Population dynamics based on ladder bosonic operators

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Abstract

We adopt an operatorial method, based on truncated bosons, to describe the dynamics of populations in a closed region with a non trivial topology. The main operator that includes the various mechanisms and interactions between the populations is the Hamiltonian, constructed with the density and transport operators. The whole evolution is derived from the Schrödinger equation, and the densities of the populations are retrieved from the normalized expected values of the density operators. We show that this approach is suitable for applications in very large domain, solving the computational issues that typically occur when using an Hamiltonian based on fermionic ladder operators.

Keyword: Operatorial models, Schrödinger dynamics, Population dynamics,

I Introduction

In the last decade it has become clear that the mathematical modelization based on quantum-inspired models should be considered a useful way to describe the dynamics occurring in various macroscopic systems. Many applications have been proposed during the years: the first models in economics, [1, 2, 3, 4], applications to population dynamics [5, 6, 7, 8], social life and decision-making processes [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19], and bio-ecological processes, [20, 21, 22, 23, 24] to cite a few.

The key aspect of the operatorial approach is that it can very easily describe the main interactions occurring between the various agents of the system; this is done by means of the raising and lowering operators very well known in quantum mechanics, and the construction of a suitable Hamiltonian operator

H, not necessarily Hermitian, which contains the operators describing all the interactions. In the cited applications, the ladder operators were of two kinds: bosonic or fermionic. This means that our system may have either an infinite number of different conditions (the various eigenstates of the bosonic number operator), or just two[1]. This key difference fixes the realm of the application of the two cases: fermionic operators introduce a bound on the computed quantities, so that they are more properly used when dealing with quantities that are bounded on macroscopic scale (for instance the density of a population in a region as originally done in [5]), whereas bosonic operators should be used for describing individual agent dynamics with counting purposes. Actually, the bosonic operator are not easy to treat computationally, so that the truncated version are generally adopted in computations (see for instance application to cancer cells proliferation in [24]).

In both cases the relevant aspect is to derive the time evolution of some observables of the macroscopic system with the Schrödinger or Heisenberg equations (depending on the self-adjointness of H and on the quantity one needs to derive, states or operators), and in particular their expected values which are phenomenologically associated to some macroscopic quantities like densities, decision functions and so on. We stress here that, although quantum mechanics is based on self-adjoint Hamiltonians, in the last decades many recent research activities have been based on the relaxation of this condition, in particular in quantum optics and in PT-quantum mechanics, [25, 26, 27, 28] and in macroscopic system described with effective Hamiltonians, [21, 20].

In this paper we adopt an operatorial method to describe the dynamics of various populations moving on a bounded 2D region. Previously, [5, 7], the framework based on fermionic operators was adopted although there were some evident drawbacks. In fact, the Hilbert space where all the operators and states live can have a very large dimension, so that it can be hardly handled from a computational point of view. For example, the migration of M populations on region with N sites requires an Hilbert space of dimension 2^{MN} in which all the operators are matrices of dimension $2^{MN} \times 2^{MN}$. This is because each site of a specific populations is seen as a fermionic mode interacting with other modes. Although this approach is reasonable and successfully adopted, it is evident that the dimension of the Hilbert space grows uncontrollably with N and M requiring a large amount of random access memory in the computer: taking $N = 2$, $M = 16$ the Hamiltonian operator requires 4 gigabyte of memory. Of course one can use the computational tricks based on sparse matrices, but this does not solve all the issues, especially when different kind of interactions are considered in the system, and the Hamiltonian has many non-zero entries. Only in the case of a self adjoint Hamiltonian, an explicit formulation of densities can be derived, but self-adjoint implies that the dynamics has some reversible mechanism that is not always required in general (e.g. a self-adjoint Hamiltonian with a death process requires the reverse birth process). Leaving the realm of the self-adjoint Hamiltonian, a formal solution is generally not easy to determine, and numerical computations are the only way; but the above computational problems can only be solved by limiting the number of sites or populations, often leading to an unwanted and forced lack of resolution in the derived dynamics. In this article we propose an approach that seeks to overcome this problem by using truncated bosonic operators instead of fermionic ones. The underlying idea is to attach to each of the $M - th$ population a (truncated) bosonic mode of dimension N [2], so that the dimension of the

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[1]This is because the fermionic number operator has just two eigenstates.

[2]Actually, as we shall explain during the paper, the dimension of each bosonic mode could be $3N + 1$ according to the dynamics included in the model.

Hilbert space is N^M . This permits to see each population in all the sites of the region as a level of a boson. The obvious advantage is that we can increase the resolution by increasing the number of sites, with a dimension of the Hilbert space that can be managed computationally. Obviously the difficulty now is the proper construction of the Hilbert space itself with the element of the basis, and the related operators to describe the dynamics; as we will see we introduce some dummy levels in the bosons which will be correlated to the state of a population in a site (alive or dead) and to the presence of reservoirs for the processes of production or birth.

The main application presented in this paper are of two different kind. The first is a prey-predator dynamics. It is clear that the mathematical investigation devoted to the prey-predator dynamics is full of established models based on ODE/PDE and stochastic models [29], game theory [30], and agent-based simulations [31]. However, as we will see, the operatorial approach simulates this dynamics very well. The advantage of our approach lies in the fact that the prey-predator mechanism is naturally implementable through a transport operator of the kind $|R\rangle\langle L|$ that Lowers the level of the prey and Raises that of the predators, following exactly the meaning of the operator in quantum mechanics. The second application is the transport of a marine population in a very large basin. In this context PDE-based simulations in the Eulerian or Lagrangian framework, see [32] for a review, are consolidated choices. Again our approach can naturally implement the advection through a transport operator, lowering the level of a specific cell, and raising that of a contiguous one. For this specific application we will perform a rough comparison with some real data presented in [7].

The paper is organized as follows. In Section II we introduce the mathematical settings in which we define the main operators used to construct the Hamiltonian of the system, and we derive the time evolution of the system. In Section III we propose a pedagogical example on a 1D domain for the the dynamics of two populations. In Section IV we explain how to extend the approach to a 2D domain, and we propose the two main applications: the prey-predator dynamics, and the marine transport dynamics in a very large domain. Finally we shall give our discussions in Section V.

II   The mathematical framework

In this section we present the mathematical framework which is at the base for the construction of the various operators and states of the system.

II.1   The Ladder operators

Here we show how to construct truncated bosonic ladder operators for a finite-dimensional Hilbert space $\mathcal{H}_N$, with $2 < N < \infty$, endowed with the scalar product $\langle\cdot,\cdot\rangle$ and norm $\|v\| = \sqrt{\langle v, v\rangle}, \quad \forall v \in \mathcal{H}_N$. Let $\mathcal{E}_N = \{e_j, \ j = 0, 1, \ldots, N-1\}$ be the canonical orthonormal (o.n.) basis of $\mathcal{H}_N$, being $e_j = (0, \ldots, \underbrace{1}_{(j+1)-th}, \ldots, 0)$. We define an operator $\hat{a}$ and its adjoint $\hat{a}^\dagger$ via their action on the e_j's given by the following ladder equations (of course the basis should be intended as column vectors):

$$\hat{a}e_0 = 0, \quad \hat{a}e_j = \sqrt{j}\,e_{j-1}, \quad \hat{a}^\dagger e_j = \sqrt{j+1}\,e_{j+1}, \quad \hat{a}^\dagger e_{N-1} = 0. \tag{2.1}$$

The two operators work essentially as the truncated versions of the lowering and raising bosonic operators in quantum mechanics: $\hat{a}$ destroys the ground state e_0 and decreases the j-th level of the state e_j to the $j-1$-th level, whereas $\hat{a}^\dagger$ works in the opposite way by increasing the $j-$th level and destroying the most excited state e_{N-1}. In terms of matrix form we have the following representation:

$$\hat{a} = \left\{ \sqrt{j}\delta_{j,k-1} \right\}_{\substack{j=1,\ldots,N \\ k=2,\ldots,N}} \quad , \quad \hat{a}^\dagger = \left\{ \sqrt{k}\delta_{j,k+1} \right\}_{\substack{j=1,\ldots,N \\ k=1,\ldots,N-1}} \quad .$$

It easy to check that these operators satisfy the commutation relation

$$[\hat{a}, \hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = \mathbb{1}_N - N P_{N-1}, \tag{2.2}$$

where $\mathbb{1}_N$ is the identity operator in $\mathcal{H}_N$, and P_{N-1} is the projection operator on e_{N-1} defined with the action $P_{N-1}v = \langle e_{N-1}, v\rangle\, e_{N-1}$, for all $v \in \mathcal{H}_N$. In term of the common $bra-ket$ symbolism of quantum mechanics, we simply write $P_{N-1} = |e_{N-1}\rangle\langle e_{N-1}|$ to indicate this operator. The above rules are nothing else that the truncated version of the CCR (canonical commutation rules) valid the standard bosonic ladder operators in quantum mechanics. We refer to [33] for more details and further properties of the truncated bosonic operators and to [34] for the extension of the commutation rules in pseudo-Hermitian quantum mechanics.

