$\bar{A}HBK$	0	1	0	1
C_3	$\bar{A}H\bar{B}K$	0	0	0	0
C_4	$\bar{A}H\bar{K}$	0	y	0	y
C_5	$\bar{H}BK$	x	1	x	1
C_0	$\bar{H}\bar{K}$	x	y	z	w

Table 2: *Possible values of* $A|H$, $B|K$, $(A|H) \wedge (B|K)$ *and* $(A|H) \vee (B|K)$, *when* $A|H \subseteq B|K$.

2.* We show the analogous of property 2 in terms of iterated conditionals. Under the hypothesis $0 \neq A|H \leq B|K$, that is $K \subseteq B$, or $AH \neq \emptyset$ and $A|H \subseteq B|K$, it can be verified that $\mathbb{P}[(B|K)|(A|H)] = 1$ and $(B|K)|(A|H) = 1$. Indeed, defining $\mathbb{P}[(B|K)|(A|H)] = \mu$, from (16) it holds that $(B|K) \wedge (A|H) = A|H$; then

$$(B|K)|(A|H) = A|H + \mu \bar{A}|H = \begin{cases} 1, & \text{if } AH \text{ is true,} \\ \mu, & \text{if } \bar{A}H \text{ is true,} \\ x + \mu(1-x), & \text{if } \bar{H} \text{ is true.} \end{cases}$$

By linearity of prevision it holds that

$$\mu = x + \mu(1-x);$$

then $(B|K)|(A|H) \in \{1, \mu\}$ and, by coherence, $\mu = 1$. For a discussion of this result in the context of connexive logic see Pfeifer and Sanfilippo 2021.

3.* Relation between disjunction and conjunction of conditional events (Gilio and Sanfilippo 2014, Section 6)

$$\mathbb{P}[(A|H) \vee (B|K)] = P(A|H) + P(B|K) - \mathbb{P}[(A|H) \wedge (B|K)],$$

and

$$(A|H) \vee (B|K) = A|H + B|K - (A|H) \wedge (B|K).$$

4.* De Morgan's laws for conjunction and disjunction of conditional events (Gilio and Sanfilippo 2019, Theorem 5).

$$\overline{(A|H) \wedge (B|K)} = (\bar{A}|H) \vee (\bar{B}|K), \quad \overline{(A|H) \vee (B|K)} = (\bar{A}|H) \wedge (\bar{B}|K),$$

where

$$\overline{(A|H) \wedge (B|K)} = 1 - (A|H) \wedge (B|K),$$

and

$$\overline{(A|H) \vee (B|K)} = 1 - (A|H) \vee (B|K).$$

5.* Inclusion-exclusion principle for conditional events.

Given n conditional events $E_1|H_1, \dots, E_n|H_n$, by recalling (13) and (14), it holds that (Gilio and Sanfilippo 2020)

$$\mathcal{D}_{1\cdots n} = \sum_{i=1}^{n} \mathcal{C}_i - \sum_{1 \le i_1 < i_2 \le n} \mathcal{C}_{i_1 i_2} + \cdots + (-1)^{n+1} \mathcal{C}_{1\cdots n},$$

where $\mathcal{C}_{i_1\cdots i_k} = (E_{i_1}|H_{i_1}) \wedge \cdots \wedge (E_{i_k}|H_{i_k})$, $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$.

6.* Fréchet-Hoeffding bounds for the conjunction of conditional events. Given a family of n conditional events $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\}$, let $\mathcal{P} = (x_1, \dots, x_n)$ be a coherent probability assessment on $\mathcal{F}$, with $x_i = P(E_i|H_i), i = 1, \dots, n$. We set $\mathbb{P}(\mathcal{C}_{1\cdots n}) = z$. Then, under logical independence of $E_1, \dots, E_n, H_1, \dots, H_n$, z is a coherent extension of $(x_1, , x_n)$ if and only if

$$\max\{x_1 + \cdots + x_n - n + 1, 0\} \le z \le \min\{x_1, \dots, x_n\}, \tag{17}$$

that is the Fréchet-Hoeffding bounds continue to hold for our notion of conjunction of conditional events. The necessity of (17) has been proved in Gilio and Sanfilippo 2019, while the sufficiency has been proved in Gilio and Sanfilippo 2021a.

7.* Probabilistic consistency of a family of conditional events. A family of n conditional events $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\}$ is defined p-consistent if the assessment $P(E_1|H_1) = \cdots = P(E_n|H_n) = 1$ is coherent. It holds that (see Gilio and Sanfilippo 2019, proof of Theorem 17)

$$P(E_1|H_1) = \cdots = P(E_n|H_n) = 1 \Longleftrightarrow \mathbb{P}[(E_1|H_1) \wedge \cdots \wedge (E_n|H_n)] = 1.$$

Then,

$$\mathcal{F} \text{ is p-consistent} \Longleftrightarrow \mathbb{P}[(E_1|H_1) \wedge \cdots \wedge (E_n|H_n)] = 1 \text{ is coherent.}$$

8.* Probabilistic entailment from a family of conditional events. Given a p-consistent family of n conditional events $\mathcal{F} = \{E_1|H_1, \dots, E_n|H_n\}$ and a further conditional event $E_{n+1}|H_{n+1}$, we say that $\mathcal{F}$ p-entails $E_{n+1}|H_{n+1}$ if and only if $P(E_i|H_i) = 1, i = 1, \dots, n$, implies $P(E_{n+1}|H_{n+1}) = 1$. It can be verified (Gilio and Sanfilippo 2019, Theorem 18) that the following three properties are equivalent:

(a) $\mathcal{F}$ p-entails $E_{n+1}|H_{n+1}$;

(b) $(E_1|H_1) \wedge \cdots \wedge (E_n|H_n) \leq E_{n+1}|H_{n+1}$;

(c) $(E_1|H_1) \wedge \cdots \wedge (E_n|H_n) \wedge E_{n+1}|H_{n+1} = (E_1|H_1) \wedge \cdots \wedge (E_n|H_n)$.

In particular, when $n = 1$, a p-consistent conditional event $E_1|H_1$ p-entails $E_2|H_2$ if and only if $(E_1|H_1) \leq (E_2|H_2)$, that is (as shown by property 2*; see also Gilio, Pfeifer and Sanfilippo 2020, Theorem 4)

$$E_1|H_1 \text{ p-entails } E_2|H_2 \iff (E_2|H_2)|(E_1|H_1) = 1, \tag{18}$$

where $E_1H_1 \neq \emptyset$. In Gilio and Sanfilippo 2019, Definition 14, the notion of iterated conditional $(E_{n+1}|H_{n+1})|((E_1|H_1) \wedge \cdots \wedge (E_n|H_n))$, with $(E_1|H_1) \wedge \cdots \wedge (E_n|H_n) \neq 0$, has been defined as the following random quantity

$$(E_1|H_1) \wedge \cdots \wedge (E_{n+1}|H_{n+1}) + \mu(1 - (E_1|H_1) \wedge \cdots \wedge (E_n|H_n)),$$

where $\mu = \mathbb{P}[(E_{n+1}|H_{n+1})|((E_1|H_1) \wedge \cdots \wedge (E_n|H_n))]$. In particular,

$$(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) = (E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3) + \\ + \mu(1 - (E_1|H_1) \wedge (E_2|H_2)),$$

where $\mu = \mathbb{P}[(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2))]$ and

$$(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3) = \begin{cases} 1, & \text{if } E_1H_1E_2H_2E_3H_3 \text{ is true,} \\ 0, & \text{if } \overline{E}_1H_1 \vee \overline{E}_2H_2 \vee \overline{E}_3H_3 \text{ is true,} \\ x_1, & \text{if } \overline{H}_1E_2H_2E_3H_3 \text{ is true,} \\ x_2, & \text{if } E_1H_1\overline{H}_2E_3H_3 \text{ is true,} \\ x_3, & \text{if } E_1H_1E_2H_2\overline{H}_3 \text{ is true,} \\ x_{12}, & \text{if } \overline{H}_1\overline{H}_2E_3H_3 \text{ is true,} \\ x_{13}, & \text{if } \overline{H}_1E_2H_2\overline{H}_3 \text{ is true,} \\ x_{23}, & \text{if } E_1H_1\overline{H}_2\overline{H}_3 \text{ is true,} \\ x_{123}, & \text{if } \overline{H}_1\overline{H}_2\overline{H}_3 \text{ is true,} \end{cases} \tag{19}$$

where $x_i = P(E_i|H_i)$, $i = 1, 2, 3$, $x_{ij} = x_{ji} = \mathbb{P}[(E_i|H_i) \wedge (E_j|H_j)]$, $i \neq j$, and $x_{123} = \mathbb{P}[(E_1|H_1) \wedge (E_2|H_2) \wedge (E_3|H_3)]$. In Gilio, Pfeifer and Sanfilippo 2020, given a p-consistent family $\mathcal{F} = \{E_1|H_1, E_2|H_2\}$ and a further event $E_3|H_3$, it has been proved that the p-entailment of $E_3|H_3$ from $\mathcal{F}$ is equivalent to the property that the iterated conditional $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2))$ is constant and equal to 1, that is

$$\{E_1|H_1, E_2|H_2\} \text{ p-entails } E_3|H_3 \iff \\ (E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) = 1, \tag{20}$$

where $\{E_1|H_1, E_2|H_2\}$ is p-consistent. For the extension of (20) to the general case see Gilio and Sanfilippo 2021b.

4. On Iterated Conditionals

In this section we deepen some aspects and applications of iterated conditionals. In the next subsection we show that some complex sentences on conditionals, which seem intuitively acceptable, can be analyzed in a rigorous way in terms of iterated conditionals.

4.1 Complex Sentences and Iterated Conditionals

Given an indicative conditional "*if H then A*", simply denoted $\mathcal{C}$, let us consider the following intuitively valid assertions:

(a) the probability of $\mathcal{C}$ is (not the probability of its truth, but) the probability of its truth, given that it is true or false;
(b) the probability of $\mathcal{C}$, given that A and H are true, is 1;
(c) the probability of $\mathcal{C}$, given that A is false and H is true, is 0;
(d) the probability of $\mathcal{C}$, given that H is false, is $P(A|H)$;
(e) the probability of $\mathcal{C}$, given that H is true, is $P(A|H)$;
(f) the probability of $\mathcal{C}$, given that "if H then A", is 1;
(g) the probability of $\mathcal{C}$, given that "if H then $\bar{A}$", is 0;
(h) is it the case that the probability of $\mathcal{C}$, given that $\overline{AH}$, is equal to 0?

We show below that the previous assertions have a clear meaning in the context of iterated conditionals.

(a) We consider the compound conditional "if $\mathcal{C}$ is true or false, then $\mathcal{C}$ is true", which can be directly represented by the conditional event $AH|(AH \vee \bar{A}H) = AH|H = A|H$, so that the probability of $\mathcal{C}$ is $P(AH|(AH \vee \bar{A}H)) = P(A|H)$. Then, the probability of $\mathcal{C}$ is the probability of its truth, given that it is true or false.
(b) We consider the compound conditional "if AH then $\mathcal{C}$" and we represent it by the iterated conditional $(A|H)|AH$. We observe that $AH \subseteq A|H$ and hence $(A|H) \wedge AH = AH$. Then,

$$(A|H)|(AH) = (A|H) \wedge AH + \mu\overline{AH} = AH + \mu\overline{AH},$$

which is equal to 1, or μ, according to whether AH is true, or false, respectively. By coherence $\mu = 1$ and hence $(A|H)|(AH) = 1$. The same result follows by exploiting the representation $A|H = AH + x\bar{H}$, where $x = P(A|H)$. We observe that $\bar{H}|AH = 0$; then

$$(A|H)|AH = (AH + x\bar{H})|AH = AH|AH = 1,$$

and hence $\mathbb{P}[(A|H)|AH] = P(AH|AH) = 1$. Thus, the conditional "if AH then $\mathcal{C}$" is the iterated conditional $(A|H)|AH$, which coincides with the constant $AH|AH = 1$ and has probability 1.
(c) We consider the compound conditional "if $\bar{A}H$ then $\mathcal{C}$" and represent it by the iterated conditional $(A|H)|\bar{A}H$. We set $P(A|H) = x$ and we observe that $AH|\bar{A}H = \bar{H}|\bar{A}H = 0$; then

$$(A|H)|\bar{A}H = (AH + x\bar{H})|\bar{A}H = AH|\bar{A}H + x\bar{H}|\bar{A}H = 0,$$

and hence $\mathbb{P}[(A|H)|\bar{A}H] = 0$. Thus, the conditional "if $\bar{A}H$ then $\mathcal{C}$" is the iterated conditional $(A|H)|\bar{A}H$, which coincides with the constant 0 and has probability 0.