In what follows, we extend the above definition to construct a family of ladder operators, one for each agent of the macro system $\mathcal{S}$ we want to describe. This requires a tensor product $\otimes$, in order to have a common functional framework. Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_K$ be the K populations of our system $\mathcal{S}$, and we attach to the generic population $\mathcal{P}_j$ the ladder operators $\hat{a}_j$ and $\hat{a}_j^\dagger$, living in the Hilbert space $\mathcal{H}_{N_j}^{(j)}$ of dimension N_j and basis $\mathcal{E}_{N_j}^{(j)} = \{e_l^{(j)},\, l = 0,1,\ldots,N_j-1\}$. These operators satisfy the ladder equations (2.1) for the elements of $\mathcal{E}_{N_j}^{(j)}$. The whole Hilbert space of the system $\mathcal{S}$ is now given by the tensor product

$$\mathcal{H}_\mathcal{S} = \mathcal{H}_{N_1}^{(1)} \otimes \ldots \otimes \mathcal{H}_{N_K}^{(K)},$$

whose dimension is clearly $\dim(\mathcal{H}_\mathcal{S}) = N = \prod_{j=1,\ldots,L} N_j$. The orthonormal basis for $\mathcal{H}_\mathcal{S}$ is then defined as:

$$\mathcal{E} = \left\{ \varphi_{\vec{n}} := e_{n_1}^{(1)} \otimes \ldots \otimes e_{n_K}^{(K)},\, \vec{n} = (n_1, n_2, \ldots, n_K),\, n_\alpha = 0, 1, \ldots, N_\alpha - 1, \alpha = 1, \ldots, K \right\},$$

so that any state Ψ of the system $\mathcal{S}$ can be expressed as a combination of these vectors:

$$\Psi = \sum_{\vec{n}} c_{\vec{n}} \varphi_{\vec{n}}, \tag{2.3}$$

where the complex scalars $c_{\vec{n}}$ do not necessarily satisfy $\sum_{\vec{n}} |c_{\vec{n}}|^2 = 1$.

A generic operator $\hat{X}_j$ on $\mathcal{H}_{N_j}^{(j)}$ is identified on $\mathcal{H}_\mathcal{S}$ with the operator $\hat{X}_j^\mathcal{S}$ defined by the following tensor product:

$$\hat{X}_j^\mathcal{S} = \mathbb{1}_{N_1} \otimes \ldots \otimes \hat{X}_j \otimes \ldots \otimes \mathbb{1}_{N_K}.$$

From now on, when no confusion arises, we will just write $\hat{X}_j$ instead of $\hat{X}_j^\mathcal{S}$ and the action is obviously intended on the whole $\mathcal{H}_\mathcal{S}$ if not differently specified. Easy checking shows that the global action of several operators in the various Hilbert spaces is defined as

$$\left(\hat{X}_1 \otimes \hat{X}_2 \otimes \ldots \otimes \hat{X}_K \right) (v_1 \otimes v_2 \otimes \ldots \otimes v_K) = (\hat{X}_1 v_1) \otimes (\hat{X}_2 v_2) \otimes \ldots \otimes (\hat{X}_K v_K),$$

for all the states $v_j \in \mathcal{H}_{N_j}^{(j)}$, $j = 1, \ldots, K$. The above constructions ensures that the commutation rules (2.2) for the ladder operators $\hat{a}_j$ and $\hat{a}_j^\dagger$ are preserved also on $\mathcal{H}_\mathcal{S}$.

II.2   Density and transport operators

In our applications we shall often make use of suitable density operators which allows to retrieve the density of a state Ψ in a particular level. To construct these operators we simply consider that each state $\Psi = \sum_{\vec{m}} c_{\vec{m}} \varphi_{\vec{m}}$ of our system is the combination of the elements of the basis $\varphi_{\vec{m}}$, and for each possible choice $\vec{n}$ we define the following projector operator via the usual action on the state Ψ:

$$P_{\vec{n}} \Psi = \left(|\varphi_{\vec{n}}\rangle\langle\varphi_{\vec{n}}| \right) \Psi = \langle \varphi_{\vec{n}}, \Psi \rangle \, \varphi_{\vec{n}} = c_{\vec{n}} \varphi_{\vec{n}}. \tag{2.4}$$

Following the number operator scheme adopted in many contexts, see [35], these operators are used to measure the density or level of a state. For that, see [36], we introduce the following expected values of the system:

$$\langle P_{\vec{n}} \rangle = \left\langle \frac{\Psi}{\|\Psi\|}, P_{\vec{n}} \frac{\Psi}{\|\Psi\|} \right\rangle = \frac{|c_{\vec{n}}|^2}{\sum_{\vec{m}} |c_{\vec{m}}|^2}. \tag{2.5}$$

Of course $0 \leq \langle P_{\vec{n}} \rangle \leq 1$ and $\sum_{\vec{n}} \langle P_{\vec{n}} \rangle = 1$, and $\langle P_{\vec{n}} \rangle$ can be phenomenologically interpreted as a measure of the density of the state Ψ in the level $\vec{n}$: this is the reason why we call $\langle P_{\vec{n}} \rangle$ *density function*. Adopting the probabilistic point of view of quantum mechanics, $\langle P_{\vec{n}} \rangle$ is nothing else that the probability of the state Ψ to be $\varphi_{\vec{n}}$, so that if $\vec{n} = (n_1, n_2, \ldots, n_K)$, $\langle P_{\vec{n}} \rangle$ is the probability that the generic population $\mathcal{P}_j$ has quantum number[3] n_j.

Actually, as better explained in the following sections, we shall consider in our applications a suitable projection $\tilde{\Psi}$ of Ψ on a subspace $\tilde{\mathcal{H}}$ of $\mathcal{H}$ and consider this projection to compute the expected values as in (2.5). In particular, if $\tilde{\mathcal{H}}$ is generated by the basis $\tilde{\mathcal{E}} \subsetneq \mathcal{E}$, and $\chi_{\tilde{\mathcal{E}}}$ is the projection operator s.t.

$$\chi_{\tilde{\mathcal{E}}} \varphi_n = \varphi_n, \quad \varphi_n \in \tilde{\mathcal{E}}, \quad \chi_{\tilde{\mathcal{E}}} \varphi_n = 0, \quad \varphi_n \notin \tilde{\mathcal{E}},$$

then

$$\langle \tilde{P}_{\vec{n}} \rangle = \left\langle \frac{\chi_{\tilde{\mathcal{E}}} \Psi}{\|\chi_{\tilde{\mathcal{E}}} \Psi\|}, P_{\vec{n}} \frac{\chi_{\tilde{\mathcal{E}}} \Psi}{\|\chi_{\tilde{\mathcal{E}}} \Psi\|} \right\rangle. \tag{2.6}$$

We finally define the *transport* operators, which will be used to project some level into another (a transport like effect). In particular these operators act on state in the following way:

$$P_{\vec{m}, \vec{n}} \Psi = \left(|\varphi_{\vec{m}}\rangle\langle\varphi_{\vec{n}}| \right) \Psi = \langle \varphi_{\vec{n}}, \Psi \rangle \, \varphi_{\vec{m}} = c_{\vec{n}} \varphi_{\vec{m}}. \tag{2.7}$$

The meaning of this action is straightforward: it projects on the level $\vec{m}$ only if the density in the level $\vec{n}$ non zero ($c_{\vec{n}} \neq 0$), and destroys the other levels. It is useful to describe some transport from the level $\vec{n}$ to the level $\vec{m}$, for instance in an advection process, but also for the birth and death processes.

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[3]Her we simply use the term quantum number as a synonymous of energy level, excitation level and so on.

II.3 Evolution of the system

To determine the time evolution $\Psi(t)$ of an initial state $\Psi(0) = \sum_{\vec{n}} c_{\vec{n}}(0)\varphi_{\vec{n}}$, and subsequently the densities we are interested in, we derive the time-evolution of the functions $c_{\vec{n}}(t)$ through the Schrödinger equation $i\frac{\partial \Psi(t)}{\partial t} = H(t)\,\Psi(t)$ [4], where $H(t)$ is the Hamiltonian operator which is the combination of the operators representing the mechanisms and interactions between the various populations. We assume that $H(t)$ explicitly depends on time and on the local densities of the various populations, and in general is not self-adjoint. Using the orthogonality conditions of the basis vectors $\varphi_{\vec{n}}$, we obtain the following ordinary differential equations (ODE) system for the coefficients $c_{\vec{n}}(t)$

$$i\frac{\partial c_{\vec{n}}(t)}{\partial t} = \langle \varphi_{\vec{n}}, H\Psi(t)\rangle, \tag{2.8}$$

which, when solved, returns the various densities (2.5). The above system, when it is not possible to provide a formal expression of the solution, is solved numerically with an explicit fourth order Runge-Kutta method with 6 steps and an adaptive time step chosen to satisfy the integration tolerance of 10^{-7} (see Dormand and Prince [37] for details on this method). All the numerical solutions were tested to be tolerance independent (almost no difference with the 10^{-8} case). Computations have been performed with Matlab. Heaviest simulations, see the marine dynamics in Section IV.2, took no more than 40 minutes, and the storage of the various operators in the pre-processing phase required 20 minutes. The peak of RAM used was just less than 3 gigabyte.