(d) We consider the compound conditional "if $\overline{H}$ then $\mathcal{C}$" and represent it by the iterated conditional $(A|H)|\overline{H}$. We set $P(A|H) = x$ and we observe that $AH|\overline{H} = 0$ and $\overline{H}|\overline{H} = 1$; then

$$(A|H)|\overline{H} = (AH + x\overline{H})|\overline{H} = x\overline{H}|\overline{H} = x,$$

and hence $\mathbb{P}[(A|H)|\overline{H}] = x = P(A|H)$. Thus, the conditional "if $\overline{H}$ then $\mathcal{C}$" is the iterated conditional $(A|H)|\overline{H}$, which coincides with the constant x and has probability $x = P(A|H)$.

(e) We consider the compound conditional "if H then $\mathcal{C}$" and represent it by the iterated conditional $(A|H)|H$. Then, defining $P(A|H) = x$ and by observing that $\overline{H}|H = 0$, it holds that

$$(A|H)|H = (AH + x\overline{H})|H = A|H,$$

and hence $\mathbb{P}[(A|H)|H] = P(A|H)$. Thus, the conditional "if H then $\mathcal{C}$" is the iterated conditional $(A|H)|H$, which coincides with $A|H$ and its probability is $P(A|H)$. We observe that the conditional "if H then $\mathcal{C}$" is equivalent to the conditional "if $\mathcal{C}$ is true or false, then $\mathcal{C}$".

(f) We consider the compound conditional "if $\mathcal{C}$ then $\mathcal{C}$" and we represent it by the iterated conditional $(A|H)|(A|H)$. We set $\mathbb{P}[(A|H)|(A|H)] = \mu$, $P(A|H) = x$ and we recall that $(A|H) \wedge (A|H) = A|H$. Then

$$(A|H)|(A|H) = (A|H) \wedge (A|H) + \mu\overline{A}|H = A|H + \mu\overline{A}|H = \begin{cases} 1, & \text{if } AH \text{ is true.} \\ \mu, & \text{if } \overline{A}H \text{ is true,} \\ x + \mu(1-x), & \text{if } \overline{H} \text{ is true.} \end{cases}$$

We observe that

$$\mu = \mathbb{P}[A|H + \mu(1 - A|H)] = P(A|H) + \mu(1 - P(A|H)) = x + \mu(1-x).$$

Then, $(A|H)|(A|H) \in \{1, \mu\}$ and by coherence it must be $\mu = 1$. Thus, the compound conditional "if $\mathcal{C}$ then $\mathcal{C}$ is the iterated conditional $(A|H)|(A|H)$, which is the constant 1 and has probability 1.

(g) We consider the compound conditional "if (if H then $\overline{A}$), then $\mathcal{C}$" and we represent it by the iterated conditional $(A|H)|(\overline{A}|H)$. We set $\mathbb{P}[(A|H)|(\overline{A}|H)] = \mu$, $P(A|H) = x$ and we observe that $(A|H) \wedge (\overline{A}|H) = 0$ and $\overline{\overline{A}|H} = 1 - \overline{A}|H = A|H$. Then

$$(A|H)|(\overline{A}|H) = (A|H) \wedge (\overline{A}|H) + \mu\overline{\overline{A}|H} = \mu A|H = \begin{cases} \mu, & \text{if } AH \text{ is true.} \\ 0, & \text{if } \overline{A}H \text{ is true,} \\ \mu x, & \text{if } \overline{H} \text{ is true,} \end{cases}$$

so that $\mu = \mathbb{P}[(A|H)|(\overline{A}|H)] = \mu P(A|H) = \mu x$. If $x < 1$, then $\mu = 0$. If $x = 1$, then $(A|H)|(\overline{A}|H) \in \{0, \mu\}$ and by coherence it must be $\mu = 0$. Thus, the compound conditional "if (if H then $\overline{A}$), then $\mathcal{C}$" is the iterated conditional $(A|H)|(\overline{A}|H)$, which is the constant 0 and has probability 0. Likewise, it holds that $(\overline{A}|H)|(A|H) = 0$.

(h) We consider the compound conditional "if $\overline{AH}$ then $\mathcal{C}$" and represent it by the iterated conditional $(A|H)|\overline{AH}$. Then, defining $P(A|H) = x$ and

$\mathbb{P}[(A|H)|\overline{AH}] = \mu$, by observing that $AH|\overline{AH} = 0$, we obtain

$$(A|H)|\overline{AH} = (AH + x\overline{H})|\overline{AH} = x\overline{H}|\overline{AH} = x\overline{H}|(\overline{A} \vee \overline{H}), \qquad (21)$$

and hence $\mu = \mathbb{P}[(A|H)|\overline{AH}] = P(A|H)P[\overline{H}|(\overline{A} \vee \overline{H})] = xP[\overline{H}|(\overline{A} \vee \overline{H})]$, which in general is not 0.

Notice that in Edgington 2020, page 51, it is observed that in Bradley's theory (Bradley 2012) the probability of "$\mathcal{C}$, given $\overline{AH}$" is 0, instead of $P(A|H)P(\overline{H}|(\overline{A} \vee \overline{H})$. This is clearly not correct; indeed, to assume $\overline{AH}$ true amounts to assuming $\overline{A} \vee \overline{H}$ true, that is $\overline{A}H \vee \overline{H}$ true. Then, based on (22), in the betting framework, the conditional prevision $\mu = \mathbb{P}(\mathcal{C}|\overline{AH}) = \mathbb{P}[x\overline{H}|(\overline{A}H \vee \overline{H})]$ is the amount that should be paid in a conditional bet in order to receive 0 (when $\overline{A}H$ is true), or x (when $\overline{H}$ is true), with probability $P[\overline{A}H|(\overline{A}H \vee \overline{H})]$, or probability $P[\overline{H}|(\overline{A}H \vee \overline{H})]$, respectively (with the bet called off when AH is true). Hence, the amount μ to be paid is (not 0, but)

$$\mu = 0 \cdot P[\overline{A}H|(\overline{A}H \vee \overline{H})] + xP[\overline{H}|(\overline{A}H \vee \overline{H})] = xP[\overline{H}|(\overline{A}H \vee \overline{H})]. \qquad (22)$$

4.2 Import-Export Principle

Given three events A, H, K, with $HK \neq \emptyset$, if the Import-Export principle (McGee 1989) were satisfied, then it would be $(A|H)|K = A|HK$. In our approach, in general, the Import-Export principle does not hold (Gilio and Sanfilippo 2014), that is $(A|H)|K \neq A|HK$; moreover $(A|H)|K \neq (A|K)|H$. The Import-Export principle holds when $H \subseteq K$, or $K \subseteq H$, in which case $(A|H)|K = (A|K)|H = A|HK$. We also observe that $A|(H|K) \neq A|HK$ (Sanfilippo, Gilio, Over et al. 2020, Remark 7). We illustrate by an example the non validity of the Import-Export principle. Let us consider the iterated conditional $(A|H)|(\overline{H} \vee A)$, where $(\overline{H} \vee A)$ is the material conditional associated to "if H then A". If the Import-Export principle were valid it would be

$$(A|H)|(\overline{H} \vee A) = (A|(H \wedge (\overline{H} \vee A)) = A|AH = 1.$$

On the contrary, defining $P(A|H) = x$ and $\mathbb{P}[(A|H)|(\overline{H} \vee A)] = \mu$, as $A|H \subseteq (\overline{H} \vee A)$ it holds that $(A|H) \wedge (\overline{H} \vee A) = A|H$; then by Definition 7 we have

$$(A|H)|(\overline{H} \vee A) = (A|H) \wedge (\overline{H} \vee A) + \mu\overline{A}H = A|H + \mu\overline{A}H = \begin{cases} 1, & \text{if } AH \text{ is true.} \\ \mu, & \text{if } \overline{A}H \text{ is true,} \\ x, & \text{if } \overline{H} \text{ is true.} \end{cases} \qquad (23)$$

By coherence, $\mu \in [x, 1]$ and hence the iterated conditional $(A|H)|(\overline{H} \vee A)$ does not coincide with the constant $A|AH = 1$, thus the Import-Export principle does not hold. Moreover, when $P(\overline{H} \vee A) > 0$ from (12) it follows that

$$\mathbb{P}[(A|H)|(\overline{H} \vee A)] = \frac{P(A|H)}{P(\overline{H} \vee A)}.$$

We observe that the iterated conditional $(A|H)|(\overline{H} \vee A)$ is associated with the inference from the disjunction $\overline{H} \vee A$ to the conditional event $A|H$ (see Section 4.4). A probabilistic analysis of constructive and non-constructive inferences from the disjunction $A \vee B$ to the conditional event $B|\bar{A}$ has been given in Gilio and Over 2012.

More in general, given two conditional events $A|H$ and $B|K$, with $A|H \subseteq B|K$, defining $P(A|H) = x$, $P(B|K) = y$ and $\mathbb{P}[(A|H)|(B|K)] = \mu$, it holds that

$$(A|H)|(B|K) = (A|H) \wedge (B|K) + \mu\bar{B}|K = A|H + \mu\bar{B}|K.$$

Then, by (12), $\mu y = x$ and when $y > 0$ it follows that $\mu = \frac{x}{y}$, i.e.,

$$\mathbb{P}[(A|H)|(B|K)] = \frac{P(A|H)}{P(B|K)}, \quad (\text{if } A|H \subseteq B|K \text{ and } P(B|K) > 0).$$

4.3 Some p-valid Inference Rules

In Gilio, Pfeifer and Sanfilippo 2020, Theorem 8, it is shown that the p-entailment of a conditional event $E_3|H_3$ from a p-consistent family $\{E_1|H_1, E_2|H_2\}$ is equivalent to the condition $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) = 1$, i.e., to the condition that the set of possible values of $(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2))$ is the singleton $\{1\}$. In Table 3 we illustrate some p-valid inference rules and the associated iterated conditionals which are equal to 1.

Inference rule	$\{E_1|H_1, E_2|H_2\} \Rightarrow_p E_3|H_3$	$(E_3|H_3)|((E_1|H_1) \wedge (E_2|H_2)) = 1$
And	$\{B|A, C|A\} \Rightarrow_p BC|A$	$(BC|A)|(BC|A) = 1$
Cut	$\{C|AB, B|A\} \Rightarrow_p C|A$	$(C|A)|(BC|A) = 1$
CM	$\{C|A, B|A\} \Rightarrow_p C|AB$	$(C|AB)|(BC|A) = 1$
Or	$\{C|A, C|B\} \Rightarrow_p C|(A \vee B)$	$(C|(A \vee B))|((C|A) \wedge (C|B)) = 1$
Modus Ponens	$\{C|A, A\} \Rightarrow_p C$	$C|AC = 1$
Modus Tollens	$\{C|A, \bar{C}\} \Rightarrow_p \bar{A}$	$\bar{A}|((C|A) \wedge \bar{C}) = 1$
Bayes	$\{E|AH, H|A\} \Rightarrow_p H|EA$	$(H|EA)|(EH|A) = 1$

Table 3: *Some p-valid inference rules and their associated iterated conditionals.*

4.4 Some Non-p-valid Inference Rules

In this section we consider some non-p-valid inference rules, by showing that the associated iterated conditionals are not equal to 1.