As the Hamiltonian is in general non self-adjoint, it follows that unitary of the evolution is not preserved, $\|\Psi(t)\| \neq \|\Psi(0)\|$, explaining the requirement of a normalization in the computation of the densities in (2.5) (see in particular [36, 24] for some applications and further details).

III   The Hamiltonian operator

Let us consider a pedagogical example to understand how the Hamiltonian operator can be constructed, and for which a formal solution in terms of the densities can be deduced and analyzed quite easily.

The system is made with only one population moving on simple domain $\mathcal{R}$ consisting of two cells, and the population can only move from the first to the second cell. We also assume that the population dies in the second cell and that the resources for production are present only in the first cell. As we will see, we can implement either the condition where resources are limited or can be continuously generated as an infinite reservoir.

The whole system is then made by two physical cells, attached to the levels 1 and 2, one level for the death process in cell 2 (the dummy level 3 which contains the dead population), and a level as a reservoir for the production in the cell 1 (the dummy level 4). Hence, following the framework adopted in the Section II.2, the Hibert space $\mathcal{H}$ of the system has dimension 5, with basis $\{\varphi_j, \quad j = 0, \ldots, 4\}$. It is important to note that, actually, the ground state φ_0 is used here only to build the other elements of the basis through the ladder equations (2.1), and corresponds to the configuration in which everything is empty, that is there is no population or resources present.

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[4]We put $\hbar = 1$.

The construction of the Hamiltonian operator is straightforward and follows the rules adopted in several other applications, [35]. The advection from cell 1 to 2 is described by the (transport) operator $|\varphi_2\rangle\langle\varphi_1|$ which destroys the level 1 and creates the level 2; the production in level 1 is described by the operator $|\varphi_1\rangle\langle\varphi_4|$, which destroys part of the resources in the level 4 and creates population in the level 1; finally the death process in level 2 is described by the operator $|\varphi_3\rangle\langle\varphi_2|$ which destroys the population in the level 2 and adds dead population in the level 3. A schematic representation of the dynamics and the related operators is shown Fig. 1.
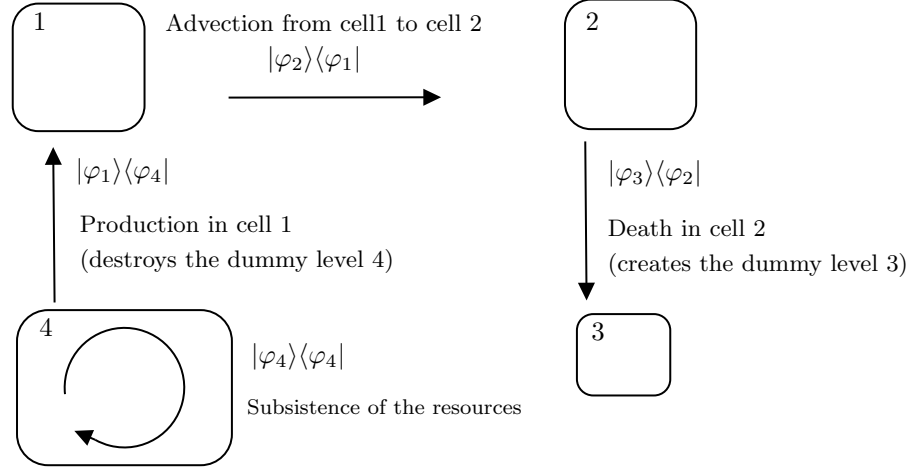
Figure 1: Schematic representation of the 1D advection from cell 1 to cell 2. Also the death in cell 2 and the production in the cell 1 are considered. The main operator governing the dynamics are also shown:$|\varphi_2\rangle\langle\varphi_1|$ for the advection, $|\varphi_3\rangle\langle\varphi_2|$ for the death in cell 2, $|\varphi_1\rangle\langle\varphi_4|$ for the production in cell 1, $|\varphi_4\rangle\langle\varphi_4|$ for subsistence of the resources.

It follows that the Hamiltonian describing the dynamics can be written as

$$H = \lambda|\varphi_2\rangle\langle\varphi_1| + \alpha|\varphi_3\rangle\langle\varphi_2| + \beta|\varphi_1\rangle\langle\varphi_4| + \omega H_p, \tag{3.1}$$

being the parameters $\lambda, \alpha, \beta \in \mathbb{R}^+$ and $\omega \in \mathbb{C}$. The term H_p is an operator to be included if one requires a continuous sustenance over time for the resources in the reservoir. Following the general rules adopted in many other applications, [35], this operator has the form $H_p = |\varphi_4\rangle\langle\varphi_4|$, which works like a standard free dynamics operator in quantum mechanics and it is responsible for inertial or gain processes if ω is real or pure imaginary, respectively. As we will see, to require a continuous sustenance we must take ω pure imaginary, a common way of describing a gain process in quantum system, [38, 39, 40].

Remark: We want to underline here a very key point concerning the meaning of the various parameters used. They should not be considered as the rates of change per unit time of the densities due to the related mechanisms of advection, production, death, subsistence respectively. The reason is that they are parameters of the Hamiltonian and, according to (2.8), they can be more properly connected with the rates of change of the coefficients of the state $\Psi(t)$ of the system.

Looking at the dynamics of the system through (2.8), we consider an initial state representing a

distribution initially concentrated in the cell 1, and the presence of resources: $\Psi(0) = c_1\varphi_1 + c_4\varphi_4$ with $c_1, c_4 \in \mathbb{C}$. As the Hamiltonian is time independent, the evolving state is easily achieved, $\Psi(t) = \exp(-iHt)\Psi(0)$, from which $c_n(t) = \langle\varphi_n, \Psi(t)\rangle$, $n = 0, 1, 2, 3, 4$ (actually $c_0(t) = 0\,\forall t$ because no terms in H works by changing it). Since we are only interested in the net densities of the population, we compute them through (2.6), adopting the Hilbert space $\tilde{\mathcal{H}}$ generated by the elements of the basis $\tilde{\mathcal{E}} = \{\tilde{\varphi}_1 = \varphi_1, \tilde{\varphi}_2 = \varphi_2, \tilde{\varphi}_3 = \varphi_3\}$: the reason is that we are only interested in the densities of the population, including of alive and dead, so that we project into the subspace that does not include the reservoir (in this case related to the dummy level 4, and hence to the element φ_4).

We consider initially $\omega > 0$, meaning that the term ωH_p behaves like an inertial term: the higher the value, the higher the tendency of the density of resources to stay closer to its initial value over a long period (see [35] for details). In this case we want to describe the situation in which there is only a finite amount of resources and we should expect, as time passes, the gradual decrease of the resources and, as consequences, of the alive population in the 2 cells. Although the simplicity of the system, the formal solution can be formally deduced but is quite cumbersome to write: in general the various densities $\langle\tilde{P}_k(t)\rangle$, $k = 1, 2, 3$ have the form

$$\sum_{k=0}^{k=4} f_k(\lambda, \alpha, \beta, \omega, c_1, c_4, \exp(i\omega t), \exp(-i\omega t))t^k, \tag{3.2}$$

being the f_k functions depending on the initial parameters and on suitable combination of the exponentials $\exp(\pm i\omega t)$. For simplicity, we report only the high order asymptotic terms in time of the densities in the various levels:

$$\langle\tilde{P}_1(t)\rangle \approx \frac{\sin^2(t\omega)(\beta c_4 - c_1\omega)^2 + (\cos(t\omega)(c_1\omega - \beta c_4) + \beta c_4)^2}{t^4\alpha^2\lambda^2\left(\frac{1}{4}c_1^2\omega^2 - \frac{1}{2}\beta c_1 c_3\omega + \frac{1}{4}\beta^2 c_4\right)}, \tag{3.3}$$

$$\langle\tilde{P}_2(t)\rangle \approx \frac{(c_1^2\omega^2 - 2\beta c_1 c_4\omega + \beta^2 c_4)}{t^2\alpha^2\left(\frac{1}{4}c_1^3\omega^2 - \frac{1}{2}\beta c_1 c_4\omega + \frac{1}{4}\beta^2 c_4\right)}, \tag{3.4}$$

$$\langle\tilde{P}_3(t)\rangle \approx 1. \tag{3.5}$$

We can deduce that, for $t \to \infty$, the densities in the cells 1 and 2 decay to zero but with a different ratio, while the density of dead population reaches asymptotically 1. This is somewhat expected because no generation of resources was considered in the Hamiltonian (3.1), and the term ωH_p determines only how fast the densities in the cells decay to zero. In order to have a continuous sustenance it is enough to introduce a gain effect which is achieved by requiring $\omega = i\omega_0$, $\omega_0 > 0$. With this choice the densities (again we show the high order asymptotic terms) are the following:

$$\langle\tilde{P}_1(t)\rangle \approx \frac{\omega_0^4}{\alpha^2\lambda^2 + \lambda^2\omega_0^2 + \omega_0^4}, \ \langle\tilde{P}_2(t)\rangle \approx \frac{\lambda^2\omega_0^2}{\alpha^2\lambda^2 + \lambda^2\omega_0^2 + \omega_0^4}, \ \langle\tilde{P}_3(t)\rangle \approx \frac{\alpha^2\lambda^2}{\alpha^2\lambda^2 + \lambda^2\omega_0^2 + \omega_0^4}. \tag{3.6}$$