Contraposition. Contraposition is not p-valid, that is the premise $\{C|A\}$ does not p-entail the conclusion $\bar{A}|\bar{C}$. Thus, from (18), $(\bar{A}|\bar{C})|(C|A) \neq 1$. Indeed, by setting, $P(C|A) = x$, $P(\bar{A}|\bar{C}) = y$, $\mathbb{P}[(C|A) \wedge (\bar{A}|\bar{C})] = z$, $\mathbb{P}[(\bar{A}|\bar{C})|(C|A)] = \mu$, it holds that

$$(\bar{A}|\bar{C})|(C|A) = (\bar{A}|\bar{C}) \wedge (C|A) + \mu(1 - C|A) = \begin{cases} y, & \text{if } AC \text{ is true,} \\ \mu, & \text{if } A\bar{C} \text{ is true,} \\ z, & \text{if } \bar{A}C \text{ is true,} \\ x + \mu(1-x), & \text{if } \bar{A}\bar{C} \text{ is true,} \end{cases}$$

which does not coincide with 1. For instance, by recalling that $\max\{x + y - 1, 0\} \leq z \leq \min\{x, y\}$, when $x = 1$ it holds that $z = y$. Then the iterated conditional becomes

$$(\bar{A}|\bar{C})|(C|A) = \begin{cases} y, & \text{if } C \text{ is true,} \\ \mu, & \text{if } A\bar{C} \text{ is true,} \\ 1, & \text{if } \bar{A}\bar{C} \text{ is true,} \end{cases}$$

with μ being coherent, for every $\mu \in [y, 1]$.

Strengthening the Antecedent. Strengthening of the antecedent is not p-valid, that is the premise $\{C|A\}$ does not p-entail the conclusion $C|AB$. Thus, from (18), $(C|AB)|(C|A) \neq 1$. Indeed, by setting $P(C|A) = x$, $P(C|AB) = y$, $\mathbb{P}[(C|AB) \wedge (C|A)] = z$, $\mathbb{P}[(C|AB)|(C|A)] = \mu$, it holds that

$$(C|AB)|(C|A) = (C|AB) \wedge (C|A) + \mu(1 - C|A) = \\ = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ \mu, & \text{if } AB\overline{C} \text{ is true,} \\ y, & \text{if } A\overline{B}C \text{ is true,} \\ \mu, & \text{if } A\overline{B}\overline{C} \text{ is true,} \\ z + \mu(1-x), & \text{if } \overline{A}BC \text{ is true,} \\ z + \mu(1-x), & \text{if } \overline{A}B\overline{C} \text{ is true,} \\ z + \mu(1-x), & \text{if } \overline{A}\overline{B} \text{ is true.} \end{cases}$$

By linearity of prevision it holds that

$$\mu = \mathbb{P}[(C|AB)|(C|A)] = \mathbb{P}[(C|AB) \wedge (C|A) + \mu(1 - C|A)] = z + \mu(1-x).$$

Then,

$$(C|AB)|(C|A) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ y, & \text{if } A\overline{B}C \text{ is true,} \\ \mu, & \text{if } \overline{A} \vee \overline{C} \text{ is true,} \end{cases}$$

with μ being coherent, for every $\mu \in [y, 1]$.

From Disjunction to Conditional. Based on (18), the inference of a conditional $C|A$ from the associated material conditional $\overline{A} \vee C$ is not p-valid because, as shown in (23), the iterated conditional $(C|A)|(\overline{A} \vee C)$ does not coincide with 1.

Transitivity. Transitivity is not p-valid, that is the set of conditionals $\{C|B, B|A\}$ does not p-entail the conclusion $C|A$. Thus, from (20), $(C|A)|((C|B) \wedge (B|A))$ does not coincide with 1. Indeed, defining $P(B|A) = x$, $P(BC|A) = y$, $\mathbb{P}[(C|B) \wedge (B|A) \wedge (C|A)] = w$, $\mathbb{P}[(C|B) \wedge (B|A)] = z$, we have

$$(C|B) \wedge (B|A) \wedge (C|A) = (C|B) \wedge (BC|A) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\overline{C} \text{ is true,} \\ 0, & \text{if } A\overline{B}C \text{ is true,} \\ 0, & \text{if } A\overline{B}\overline{C} \text{ is true,} \\ y, & \text{if } \overline{A}BC \text{ is true,} \\ 0, & \text{if } \overline{A}B\overline{C} \text{ is true,} \\ w, & \text{if } \overline{A}\overline{B} \text{ is true,} \end{cases} \quad (24)$$

and

$$(C|B) \wedge (B|A) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } AB\overline{C} \text{ is true,} \\ 0, & \text{if } A\overline{B}C \text{ is true,} \\ 0, & \text{if } A\overline{B}\overline{C} \text{ is true,} \\ x, & \text{if } \overline{A}BC \text{ is true,} \\ 0, & \text{if } \overline{A}B\overline{C} \text{ is true,} \\ z, & \text{if } \overline{A}\overline{B} \text{ is true.} \end{cases} \quad (25)$$

Then, by setting $\mu = \mathbb{P}[(C|A)|((C|B) \wedge (B|A))]$, it holds that

$$(C|A)|((C|B) \wedge (B|A)) = (C|B) \wedge (B|A) \wedge (C|A) + \mu(1 - (C|B) \wedge (B|A)) =$$
$$= \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ \mu, & \text{if } AB\overline{C} \text{ is true,} \\ \mu, & \text{if } A\overline{B}C \text{ is true,} \\ \mu, & \text{if } A\overline{B}\overline{C} \text{ is true,} \\ y + \mu(1-x), & \text{if } \overline{A}BC \text{ is true,} \\ \mu, & \text{if } \overline{A}B\overline{C} \text{ is true,} \\ w + \mu(1-z), & \text{if } \overline{A}\overline{B} \text{ is true.} \end{cases} = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ y + \mu(1-x), & \text{if } \overline{A}BC \text{ is true,} \\ \mu, & \text{if } \overline{B} \vee \overline{C} \text{ is true} \end{cases}$$

because, by linearity of prevision, $\mu = w + \mu(1 - z)$. As we can see the iterated conditional $(C|A)|((C|B) \wedge (B|A))$ does not coincide with 1. Indeed, when $\overline{A}BC$ is true, the iterated conditional assumes the value $y + \mu(1 - x)$ which is equal to 0 when $(x, y) = (1, 0)$; then

$$(C|A)|((C|B) \wedge (B|A)) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } \overline{A}BC \text{ is true,} \\ \mu, & \text{if } \overline{B} \vee \overline{C} \text{ is true,} \end{cases}$$

with μ being coherent, for every $\mu \in [0, 1]$.

On Combining Evidence (Boole). We illustrate an example introduced in Boole 1857. In Hailperin 1996 it is shown that given a coherent assessment (x, y) on $\{C|A, C|B\}$, the extension $P(C|AB) = \xi$ is coherent for every $\xi \in [0, 1]$. Thus, the inference of $C|AB$ from the p-consistent family $\{C|A, C|B\}$ is not p-valid (see also Gilio and Sanfilippo 2019). We verify below that the associated iterated conditional $(C|AB)|((C|A) \wedge (C|B))$ does not coincide with 1. We set $P(C|A) = x$, $P(C|B) = y$, $\mathbb{P}((C|A) \wedge (C|B)) = z$. We obtain

$$(C|A) \wedge (C|B) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } (A \vee B)\overline{C} \text{ is true,} \\ x, & \text{if } \overline{A}BC \text{ is true,} \\ y, & \text{if } A\overline{B}C \text{ is true,} \\ z, & \text{if } \overline{A}\overline{B} \text{ is true.} \end{cases} \tag{26}$$

Moreover, by defining $\mathbb{P}[(C|A) \wedge (C|AB)] = u$, $\mathbb{P}[(C|B) \wedge (C|AB)] = v$ and $\mathbb{P}[(C|A) \wedge (C|B) \wedge (C|AB)] = t$, we obtain

$$(C|A) \wedge (C|B) \wedge (C|AB) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ 0, & \text{if } (A \vee B)\overline{C} \text{ is true,} \\ u, & \text{if } \overline{A}BC \text{ is true,} \\ v, & \text{if } A\overline{B}C \text{ is true,} \\ t, & \text{if } \overline{A}\overline{B} \text{ is true.} \end{cases}$$

Then, by setting $\mu = \mathbb{P}[(C|AB)|((C|A) \wedge (C|B))]$, it holds that

$$(C|AB)|((C|A) \wedge (C|B)) = (C|A) \wedge (C|B) \wedge (C|AB) + \mu(1 - (C|A) \wedge (C|B)) =$$
$$= \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ \mu, & \text{if } (A \vee B)\overline{C} \text{ is true,} \\ u + \mu(1-x), & \text{if } \overline{A}BC \text{ is true,} \\ v + \mu(1-y), & \text{if } A\overline{B}C \text{ is true,} \\ t + \mu(1-z), & \text{if } \overline{A}\overline{B} \text{ is true.} \end{cases}$$

By linearity of prevision $\mu = t + \mu(1-z)$; then

$$(C|AB)|((C|A) \wedge (C|B)) = \begin{cases} 1, & \text{if } ABC \text{ is true,} \\ u + \mu(1-x), & \text{if } \overline{A}BC \text{ is true,} \\ v + \mu(1-y), & \text{if } A\overline{B}C \text{ is true,} \\ \mu, & \text{if } A\overline{C} \vee B\overline{C} \vee \overline{A}\overline{B} \text{ is true,} \end{cases}$$

which does not coincide with 1. We observe that, if we replace the conclusion $C|AB$ by $C|(A \vee B)$, we obtain the well-known p-valid Or rule.

We recall that the *Affirmation of the Consequent* and the *Denial of the Antecedent* are other non p-valid inference rules; thus, the associated iterated conditionals are not equal to 1 (Gilio, Pfeifer and Sanfilippo 2020). In Table 4 we list the previous non-p-valid inference rules and their associated iterated conditionals which do not coincide with 1. We denote by $\mathcal{C}(\mathcal{F})$ the conjunction of the conditional events in the set of premises $\mathcal{F}$.

Inference rule	$\mathcal{F} \not\Rightarrow_p E|H$	$(E|H)|\mathcal{C}(\mathcal{F})) \neq 1$
Contraposition	$C|A \not\Rightarrow_p \overline{A}|\overline{C}$	$(\overline{A}|\overline{C})|(C|A) \neq 1$
Strengthening the antecedent	$C|A \not\Rightarrow_p C|AB$	$(C|AB)|(C|A) \neq 1$
From disjunction to conditional	$\overline{A} \vee C \not\Rightarrow_p C|A$	$(C|A)|(\overline{A} \vee C) \neq 1$
Transitivity	$\{C|B, B|A\} \not\Rightarrow_p C|A$	$(C|A)|((C|B) \wedge (B|A)) \neq 1$
Combining evidence	$\{C|A, C|B\} \not\Rightarrow_p C|AB$	$(C|AB)|((C|A) \wedge (C|B)) \neq 1$
Affirmation of the Consequent	$\{C|A, C\} \not\Rightarrow_p A$	$(A)|((C|A) \wedge C) \neq 1$
Denial of the antecedent	$\{C|A, \overline{A}\} \not\Rightarrow_p \overline{C}$	$(\overline{C})|((C|A) \wedge \overline{A}) \neq 1$

Table 4: *Some non p-valid inference rules and their associated iterated conditionals.*

5. Conclusions

In this paper we have illustrated the notions of conjoined, disjoined and iterated conditionals introduced in recent papers, in the setting of coherence. These objects are defined as suitable conditional random quantities, with a finite set of possible values in the interval $[0, 1]$. We have motivated our definitions by examining the experiment of flipping a coin twice. We have shown that the well known probabilistic properties valid for unconditional events continue to hold when replacing events by conditional events. We have examined several, intuitively acceptable, compound sentences on conditionals, by providing for them a formal interpretation in terms of iterated conditionals. We have discussed the Import-Export principle, which is not valid in our approach, by examining in particular the iterated conditional $(A|H)|(\overline{H} \vee A)$. Finally, we have illustrated, in terms of suitable iterated conditionals, several well known, p-valid and non p-valid, inference rules. With each inference rule, denoting by $E|H$ the conclusion and by $\mathcal{F}$ the set of premises, we have associated the iterated conditional $(E|H)|\mathcal{C}(\mathcal{F})$ where $\mathcal{C}(\mathcal{F})$ is the conjunction of the conditional events in $\mathcal{F}$. In Table 3 we recalled some well known p-valid inference rules, characterized by the property that $(E|H)|\mathcal{C}(\mathcal{F}) = 1$. Finally, we have examined some non p-valid inference rules, listed in Table 4, by verifying that for each of them the associated iterated conditional does not coincide with the constant 1.[1]

References

Adams, E.W. 1975, *The Logic of Conditionals*, Dordrecht: Reidel.