As one can see, the densities reach an asymptotic equilibrium which is dependent only on the parameters of the Hamiltonian. In particular, a strong effect of resources production, corresponding to a value of ω_0 larger than α and λ ($\omega_0 \gg \alpha, \lambda$), ensures a high density in the cell 1; a strong advection, $\lambda \gg \alpha, \omega_0$, induces greater density in the cell 2, being the densities of the dead population dependent on the balance

of the parameters α and ω; finally, a strong death process given by the condition $\alpha \gg \lambda, \omega_0$, leads to a large amount of dead population. As we can see, the only way to maintain a sustainable environment is to take ω pure imaginary.

III.1   Prey-Predator dynamics

It is clear that with the operatorial approach and the use of the density and transport operators we can easily implement many other mechanisms. In this application we fix the first population as the predators and the second as prey. The element of the basis are $\{\varphi_{0,0}, \varphi_{1,0}, \varphi_{0,1}, \varphi_{1,1}, \varphi_{1,2}\}$, where $\varphi_{0,0}$ is the vacuum, $\varphi_{1,0}$ is the state related to predator and no prey, $\varphi_{0,1}$ only prey, $\varphi_{1,1}$ both prey and predator, and $\varphi_{1,2}$ is the state attached with the dead prey. The prey-predator mechanism is hence implemented with an operator $\mathcal{O}_{PP} = \delta|\varphi_{1,2}\rangle\langle\varphi_{1,1}|$: it acts on the generic state in the following way

$$\mathcal{O}_{PP}\Psi = \delta c_{1,1}\varphi_{1,2}, \tag{3.7}$$

meaning that in works by creating a state in which the predator are alive, and prey are dead. It is very easy in this case to retrieve the formal solution of the Schrödinger equation with the Hamiltonian $H = \mathcal{O}_{PP}$, and consequently the densities in each level. In particular, the densities of prey and predator are given respectively by

$$\langle P_{0,1} + P_{1,1}\rangle = \left(|c_{0,1}(0)|^2 + |c_{1,1}(0)|^2\right)/D \tag{3.8}$$

$$\langle P_{1,0} + P_{1,1} + P_{1,2}\rangle = \left(|c_{1,0}(0)|^2 + |c_{1,1}(0)|^2 + |(c_{1,0}(0) + c_{1,1}(0))\delta t|^2\right)/D, \tag{3.9}$$

$$D = |c_{0,1}(0)|^2 + |c_{1,0}(0)|^2 + |c_{1,1}(0)|^2 + |(c_{1,0}(0) + c_{1,1}(0))\delta t|^2, \tag{3.10}$$

from which we can deduce that, asymptotically,

$$\langle P_{0,1} + P_{1,1}\rangle \to 0, \langle P_{1,0} + P_{1,1} + P_{1,2}\rangle \to 1, \tag{3.11}$$

that is, as expected, all the prey are dead. Of course with the same procedure we can enrich the dynamics with death or birth processes for both populations by adding the operators introduced above in the previous example.

IV   The 2D model

In view of the explained the 1D case, we now explain how to apply the operatorial approach to a system of populations living in a 2D region $\mathcal{R}$. Let us assume that $\mathcal{R}$ divided in N cells, in which the various populations $\mathcal{P}_j$ are distributed. $\mathcal{R}$ can have in principle any geometry, and we attach to all the cells of $\mathcal{R}$ an index ranging from 1 to N. As done in Section II.1 we relate to the generic population $\mathcal{P}_j$ the ladder operator $\hat{a}_j$, and each cell, with its index, can be seen as a bosonic level of $\mathcal{P}_j$. To add some birth/death process, we can also add various dummy levels in the same way done for the 1D case. Of course, the possibility that the various populations may occupy or not a specific cell/level depends on the dynamics of the system and how the Hamiltonian operator is defined.

To better understand how the link between the cells of $\mathcal{R}$ and the bosonic operators works, let us consider a very simple configuration, in which $\mathcal{R}$ is a 3×3 lattice, and only one population $\mathcal{P}_1$ is present in the system (see Fig. 2). Following the procedure described in Section II.1, the Hilbert space of the system has dimension $\dim(\mathcal{H}) = 28$, taking into account the 9 physical cells of $\mathcal{R}$ and related dummy levels for the possible death processes and resources (the maximum amount is $9+9$ cells), and the ground level 0. Hence, a generic state $\Psi = \sum_{k=0}^{k=27} c_k \varphi_k$ of the system represents a specific configuration made of contributions given by the distribution of the population in $\mathcal{R}$, by the presence of dead population, and by the presence of resources for the production. In particular the pure states $\{\varphi_j, j = 1, \ldots, 9\}$ of the basis are related to specific configuration in which the population is concentrated only in the $j-$th cell, whereas a combination of them gives a distribution on the $\mathcal{R}$ with densities depending on the coefficients c_j according to (2.5). The elements $\{\varphi_j, j = 10, \ldots, 18\}$ are instead related to presence of dead population in the cells of $\mathcal{R}$ due to the death process in the system, and the elements $\{\varphi_j, j = 19, \ldots, 27\}$ are related to the presence of resources in the cells. Of course, if no death/birth process is present in a specific cell, we remove the attached dummy cells to avoid a not useful increment in the dimension of the Hilbert space.

In order to retrieve a formal expression of density that can be easily interpreted by the reader, we assume that the population moves leftward and a birth/death process is also present (see 2). Looking for instance at the dynamics in only the central cell (level number 5), the population can move from 5 to the cells $1, 4, 7$, and from $9, 6, 3$ to 5. In order to simulate the birth/death process we add two dummy levels engaged for the death (level 11) and birth (level 10): the death can be seen as a transport from 5 to 11, while birth as a transport from 10 to 5. In view of the processes above described, the Hamiltonian operator describing this dynamics contains the following terms (and the other similar for the other cells):

$$
\begin{aligned}
H &= H_o + H_i + H_b + H_d, \\
H_o &= \lambda_1 |\varphi_1\rangle\langle\varphi_5| + \lambda_4 |\varphi_4\rangle\langle\varphi_5| + \lambda_7 |\varphi_7\rangle\langle\varphi_5|, \\
H_i &= \lambda_9 |\varphi_5\rangle\langle\varphi_9| + \lambda_6 |\varphi_5\rangle\langle\varphi_6| + \lambda_3 |\varphi_5\rangle\langle\varphi_3|, \\
H_b &= \lambda_{10} |\varphi_5\rangle\langle\varphi_{10}|, \\
H_d &= \lambda_{11} |\varphi_{11}\rangle\langle\varphi_5|,
\end{aligned}
$$

where the various coefficients λ_i tune the strength of the related operatorial effect. The meaning of the various operators $|\varphi_i\rangle\langle\varphi_j|$ is straightforward and follows the construction adopted in the 1D example in Section III: it destroys the $j-$th level and creates the $i-$th level, modifying the densities in the two levels. In fact, given the generic state $\Psi = \sum_{k=0}^{k=11} c_k \varphi_k$, we have

$$
H\Psi = \lambda_1 c_5 \varphi_1 + \lambda_4 c_5 \varphi_4 + \lambda_7 c_5 \varphi_7 + \lambda_{11} c_5 \varphi_{11} + (\lambda_9 c_9 + \lambda_6 c_6 + \lambda_3 c_3 + \lambda_{10} c_{10})\, \varphi_5,
$$

and the various densities of the state $H\Psi$, computed with (2.5), are

$$
\langle P_1 \rangle = \frac{|\lambda_1 c_5|^2}{D}, \langle P_4 \rangle = \frac{|\lambda_4 c_5|^2}{D}, \langle P_7 \rangle = \frac{|\lambda_7 c_5|^2}{D}, \langle P_{11} \rangle = \frac{|\lambda_{11} c_5|^2}{D},
$$

$$
\langle P_5 \rangle = \frac{|\lambda_8 c_8 + \lambda_6 c_6 + \lambda_4 c_4 + \lambda_{10} c_{10}|^2}{D},
$$

$$
D = |\lambda_1 c_5|^2 + |\lambda_4 c_5|^2 + |\lambda_7 c_5|^2 2 + |\lambda_{11} c_5|^2 + |\lambda_8 c_8 + \lambda_6 c_6 + \lambda_4 c_4 + \lambda_{10} c_{10}|^2.
$$

It is evident that the action of H on Ψ induces a growth of density in the cells where the population can move (we recall that $H\Psi$ appears in the r.h.s of (2.8)), given that the various coefficients c_i should be different from zero.
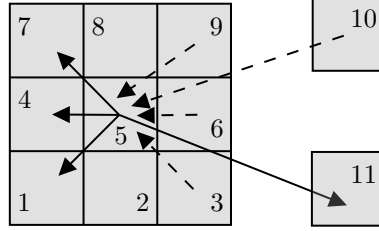

Figure 2: Schematic representation of the dynamics in the central cell for a 3×3 region.