Baratgin, J., Politzer, G., Over, D. et al. 2018, "The Psychology of Uncertainty and Three-Valued Truth Tables", *Frontiers in Psychology*, 9, 1479.

Benferhat, S., Dubois, D. and Prade, H. 1997, "Nonmonotonic Reasoning, Conditional Objects and Possibility Theory", *Artificial Intelligence*, 92, 259-276.

Biazzo, V., Gilio, A., Lukasiewicz, T. et al. 2005, "Probabilistic Logic under Coherence: Complexity and Algorithms", *Annals of Mathematics and Artificial Intelligence*, 45, 1-2, 35-81.

Boole, G. 1857, "On the Application of the Theory of Probabilities to the Question of the Combination of Testimonies or Judgments", *Transactions of the Royal Society of Edinburgh*, 21, 4, 597-653.

Bradley, R. 2012, "Multidimensional Possible-World Semantics for Conditionals", *The Philosophical Review*, 121, 4, 539-71.

Calabrese, P. 1987, "An Algebraic Synthesis of the Foundations of Logic and Probability", *Information Sciences*, 42, 3, 187-237.

Calabrese, P. 2017, *Logic and Conditional Probability: A Synthesis*, London: College Publications.

Ciucci, D. and Dubois, D. 2012, "Relationships between Connectives in Three-Valued Logics", in Greco, S., Bouchon-Meunier, B., Coletti, G. et al. (eds.), *Advances in Computational Intelligence*, Proceedings of the 14th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, IPMU2012, Catania, Italy, July 2012, CCIS 297, Berlin: Springer, 633-42.

Ciucci, D. and Dubois, D. 2013, "A Map of Dependencies among Three-Valued Logics", *Information Sciences*, 250, 162-77.

Coletti, G., Scozzafava, R. and Vantaggi, B. 2013, "Coherent Conditional Probability, Fuzzy Inclusion and Default Rules", in Yager, R.R., Abbasov, A.M., Reformat, M.Z. et al. (eds.), *Soft Computing: State of the Art Theory and Novel Applications*, Berlin: Springer, 193-208.

Coletti, G., Scozzafava, R. and Vantaggi, B. 2015, "Possibilistic and Probabilistic Logic under Coherence: Default Reasoning and System P", *Mathematica Slovaca*, 65, 4, 863-90.

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Cruz, N. 2020, "Deduction from Uncertain Premises?", in Elqayam, Douven, Evans et al. 2020, 27-41.

de Finetti, B. 1936, "La logique de la probabilité", in, *Actes du Congrès International de Philosophie Scientifique*, Vol. IV, Paris: Hermann et C.ie, 1-8.

de Finetti, B. 1980, "Foresight: Its Logical laws, Its Subjective Sources", in Kyburg Jr., H.E. and Smokler, H.E. (eds.), *Studies in Subjective Probability*, 2^{nd} edition, Huntington, NY: Krieger, 93-158.

de Finetti, B. 1995, "The Logic of Probability", *Philosophical Studies*, 77, 181-90.

Douven, I. and Dietz, R. 2011, "A Puzzle about Stalnaker's Hypothesis", *Topoi*, 30, 31-7.

Douven, I., Elqayam, S., Singmann, H. et al. 2019, "Conditionals and Inferential Connections: Toward a New Semantics", *Thinking & Reasoning*, 26, 1-41.

ECSQARU 2021, *Symbolic and Quantitative Approaches to Reasoning with Uncertainty: Proceedings of the 16^{th} European Conference*, Lecture Notes in Computer Science, Berlin-Heidelberg: Springer, forthcoming.

Edgington, D. 1995, "On Conditionals", *Mind*, 104, 235-329.

Edgington, D. 2020, "Compounds of Conditionals, Uncertainty, and Indeterminacy", in Elqayam, Douven, Evans et al. 2020, 42-56.

Elqayam, S., Douven, I., Evans, J.S.B.T. et al. (eds.) 2020, *Logic and Uncertainty in the Human Mind: A Tribute to David E.Over*, London: Routledge.

Flaminio, T., Godo, L. and Hosni, H. 2020, "Boolean Algebras of Conditionals, Probability and Logic", *Artificial Intelligence*, 286, URL: https://www.sciencedirect.com/science/article/pii/S000437022030103X.

Gilio, A. 2002, "Probabilistic Reasoning under Coherence in System P", *Annals of Mathematics and Artificial Intelligence*, 34, 5-34.

Gilio, A. and Over, D. 2012, "The Psychology of Inferring Conditionals from Disjunctions: A Probabilistic Study", *Journal of Mathematical Psychology*, 56, 118-31.

Gilio, A., Over, D., Pfeifer, N. et al. 2017, "Centering and Compound Conditionals under Coherence", in Ferraro, M.B., Giordani, P., Vantaggi, B. et al. (eds.), *Soft Methods for Data Science*, Collection of selected peer reviewed papers presented at the 8th International Conference on Soft Methods in Probability and Statistics (SMPS 2016) held in Rome (Italy), Advances in Intelligent Systems and Computing 456, Berlin: Springer, 253-60.

Gilio, A., Pfeifer, N. and Sanfilippo, G. 2020, "Probabilistic Entailment and Iterated Conditionals", in Elqayam, Douven, Evans et al. 2020, 71-101.

Gilio, A. and Sanfilippo, G. 2013a, "Conjunction, Disjunction and Iterated Conditioning of Conditional Events", in Kruse, R., Berthold, M.R., Moewes, C. et al. (eds.), *Synergies of Soft Computing and Statistics for Intelligent Data Analysis*, Vol. 190, Berlin: Springer, 399-407.