IV.1 First application: prey-predator dynamics

We now present some applications for the dynamics in medium-large domains.

The first application is actually a toy-model for the dynamics of two populations (predator and prey) inside a square lattice $\mathcal{R}$ made by $L \times L$ cells, where not all the cells are accessible. The main purpose of this application is to implement all the mechanisms described in previous examples in a non trivial domain. We therefore make some basic assumptions on the population dynamics in order to derive an evolution that can be easily interpreted following these assumptions. We suppose that the populations can move only *rightward and upward*, the prey-predator dynamics occurs in every cell, prey can reproduce, and predators die in absence of prey. Following the framework of Section II.1 and the operator construction adopted in Sections III-IV, the elements of the basis are the states $\mathcal{E} = \{\varphi_{n_1,n_2}, n_1 = 0, 2L^2, n_2 = 0, 3L^2\}$. In particular the first index n_1 is related to the L^2 levels attached to each physical cell of $\mathcal{R}$ for the predators, and to the L^2 dummy levels for the dead predators; similarly for the second population (prey), but with the additional L^2 levels for the prey resources (predators have only the prey as resource). Of course we also have also the ground $\varphi_{0,0}$ which is related to an empty system. The Hamiltonian is easily written:

$$H = H_1 + H_2 + H_3 + H_4, \tag{4.1}$$

$$H_1 = \sum_{j,k=1,\ldots,L^2,\, m=0,\ldots,2L^2} \lambda^{(1)}_{j,k}|\varphi_{j,m}\rangle\langle\varphi_{k,m}| + \lambda^{(2)}_{j,k}|\varphi_{m,j}\rangle\langle\varphi_{m,k}|, \tag{4.2}$$

$$H_2 = \sum_{j=1,\ldots,L^2,\, m=0,\ldots,2L^2} \alpha_j|\varphi_{j+L^2,m}\rangle\langle\varphi_{j,m}|, \tag{4.3}$$

$$H_3 = \sum_{j=1,\ldots,L^2,\, m=0,\ldots,2L^2} \omega_j|\varphi_{m,j}\rangle\langle\varphi_{m,j+2L^2}| \tag{4.4}$$

$$H_4 = \sum_{j=1}^{L^2} \delta_j|\varphi_{j,j+L^2,}\rangle\langle\varphi_{j,j}|. \tag{4.5}$$

The terms in H_1 are responsible for the advection of the two populations, $\lambda^{(1)}_{j,k}, \lambda^{(2)}_{j,k}$ are non negative coefficients different form zero only if the related population can move from cell k to cell j. The terms

in H_2, with α_j non negative, are related to the death of the predators, whereas those in H_3, with $\omega_j > 0$ or pure imaginary, are the operators for the sustenance of the prey. Finally H_4 with δ_j non negative, is responsible for the prey-predator dynamics in each level j attached to physical cells of $\mathcal{R}$: notice that the index in the summation starts from 1, simply because the prey predator dynamics requires that both populations are alive in the physical cells (the ground therefore is not used). It is clear that combining the above operators differently or using density-dependent parameters as done in [7] for a pure migration model, one could derive an Hamiltonian that can simulate different scenarios. For instances, we can include some chemotactic responses by assuming that the predators advection can be directed only where densities of prey are above a specific threshold and defining the various $\lambda_{j,k}^1$ density dependent and time varying. Similarly, diffusion from one cell can be varied based on the density in the cell or in its neighbourhood. In the same way also the other parameters can be defined all densities dependent, adding in the model a great variety of prey-predator dynamics.

The densities $P_{n_1}^{(1)}(t), P_{n_2}^{(2)}(t)$ of the populations in all the physical cells $n_1, n_2 = 1, \ldots, L^2$, are easily computed by means of (2.6) as

$$P_{n_1}^{(1)}(t) = \sum_{n_2 = 0, \ldots 2L^2} \langle P_{n_1, n_2}(t) \rangle, \; n_1 = 1, \ldots, L^2, \tag{4.6}$$

$$P_{n_2}^{(2)}(t) = \sum_{n_1 = 0, \ldots 2L^2} \langle P_{n_1, n_2}(t) \rangle, \; n_2 = 1, \ldots, L^2, \tag{4.7}$$

where the various contributions $\langle P_{n_1, n_2} \rangle$ are obtained considering the subspace $\tilde{\mathcal{E}} = \{\varphi_{n_1, n_2}, n_1 = 0, 2L^2, n_2 = 0, 2L^2\}$ related to the populations levels (excluding the resources). We show in Figs.3-4 the densities in each cell for the initial time (assigned initial condition) and subsequent times for the following choice of the parameters: $L = 20$, $\lambda_{j,k}^{(1)} = \lambda_{j,k}^{(2)} = 0.2$ if the populations can move from cell k to j, and 0 otherwise (we recall that the movements are rightward and upward); $\alpha_j = 0.05$, $\omega_j = 0.05i$, $\delta_j = 0.3$ for all j, and the cells corresponding to the lattice $[10 : 14] \times [10 : 14]$ are not accessible. The outcome of the simulation follows straightforwardly from the assumptions made; (i) the prey and the predators move towards the top right corner, with a crowding effect for the predators (ii) due to the prey-predator dynamics the prey can only survive in the cells where the predators have low density, this explains why no prey are found in the upper right cell (iii) the sustainment of the prey allows for their birth in the cells where no predators are found. As done in the 1D example, taking the ω_j pure imaginary means that we are supposing continuous sustainment for the prey in each cell. In Fig.5(a) the time evolutions of the global densities

$$P^{(1)}(t) = \sum_{n_1} P_{n_1}^{(1)}(t), \; P^{(2)}(t) = \sum_{n_2} P_{n_2}^{(2)}(t) \tag{4.8}$$

for both populations are shown for the same choice of parameters, while in Fig.5(b) the global densities of the predators are shown for different values of the parameter δ: as expected if no prey-predator dynamics exists ($\delta = 0$) all the predators die, whereas for increasing value of δ their global density is on average higher.
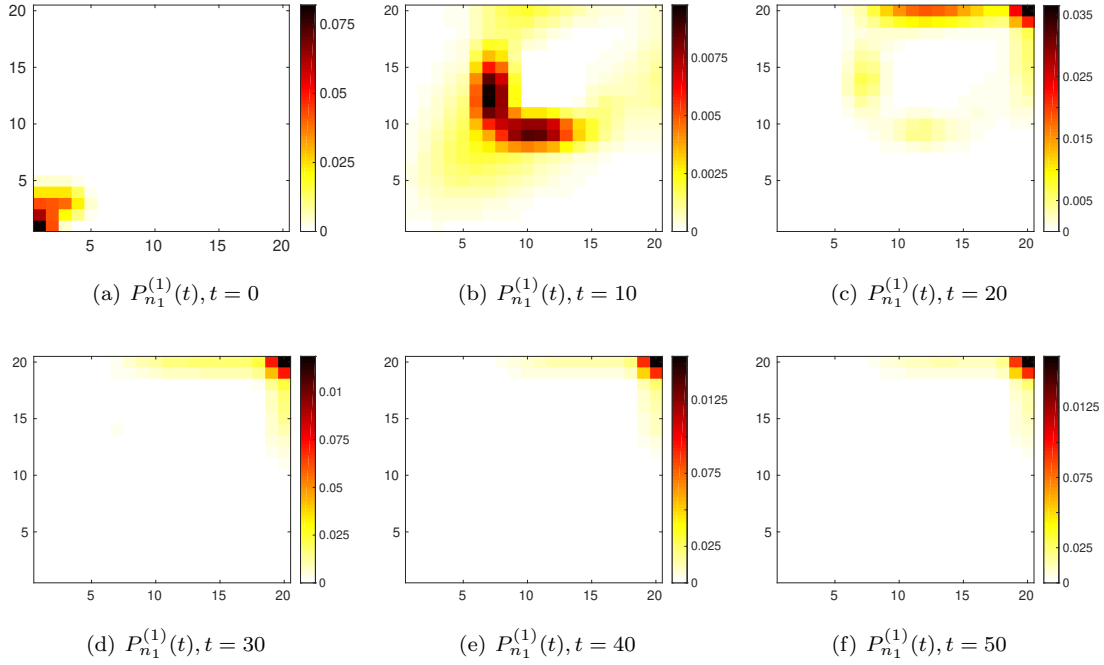
Figure 3: The densities $P_{n_1}^{(1)}(t)$, $n_1 = 1, \ldots, L^2$ at various times for the first population (predator).

IV.2   Second application: a large scale marine dynamics

The aim of this second application is to apply our framework to the dynamics of a population in a very large domain, usually not computationally solvable with the fermionic operators. The system consists of a population of drifters passively transported in the Mediterranean basin[5] through the marine current. The whole basin is dived in squared cells of amplitude $1/8° \times 1/8° (\approx 14km \times 14km)$, and to each cell j we associate a steady marine current field $\mathbf{u}(j) = (u(j), v(j))$, corresponding to the real monthly mean field[6] on the surface in August 2015. This velocity field is obtained from the Mediterranean Sea Forecasting System dataset (freely available at https://www.copernicus.eu/en/myocean, see also [41] for details), and it is shown in Fig.6. The total number of cells, excluding the inland regions to avoid unwanted increment of the Hilbert space dimension, is $N = 27247$, and they are counted following the latitude order starting from the south-west corner, so that the whole Hilbert space has dimension 27248 (we include as usually the ground state).

The Hamiltonian we consider is made by two main terms, and takes into account for two passive

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[5]North-east degree for the longitudinal-latitudinal domain is $[-6, 36.25] \times [30.185, 45.9375]$.
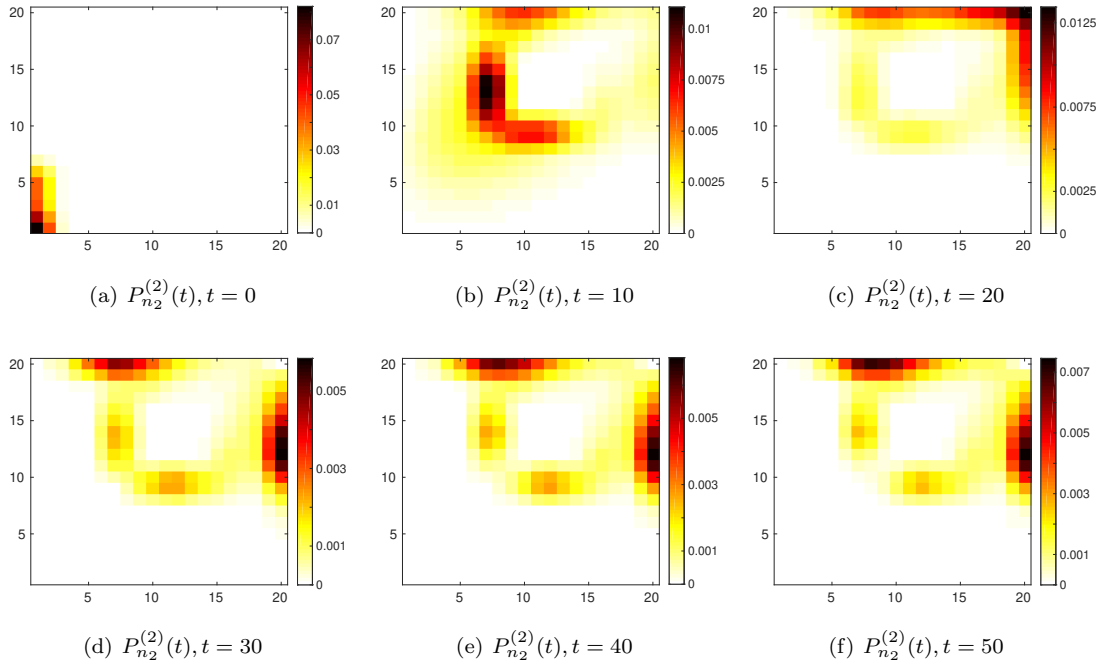
[6]The mean velocity field is measured in meter per second.

(a) $P_{n_2}^{(2)}(t), t = 0$

(b) $P_{n_2}^{(2)}(t), t = 10$

(c) $P_{n_2}^{(2)}(t), t = 20$

(d) $P_{n_2}^{(2)}(t), t = 30$

(e) $P_{n_2}^{(2)}(t), t = 40$

(f) $P_{n_2}^{(2)}(t), t = 50$

Figure 4: The densities $P_{n_2}^{(2)}(t)$, $n_2 = 1, \ldots, L^2$ at various times for the second population (prey).
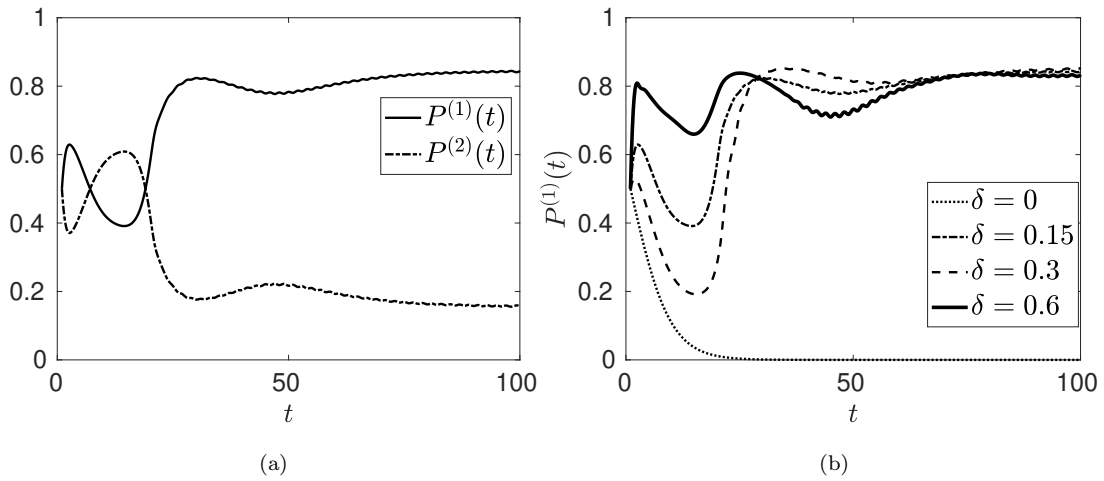

(a)

(b)

Figure 5: (a) The time evolutions of the global densities $P^{(1)}(t)$ (predator) and $P^{(2)}(t)$ (prey): parameters as in Figs.3-4. (b) The time evolutions of the global density of the predators for different values of the parameter δ.

14

transport dynamics:

$$H = H_{lc} + H_{sc}, \tag{4.9}$$

$$H_{lc} = \sum_{j,k=1}^{N} \lambda_{k,j}^{lc} |\varphi_k\rangle\langle\varphi_j|, \; H_{sc} = \sum_{j,k=1}^{N} \lambda_{k,j}^{sc} |\varphi_k\rangle\langle\varphi_j|. \tag{4.10}$$

The two terms H_{lc} and H_{sc} both describe a transport process, although they have different meaning and act on a different spatio-temporal scale. The term H_{lc} is responsible for the large scale circulation induced by the velocity field $\mathbf{u}$. The parameters $\lambda_{k,j}^{lc}$ are zero if the cell j and k are not contiguous; if they are contiguous we check for the angle formed by the cell k and the direction of the velocity field $\mathbf{u}(j)$. In particular (see Fig.7) let be k_1, k_2 the two contiguous cells pointing to the direction of $\mathbf{u}(j)$. Given $\alpha(k_1), \alpha(k_2)$ the two angles formed between the direction of the cells k_1, k_2 and the direction of $\mathbf{u}(j)$, we define

$$\lambda_{k_1,j}^{lc} = w_{k_1}|\mathbf{u}(j)|, \; \lambda_{k_2,j}^{lc} = w_{k_2}|\mathbf{u}(j)|, \tag{4.11}$$

where $w_{k_1} = \alpha(k_2)/\alpha_{\max}$, $w_{k_2} = \alpha(k_1)/\alpha_{\max}$ are the weights of the cells, given that the maximum values admissible of the two angles is $\alpha_{\max} = \pi/4$: in this way we distribute the transport in the two cells, instead of only one. Of course there are some special case to consider. For instance when one of the two angle is 0: in that case the velocity field points exactly towards one cell (say k_1), and it results $\alpha(k_1) = 0$, $\alpha(k_2) = \pi/4$, $w_{k_1} = 1$, $w_{k_2} = 0$, and consequently we put $\lambda_{k_1,j}^{lc} = |\mathbf{u}(j)|$, $\lambda_{k_2,j}^{lc} = 0$. The other special case is when $\mathbf{u}(j)$ points toward one or two inland cells (it actually happens in very few cases): in this case we search for the two (or one) non inland cells having direction closest to that of $\mathbf{u}(j)$, and we define the parameters as before but with the weights defined as the angle formed between the two cells and the direction of $\mathbf{u}(j)$.