Gilio, A. and Sanfilippo, G. 2013b, "Conditional Random Quantities and Iterated Conditioning in the Setting of Coherence", in van der Gaag, L.C. (ed.), *Symbolic and Quantitative Approaches to Reasoning with Uncertainty*, Proceedings of the 12th European Conference, ECSQARU 2013, Lecture Notes in Computer Science 7958, Berlin-Heidelberg: Springer, 218-29.

Gilio, A. and Sanfilippo, G. 2013c, "Probabilistic Entailment in the Setting of Coherence: The Role of Quasi Conjunction and Inclusion Relation", *International Journal of Approximate Reasoning*, 54, 4, 513-25.

Gilio, A. and Sanfilippo, G. 2013d, "Quasi Conjunction, Quasi Disjunction, T-norms and T-conorms: Probabilistic Aspects", *Information Sciences*, 245, 146-67.

Gilio, A. and Sanfilippo, G. 2014, "Conditional Random Quantities and Compounds of Conditionals", *Studia Logica*, 102, 4, 709-29.

Gilio, A. and Sanfilippo, G. 2017, "Conjunction and Disjunction Among Conditional Events", in Benferhat, S., Tabia, K. and Ali, M. (eds.), *Advances in Artificial Intelligence: From Theory to Practice*, Proceedings of the 30th International Conference on Industrial Engineering and Other Applications of Applied Intelligent Systems, IEA/AIE 2017, Arras, France, June 27-30, 2017, Part II, Lecture Notes in Computer Science 10351, Springer, 85-96, URL: https://doi.org/10.1007/978-3-319-60045-1%5C_11.

Gilio, A. and Sanfilippo, G. 2019, "Generalized Logical Operations among Conditional Events", *Applied Intelligence*, 49, 1, 79-102.

Gilio, A. and Sanfilippo, G. 2020, "Algebraic Aspects and Coherence Conditions for Conjoined and Disjoined Conditionals", *International Journal of Approximate Reasoning*, 126, 98-123.

Gilio, A. and Sanfilippo, G. 2021a, "Compound Conditionals, Fréchet-Hoeffding Bounds, and Frank t-Norms", *International Journal of Approximate Reasoning*, 36, 168-200.

Gilio, A. and Sanfilippo, G. 2021b, "Iterated Conditionals and Characterization of P-Entailment", in ECSQARU 2021.

Goodman, I.R. and Nguyen, H.T. 1988, "Conditional Objects and the Modeling of Uncertainties", in Gupta, M.M. and Yamakawa, T. (eds.), *Fuzzy Computing: Theory, Hardware, and Applications*, Amsterdam: North-Holland Publishing Company, 119-38.

Goodman, I.R., Nguyen, H.T. and Walker, E.A. 1991, *Conditional Inference and Logic for Intelligent Systems: A Theory of Measure-Free Conditioning*, Amsterdam: North-Holland Publishing Company.

Hailperin, T. 1996, *Sentential Probability Logic: Origins, Development, Current Status, and Technical Applications*, Bethlehem: Lehigh University Press.

Kaufmann, S. 2009, "Conditionals Right and Left: Probabilities for the Whole Family", *Journal of Philosophical Logic*, 38, 1-53.

Knauff, M. and Spohn, W. (eds.) 2021, *Handbook of Rationality*, Cambridge, MA: The MIT Press, forthcoming.

Lewis, D. 1976, "Probabilities of Conditionals and Conditional Probabilities", *The Philosophical Review*, 85, 3, 297-315.

McGee, V. 1989, "Conditional Probabilities and Compounds of Conditionals", *Philosophical Review*, 98, 485-541.

Milne, P. 1997, "Bruno de Finetti and the Logic of Conditional Events", *British Journal for the Philosophy of Science*, 48, 195-232.

Mura, A. 2011, "Towards a New Logic of Indicative Conditionals", *Logic and Philosophy of Science*, 9, 17-31.

Nguyen, H.T. and Walker, E.A. 1994, "A History and Introduction to the Algebra of Conditional Events and Probability Logic", *IEEE Transactions on Systems, Man, and Cybernetics*, 24, 12, 1671-5.

Over, D. and Cruz, N. 2021, "Suppositional Theory of Conditionals and Rationality", in Knauff and Spohn 2021.

Pelessoni, R. and Vicig, P. 2014, "The Goodman-Nguyen Relation within Imprecise Probability Theory", *International Journal of Approximate Reasoning*, 55, 8, 1694-707.

Pfeifer, N. 2021, "Probability Logic", in Knauff and Spohn 2021.

Pfeifer, N. and Sanfilippo, G. 2021, "Interpreting Connexive Principles in Coherence-Based Probability Logic", in ECSQARU 2021.

Ramsey, F. 1990, "General Propositions and Causality", in Mellor, D.H. (ed.), *Philosophical Papers*, Cambridge: Cambridge University Press, 145-63.

Sanfilippo, G. 2018, "Lower and Upper Probability Bounds for Some Conjunctions of Two Conditional Events", in Ciucci, D., Pasi, G. and Vantaggi, B. (eds.), *Scalable Uncertainty Management*, Proceedings of the 12th International Conference, SUM 2018, Milan, Italy, October 3-5, 2018, Lecture Notes in Computer Science 11142, Heidelberg: Springer, 260-75.

Sanfilippo, G., Gilio, A., Over, D. et al. 2020, "Probabilities of Conditionals and Previsions of Iterated Conditionals", *International Journal of Approximate Reasoning*, 121, 150-73.

Sanfilippo, G., Pfeifer, N. and Gilio, A. 2017, "Generalized Probabilistic Modus Ponens", in A., A., L., C. and Papini, O. (eds.), *Symbolic and Quantitative Approaches to Reasoning with Uncertainty*, Proceedings of the 14th European Conference, ECSQARU 2017, Lugano, Switzerland, July 10-14, 2017, Lecture Notes in Computer Science 10369, Springer, 480-90, URL: https://doi.org/10.1007/978-3-319-61581-3%5C_43.

Sanfilippo, G., Pfeifer, N., Over, D. et al. 2018, "Probabilistic Inferences from Conjoined to Iterated Conditionals", *International Journal of Approximate Reasoning*, 93, Supplement C, 103-118.