Regarding the Hamiltonian H_{sc}, it is simply added to partially restore the typical small scale variability of the marine current circulation which is unavoidably removed from the averaging nature of velocity field $\mathbf{u}$. In particular we consider the small scale velocity field $\mathbf{u}_{sc} = (u_{sc}, v_{sc})$, similar to that introduced in [42], and whose components are obtained from the stream function, i.e., $\partial_x \Psi_{sc} = v_{sc}$, $\partial_y \Psi_{sc} = -u_{sc}$,

$$\Psi_{sc} = \frac{A}{k} \sin[k(x - \varepsilon\sin(\omega t)] \sin[k(y - \varepsilon\sin(\omega t)]$$

where $A = 0.1\mathrm{ms}^{-1}$, $\varepsilon = 0.1I_0$, $\omega = 2\pi\mathrm{A}_0/\mathrm{I}_0$ and $k = 2\pi/\mathrm{I}_0$, $I_0 = 20\mathrm{km}$, and x, y are the longitude and latitude coordinates of the center of the generic cell. This stream function defines an incompressible flow of non-steady 2D lattice of vortices subject to time-periodic oscillations around their mean positions ($\varepsilon \neq 0$), and whose vorticity is

$$\omega_{sc} = 2Ak \sin[k(x - \varepsilon\sin(\omega t))] \sin[k(y - \varepsilon\sin(\omega t))].$$

The physical meaning of the parameters are easily deducible: A is the maximum attainable velocity, k is the spatial wave number associated to the wavelength I_0, ε is the maximum oscillation amplitude, ω is the pulsation, with $T = 2\pi/\omega$ the period. This field permits the transport in all the contiguous cells, and was successfully used in several numerical models concerning the passive transport in the Mediterranean basin with the goal of restoring the small scale variability of the marine circulation, see [43, 44]. The
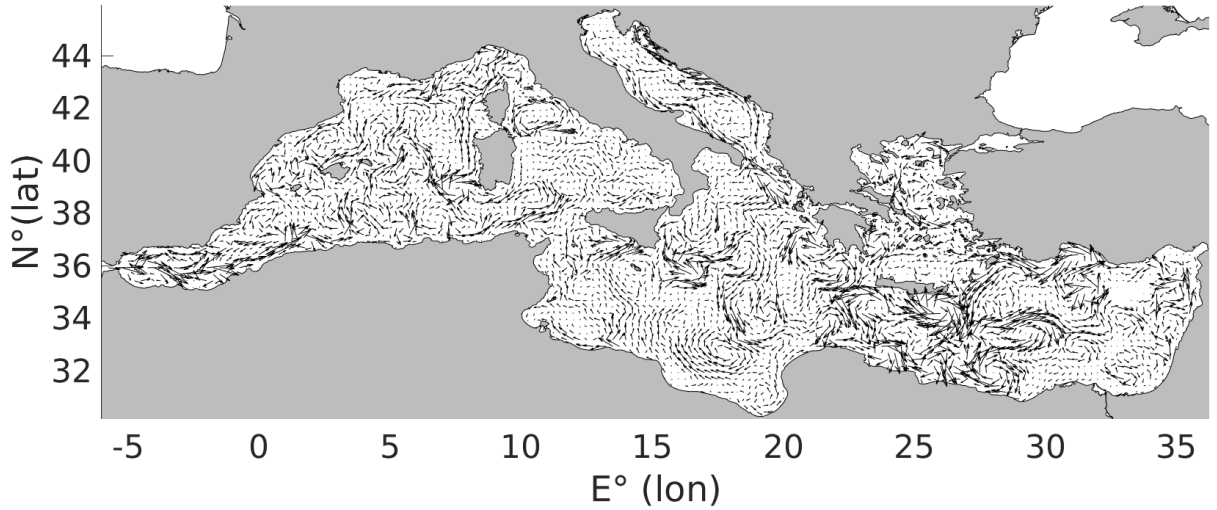
15

Figure 6: The domain of the second application made of 27247 (non-inland) cells. The direction of mean marine current velocity field **u** at the surface in the month August 2015 is superimposed. The maximum intensity value of **u** is $0.822m/s$.

choice of the parameters value are based on the typical hydrological features of the basin, in particular I_0 has been chosen twice the Rossby radius[7] in the Mediterranean basin. We refer to [42] for further details and discussion on the derivation of this vorticity field. Once we have defined the velocity field $\mathbf{u}_{sc}$ the parameters $\lambda_{k,j}^{sc}$ are retrieved in the same way we did for the parameters $\lambda_{k,j}^{lc}$ in (4.11) by considering $\mathbf{u}_{sc}$ in place of **u**.

We now show the outcomes of some numerical simulations starting from three different initial conditions (IC1,IC2,IC3): the IC1 condition covers the large subregion close the strait of Gibraltar, the IC2 the coastal subregion of Algeria and Tunisia, the IC3 the coastal subregion of the south Sicily. The densities for the three experiment are shown in Figs.8-9-10 at different times (full multimedia view IC1,IC2,IC3). The results show that, in all cases, the main transport is dictated by the large scale field **u** with dispersive

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[7]This is the length scale at which rotational effects become as important as buoyancy or gravity wave effects.
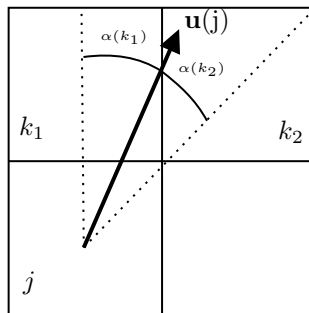
Figure 7: Schematic representation of the two main cells for the transport from the cell j used for the determination coefficients $\lambda_{k,j}^{lc}$ of H_{lc} in (4.10).

effects due to the small scale field $\mathbf{u}_{sc}$. In the experiment IC1 densities changes over time following the African coastal areas where the velocity field is intense and rightward-directed. In the experiment IC2 densities are mainly concentrated in the north side of the strait of Sicily, while in IC3 experiment the recirculation features of $\mathbf{u}$ in the south Sicily coastal area are responsible for the almost permanent patch visible in all figures in that area. We notice that the initial condition area for the experiment IC3 covers the same area used for lagrangian simulations in [44] and partially in [42]. Although several limitations are imposed in our model, the outcomes of the simulations are capable to detect the same high-density retention area found in the central-south eastern side of the Sicily, due to the recirculation imposed by the semi-permanent cyclonic Adventure Bank Vortex, [44]. We stress however that in [44, 42] a daily time-varying large scale velocity field was adopted. Of course, this refinement can also be used in our operatorial method by defining the coefficients in (4.11) which vary over time daily, although this would be only justified once the time scale has been correctly determined, a point that still needs to be analysed and deserves a dedicated investigation.

We finally consider that these simulations are inevitably cell-size dependent. In fact, we have used the maximum resolution attainable from the dataset used to define the various transport parameters, and since the transport from on cell to another follows the direction of the velocity field, therefore cell dependent, the use of a coarser grid could remove some dynamics captured with a finer resolution. This problem, of course, is not directly related to the operatorial approach, and would also be present in other classical models based on the Eulerian or Lagrangian approaches. The cell-size dependence is not present in the prey-predator dynamics model we have considered, where the advection follows a specific direction (up-rightwards) irrespective of the cell-size.

V   Discussions

In this paper we have considered the macroscopic dynamics of populations in a closed region, solved with the operatorial method based on truncated bosonic operators. In the very first attempt to solve population dynamics with an operatorial approach,[5], the fermionic operators were used and their main drawback was the impossibility to solve the problem with non reversible effects in a very large domain,
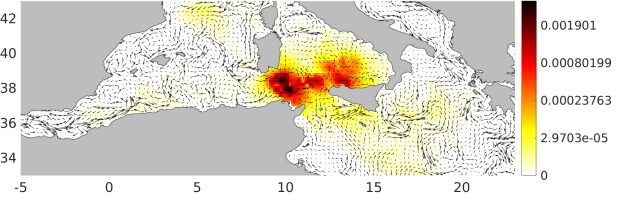
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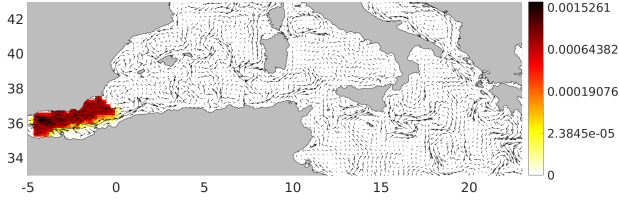
(a) Experiment IC1, $t = 2$
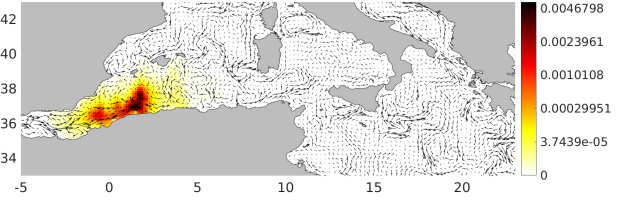
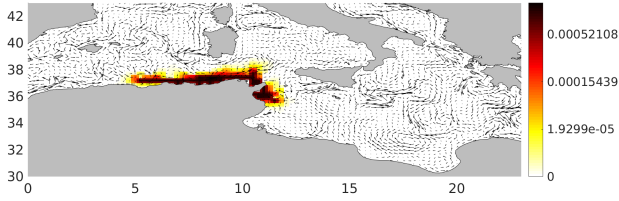(b) Experiment IC1, $t = 50$
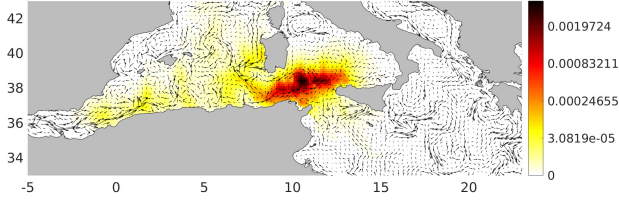
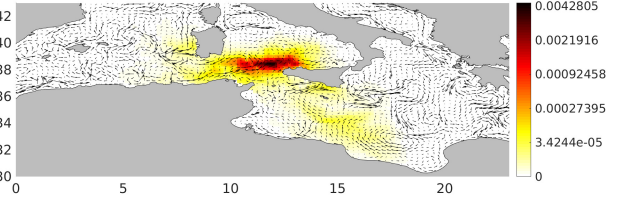(c) Experiment IC1, $t = 150$

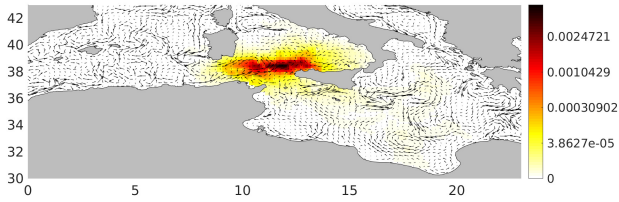(d) Experiment IC1, $t = 370$

Figure 8: The densities in the various cells computed with the initial condition IC1. Colormaps are nonlinear to highlight the low density cells. Full multimedia view IC1.
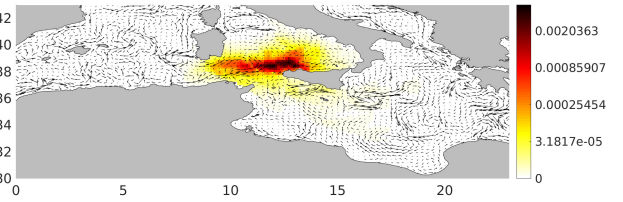

(a) Experiment IC2, $t = 2$

(b) Experiment IC2, $t = 50$

(c) Experiment IC2, $t = 150$

(d) Experiment IC2, $t = 370$

Figure 9: The densities in the various cells computed with the initial condition IC2. Colormaps are nonlinear to highlight the low density cells. Full multimedia view IC2.

due to computational request needed to deal with the large dimension of the Hilbert space and the related dimension of the operators. Here we have seen that very large domain dynamics can be handled with the operatorial approach based on truncated bosons. In particular, we can successfully use bosons levels

(a) Experiment IC3, $t = 2$

(b) Experiment IC3, $t = 50$

(c) Experiment IC3, $t = 150$
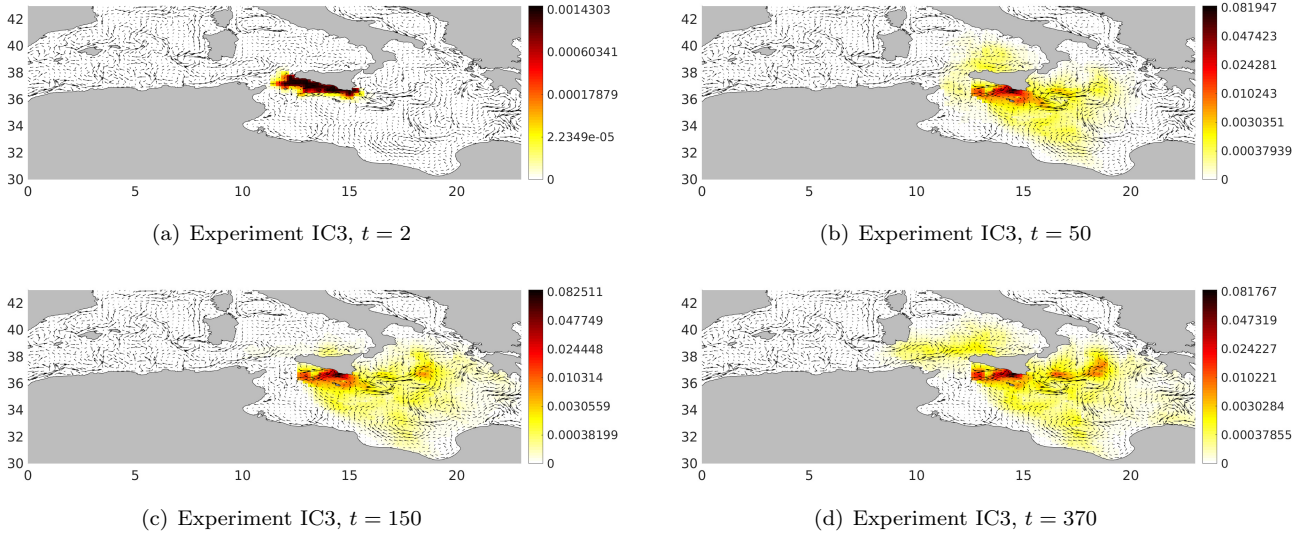
(d) Experiment IC3, $t = 370$

Figure 10: The densities in the various cells computed with the initial condition IC3. Colormaps are nonlinear to highlight the low density cells. Full multimedia view IC3.

by attaching to them the various compartments (dead or alive population, and resources) in a suitable way. The whole dynamics has been deduced by solving the Schrödinger equations with a non self-adjoint Hamiltonian. The densities of the populations in the various sites of the domain were retrieved from the expected values of the various density level operators on the normalized state, a standard approach when dealing with non self-adjoint Hamiltonian, [25]. We have seen that it is possible to easily add different kinds of mechanisms in the Hamiltonian of the system with the aid of the transport and density operators: advection, birth/death process, prey-predator dynamics, resources maintenance.

The main advantage of the operatorial approach is that all the interactions and mechanisms can be easily described with operators having the same nature. In fact, all operators in the Hamiltonian have been defined in the form of a transport operator $c|\varphi\rangle\langle\psi|$, being c a suitable parameter, and φ, ψ specific elements of the basis attached to bosonic levels: when $\psi = \varphi$ we have the density operator. This approach has the advantage to be easy to implement, and the construction of the Hamiltonian is intuitive. Moreover, as we have seen for the very basic examples shown in Sections III-IV, it can be very easy to derive a formal expressions of the solution without requiring the numerical simulations. However, we are conscious that much remains to be done, in particular for possible applications to real scenarios. The applications we have proposed were in fact intended to show how the operatorial approach can be successfully adopted to mimic different kind of population dynamics, especially in very large domains. With this in mind the next step is to apply the proposed methodology to real problems as done in [7] for a small domain. One key point is of course to correctly define the values of the various parameters and the time scale of the system. For instances, looking at the marine dynamics, it is clear that the parameters in (4.11) are irrespective of the time step dt chosen to numerically solve the system (2.8): in advection models governed by ODE/PDE derived from conservation laws, the term $\lambda_{k,j}^{lc} dt$ is the distance travelled

19

that, according dt and supposing that the unit of time is the second, could exceed the distance with the contiguous cell. One way to overcome this problem is to define the weights based on the time step or, equivalently, reducing the time step so that the maximum value of $\lambda_{k,j}^{lc} dt$ does not exceed the distance between contiguous cells (this is actually connected with the meaning of the very well known Courant-Friedrichs-Lewy condition for the stability in finite difference schemes, [45]). It remains to be understood what the time scale represents and, consequently, whether the terms $\lambda_{k,j}^{lc} dt$ could indeed considered as a kind of distance. In fact, the $\lambda_{k,j}^{lc} dt$ terms do not have the same interpretation that usually is given in a classical advection model based on conservation laws, because $\lambda_{k,j}^{lc}$ are used as parameters of the Hamiltonian operator and are not physical velocities, but more properly rates of change of the modes due to the interactions of the bosonic levels. The strong non-linearities appearing in the density function make it difficult to determine the time scale and the various parameters, and nonlinear fitting procedures should be implemented and applied in comparison with real density data to determine the time scale. We reserve to investigate these key points in the near future to make our model more reliable.

We conclude by considering the possibility of extending our approach based on truncated bosonic operator with the Markovian approximation of the quantum master equation, the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL), already used to derive macrosystem dynamics with fermions (the classical annihilation and creation operators for a two level system), see [14, 12, 18, 4, 22]: obviously this would require the definition of the correct master equation for the density operators in the hypothesis of non self-adjoint Hamiltonian, and this is part of the future applications of the proposed operatorial approach.

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